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ORIGINAL PAPER

On the Structure of Modules Defined by Opposites of FP Injectivity

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Abstract

Let R be a ring with unity and let M_R and ${}_RN$ be right and left modules, respectively. The module M_R is said to be absolutely ${}_RN$-pure if $M \otimes N \to L \otimes N$ is a monomorphism for every extension L_R of M_R. For a module M_R, the subpurity domain of M_R is defined to be the collection of all modules ${}_RN$, such that M_R is absolutely ${}_RN$-pure. Clearly, M_R is absolutely ${}_RF$-pure for every flat module ${}_RF$ and that M_R is FP-injective if the subpurity domain of M is the entire class of left modules. As an opposite of FP-injective modules, M_R is said to be a test for flatness by subpurity (or t.f.b.s. for short) if its subpurity domain is as small as possible, namely, consisting of exactly the flat left modules. We characterize the structure of t.f.b.s. modules over commutative hereditary Noetherian rings. We prove that a module M is t.f.b.s. over a commutative hereditary Noetherian ring if and only if $M/Z(M)$ is t.f.b.s. if and only if $\mathrm{Hom}(M/Z(M), S) \neq 0$ for each singular simple module S. Prüfer domains are characterized as those domains all of whose nonzero finitely generated ideals are t.f.b.s.

Keywords Subpurity domain · Flat modules

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1 Introduction and Preliminaries

Throughout this paper, rings are associative with unity and modules are unitary right modules. Given right module M and a left module N, M is said to be absolutely N-pure if $M \otimes N \to L \otimes N$ is a monomorphism for every extension L_R of M_R. The subpurity domain of M, denoted as $\mathcal{S}p(M)$, is collection of all left modules N, such that M is absolutely N-pure. It is clear that M is FP-injective if and only if $\mathcal{S}p(M) = R\text{-}\mathcal{M}od$. Flat left modules are contained in the subpurity domain of each right module. As in [6], M is called test for flatness by subpurity (or, t.f.b.s. for short) if its subpurity domain of M is exactly the class of flat left modules. For further results about t.f.b.s. modules, we refer to [6].

In this paper, we characterize t.f.b.s. modules over commutative hereditary Noetherian rings. We prove that over a commutative hereditary Noetherian ring M is t.f.b.s. if and only if $M/Z(M)$ is t.f.b.s. if and only if $\text{Hom}(M/Z(M), S) \neq 0$ for every singular simple R-module S. A commutative domain R is Prüfer if and only if every nonzero finitely generated ideal of R is t.f.b.s. if and only if every finitely generated module M with $\text{Hom}(M, R) \neq 0$ is t.f.b.s. In particular, over a Prüfer domain, a finitely generated R-module M is t.f.b.s. if and only if $T(M) \neq M$, where $T(M)$ is the torsion part of M.

A module M is said to be A-subinjective if for every extension B of A any homomorphism $\varphi : A \to M$ can be extended to a homomorphism $\phi : B \to M$ (see [5]). It is easy to see that M is injective if and only if M is A-subinjective for each module A. In [1], a module A is said to be a test for injectivity by subinjectivity (or t.i.b.s.) if whenever a module M is A-subinjective implies M is injective. It is known that every t.i.b.s. module is t.f.b.s. by [6, Proposition 3.9]. We prove that a finitely generated abelian group G is t.i.b.s. if and only if G is t.f.b.s.

In [7], the author investigates the absolutely pure domain of a left module N as the collection of all right modules M, such that M is absolutely N-pure. Absolutely pure domain of any module consists of the class of FP-injective modules. A left module N is said to be f-indigent if its absolutely pure domain is exactly the class of FP-injective right modules. We proved that if R is a left Noetherian, right and left IF-ring, then a right module M is f-indigent if and only if M is t.f.b.s. Following [3], a right module M is called f-test module if for every left module N, and $Tor_1(M, N) = 0$ implies N is flat. If R is a right IF-ring, then a right R-module N is t.f.b.s. if and only if $E(N)/N$ is f-test. We showed that t.f.b.s., f-indigent, and f-test modules are not comparable, in general.

