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Level Set Estimates for the Discrete Frequency Function

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Abstract We introduce the discrete frequency function as a possible new approach to understanding the discrete Hardy–Littlewood maximal function. Considering that the discrete Hardy–Littlewood maximal function is given at each integer by the supremum of averages over intervals of integer length, we define the discrete frequency function at that integer as the value at which the supremum is attained. After verifying that the function is well-defined, we investigate size and smoothness properties of this function.

Keywords Hardy–Littlewood maximal function · Frequency function · Averaging operators · Integral operators · Optimal intervals

Mathematics Subject Classification Primary 42B25 · Secondary 46E35

1 Introduction

Let $\mathbb{Z}$ be the set of integers, and let $\mathbb{Z}^+$ denote the set of non-negative integers. Let $f \in l^1(\mathbb{Z})$. For real numbers $a \leq b$, let $[a, b]$ denote the set of integers n such that $a \leq n \leq b$. We will call $[a, b]$ an interval in $\mathbb{Z}$. We define the average of f over an interval of radius $r \in \mathbb{Z}^+$ by

$$\mathcal{A}_r f(n) := \frac{1}{2r+1} \sum_{k=-r}^{r} |f(n+k)|.$$

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The discrete Hardy–Littlewood maximal function is defined as

$$\mathcal{M}f(n) := \sup_{r\in\mathbb{Z}^+} \mathcal{A}_r f(n). \tag{1}$$

This is the discrete analogue of the Hardy–Littlewood maximal function on Euclidean spaces

$$Mf(x) := \sup_{r>0} \fint_{B(x,r)} |f(y)|dy, \tag{2}$$

where $B(x,r)$ denotes ball of center x and radius r, and dashed integral denotes average. Our aim in this work is to study the distribution of the values r for which $\mathcal{M}f(n) = \mathcal{A}_r f(n)$. More precisely, we define the discrete frequency function as

$$\mathcal{T}f(n) := \inf E_{f,n} \quad \text{where} \quad E_{f,n} := \{r \in \mathbb{Z}^+ : \mathcal{M}f(n) = \mathcal{A}_r f(n)\}. \tag{3}$$

This function is well defined, for the set $E_{f,n}$ is non-empty, and when f is not identically zero it is actually finite; we will prove these in the next section. We also remark that $\mathcal{T}f(n) \in E_{f,n}$.

We take two very simple functions f, g and calculate $\mathcal{T}f, \mathcal{T}g$ for these two functions. These calculation will also motivate the theorems below. Let

$$f(n) := \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}, \qquad g(n) := \begin{cases} 3/5 & n = 0 \\ 1/5 & |n| = 5 \\ 0 & n \neq 0, 5, -5. \end{cases}$$

Notice that both functions have the same l^1 norm. Then we have for maximal functions

$$\mathcal{M}f(n) = \frac{1}{2|n|+1}, \quad \mathcal{M}g(n) = \begin{cases} \frac{1}{2|n|+1}\frac{3}{5} & |n| \leq 4 \\ \frac{1}{2(|n|-5)+1}\frac{1}{5} & |n| = 5, 6 \\ \frac{1}{2|n|+1}\frac{4}{5} & 7 \leq |n| \leq 19 \\ \frac{1}{2(|n|+5)+1} & 20 \leq |n|, \end{cases}$$

and for frequency functions

$$\mathcal{T}f(n) = |n|, \qquad \mathcal{T}g(n) = \begin{cases} |n| & |n| \leq 4 \\ |n| - 5 & |n| = 5, 6 \\ |n| & 7 \leq |n| \leq 19 \\ |n| + 5 & 20 \leq |n|. \end{cases}$$

Notice that for both f, g the frequency functions $\mathcal{T}f(n)/|n|, \mathcal{T}g(n)/|n|$ is very close to 1 for $|n|$ large enough. For any summable function a behaviour of this type should be expected, for a summable function must have most of its mass on an interval of

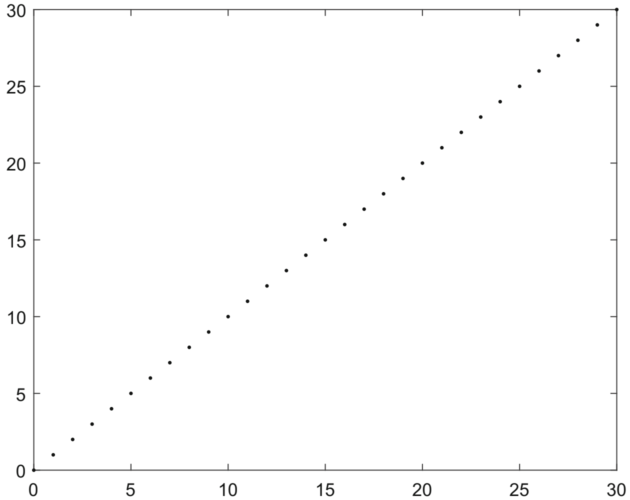

Fig. 1 $\mathcal{T}f(n)$ for non-negative n

finite length centered at the origin. Therefore it is natural to expect the frequency function to take a value like $|n|$ at $|n|$ large enough. Because of this, it is most natural to consider level set estimates obtained by comparing $\mathcal{T}f(n)$ to $|n|$. The example g shows existence of local maxima that are distant from each other may lead to the frequency function dropping suddenly to 0. Our third theorem will investigate how often this dropping to 0 can happen for large $|n|$ (Fig. 1).

