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ABSTRACT

Time evolution of squeezed coherent states for a quantum parametric oscillator with the most general self-adjoint quadratic Hamiltonian is found explicitly. For this, we use the unitary displacement and squeeze operators in coordinate representation and the evolution operator obtained by the Wei-Norman Lie algebraic approach. Then, we analyze squeezing properties of the wave packets according to the complex parameter of the squeeze operator and the time-variable parameters of the Hamiltonian. As an application, we construct all exactly solvable generalized quantum oscillator models classically corresponding to a driven simple harmonic oscillator. For each model, defined according to the frequency modification in position space, we describe explicitly the squeezing and displacement properties of the wave packets. This allows us to see the exact influence of all parameters and make a basic comparison between the different models.

I. INTRODUCTION

Coherent states (CS) and squeezed coherent states (SCS) of the quantum harmonic oscillator are known since the beginning of quantum mechanics[1,2] and have many applications in different branches of physics and engineering.[3–6] For the standard harmonic oscillator (SHO) defined by the Hamiltonian

$$\hat{H}_0 = \frac{\hat{p}^2}{2} + \frac{\omega_0^2}{2}\hat{q}^2,$$

coherent states are minimum uncertainty states with equal uncertainties in both quadratures, whose motion follows the classical trajectory, and they are the closest analogs to the classical states. On the other hand, the squeezed coherent states are generalizations of the coherent states, which in the simplest case obey the minimum uncertainty principle, but have less uncertainty in one quadrature at the expense of increased uncertainty in the other. Essential properties of squeezed states were derived by Stoler[7,8] and Yuen[9] and then extensively investigated by many authors.[10–17] The study of squeezed states is motivated mainly by their nonclassical properties and possible applications in different areas such as nonlinear optical processes, optical communications using lasers, the detection of gravitational waves, and electromagnetism.[9,18–20]

As known, the coherent and squeezed states of SHO can be generated using different but equivalent approaches. One way to obtain coherent states is to apply the unitary displacement operator $\hat{D}(\alpha)$ to the ground state. On the other hand, squeezed states can be found by application of the unitary squeeze operator $\hat{S}(z)$. Using this formalism, the displaced and squeezed number states of SHO and their time evolution were derived explicitly by Nieto.[21–23] It was shown that the time-evolved squeezed coherent states of SHO correspond to wave packets whose width oscillates with time, the minimum uncertainty relation is no longer preserved during time evolution, and their peak follows the classical trajectory.

Another possibility of generating displaced squeezed states is by adding to the standard Hamiltonian $\hat{H}_0$ a mixed term $B(t)(\hat{q}\hat{p} + \hat{p}\hat{q})$ with squeezing (two-photon) parameter $B(t)$ and an external linear term $D(t)\hat{q}$ with displacement parameter $D(t)$. This was realized by Yuen in the context of two-photon coherent states of the radiation field.[9] He found that in the presence of a constant two-photon parameter, the frequency of the oscillator changes and quantum noise becomes purely oscillatory. Then, in Refs. 24 and 25, it was discussed that another simple way of realizing the displaced and squeezed number states is to add to $\hat{H}_0$, at some moment of time, a term of the form $\frac{1}{2}\omega_1^2 q^2 - f_0 q$, which clearly corresponds to the change in frequency and displacement.

On the other hand, when the oscillator has time-dependent mass $\mu(t)$ or/and frequency $\omega(t)$ similar to the Caldirola-Kanai model, squeezing effects appear naturally due to the time-variable parameters. As a consequence, it was noticed in Ref. 15 that the evolution operator of the quadratic parametric oscillator can be considered as some kind of generalized squeezing operator. In other words, evolution itself is a displacement and squeezing process.

In this work, we consider a quantum parametric oscillator with the most general self-adjoint quadratic Hamiltonian

$$\hat{H}_g(t) = \frac{\hat{p}^2}{2\mu(t)} + \frac{\mu(t)\omega^2(t)}{2}\hat{q}^2 + \frac{B(t)}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}) + D(t)\hat{q} + E(t)\hat{p} + F(t)\hat{I},$$

where $\hat{q}$ is position operator and $\hat{p} = -i\hbar\partial/\partial q$ is the momentum operator. The coherent and squeezed states of this generalized oscillator were investigated in Refs. 27–29, using the Lewis-Riesenfeld invariant approach.[26] Recently, using the Wei-Norman Lie algebraic approach,[30] we found the exact evolution operator for the quantum system described by $H_g(t)$ and we explicitly obtained time development of the eigenstates and coherent states.[31] Then, in Ref. 32, we investigated the squeezing and resonance properties of coherent states for generalized Caldirola-Kanai type models. Here, we obtain time development of the squeezed coherent states, using the evolution operator approach, and analyze in detail their squeezing and displacement properties. More precisely, this paper is organized as follows.