For a ring R and a right module M, $E(M)$, $\text{Rad}(M)$, $\text{Soc}(M)$, $Z(M)$ will, respectively, denote the injective hull, Jacobson radical, socle, and singular submodule of M. The character module $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ will be denoted by M^+. By $N \leq M$, we mean that N is a submodule of M. For additional terminology, concepts, and results not mentioned here, we refer the reader to [4,12,16].

2 Preliminaries

In this section, we recall some known results that will be used in the sequel.

Proposition 2.1 *Let R be a ring and M, N be right modules. The following are hold.*

(1) [15, Theorem 3] *R is right Noetherian if and only if each FP-injective right module is injective.*
(2) [10, Proposition 2.3] *If R is nonsingular commutative ring, then all nonsingular modules are flat if and only if R is semihereditary.*
(3) [8, Theorem 3.2.10] *M is flat if and only if M^+ is injective.*
(4) [8, Theorem 3.2.16] *If R is right Noetherian, M is injective if and only if M^+ is flat.*
(5) [8, Theorem 3.2.11] *If M is finitely presented then $M \otimes_R N^+ \cong \mathrm{Hom}_R(M_R, N_R)^+$.*
(6) [11, Exercise 12, pp. 139] *If R is right nonsingular and right finite-dimensional ring, then all flat right modules are nonsingular.*

Proposition 2.2 [6, Proposition 2.8] *Let F be a flat right module. Suppose that F is absolutely M-pure for some left module M. Then, F is absolutely K-pure for any submodule K of M. In other words, the subpurity domain of any flat right module is closed under submodules.*

Proposition 2.3 [6, Proposition 2.10] *A ring R is right semihereditary if and only if whenever a right module M is absolutely N-pure for some left module N, then M/K is absolutely N-pure for each $K \leq M$.*

Corollary 2.4 *Let R be a right semihereditary ring and M be a right R-module. If M/K is t.f.b.s. for some submodule K of M, then M is t.f.b.s.*

Proposition 2.5 [6, Proposition 3.2] *The following hold for a right R-module M.*

(1) *If M has a pure submodule N which is t.f.b.s., then M is t.f.b.s.*
(2) *If M is t.f.b.s., then $M \oplus N$ is t.f.b.s. for any module N.*
(3) *If A be an FP-injective right module, then $M \oplus A$ is t.f.b.s. if and only if M is t.f.b.s.*
(4) *M is t.f.b.s. if and only if M^n is t.f.b.s. for some $n \geq 1$*

3 t.f.b.s. Modules Over Commutative Rings

In this section, we deal with t.f.b.s. modules over commutative rings. It is shown that a commutative domain is Prüfer if and only if each finitely generated ideal is t.f.b.s. We also give a complete characterization of t.f.b.s. modules over commutative hereditary Noetherian rings.

Theorem 3.1 *The following are equivalent for a commutative domain R.*

(1) *R is Prüfer.*
(2) *R is t.f.b.s.*
(3) *Every nonzero finitely generated ideal is t.f.b.s.*
(4) *A finitely generated R-module M is t.f.b.s. when* $\mathrm{Hom}(M, R) \neq 0$.

Proof (1) $\Leftrightarrow$ (2) By [6, Corollary 3.7].

(1) $\Rightarrow$ (3) Let I be a nonzero finitely generated ideal of R. Since R is Prüfer and I is finitely generated, I is projective by [9, Theorem 2.7]. We shall first prove that $Q.I \neq I$ for each maximal ideal of Q of R. Suppose the contrary that $P.I = I$ for some maximal ideal P of R. Then, the localization at P gives $I_P = (I.P)_P = I_P.P_P$. Note that R_P is a local ring with unique maximal ideal P_P. Since I_P is a finitely generated ideal of R_P, $I_P = I_P.P_P$ implies $I_P = 0$ by Nakayama's Lemma. As R is a domain, $I_P = 0$ implies $I = 0$. Contradiction. Therefore, we have $I.Q \neq I$ for each maximal ideal Q of R. Therefore, $I/(Q.I)$ is nonzero and semisimple both as an R/Q -module and as an R-module. From $I/(Q.I) \cong (R/Q)^n$, $n \geq 1$, we conclude that $\mathrm{Hom}(I, R/Q) \neq 0$ for all maximal ideals Q of R. Thus, I is a projective generator by [4, Proposition 17.9]. Therefore, there is an epimorphism $f : I^k \rightarrow R$. Then, $I^k \cong R \oplus L$ for some $L \leq I^k$, by projectivity of R. Now, the hypothesis (2) and Proposition 2.5(2) together imply that I^k is a t.f.b.s. R-module. Hence, I is t.f.b.s. by Proposition 2.5(4). This proves (3).