The motivation behind our definition of the discrete frequency function, and our naming it so comes from the works [9,10]. In [7] Kinnunen proved that the Hardy–Littlewood maximal operator is bounded on the Sobolev space $W^{1,p}(\mathbb{R}^n)$ for $1 < p \leq \infty$. Since the maximal function of a non-trivial function is never integrable, this cannot be true for $p = 1$. But, as Hajlasz and Onninen asked in [6], it may still be that $f \mapsto \nabla Mf$ is a bounded operator from $W^{1,p}(\mathbb{R}^n)$ to $L^1(R^n)$. In [9] Kurka showed that if $n = 1$, actually the following stronger result holds

$$\text{var } Mf \leq C \text{ var } f,$$

and in [10] the author showed that this can be extended to the discrete case, i.e.

$$\text{var } \mathcal{M}f \leq C \text{ var } f.$$

In both works [9,10], a decomposition of the domain of definition of the function f with respect to lengths of intervals on which average of f is equal to value of the maximal function is used to great effect. As this decomposition reminds us the decomposition of linear operators using eigenvalues we find it appropriate to call our operator frequency function. Effectiveness of the decomposition with respect to frequencies used in [9,10] motivates a more systematic study of the frequency function. The only investigation of this function that the author could find is [11], where it is proved that if the frequency

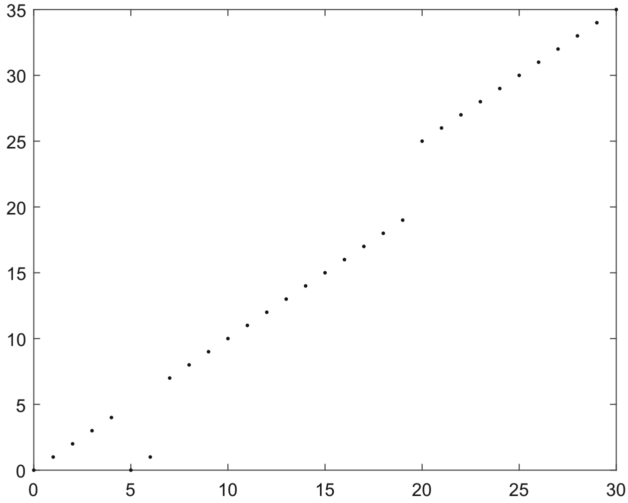

Fig. 2 $\mathcal{T}g(n)$ for non-negative n

function $\mathcal{T}f$ takes only a few values then f must be a sine type function, although we note that in [11] the functions $\mathcal{M}$ and $\mathcal{T}$ are defined somewhat differently. In this work we will explore aspects of this function quite different from those in [11], and we will mainly concentrate on its size and smoothness properties (Fig. 2).

This work is broadly a part of more than two decades long study of regularity aspects of the Hardy–Littlewood maximal function and its variants. The reader interested in these topics can consult the excellent survey [1], and original research articles [2–5,8,12]. We now state our theorems.

Theorem 1 *For any $f \in l^1(\mathbb{Z})$ and $C > 1$ the following set is finite*

$$\left\{ n : \frac{|n|}{2C} \le \mathcal{T}f(n) \le \frac{|n|}{C} \right\}. \tag{4}$$

Theorem 2 *For any $0 \neq f \in l^1(\mathbb{Z})$ and $C > 1$ we have*

$$\lim_{N\to\infty} \frac{\left|\left\{ n : |n| \le N,\ \mathcal{T}f(n) \le \frac{|n|}{C} \right\}\right|}{N} = 0. \tag{5}$$

The statement fails if we replace the denominator N with $N/\log^{1+\varepsilon} N$ for any positive ε. That is, for every fixed $\varepsilon > 0$ there exist an $f \neq 0$ and a $C > 1$ such that

$$\lim_{N\to\infty} \frac{\left|\left\{ n : |n| \le N,\ \mathcal{T}f(n) \le \frac{|n|}{C} \right\}\right|}{N/\log^{1+\varepsilon} N} \neq 0. \tag{6}$$

We point out that Theorem 2 is sharp in another way. It is not possible to have

$$\lim_{N\to\infty} \frac{|\{n : |n| \le N,\ \mathcal{T} f(n) \le \theta(n)\}|}{N/\log^{1+\varepsilon} N} = 0. \tag{7}$$

where $\theta : \mathbb{Z} \to \mathbb{Z}^+$ is any function satisfying $\theta(n) \le n/C$. Equivalently, we have the following theorem.

Theorem 3 *For every $\varepsilon > 0$, there exists a function $f \in l^1(\mathbb{Z})$ such that*

$$|\{n : |n| \le N,\ \mathcal{T} f(n) = 0\}| \ge \frac{1}{8} N/\log^{1+\varepsilon} N \tag{8}$$

for infinitely many values of N.

Thus (7) fails even if $\theta(n) = 0$ for all n. Therefore strengthening the requirement $\mathcal{T} f(n) \le |n|/C$ does not give us a better estimate.