In Sec. II, we provide the coordinate representation of the coherent states and squeezed coherent states of the standard harmonic oscillator (SHO) with Hamiltonian $\hat{H}_0$ and find their time evolution. For this, we use the displacement, squeeze, and evolution operators expressed as a product of group generators, associated with the corresponding Lie algebras.

Then, in Sec. III, using the exact evolution operator of the generalized quantum parametric oscillator with Hamiltonian $\hat{H}_g(t)$, we explicitly find time evolution of the squeezed coherent states, their probability densities, expectations, and uncertainties of position and momentum. This allows us to discuss the properties of the quantum states according to the complex parameter α of the displacement operator $\hat{D}(\alpha)$, the complex parameter z of the squeeze operator $\hat{S}(z)$, and the time-dependent parameters of the Hamiltonian $\hat{H}_g(t)$.

As an application of the general results obtained in Sec. III, in Sec. IV, we construct all exactly solvable generalized quantum oscillator models with different parameters $B(t)$, for which the corresponding classical equation of motion in position space describes simple harmonic motion. Then, for each model, we consider the time evolution of squeezed coherent states, discuss their properties explicitly both in position and momentum space, and construct many illustrative figures. Section V includes a brief summary and a basic comparison of the different models.

II. STANDARD QUANTUM HARMONIC OSCILLATOR

In this section, for completeness and later use of the results, we briefly reconsider a derivation of the time-evolved coherent and squeezed coherent states of the standard harmonic oscillator (SHO). Precisely, we have an initial value problem for the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(q,t) = \hat{H}_0\Psi(q,t), \quad t > 0, \tag{1}$$

$$\Psi(q,t_0) = \Psi_0(q), \quad -\infty < q < \infty, \tag{2}$$

with standard harmonic oscillator Hamiltonian

$$\hat{H}_0 = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2} + \frac{\omega_0^2}{2}q^2, \tag{3}$$

where $\omega_0 > 0$ is the natural frequency and mass is $m = 1$. The evolution operator for this problem is $\hat{U}_0(t) = \exp(-it\hat{H}_0/\hbar)$, and using that $\hat{H}_0$ is a linear combination of the generators of the $su(1,1)$ Lie algebra

$$\hat{K}_- = -\frac{i}{2}\frac{\partial^2}{\partial q^2}, \qquad \hat{K}_+ = \frac{i}{2}q^2, \qquad \hat{K}_0 = \frac{1}{2}(q\frac{\partial}{\partial q} + \frac{1}{2}), \tag{4}$$

in coordinate representation, we have

$$\hat{U}_0(t) = \exp\left[\frac{i}{2}f_0(t)q^2\right]\exp\left[h_0(t)\left(q\frac{\partial}{\partial q} + \frac{1}{2}\right)\right]\exp\left[-\frac{i}{2}g_0(t)\frac{\partial^2}{\partial q^2}\right], \tag{5}$$

where $f_0(t)$, $g_0(t)$, and $h_0(t)$ are real-valued functions found as

$$f_0(t) = \frac{\dot{x}_1(t)}{\hbar x_1(t)}, \quad g_0(t) = -\hbar x_0^2\frac{x_2(t)}{x_1(t)}, \quad h_0(t) = -\ln\left|\frac{x_1(t)}{x_0}\right|,$$

with $x_1(t) = x_0\cos(\omega_0 t)$ and $x_2(t) = (1/\omega_0 x_0)\sin(\omega_0 t)$ being solutions of the simple harmonic oscillator $\ddot{x} + \omega_0^2 x = 0$, satisfying the initial conditions $x_1(0) = x_0$, $\dot{x}_1(0) = 0$ and $x_2(0) = 0$, $\dot{x}_2(0) = 1/x_0$, $x_0 \neq 0$. Then, the operator[5] in explicit form becomes

$$\hat{U}_0(t) = \frac{1}{\sqrt{|\cos(\omega_0 t)|}} \exp\left[\frac{-i\omega_0}{2\hbar}\tan(\omega_0 t)q^2\right]\exp\left[-\ln|\cos(\omega_0 t)|q\frac{\partial}{\partial q}\right]\exp\left[\frac{i\hbar}{2\omega_0}\tan(\omega_0 t)\frac{\partial^2}{\partial q^2}\right], \tag{6}$$

and this form of the evolution operator is used to find the time-evolved coherent states (CS) and squeezed coherent states (SCS) of SHO.