(3) $\Rightarrow$ (2) is clear.

(3) $\Rightarrow$ (4) Let M be a finitely generated module. Let $0 \neq f \in \mathrm{Hom}(M, R)$. Then, $f(M)$ is a nonzero finitely generated ideal of R, and hence, $f(M)$ is projective by the equivalence (1) $\Leftrightarrow$ (3). Therefore, $M \cong f(M) \oplus K$ for some $K \leq M$. Since $f(M)$ is t.f.b.s. by (3), the module M is t.f.b.s. by Proposition 2.5(2).

(4) $\Rightarrow$ (2) is clear. □

Over a Prüfer domain, each finitely generated module can be written as a direct sum of its torsion submodule and a projective submodule by [9, Corollary 2.9]. Hence, the following is clear by Theorem 3.1.

Corollary 3.2 *Let R be a Prüfer domain and M be a finitely generated R-module. M is t.f.b.s. if and only if $T(M) \neq M$.*

Corollary 3.3 *Let R be a Prüfer Domain and M be an R-module. If $M/T(M)$ is t.f.b.s., then M is t.f.b.s.*

Remark 3.4 Let R be a commutative Noetherian ring and S be a simple R-module. Then, being injective, flat and projective are equivalent for S see, for example [2, Lemma 3.4].

Theorem 3.5 *Let R be a commutative hereditary Noetherian ring and F be a flat R-module. The following are equivalent.*

(1) *F is a t.f.b.s. R-module.*
(2) $\mathrm{Hom}(F, S) \neq 0$ *for each singular simple R-module S.*
(3) *$F \cdot Q \neq F$ for each essential maximal ideal Q of R.*

Proof (1) $\Rightarrow$ (2) Suppose F is a t.f.b.s. R-module and $S \cong R/I$ is a singular simple R-module, where I is a maximal ideal of R. Then, S is non injective by Remark 3.4. Thus, F is not absolutely S-pure, and so, in particular $F \otimes S \neq 0$. This implies that

$$F \otimes S \cong F \otimes R/I \cong F/FI \neq 0.$$

Therefore, F has a maximal submodule K, such that $F/K \cong R/I$. This implies that $\mathrm{Hom}(F, S) \neq 0$.

(2) $\Rightarrow$ (1) Assume the contrary that F is not t.f.b.s. Then, there is a non-flat R-module M, such that F is absolutely M-pure. Since M is not flat and the ring is hereditary, $Z(M) \neq 0$ by Proposition 2.1(2). Therefore, $Z(M)$ contains a (singular) simple R-module, say S, by [14, Proposition 4.5, pp. 161]. Set $E = E(F)$. As F is flat and absolutely M-pure, $F \otimes S \to E \otimes S$ is a monomorphism by Proposition 2.2. Since E is injective and R is Noetherian, E^+ is flat by Proposition 2.1(4). Then, E^+ is nonsingular by Proposition 2.1(6). Then, $(E \otimes S)^+ \cong \mathrm{Hom}(S, E^+) = 0$, because S is singular and E^+ is nonsingular. Therefore, $E \otimes S = 0$ and $F \otimes S = 0$. This implies that $\mathrm{Hom}(F, S) = 0$. Contradiction. Hence, M is nonsingular, i.e., flat. This implies that F is t.f.b.s.