We will also investigate the variational behavior of the frequency function, and show that in fares poorly in this aspect. We will show that for any $C > 0$ we can find a function f_C such that $\mathcal{T} f_C(1) - \mathcal{T} f_C(0) > C$. By a more elaborate construction we will also exhibit a function f such that

$$\sup_{n\in\mathbb{Z}} |\mathcal{T} f(n+1) - \mathcal{T} f(n)| = \infty.$$

We can define and investigate similar concepts for the discrete bilinear maximal function as well. Let $f, g \in l^1(\mathbb{Z})$. We define for $r \in \mathbb{Z}^+$

$$\mathcal{B}_r(f, g)(n) := \frac{1}{2r+1} \sum_{k=-r}^{r} |f(n-k)g(n+k)|.$$

The bilinear maximal function is defined as

$$\mathcal{B}(f, g)(n) = \sup_{r\in\mathbb{Z}^+} \mathcal{B}_r(f, g)(n). \tag{9}$$

We introduce the function

$$\mathcal{T}(f, g)(n) := \inf E_{f,g,n} \quad \text{where} \quad E_{f,g,n} := \{r : \mathcal{B}(f, g)(n) = \mathcal{B}_r(f, g)(n)\}. \tag{10}$$

This function is also well defined, as will be discussed in the next section. It seems reasonable to expect a result analogous to Theorem 2 to hold for this case as well, but we are not able to prove this. What we are able to show is the analogue of Theorem 3.

Theorem 4 *For every $\varepsilon > 0$, there are functions $f, g \in l^1(\mathbb{Z})$ such that*

$$|\{n : |n| \le N,\ \mathcal{T}(f, g)(n) = 0\}| \ge \frac{1}{8} N/\log^{1+\varepsilon} N \tag{11}$$

for infinitely many values of N.

The rest of the paper proceeds as follows. In the next section we show that both the discrete frequency function and the discrete bilinear frequency function are well defined. In the third section we prove the three theorems on the discrete frequency function and in the fourth section we investigate the its variational properties. In the fifth section we discuss the bilinear discrete frequency function. Finally, in the sixth section we mention some open problems that emerge from the investigations undertaken in this work.

2 Well-definedness of the Discrete Frequency Functions

In this section we will show that the discrete frequency functions given by (3) and (10) are both well defined by showing that the sets therein are non-empty. We start with the function in (3). We first note that if the function f is zero everywhere, then $E_{f,n}$ obviously is not empty. So we may assume that f is not zero everywhere. In this case for any point n the value $\mathcal{M}f(n)$ is positive. By (1) we can find a non-negative integer r_1 such that $\mathcal{M}f(n) - \mathcal{A}_{r_1}f(n) \leq 1$. Let d_1 denote the difference $\mathcal{M}f(n) - \mathcal{A}_{r_1}f(n)$. Then we can find $r_2 \in \mathbb{Z}^+$ such that $\mathcal{M}f(n) - \mathcal{A}_{r_2}f(n) \leq d_1/2$. We thus obtain a sequence $r_1, r_2, r_3, \ldots$ of non-negative integers, and a sequence of differences $d_1, d_2, d_3, \ldots$ induced by them that satisfy the relation $d_{i+1} \leq d_i/2$. Thus there exist $j \in \mathbb{N}$ such that for $i \geq j$ we have $\mathcal{A}_{r_2}f(n) \geq \mathcal{M}f(n)/2$. But then for such i

$$\|f\|_1 \geq r_i\mathcal{A}_{r_i}f(n) \geq r_i\frac{\mathcal{M}f(n)}{2},$$

and hence

$$r_i \leq \frac{2\|f\|_1}{\mathcal{M}f(n)},$$

which means the set of integers $\{r_i : i \in \mathbb{N}\}$ must be finite. Hence for some r_i we must have $\mathcal{M}f(n) = \mathcal{A}_{r_i}f(n)$, for otherwise $d_i \to 0$ as $i \to \infty$ would be impossible.

We now show that $E_{f,n}$ is finite if f is not identically zero. In this case $\mathcal{M}f(n)$ is strictly positive. If we assume the set to be infinite then we can list its elements to obtain a sequence $r_1, r_2, r_3, \ldots$ such that $r_1 < r_2 < r_3 < \ldots$. But then

$$\mathcal{M}f(n) = \frac{1}{2r_i+1}\sum_{j=-r_i}^{r_i}|f(n+j)| \leq \frac{\|f\|_1}{2r_i+1}.$$

Since elements of the set are integers, $r_i \to \infty$ as $i \to \infty$. Thus we have a contradiction, and our set is finite.

Proof of well-definedness of the bilinear discrete frequency function follows the same lines. We here wish to prove that $E_{f,g,n}$ is not empty. If $\mathcal{B}(f,g)(n)$ is zero then of course $\mathcal{B}_r(f,g)(n)$ is zero for any non-negative r, and thus our set is not empty. So we may assume that $\mathcal{B}(f,g)(n)$ is strictly positive. By (9) we can find a non-negative integer r_1 such that $\mathcal{B}(f,g)(n) - \mathcal{B}_{r_1}(f,g)(n) \leq 1$. Let d_1 denote the difference $\mathcal{B}(f, g)(n)-\mathcal{B}_{r_1}(f, g)(n)$. Then we can find $r_2 \in \mathbb{Z}^+$ such that $\mathcal{B}(f, g)(n)-\mathcal{B}_{r_2}(f, g)(n) \le d_1/2$. We thus obtain a sequence $r_1, r_2, r_3, \ldots$ of non-negative integers, and a sequence of differences $d_1, d_2, d_3, \ldots$ induced by them that satisfy the relation $d_{i+1} \le d_i/2$. Thus there exist $j \in \mathbb{N}$ such that for $i \ge j$ we have $\mathcal{B}_{r_i}(f, g)(n) \ge \mathcal{B}(f, g)(n)/2$. But then for such i we have

$$\|f\|_1\|g\|_1 \ge r_i\mathcal{B}_{r_i}(f, g)(n) \ge r_i\frac{\mathcal{B}(f, g)(n)}{2},$$

and hence

$$r_i \le \frac{2\|f\|_1\|g\|_1}{\mathcal{B}(f, g)(n)}$$

which means the set of integers $\{r_i : i \in \mathbb{N}\}$ must be finite, and our set is non-empty.