A. Time evolution of coherent states of SHO

The coherent states of SHO are usually defined using the unitary displacement operator

$$\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}), \quad \alpha = \alpha_1 + i\alpha_2,\ \alpha_1,\ \alpha_2 \in R, \tag{7}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators, respectively. Using that

$$\hat{a} = \sqrt{\frac{\omega_0}{2\hbar}}q + \sqrt{\frac{\hbar}{2\omega_0}}\frac{\partial}{\partial q}, \quad \hat{a}^\dagger = \sqrt{\frac{\omega_0}{2\hbar}}q - \sqrt{\frac{\hbar}{2\omega_0}}\frac{\partial}{\partial q} \tag{8}$$

in coordinate representation, this operator becomes

$$\hat{D}(\alpha) = \exp\left(-\sqrt{\frac{2\hbar}{\omega_0}}\alpha_1\frac{\partial}{\partial q} + i\sqrt{\frac{2\omega_0}{\hbar}}\alpha_2 q\right),$$

and it can be written as a product of exponential operators, which are group generators associated with the Heisenberg-Weyl algebra defined by

$$\hat{E}_1 = iq, \quad \hat{E}_2 = \frac{\partial}{\partial q}, \quad \hat{E}_3 = i\hat{I}. \tag{9}$$

Indeed, the disentangled form of the displacement operator becomes

$$\hat{D}(\alpha) = \exp(-i\alpha_1\alpha_2)\exp\left(i\sqrt{\frac{2\omega_0}{\hbar}}\alpha_2 q\right)\exp\left(-\sqrt{\frac{2\hbar}{\omega_0}}\alpha_1\frac{\partial}{\partial q}\right),$$

and its application to the ground state $\varphi_0(q) = (\omega_0/\pi\hbar)^{1/4}e^{-\frac{\omega_0}{2\hbar}q^2}$ gives the well-known coherent states of SHO,

$$\phi_\alpha(q) = \left(\frac{\omega_0}{\pi\hbar}\right)^{\frac{1}{4}} \times \exp\left[-i\alpha_1\alpha_2\right]\exp\left[i\sqrt{\frac{2\omega_0}{\hbar}}\alpha_2 q\right] \times \exp\left[-\frac{\omega_0}{2\hbar}\left(q - \sqrt{\frac{2\hbar}{\omega_0}}\alpha_1\right)^2\right]. \tag{10}$$

Then, applying the evolution operator $\hat{U}_0(t)$ given by (6)–(10), the time-evolved coherent states in closed form become

$$\Phi_\alpha^0(q,t) = \left(\frac{\omega_0}{\pi\hbar}\right)^{\frac{1}{4}} \times \exp\left(-\frac{i\omega_0 t}{2}\right) \times \exp\left(\frac{\alpha^2 e^{-2i\omega_0 t} - |\alpha|^2}{2}\right) \times \exp\left(-\left(\sqrt{\frac{\omega_0}{2\hbar}}q - \alpha e^{-i\omega_0 t}\right)^2\right),$$

where superscripts "0" will be used to denote the results for SHO. Probability density is then

$$\rho_\alpha^0(q,t) = \sqrt{\frac{\omega_0}{\pi\hbar}} \times \exp\left\{-\left(\sqrt{\frac{\omega_0}{\hbar}}\left(q - \langle\hat{q}\rangle_\alpha^0(t)\right)\right)^2\right\}, \tag{11}$$

with expectation values

$$\langle\hat{q}\rangle_\alpha^0(t) = \sqrt{\frac{2\hbar}{\omega_0}}\left(\alpha_1\cos(\omega_0 t) + \alpha_2\sin(\omega_0 t)\right), \tag{12}$$

$$\langle\hat{p}\rangle_\alpha^0(t) = \sqrt{2\hbar\omega_0}\left(-\alpha_1\sin(\omega_0 t) + \alpha_2\cos(\omega_0 t)\right) \tag{13}$$

and uncertainties

$$(\Delta\hat{q})^0_\alpha(t) = \sqrt{\frac{\hbar}{2\omega_0}}, \quad (\Delta\hat{p})^0_\alpha(t) = \sqrt{\frac{\omega_0\hbar}{2}}, \quad (\Delta\hat{q}\Delta\hat{p})^0_\alpha(t) = \frac{\hbar}{2}.$$

Thus, we recall that the coherent states of SHO remain coherent under time evolution, there is no squeeze or spread of the wave packets, and the center of the wave packets follows the classical trajectory.