(2) $\Leftrightarrow$ (3) This implication follows from the fact that R/I is singular for some ideal I of R if and only if I is essential in R by [11, Proposition 1.21]. □

Lemma 3.6 *Let R be a commutative Noetherian ring and M be an R-module. If $M \otimes R/P = 0$ for some maximal ideal P of R, then $M \otimes E(R/P) = 0$, where $E(R/P)$ is the injective hull of R/P.*

Proof For each $i \in \mathbb{Z}^+$, let $A_i = \{x \in E(R/P) | P^i x = 0\}$. Then, A_i is finitely generated for each $i \in \mathbb{Z}^+$ and $E(R/P) = \bigcup_{i \in \mathbb{Z}^+} A_i$ by [13, Theorem 3.4]. Then, A_i is a finitely generated module over the Artinian ring R/P^i. Therefore, A_i has a finite composition length for each $i \in \mathbb{Z}^+$. Let

$$0 = T_0 \leq T_1 \leq \cdots \leq T_n = A_i$$

be a composition series of A_i. Then $T_{k+1}/T_k \cong R/P$ for each $k = 0, \ldots, i-1$. Consider the sequence

$$M \otimes T_1 \to M \otimes T_2 \to M \otimes (T_2/T_1).$$

Now, $M \otimes R/P = 0$ implies that $M \otimes T_1 = M \otimes (T_2/T_1) = 0$, and so, $M \otimes T_2 = 0$. In the next step, from the sequence

$$M \otimes T_2 \to M \otimes T_3 \to M \otimes (T_3/T_2),$$

we obtain $M \otimes T_3 = 0$. Continuing in this way, at the last step, we shall get $M \otimes A_i = 0$. This fact together with $E(R/P) = \bigcup_{i \in \mathbb{Z}^+} A_i$ implies that $M \otimes E(R/P) = 0$. This completes the proof. □

Now, we are in a position to prove our main theorem. Note that for every module M over a nonsingular ring, the module $M/Z(M)$ is nonsingular (see [11, Proposition 1.23(a)]).

Theorem 3.7 *Let R be a commutative hereditary Noetherian ring and N be an R-module. The following are equivalent.*

(1) *N is t.f.b.s.*

(2) *$N/Z(N)$ is t.f.b.s.*
(3) $\mathrm{Hom}(N/Z(N), S) \neq 0$ *for every singular simple R-module S.*
(4) *$N/Z(N) \otimes S \neq 0$ for every singular simple R-module S.*

Proof (1) $\Rightarrow$ (4) Assume (1), and suppose the contrary that $N/Z(N) \otimes S = 0$ for some singular simple R-module S. Then, $N/Z(N) \otimes E(S) = 0$ by Lemma 3.6. On the other hand

$$(Z(N) \otimes E(S))^+ \cong \mathrm{Hom}(Z(N), E(S)^+) = 0,$$

because $Z(N)$ is singular, and $E(S)^+$ is nonsingular by Propositions 2.1(4) and 2.1(6). Thus, $Z(N) \otimes E(S) = 0$. Therefore, from the sequence

$$Z(N) \otimes E(S) \to N \otimes E(S) \to N/Z(N) \otimes E(S),$$

we obtain that $N \otimes E(S) = 0$. This means that N is absolutely $E(S)$-pure, and so, $E(S)$ is flat by (1). Then, $E(S)$ is nonsingular by Proposition 2.1(6). This contradicts with the fact that $E(S)$ is singular. Therefore, we must have $N/Z(N) \otimes S \neq 0$.

(2) $\Rightarrow$ (1) Suppose N is an absolutely A-pure module for some R-module A. Then, $N/Z(N)$ is absolutely A-pure by Proposition 2.3. By (2) $N/Z(N)$ is t.f.b.s., so A is flat. This implies that N is t.f.b.s.

(2) $\Leftrightarrow$ (3) The module $N/Z(N)$ is nonsingular, i.e., flat by Proposition 2.1(2). Therefore, the proof is clear by Theorem 3.5.

(3) $\Leftrightarrow$ (4) Clear. □

Recall that a nonzero element a of a Principal ideal domain is irreducible if whenever $a = b.c$ for some $b, c \in R$, then either b or c is a unit in R.

Corollary 3.8 *Let R be a Principal Ideal Domain. Then, an R-module G is t.f.b.s. if and only if $G/T(G) \neq p(G/T(G))$ for every irreducible element p in R.*

By [1, Theorem 26], an abelian group G is t.i.b.s. if and only if G contains a direct summand isomorphic to $\mathbb{Z}$. Now, the following is clear by Corollary 3.8 and [1, Theorem 26].