We also prove that if $\mathcal{B}(f, g)(n)$ is not zero then $E_{f,g,n}$ is finite. If we assume it to be infinite then we can list its elements to obtain a sequence $r_1, r_2, r_3, \ldots$ such that $r_1 < r_2 < r_3 < \ldots$. Then

$$\mathcal{B}(f, g)(n) = \frac{1}{2r_i + 1}\sum_{j=-r_i}^{r_i} |f(n-j)g(n+j)| \le \frac{\|f\|_1\|g\|_1}{2r_i + 1}.$$

Since elements of our set are integers, $r_i \to \infty$ as $i \to \infty$. Thus we have a contradiction, and our set is finite.

3 Proofs of Theorems 1, 2, 3

3.1 Proof of Theorem 1

The main idea of the proof is that if the set in the theorem, which we will denote by S, were infinite, then it would contain an infinite number of points distant enough from each other such that at each of these points average over an interval not containing the other points would be comparable to $\|f\|_1$. This contradicts f being summable.

Proof If f is identically zero then we have our result. So we will assume f is not identically zero. Assume to the contrary that S is not finite. Then we have two cases: either positive elements of S are infinite, or negative elements of S are infinite. We will show the impossibility of the first case, that the second is not possible either can be shown following exactly the same arguments. Let

$$A := \frac{C+1}{C-1}, \quad B := \frac{C+1}{C}, \quad D := \frac{C-1}{C}.$$

Since $f \in l^1(\mathbb{Z})$ we must have some $m \in \mathbb{N}$ such that

$$\sum_{j=-m}^{m} |f(j)| \ge \frac{\|f\|_1}{2}.$$

Let $n_1 > m$ be a positive element of the set, we can find such an element since we assumed our set to have infinitely many positive elements. For the same reason we can find n_2 such that $n_2 > 2An_1$. Proceeding thus we obtain a sequence $\{n_i\}_{i\in\mathbb{N}}$ with $n_{i+1} > 2An_i$ for each natural number i. Then we observe that

$$\begin{aligned}\mathcal{M}f(n_i) = \mathcal{A}_{\mathcal{T}f(n_i)}f(n_i) &= \frac{1}{2\mathcal{T}f(n_i)+1}\sum_{j=-\mathcal{T}f(n_i)}^{\mathcal{T}f(n_i)}|f(n_i+j)| \\ &\leq \frac{1}{2\mathcal{T}f(n_i)+1}\sum_{j\in[Dn_i,Bn_i]}|f(j)|.\end{aligned}$$

Thus we have

$$\left(\frac{n_i}{C}+1\right)\mathcal{M}f(n_i) \leq \sum_{j\in[Dn_i,Bn_i]}|f(j)|.$$

But notice that since $A = B/D$, we have $Dn_{i+1} > 2Bn_i$, and therefore the intervals $[Dn_i, Bn_i]$ never intersect. Hence we must have

$$\sum_{i\in\mathbb{N}}\left(\frac{n_i}{C}+1\right)\mathcal{M}f(n_i) \leq \sum_{i\in\mathbb{N}}\sum_{j\in[Dn_i,Bn_i]}|f(j)| \leq \|f\|_1. \tag{12}$$

On the other hand, since $n_i > m$ we must have

$$\begin{aligned}\mathcal{M}f(n_i) \geq A_{2n_i}f(n_i) &= \frac{1}{4n_i+1}\sum_{j=-2n_i}^{2n_i}|f(n_i+j)| \\ &= \frac{1}{4n_i+1}\sum_{j=-n_i}^{3n_i}|f(j)| \geq \frac{\|f\|_1}{8n_i+2}.\end{aligned}$$

Thus the inequality (12) implies

$$\frac{\|f\|_1}{C}\sum_{i\in\mathbb{N}}\frac{n_i+C}{8n_i+2} \leq \|f\|_1.$$

Since $C > 1$ this implies

$$\frac{1}{C}\sum_{i\in\mathbb{N}}\frac{1}{8} \leq 1,$$

which is not possible. Thus the set cannot contain infinitely many positive elements. $\square$

3.2 Proof of Theorem 2

We move to the proof of Theorem 2. We will use the Vitali covering lemma. For the sake of completeness we will give a proof.

Lemma 1 (Vitali Covering Lemma) *Let $\{B_i\}_{i=1}^m$ be a finite collection of intervals with finite length. Let E be a subset of integers covered by these intervals. Then we can find a disjoint subcollection $\{B_{i_k}\}_{k=1}^n$ of $\{B_i\}_{i=1}^m$ such that*

$$\sum_{k=1}^{n} |B_{i_k}| \geq \frac{|E|}{3}.$$

Proof Let B_{i_1} be the longest of our intervals. Let B_{i_2} be the longest interval that does not intersect B_{i_1}. We choose B_{i_3} to be the longest of the intervals that does not intersect either B_{i_1} or B_{i_2}. We proceed thus to obtain a subcollection, which is disjoint. Also observe that any B_i for $1 \leq i \leq m$ must intersect an interval in the subcollection that has at least the same length as itself. For if an interval does not intersect intervals of at least the same length then it must be a member of the collection, which leads to a contradiction. Therefore if we consider the collection $\{3B_{i_k}\}_{k=1}^n$ where $3B_{i_k}$ is the interval obtained by adding a translate of B_{i_k} to its left and another to its right, this collection must cover E. Therefore

$$|E| \leq \sum_{k=1}^{n} |3B_{i_k}| = 3\sum_{k=1}^{n} |B_{i_k}|$$

which implies what we wish. □

We can start the proof of Theorem 2. We use the Vitali covering lemma to refine the ideas used for the proof of Theorem 1.