B. Time evolution of squeezed coherent states of SHO

The squeeze operator is a unitary operator mostly known in the form

$$\hat{S}(z) = \exp\left[\frac{1}{2}\left(z\hat{a}^{\dagger 2} - z^*\hat{a}^2\right)\right], \quad z = z_1 + iz_2, z_1, z_2 \in R, \tag{14}$$

where operators $\hat{a}$ and $\hat{a}^\dagger$ are given by (8).[7,22] In coordinate representation, it becomes

$$\hat{S}(z_1, z_2) = \exp\left[-\frac{i}{\hbar\omega_0}z_2\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2} - \frac{\omega_0^2}{2}q^2\right) - z_1\left(q\frac{\partial}{\partial q} + \frac{1}{2}\right)\right]. \tag{15}$$

We note that, when z is real, that is, $z_2 = 0$, one has the usual squeeze operator whose action on analytic functions is well-known, that is,

$$\hat{S}(z_1, 0)f(q) = \exp\left[-z_1\left(q\frac{\partial}{\partial q} + \frac{1}{2}\right)\right]f(q) = e^{-\frac{z_1}{2}}f\left(e^{-z_1}q\right).$$

On the other hand, when z is pure imaginary, i.e., $z_1 = 0$, we have the operator

$$\hat{S}(0, z_2) = \exp\left[-\frac{iz_2}{\hbar\omega_0}\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2} - \frac{\omega_0^2}{2}q^2\right)\right] = \exp\left[-\frac{iz_2}{\hbar\omega_0}\hat{H}_{inv}\right], \tag{16}$$

where $\hat{H}_{inv}$ is the inverted Hamiltonian or the Lagrangian of the harmonic oscillator. The action of this operator on the ground state is

$$\hat{S}(0, z_2)\varphi_0(q) = \left(\frac{\omega_0}{\pi\hbar}\right)^{1/4}\frac{1}{(\cosh 2z_2)^{1/4}} \times \exp\left(-\frac{i}{2}\int_0^{z_2}\frac{dz_2}{\cosh 2z_2}\right)$$
$$\times \exp\left(\frac{i\omega_0}{2\hbar}\tanh z_2\left(1 + \frac{1}{\cosh 2z_2}\right)q^2\right) \times \exp\left(-\frac{\omega_0}{2\hbar}\left(\frac{q}{\sqrt{\cosh 2z_2}}\right)^2\right),$$

which also gives a squeezed state, but with a more complicated phase factor.

Now, using polar representation $z = re^{i\theta}$, with $r \geq 0$, $\theta \in [0, 2\pi)$, the squeeze operator becomes

$$\hat{S}(r, \theta) = \exp\left[r\left(i\frac{\omega_0}{2\hbar}(\sin\theta)q^2 - (\cos\theta)(q\frac{\partial}{\partial q} + \frac{1}{2}) + i\frac{\hbar}{2\omega_0}\sin\theta\frac{\partial^2}{\partial q^2}\right)\right]. \tag{17}$$

Then, it can be disentangled as a product of exponential operators, which are generators of the SU(1, 1) group corresponding to Lie algebra defined by (4), that is,

$$\hat{S}(r, \theta) = \exp\left[\frac{i}{2}f_\theta(r)q^2\right] \times \exp\left[h_\theta(r)\left(q\frac{\partial}{\partial q} + \frac{1}{2}\right)\right] \times \exp\left[-\frac{i}{2}g_\theta(r)\frac{\partial^2}{\partial q^2}\right], \tag{18}$$

where $f_\theta(r)$, $g_\theta(r)$ and $h_\theta(r)$ are real-valued functions. Indeed, taking derivative with respect to r of (17) and (18) and comparing the results, we find that

$$f_\theta(r) = \frac{\omega_0}{\hbar\sin\theta}\left(\frac{y'_{1,\theta}(r)}{y_{1,\theta}(r)}\right), \; f_\theta(0) = 0,$$
$$g_\theta(r) = -\frac{\hbar\sin\theta}{\omega_0}y^2_{1,\theta}(0)\left(\frac{y_{2,\theta}(r)}{y_{1,\theta}(r)}\right), \; g_\theta(0) = 0,$$
$$h_\theta(r) = -\left(r\cos\theta - \ln\left|\frac{y_{1,\theta}(r)}{y_{1,\theta}(0)}\right|\right), \; h_\theta(0) = 0,$$

where $y_{1,\theta}(r)$, $y_{2,\theta}(r)$ are two independent solutions of the classical inverted oscillator

$$y''_{\theta}(r) + 2(\cos\theta)y'_{\theta}(r) - (\sin^2\theta)y_{\theta}(r) = 0, \quad r \geq 0, \quad 0 \leq \theta < 2\pi, \tag{19}$$

satisfying the initial conditions $y_{1,\theta}(0) = y_0 \neq 0$, $y'_{1,\theta}(0) = 0$; $y_{2,\theta}(0) = 0$, $y'_{2,\theta}(0) = 1/y_0$ (prime denotes the derivative with respect to r). In terms of solutions of this differential equation, the squeeze operator becomes

$$\hat{S}(r,\theta) = \sqrt{\frac{y_0}{e^{r\cos\theta}y_{1,\theta}(r)}} \times \exp\left[\frac{i\omega_0}{2\hbar\sin\theta}\left(\frac{y'_{1,\theta}(r)}{y_{1,\theta}(r)}\right)q^2\right]$$
$$\times \exp\left[-\left(r\cos\theta - \ln\left|\frac{y_{1,\theta}(r)}{y_0}\right|\right)q\frac{\partial}{\partial q}\right] \times \exp\left[-\frac{i\hbar\sin\theta}{2\omega_0}y_0^2\left(\frac{y_{2,\theta}(r)}{y_{1,\theta}(r)}\right)\frac{\partial^2}{\partial q^2}\right],$$