Corollary 3.9 *Let G be a finitely generated abelian group. Then, the following are equivalent.*

(1) *G is t.f.b.s.*
(2) *G is t.i.b.s.*
(3) *$T(G) \neq G$.*

Every t.i.b.s. module is t.f.b.s. by [6, Proposition 3.9]. The following example shows that there are t.f.b.s. $\mathbb{Z}$-modules which are not t.i.b.s.

Example 3.10 Consider the abelian group $G = \sum \mathbb{Z}.\frac{1}{p}$, where p ranges over the set of all prime integers. Then, it is clear that $G \neq pG$ for each prime p. Thus, G is t.f.b.s. by Corollary 3.9. Note that G is indecomposable and not isomorphic to $\mathbb{Z}$. Therefore, G is not t.i.b.s. by [1, Theorem 26].

Following [3], a right module M is called f-test module if for every left module N, $Tor_1(M, N) = 0$ implies that N is flat. A ring R is called right IF if every injective right R-module is flat. Every right QF-ring is right IF.

Proposition 3.11 *Let R be a right IF-ring. A right R-module N is t.f.b.s. if and only if $E(N)/N$ is f-test.*

Proof Let K be a left module. Since R is right IF, $Tor_1(E(N), K) = 0$. Therefore, applying the functor $-\otimes K$ to the the short exact sequence $0 \to N \to E(N) \to E(N)/N \to 0$, we obtain the exact sequence

$$\begin{aligned} 0 = Tor_1(E(N), K) &\to Tor_1(E(N)/N, K) \to N \otimes K \\ &\to E(N) \otimes K \to E(N)/N \otimes K \to 0. \end{aligned}$$

From which it is easy to see that N is t.f.b.s. if and only if $E(N)/N$ is f-test. □

A right module M is said to be f-indigent if whenever a left module N is absolutely M-pure, then N is FP-injective.

Proposition 3.12 *If R is a left Noetherian, right and left IF-ring, then a right module M is f-indigent if and only if M is t.f.b.s.*

Proof Suppose that M_R is absolutely ${}_RN$-pure for any left module ${}_RN$, i.e., the sequence $0 \to M \otimes N \to E(M) \otimes N$ is monic. Then, we get the following commutative diagram

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\ h\ } & E(M) \otimes N \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle t} \\ M \otimes E(N) & \xrightarrow{\ g\ } & E(M) \otimes E(N) \end{array}$$

induced by the inclusions $M \to E(M)$ and $N \to E(N)$. Since R is right IF-ring, t is monic. Then, by commutativity of the diagram, $gf = th$ is a monomorphism. Then, f is a monomorphism, and so, ${}_RN$ is absolutely M_R-pure by [7, Proposition 2.2]. Since M_R is f-indigent, ${}_RN$ is FP-injective. Since R is left Noetherian and left IF-ring, ${}_RN$ is flat. Conversely, suppose that ${}_RN$ is absolutely M_R-pure for some left module ${}_RN$, i.e., $0 \to M \otimes N \to M \otimes E(N)$ is monic. Since R is left IF-ring, g is monic. Then, by the commutativity of diagram, $gf = th$ is a monomorphism. Then, h is a monomorphism, and so, M_R is absolutely ${}_RN$-pure by [6, Lemma 2.3]. Since M_R is t.f.b.s., ${}_RN$ is flat. Then, ${}_RN$ is FP-injective by Corollary [7, Corollary 3.1]. □

The following example shows that t.f.b.s., f-indigent, and f-test modules are not comparable, in general.

Example 3.13 Consider the semisimple $\mathbb{Z}$-module $\oplus\mathbb{Z}_p$, where p ranges over all primes and $\mathbb{Z}_p$ is the simple $\mathbb{Z}$-module of order p. Then, $\oplus\mathbb{Z}_p$ is both f-indigent and f-test, by [7, Corollary 5.1] and [3, Corollary 4.20], respectively. The module $\mathbb{Z}_p$ is not t.f.b.s. by Theorem 3.7. On the other hand, the ring of integers $\mathbb{Z}$ is t.i.b.s. by Theorem 3.7. However, $\mathbb{Z}$ is neither f-indigent nor f-test again by [7, Corollary 5.1] and [3, Corollary 4.20], respectively.

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