Proof Let A, B, D be defined exactly as in the proof of Theorem 1. We introduce the following notation for the set in the theorem

$$K_N = \left\{ n : |n| \leq N,\ \mathcal{T} f(n) \leq \frac{|n|}{C} \right\}. \tag{13}$$

Let K_N^+ denote the positive elements of K_N, and K_N^- denote its negative elements. We will show that

$$\lim_{N\to\infty} \frac{|K_N^+|}{N} = 0,$$

and it will be clear to the reader that the same arguments give this result for K_N^- as well. Our theorem follows from combining these two results.

We assume to the contrary that

$$\lim_{N\to\infty} \frac{|K_N^+|}{N} \neq 0.$$

This means there exists a small, positive ϵ such that $|K_{N_i}^+|/N_i \geq \epsilon$ for a strictly increasing sequence $\{N_i\}_{i\in\mathbb{N}}$ of natural numbers. So we have $|K_{N_i}^+| \geq \epsilon N_i$ for such N_i. We let $M > 10^{10^{10A\epsilon^{-10}}}$ be a natural number such that

$$\sum_{j=-M}^{M} |f(j)| \geq \frac{\|f\|_1}{2}.$$

We choose a subsequence $\{N_{i_k}\}_{k\in\mathbb{N}}$ of $\{N_i\}_{i\in\mathbb{N}}$ as follows. Let N_{i_1} be such that $N_{i_1} \geq M$, and let $N_{i_{k+1}} \geq 10A\epsilon^{-1}N_{i_k}$ for every $k \geq 1$. Now we fix $k \geq 1$. We have

$$\left|K_{N_{i_{2k}}}^+ \setminus K_{N_{i_{2k-1}}}^+\right| \geq \frac{9\epsilon N_{i_{2k}}}{10} \geq 9N_{i_{2k-1}}$$

Let $n \in K_{N_{i_{2k}}}^+ \setminus K_{N_{i_{2k-1}}}^+$. We have

$$\mathcal{M}f(n) = \frac{1}{2\mathcal{T}f(n)+1} \sum_{j=-\mathcal{T}f(n)}^{\mathcal{T}f(n)} |f(n+j)|$$

but also since no element of the set $K_{N_{i_{2k}}}^+ \setminus K_{N_{i_{2k-1}}}^+$ is in $\left[-N_{i_{2k-1}}, N_{i_{2k-1}}\right]$,

$$\mathcal{M}f(n) \geq \mathcal{A}_{2n}f(n) = \frac{1}{4n+1}\sum_{j=-2n}^{2n}|f(n+j)| = \frac{1}{4n+1}\sum_{j=-n}^{3n}|f(j)| \geq \frac{\|f\|_1}{8n+2}.$$

Thus combining these two we obtain the fundamental result

$$\sum_{j=-\mathcal{T}f(n)}^{\mathcal{T}f(n)} |f(n+j)| \geq \frac{2\mathcal{T}f(n)+1}{8n+2}\|f\|_1.$$

We consider a covering of $K_{N_{i_{2k}}}^+ \setminus K_{N_{i_{2k-1}}}^+$ by such $[n-\mathcal{T}f(n), n+\mathcal{T}f(n)]$. By our covering lemma we have a subset $n_1, n_2, \ldots n_{p_k}$ for which the intervals $[n_i - \mathcal{T}f(n_i), n_i + \mathcal{T}f(n_i)]$, $1 \leq i \leq p_k$ are disjoint, and

$$\sum_{i=1}^{p_k} 2\mathcal{T}f(n_i)+1 \geq \frac{1}{3}\left|K_{N_{i_{2k}}}^+ \setminus K_{N_{i_{2k-1}}}^+\right| \geq \frac{9\epsilon N_{i_{2k}}}{30}.$$

We combine this result with the fundamental result above to obtain

$$
\begin{aligned}
\sum_{i=1}^{p_k} \sum_{j=-\mathcal{T} f(n_i)}^{\mathcal{T} f(n_i)} |f(n_i+j)| &\geq \sum_{i=1}^{p_k} \frac{2\mathcal{T} f(n_i)+1}{8n_i+2} \|f\|_1 \\
&\geq \frac{\|f\|_1}{8N_{i_{2k}}+2} \sum_{i=1}^{p_k} 2\mathcal{T} f(n_i)+1 \\
&\geq \frac{\|f\|_1}{8N_{i_{2k}}+2} \frac{9\epsilon N_{i_{2k}}}{30} \\
&\geq \frac{\epsilon \|f\|_1}{30}
\end{aligned}
$$

But since $[n_i - \mathcal{T} f(n_i), n_i + \mathcal{T} f(n_i)]$ are disjoint, we have

$$
\sum_{j \in [DN_{i_{2k-1}}, BN_{i_{2k}}]} |f(j)| \geq \sum_{i=1}^{p_k} \sum_{j=-\mathcal{T} f(n_i)}^{\mathcal{T} f(n_i)} |f(n_i+j)| \geq \frac{\epsilon \|f\|_1}{30}.
$$

Owing to our choice of the subsequence $\{N_{i_k}\}_{k\in\mathbb{N}}$ the intervals $[DN_{i_{2k-1}}, BN_{i_{2k}}]$ are disjoint for each natural number k, and therefore summing over k we have

$$
\|f\|_1 \geq \sum_{k\in\mathbb{N}} \sum_{j \in [DN_{i_{2k-1}}, BN_{i_{2k}}]} |f(j)| \geq \sum_{k\in\mathbb{N}} \frac{\epsilon \|f\|_1}{30}
$$

which is a contradiction.