and since

$$y_{1,\theta}(r) = y_0 e^{-r\cos\theta}\big(\cosh r + \cos\theta\sinh r\big), \quad y_{2,\theta}(r) = \frac{1}{y_0}e^{-r\cos\theta}\sinh r, \tag{20}$$

we have explicitly

$$\hat{S}(r,\theta) = \frac{1}{\sqrt{\cosh r + \cos\theta\sinh r}} \times \exp\left[\frac{i\omega_0}{2\hbar}\left(\frac{\sin\theta\sinh r}{\cosh r + \cos\theta\sinh r}\right)q^2\right]$$
$$\times \exp\left[-\ln(\cosh r + \cos\theta\sinh r)q\frac{\partial}{\partial q}\right] \times \exp\left[\frac{i\hbar}{2\omega_0}\left(\frac{\sin\theta\sinh r}{\cosh r + \cos\theta\sinh r}\right)\frac{\partial^2}{\partial q^2}\right]. \tag{21}$$

This form of the operator coincides with the squeeze operator derived by Nieto in Ref. 22, but with a slightly different approach.

As a result, the squeezed coherent states (SCS) of SHO, which we denote by $\chi^0_{\alpha,r,\theta}(q)$, are obtained by applying the squeeze and displacement operators to the ground state, that is,

$$\chi^0_{\alpha,r,\theta}(q) = \hat{D}(\alpha)\hat{S}(r,\theta)\varphi_0(q),$$

and explicitly, we get

$$\chi^0_{\alpha,r,\theta}(q) = \sqrt{\frac{\omega_0}{\pi\hbar}} \times \frac{1}{\sqrt{S^0_{r,\theta}}} \times \exp\big[-i\alpha_1\alpha_2\big] \times \exp\left[-\frac{i}{2}\int_0^r \frac{\sin\theta}{(S^0_{r,\theta})^2}dr\right] \times \exp\left[i\alpha_2\sqrt{\frac{2\omega_0}{\hbar}}q\right]$$
$$\times \exp\left[\frac{i\omega_0}{2\hbar}\sin\theta\sinh(2r)\left(\frac{q-\alpha_1\sqrt{2\hbar/\omega_0}}{S^0_{r,\theta}}\right)^2\right] \times \exp\left[-\frac{\omega_0}{2\hbar}\left(\frac{q-\alpha_1\sqrt{2\hbar/\omega_0}}{S^0_{r,\theta}}\right)^2\right], \tag{22}$$

where

$$S^0_{r,\theta} = e^{r\cos\theta}\sqrt{\frac{(y_{1,\theta}(r))^2 + y_0^4\sin^2\theta(y_{2,\theta}(r))^2}{y_0^2}} = \sqrt{\cosh^2 r + \cos\theta\sinh 2r + \sinh^2 r} \tag{23}$$

denotes the initial squeezing, that is, $S^0_{r,\theta}$ is the squeezing coefficient due to the action of the squeeze operator $\hat{S}(r,\theta)$ on the ground state. We note that for $\alpha = 0$, Eq. (22) gives the squeezed ground state, which we denote by $\chi^0_{r,\theta}(q)$.

Then, it is not difficult to show that uncertainties at $\chi^0_{\alpha,r,\theta}(q)$ are

$$(\Delta\hat{q})^0_{r,\theta} = \sqrt{\frac{\hbar}{2\omega_0}}S^0_{r,\theta}, \quad (\Delta\hat{p})^0_{r,\theta} = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{S^0_{r,\theta}}\sqrt{1+\sin^2\theta\sinh^2(2r)},$$

$$(\Delta\hat{q}\Delta\hat{p})^0_{r,\theta} = \frac{\hbar}{2}\sqrt{1+\sin^2\theta\sinh^2(2r)}.$$

From these results, it can be seen that the SCS are minimum uncertainty states only when z is real. More precisely, according to some special values of the phase θ in $z = r\exp(i\theta)$, uncertainties are as follows:

(1) If $\theta = 0$ and $\theta = \pi$, $(z = \pm r)$, then $S^0_{r,\theta} = e^{\pm r}$, and

$$(\Delta\hat{q})^0 = \sqrt{\frac{\hbar}{2\omega_0}}e^{\pm r}, \quad (\Delta\hat{p})^0 = \sqrt{\frac{\omega_0\hbar}{2}}e^{\mp r}, \quad (\Delta\hat{q}\Delta\hat{p})^0 = \frac{\hbar}{2}.$$

Thus, when $z = r$, one has a minimum uncertainty state, which for large values of r spreads in position space and is highly localized in momentum space. When $z = -r$, the minimum uncertainty state is highly localized in position for large r, at the expense of spreading in momentum.