We give two examples that show the sharpness of the estimate. The following is our most basic example, and the next one will improve upon the same ideas. We let for a small, positive ε

$$
f(n) := \begin{cases} \dfrac{1}{m^{1+\varepsilon}} & \text{if } \quad n = m^2, \ m \in \mathbb{N}, \\ 0 & \text{elsewhere} \end{cases}
$$

Now let $N = M^2$ for $M > 10^{10^{10A\varepsilon^{-10}}}$. We have $M^2 - (M-1)^2 = 2M - 1$. Let n satisfy $M^2 - M^{1-2\varepsilon}/4 < n < M^2$. We will calculate the maximal function at this point n. If we take r to be a natural number satisfying $M^{1-2\varepsilon}/2 < r < M^{1-2\varepsilon}$, then

$$
\mathcal{A}_r f(n) = \frac{1}{2r+1} \frac{1}{M^{1+\varepsilon}} \geq \frac{1}{3M^{1-2\varepsilon} M^{1+\varepsilon}} = \frac{1}{3M^{2-\varepsilon}}.
$$

Obviously taking $M^{1-2\varepsilon} \leq r < n - (M-1)^2$ cannot give a larger average. We claim that this is not possible for $r \geq n - (M-1)^2$ either. The key idea is to note that: as we approach to the origin from the right hand side the function attains nonzero values with increasing frequency, and moreover these nonzero values grow. In technical terms, we must have average of f over the interval $[(m-2)^2, (m-1)^2-1]$ larger than its average on $[(m-1)^2, m^2-1]$, that is

$$\frac{1}{2m-3}\sum_{j=(m-2)^2}^{(m-1)^2-1} f(j) = \frac{1}{(2m-3)(m-2)^{1+\varepsilon}}$$
$$\geq \frac{1}{(2m-1)(m-1)^{1+\varepsilon}}$$
$$= \frac{1}{2m-1}\sum_{j=(m-1)^2}^{m^2-1} f(j).$$

Obviously due to this phenomenon r cannot exceed n too much. Indeed, a moment's consideration makes it clear that we must have $r < 2n$. With such r we must have

$$\mathcal{A}_r f(n) = \frac{1}{2r+1}\sum_{j=-r}^{r} f(n+j) \leq \frac{3}{2r+1}\sum_{j=-r}^{-1} f(n+j) \leq \frac{3}{2}\frac{1}{r}\sum_{j=-r}^{-1} f(n+j).$$

Thus at the end we have average over $[n-r, n-1]$ of f, and we can write,

$$\frac{1}{r}\sum_{j=-r}^{-1} f(n+j) = \frac{1}{r}\sum_{j=n-r}^{n-1} f(j) \leq 2\frac{1}{M^2-n+r}\sum_{j=n-r}^{M^2-1} f(j)$$

Now we have average over $[n-r, M^2-1]$ of f at the end, and we wish to know the greatest value that this average can attain. Of course if $(m-1)^2 < n-r \leq m^2$ for some natural number m, taking r so that $n-r = m^2$ makes this average largest. Then using our observation we conclude that we better take $m = 1$. Thus this average is at most C_ε/M^2 where C_ε is the constant given by

$$C_\varepsilon = \sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}} \leq 2\varepsilon^{-1}.$$

Therefore $\mathcal{A}_r f(n) \leq 3C_\varepsilon/M^2$. This, given our choice of M, is less than $1/3M^{2-\varepsilon}$. Thus we must have $\mathcal{T} f(n) \leq |n|/C$. And from amongst $2M-2$ values of n between $(M-1)^2$ and M^2, at least $M^{1-2\varepsilon}/8$ satisfy this property. If we apply this to each interval $[(M-k-1)^2, (M-k)^2]$ for $k \in [0, M/2]$, we similarly obtain $(M-k)^{1-2\varepsilon}/8$ values of n satisfying $\mathcal{T} f(n) \leq |n|/C$. Thus in $[-N, N]$ we have at least

$$\frac{M}{2}\frac{(M-M/2)^{1-2\varepsilon}}{8} \geq \frac{N^{1-\varepsilon}}{50}$$

such elements. Therefore K_N has at least this cardinality, which makes

$$\lim_{N\to\infty}\frac{|K_N|}{N^{1-\varepsilon}} = 0$$

impossible.

We now give our second example. With exactly the same arguments we can use the function

$$f(n) := \begin{cases} \dfrac{1}{m \log^{1+\varepsilon/2} m} & \text{if} \quad n = m^2, \ m \in \mathbb{N}, \ m \geq 10 \\ 0 & \text{elsewhere} \end{cases}$$

to show that

$$\lim_{N\to\infty} \frac{|K_N|}{N/\log^{1+\varepsilon} N} = 0$$

is not possible. □

3.3 Proof of Theorem 3

We refine the ideas used in the proof of Theorem 2.

Proof We define for a small positive ε

$$f(n) := \begin{cases} \dfrac{1}{m \log^{1+\varepsilon/2} m} & \text{if} \quad n = \lceil m \log^{1+\varepsilon} m \rceil, \ m \in \mathbb{N}, \ m \geq 10 \\ 0 & \text{elsewhere} \end{cases}$$

Here for some real number x the expression $\lceil x \rceil$ denotes the smallest integer that is not less than x.

Let $M > 10^{10^{10A\varepsilon^{-10}}}$, and let $N = \lceil M \log^{1+\varepsilon} M \rceil$. Consider $m \in [M/2, M]$ and values of $n = \lceil m \log^{1+\varepsilon} m \rceil$ that correspond to these m. For such m we of course have

$$\mathcal{A}_0 f(n) = f(n) = \frac{1}{m \log^{1+\varepsilon/2} m}.$$

We will show that $\mathcal{A}_r f(n)$ cannot be larger than this for any r. Obviously for $r \gg n$ this is true, indeed a moment's consideration makes it clear that $r < 2n$. For such r we have

$$\mathcal{A}_r f(n) = \frac{1}{2r+1} \sum_{j=-r}^{r} f(n+j) \leq \frac{f(n)}{3} + \frac{2}{r} \sum_{j=-r}^{-1} f(n+j).$$

Thus the last term is average over $[n-r, n-1]$, and by the same reasoning as in the first example this average is largest when $r = n - \lceil 10 \log^{1+\varepsilon} 10 \rceil$, for the function attains ever growing non-zero values with ever increasing frequency as we approach to the origin from the right hand side. Therefore

$$\frac{2}{r}\sum_{j=-r}^{1} f(n+j) \le \frac{4}{n}\sum_{j=10}^{\infty} j\log^{1+\varepsilon} j \le \frac{4C_\varepsilon}{n}$$

where

$$\sum_{j=10}^{\infty} j\log^{1+\varepsilon/2} j = C_\varepsilon \le \frac{2}{\epsilon}.$$

But obviously

$$\frac{4C_\varepsilon}{n} \le \frac{16C_\varepsilon}{M\log^{1+\varepsilon} M} \le \frac{32\varepsilon^{-1}}{m\log^{1+\varepsilon} m} < \frac{1}{3m\log^{1+\varepsilon/2} m} = \frac{f(n)}{3}$$

Therefore $\mathcal{A}_r f(n) < \mathcal{A}_0 f(n)$. Thus our set contains at least $M/4$ elements, hence

$$|\{n : |n| \le N,\ \mathcal{T} f(n) = 0\}| \ge \frac{M}{4} \ge \frac{1}{8} N/\log^{1+\varepsilon} N$$

establishing our claim. □

4 Variational Results

For each C positive real number we will show a function f_C such that $\mathcal{T} f_C(1) - \mathcal{T} f_C(0) > C$. Obviously it is enough to find such functions for all $C \in \mathbb{N},\ C \ge 100$. We define for such a C

$$f_C(n) := \begin{cases} 1 & \text{if } \ n = 0, \\ 2C & \text{if } \ |n| = 3C, \\ 0 & \text{elsewhere} \end{cases}$$

Now consider the only reasonable candidates that may be the value $\mathcal{T} f_C(0)$: the values $0, 3C$. We have $\mathcal{A}_0 f_C(0) = 1$ while $\mathcal{A}_{3C} f_C(0) = (4C+1)/(6C+1) < 1$. Therefore $\mathcal{T} f_C(0) = 1$. On the other hand the only reasonable candidates that may be the value $\mathcal{T} f_C(1)$ are $1, 3C-1, 3C+1$. We have $\mathcal{A}_1 f_C(1) = 1/3$ while

$$\mathcal{A}_{3C-1} f_C(1) = (2C+1)/(6C-1), \quad \mathcal{A}_{3C+1} f_C(1) = (4C+1)/(6C+3).$$

Thus given our large values of C we have $\mathcal{T} f_C(1) = 3C+1$ which proves our claim.

We now consider the function

$$f(n) = \sum_{C=100}^{\infty} 2^{-C} f_C\left(n - 4^C\right)$$

Let $n = 4^C$ for some $C \geq 200$. Then obviously only reasonable values for $\mathcal{T} f(n)$ are $0, 3C$ or values $r > 4^{C-1}$ due to the sparse structure of f. We have again $\mathcal{A}_0 f(n) = 2^{-C}$ while $\mathcal{A}_{3C} f(n) = 2^{-C}(4C+1)/(6C+1) < 2^{-C}$. On the other hand for $r > 4^{C-1}$ we have

$$\mathcal{A}_r f(n) = \frac{1}{2r+1} \sum_{j=-r}^{r} f(n+j) \leq \frac{1}{4^{C-1}} \sum_{j=100}^{\infty} \frac{4j+1}{2^j}$$
$$\leq \frac{1}{4^{C-1}} \sum_{j=1}^{\infty} \frac{1}{(\sqrt{2})^j} \leq \frac{5}{4^{C-1}}$$

which means that $\mathcal{T} f(n) = 0$. Similarly only reasonable values for $\mathcal{T} f(n+1)$ are $0, 3C-1, 3C+1$ or values $r > 4^{C-1}$. Applying exactly the same arguments shows that $\mathcal{A}_{3C+1} f(n+1)$ is the largest, and therefore $\mathcal{T} f(n+1) = 3C+1$. Now since C can be arbitrarily large

$$\sup_{n \in \mathbb{Z}} |\mathcal{T} f(n+1) - \mathcal{T} f(n)| = \infty.$$

5 The Bilinear Discrete Frequency Function

In this section we prove Theorem 4 by constructing appropriate functions f, g. The functions we construct are similar to those in the proofs of Theorems 2 and 3.