(2) If $\theta = \pi/2$ and $\theta = 3\pi/2$ $(z = \pm ir)$, then $S^0_{r,\theta} = \sqrt{\cosh 2r}$ and

$$(\Delta\hat{q})^0 = \sqrt{\frac{\hbar}{2\omega_0}}\sqrt{\cosh 2r}, \quad (\Delta\hat{p})^0 = \sqrt{\frac{\omega_0\hbar}{2}}\sqrt{\cosh 2r}, \quad (\Delta\hat{q}\Delta\hat{p})^0 = \frac{\hbar}{2}\cosh 2r.$$

In that case, the state is not minimum uncertainty and uncertainties increase with increasing r.

Next, the time-evolved squeezed coherent states of SHO are found according to the definition

$$\chi^0_{\alpha,r,\theta}(q,t) = \hat{U}_0(t)\hat{D}(\alpha)\hat{S}(r,\theta)\varphi_0(q) = \hat{U}_0(t)\chi^0_{\alpha,r,\theta}(q), \tag{24}$$

and they become

$$\begin{aligned}\chi^0_{\alpha,r,\theta}(q,t) = \left(\frac{\omega_0}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{Q^0_{r,\theta}(t)}}\exp\left[-\frac{i}{2}\int_0^r\frac{\sin\theta}{(S^0_{r,\theta})^2}dr\right]\times\exp\left[-\frac{i}{2}\int_0^t\frac{\omega_0}{(Q^0_{r,\theta}(t))^2}dt\right]\\ \times\exp\left[\frac{i\omega_0}{2\hbar}\left[\sin\theta\sinh(2r)+\tan(\omega_0 t)\left(\frac{1+\sin^2\theta\sinh^2(2r)}{S^0_{r,\theta}}\right)\right]\left(\frac{q-\sqrt{\frac{2\hbar}{\omega_0}}\cos(\omega_0 t)\lambda(\alpha,r,\theta)}{Q^0_{r,\theta}(t)}\right)^2\right]\\ \times\exp\left[i\lambda(\alpha,r,\theta)\alpha_2\right]\exp\left[-\frac{i\omega_0}{2\hbar}\tan(\omega_0 t)q^2\right]\exp\left[-\frac{\omega_0}{2\hbar}\left(\frac{q-\sqrt{2\hbar/\omega_0}\cos(\omega_0 t)\lambda(\alpha,r,\theta)}{Q^0_{r,\theta}(t)}\right)^2\right].\end{aligned} \tag{25}$$

Here, we use the notation $\lambda(\alpha, r, \theta) = \lambda_1(\alpha, r, \theta) + i\lambda_2(\alpha, r, \theta)$ with

$$\lambda_1(\alpha,r,\theta) = \alpha_1 - \frac{\sin\theta\sinh(2r)S^0_{r,\theta}}{1+\sin^2\theta\sinh^2(2r)}\alpha_2, \quad \lambda_2(\alpha,r,\theta) = \frac{S^0_{r,\theta}}{1+\sin^2\theta\sinh^2(2r)}\alpha_2, \tag{26}$$

and the squeezing coefficient of the time-evolved SCS for SHO is

$$Q^0_{r,\theta}(t) = \sqrt{\left(S^0_{r,\theta}\cos(\omega_0 t) + \frac{\sin\theta\sinh(2r)}{S^0_{r,\theta}}\sin(\omega_0 t)\right)^2 + \left(\frac{\sin(\omega_0 t)}{S^0_{r,\theta}}\right)^2}. \tag{27}$$

Consequently, the time-dependent squeezing coefficient $Q^0_{r,\theta}(t)$ of SHO is periodic and oscillatory in time, and at initial time $t = 0$, we have $Q^0_{r,\theta}(0) = S^0_{r,\theta}$, as expected.

The probability density for SCS is therefore

$$\rho^0_{\alpha,r,\theta}(q,t) = \sqrt{\frac{\omega_0}{\pi\hbar}}\times\frac{1}{Q^0_{r,\theta}(t)}\exp\left\{-\left(\sqrt{\frac{\omega_0}{\hbar}}\left(\frac{q-\langle\hat{q}\rangle^0_\alpha(t)}{Q^0_{r,\theta}(t)}\right)\right)^2\right\}, \tag{28}$$

where the expectation values of position and momentum at SCS are the same as for the coherent states, and given by (12) and (13), but the uncertainties and the uncertainty relation are as follows:

$$(\Delta\hat{q})^0_{r,\theta}(t) = \sqrt{\frac{\hbar}{2\omega_0}}Q^0_{r,\theta}(t), \quad (\Delta\hat{p})^0_{r,\theta}(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{Q^0_{r,\theta}(t)}\sqrt{1+\left(\frac{Q^0_{r,\theta}(t)\dot{Q}^0_{r,\theta}(t)}{\omega_0}\right)^2},$$

$$(\Delta\hat{q}\Delta\hat{p})^0_{r,\theta}(t) = \frac{\hbar}{2}\sqrt{1+\left(\frac{Q^0_{r,\theta}(t)\dot{Q}^0_{r,\theta}(t)}{\omega_0}\right)^2}.$$