Proof We let f, g to be the same function

$$f(n) = g(n) := \begin{cases} \dfrac{1}{m \log^{1+\varepsilon/2} m} & \text{if} \quad n = \lceil m \log^{1+\varepsilon} m \rceil,\ m \in \mathbb{N},\ m \geq 10, \\ 0 & \text{elsewhere.} \end{cases}$$

Let $M > 10^{10^{10A\varepsilon^{-10}}}$, and let $N = \lceil M \log^{1+\varepsilon} M \rceil$. Consider $m \in [M/2, M]$ and values of $n = \lceil m \log^{1+\varepsilon} m \rceil$ that correspond to these m. For such m we have of course have

$$\mathcal{B}_0(f, g)(n) = f(n) g(n) = \frac{1}{m^2 \log^{2+\varepsilon} m}$$

We wish to estimate $\mathcal{B}_r(f, g)(n)$ for r other than zero. Obviously taking $r > n$ is not reasonable. So assuming $0 < r \leq n$ we have

$$\mathcal{B}_r(f, g)(n) = \frac{1}{2r+1} \sum_{j=-r}^{r} f(n-j) g(n+j) = \frac{1}{2r+1} \sum_{j=-r}^{r} f(n-j) f(n+j).$$

We can write the last sum as

$$\frac{1}{2r+1}\left[f^2(n)+\sum_{j=-r}^{-1} f(n-j)f(n+j)+\sum_{j=1}^{r} f(n-j)f(n+j)\right].$$

The last two sums are the same, so we have

$$\frac{1}{2r+1}\left[f^2(n)+2\sum_{j=1}^{r} f(n-j)f(n+j)\right].$$

So it is enough to show that

$$\frac{1}{2r+1}\sum_{j=1}^{r} f(n-j)f(n+j) < \frac{f^2(n)}{3}.$$

We have for $j > 1$

$$f(n+j) \le \frac{1}{m\log^{1+\varepsilon/2} m}.$$

Therefore we have

$$\frac{1}{2r+1}\sum_{j=1}^{r} f(n-j)f(n+j) \le \frac{1}{m\log^{1+\varepsilon/2} m}\frac{1}{r}\sum_{j=1}^{r} f(n-j)$$

Thus we again have the average of f taken over $[n-r, n-1]$ and as explained before this becomes largest when $r = n - \lceil 10\log^{1+\varepsilon} 10\rceil$, thus we have

$$\frac{1}{m\log^{1+\varepsilon/2} m}\frac{1}{r}\sum_{j=1}^{r} f(n-j) \le \frac{2C_\varepsilon}{nm\log^{1+\varepsilon/2} m} \le \frac{2C_\varepsilon}{m^2\log^{2+3\varepsilon/2} m} < \frac{f^2(n)}{3}$$

where

$$\sum_{j=10}^{\infty} j\log^{1+\varepsilon/2} j = C_\varepsilon \le \frac{2}{\varepsilon}.$$

Hence we must have at least $M/4$ elements in $\{n : |n| \le N,\ \mathcal{T}(f,g)(n) = 0\}$, and thus

$$|\{n : |n| \le N,\ \mathcal{T}(f,g)(n) = 0\}| \ge \frac{M}{4} \ge \frac{1}{8}N\log^{1+\varepsilon} N$$

establishing our claim. $\square$

6 Open Problems

The proof of the second theorem uses the Vitali covering lemma which is also used to prove the classical weak type boundedness result for the maximal function, therefore we suspect that it may be possible to relate this theorem to that result in a relatively short way, although we could not find it.

Open Problem 1 Let $f \in l^1(\mathbb{Z})$. Is it possible to deduce the fact that for every real number $\lambda > 0$ we have

$$|\{n \in \mathbb{Z} | \mathcal{M} f(n) > \lambda\}| \leq C \frac{\|f\|_{l^1(\mathbb{Z})}}{\lambda}$$

from Theorem 2? Is it possible to deduce Theorem 2 from this result?

Another important question is having seen that we cannot replace N in the denominator of (5) with $N/\log^{1+\varepsilon} N$, whether it is possible to replace it with $N/\log N$.

Open Problem 2 Let $f \in l^1(\mathbb{Z})$ be a function that is not identically zero. Let $C > 1$ be a real number. Is the following statement true?

$$\lim_{N\to\infty} \frac{\left|\left\{n : |n| \leq N,\ \mathcal{T} f(n) \leq \frac{|n|}{C}\right\}\right|}{N/\log N} = 0.$$

It should be possible to define an analogue of the discrete frequency function for the usual Hardy–Littlewood maximal function that acts on functions on the real line, but since analogues of the sets in (3) can be empty in that case, the definition needs to be more delicate. Furthermore, to prove any kind of level set estimate we need to deal with the issue of Lebesgue measurability.

Open Problem 3 Is it possible to define an analogue of the discrete frequency function for the usual Hardy–Littlewood maximal function that acts on functions on the real line? Is it possible to extend the Theorems 1–3 to that case?

Analogues of the discrete frequency function for higher dimensional discrete and continuous maximal functions should be possible.

Open Problem 4 Is it possible to define analogues of the discrete frequency function for these higher dimensional cases? Is it possible to extend the Theorems 1–3 to those cases?

Finally, defining analogues of the discrete frequency function for non-centered maximal functions would be a very interesting open problem, as in this case to find values of maximal functions supremums of averages are taken over more than one parameters. Consider for example the discrete one-dimensional Hardy–Littlewood maximal function:

$$\mathcal{M}^* f(n) = \sup_{r,s\in\mathbb{Z}^+} \frac{1}{r+s+1} \sum_{j=-r}^{s} |f(n+j)|. \tag{14}$$

Here we have two parameters r, s. Thus the analogues of the discrete frequency function may need to be vector valued.

Open Problem 5 Is it possible to define analogues of the discrete frequency function for non-centered maximal functions? Is it possible to extend the Theorems 1–3 to those cases?

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