For some special choices of the phase, we have the following cases:

(1) If $\theta = 0$ and $\theta = \pi$, then the squeezing coefficient becomes

$$Q^0_{r,\theta}(t) = \sqrt{e^{\pm 2r}\cos^2(\omega_0 t) + e^{\mp 2r}\sin^2(\omega_0 t)},$$

and the uncertainties at time-evolved SCS for SHO are

$$(\Delta\hat{q})^0_{r,\theta}(t) = \sqrt{\frac{\hbar}{2\omega_0}}\sqrt{e^{\pm 2r}\cos^2(\omega_0 t) + e^{\mp 2r}\sin^2(\omega_0)},$$

$$(\Delta\hat{p})^0_{r,\theta}(t) = \sqrt{\frac{\omega_0\hbar}{2}\left(\frac{1+\left(\sin(2\omega_0 t)\sinh(2r)\right)^2}{e^{\pm 2r}\cos^2(\omega_0 t) + e^{\mp 2r}\sin^2(\omega_0 t)}\right)},$$

$$(\Delta\hat{q}\Delta\hat{p})^0_{r,\theta}(t) = \frac{\hbar}{2}\sqrt{1+\left(\sinh(2r)\sin(2\omega_0 t)\right)^2}.$$

(2) For $\theta = \pi/2$ and $\theta = 3\pi/2$, the squeezing coefficient is

$$Q^0_{r,\theta}(t) = \sqrt{e^{\pm 2r}\cos^2(\frac{\pi}{4} - \omega_0 t) + e^{\mp 2r}\sin^2(\frac{\pi}{4} - \omega_0 t)},$$

and uncertainties become

$$(\Delta\hat{q})^0_{r,\theta}(t) = \sqrt{\frac{\hbar}{2\omega_0}}\sqrt{e^{\pm 2r}\cos^2(\frac{\pi}{4} - \omega_0 t) + e^{\mp 2r}\sin^2(\frac{\pi}{4} - \omega_0 t)},$$

$$(\Delta\hat{p})^0_{r,\theta}(t) = \sqrt{\frac{\omega_0\hbar}{2}}\sqrt{\frac{1+\left(\sinh(2r)\cos(2\omega_0 t)\right)^2}{e^{\pm 2r}\cos^2(\frac{\pi}{4} - \omega_0 t) + e^{\mp 2r}\sin^2(\frac{\pi}{4} - \omega_0 t)}},$$

$$(\Delta\hat{q}\Delta\hat{p})^0_{r,\theta}(t) = \frac{\hbar}{2}\sqrt{1+\left(\sinh(2r)\cos(2\omega_0 t)\right)^2}.$$

Thus, since $Q^0_{r,\theta}(t)$ is periodic and oscillatory in time, so are the uncertainties, and the time-evolved SCS are not minimum uncertainty states. Note that, in all cases, the uncertainty product oscillates with frequency, which is twice the natural frequency ω_0. But, due to phase differences, when z is real, the uncertainty products will be minimum at times $t = (n\pi/2\omega_0)$, $n = 0, 1, 2, \ldots$, and when z is pure imaginary, the uncertainty products attain minimum at times $t = ((2n+1)\pi/4\omega_0)$, $n = 0, 1, 2, \ldots$.

III. THE GENERALIZED QUANTUM PARAMETRIC OSCILLATOR

In this section, we consider the evolution problem for the quantum harmonic oscillator with the most general self-adjoint quadratic Hamiltonian

$$\hat{H}_g(t) = \frac{-\hbar^2}{2\mu(t)}\frac{\partial^2}{\partial q^2} + \frac{\mu(t)\omega^2(t)}{2}q^2 - i\hbar\frac{B(t)}{2}(q\frac{\partial}{\partial q} + \frac{\partial}{\partial q}q) + D(t)q - i\hbar E(t)\frac{\partial}{\partial q} + F(t), \tag{29}$$

where $\mu(t) > 0$, $\omega^2(t) > 0$, $B(t)$, $D(t)$, $E(t)$, and $F(t)$ are real-valued parameters depending on time. Since the Hamiltonian $\hat{H}_g(t)$ is a linear combination of generators of the $su(1, 1)$ and the Heisenberg-Weyl Lie algebras, the evolution operator for the Schrödinger equation can be obtained using the Wei-Norman algebraic approach, and for details, one can see Ref. 31. It is a product of exponential operators, corresponding to multiplication, displacement, squeeze, and generalized rotation as follows:

$$\hat{U}_g(t,t_0) = \exp\left(\frac{i}{\hbar}\int_{t_0}^{t}\left[\frac{-1}{2\mu(s)}p_p^2(s) - E(s)p_p(s) + \frac{\mu(s)\omega^2(s)}{2}x_p^2(s) - F(s)\right]ds\right)$$
$$\times \exp(ip_p(t)q) \times \exp\left(-x_p(t)\frac{\partial}{\partial q}\right) \times \exp\left(i\frac{\mu(t)}{2\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right)q^2\right)$$
$$\times \exp\left(\ln\left|\frac{x_1(t_0)}{x_1(t)}\right|\left(q\frac{\partial}{\partial q} + \frac{1}{2}\right)\right) \times \exp\left(\frac{i}{2}\hbar x_1^2(t_0)\left(\frac{x_2(t)}{x_1(t)}\right)\frac{\partial^2}{\partial q^2}\right). \tag{30}$$

Here, $x_1(t)$, $x_2(t)$ are linearly independent homogeneous solutions of the classical equation of motion

$$\ddot{x} + \frac{\dot{\mu}}{\mu}\dot{x} + \left(\omega^2(t) - \left(\dot{B} + B^2 + \frac{\dot{\mu}}{\mu}B\right)\right)x = -\frac{1}{\mu}D + \dot{E} + \left(\frac{\dot{\mu}}{\mu} + B\right)E, \tag{31}$$

satisfying the initial conditions

$$x_1(t_0) = x_0 \neq 0,\ \dot{x}_1(t_0) = x_0 B(t_0), \qquad x_2(t_0) = 0,\ \dot{x}_2(t_0) = 1/\mu(t_0)x_0, \tag{32}$$

and $x_p(t)$ is a particular solution of (31) satisfying

$$x_p(t_0) = 0, \qquad \dot{x}_p(t_0) = E(t_0). \tag{33}$$

The corresponding equation for momentum is

$$\ddot{p} - \frac{(\dot{\mu\omega^2})}{\mu\omega^2}\dot{p} + \left(\omega^2 + \left(\dot{B} - B^2 - \frac{(\dot{\mu\omega^2})}{\mu\omega^2}B\right)\right)p = -\dot{D} + \left(\frac{(\dot{\mu\omega^2})}{\mu\omega^2} + B\right)D - \mu\omega^2 E \tag{34}$$

with homogeneous solutions $p_1(t) = \mu(t)\big(\dot{x}_1(t) - B(t)x_1(t)\big)$, $p_2(t) = \mu(t)\big(\dot{x}_2(t) - B(t)x_2(t)\big)$ and particular solution $p_p(t) = \mu(t)\big(\dot{x}_p(t) - B(t)x_p(t) - E(t)\big)$.

A. Time evolution of coherent states

The generalized time-evolved coherent states are found by applying the displacement and evolution operators to the ground state, that is, $\Phi_\alpha(q,t) = \hat{U}_g(t,t_0)\hat{D}(\alpha)\varphi_0(q)$. To be able to compare with the generalized squeezed coherent states derived in Sec. III B, we recall their explicit representation as found in Ref. 31, that is,

$$\Phi_\alpha(q,t) = \left(\frac{\omega_0}{\pi\hbar}\right)^{1/4}\frac{1}{\sigma(t)} \times \sqrt{\frac{x_1(t)}{x_0} - i(\omega_0 x_0)x_2(t)}$$
$$\times \exp\left[\frac{i}{\hbar}\int_{t_0}^{t}\left(\frac{-1}{2\mu(s)}p_p^2(s) - E(s)p_p(s) + \frac{\mu(s)\omega^2(s)}{2}x_p^2(s) - F(s)\right)ds\right]$$
$$\times \exp\left[-i(\omega_0)x_2(t)\frac{1}{\sigma^2(t)}\left(x_1(t) - i(\omega_0 x_0^2)x_2(t)\right)\alpha^2 + \frac{\alpha^2 - |\alpha|^2}{2}\right]$$
$$\times \exp\left(\frac{i}{\hbar}p_p(t)q\right) \times \exp\left[\frac{-i}{2\hbar}\mu(t)\left(B(t) - \frac{\dot{\sigma}(t)}{\sigma(t)}\right)(q - x_p(t))^2\right]$$
$$\times \exp\left\{-\frac{1}{\sigma^2(t)}\left[\sqrt{\frac{\omega_0}{2\hbar}}(q - x_p(t)) - \left(\frac{x_1(t)}{x_0} - i(\omega_0 x_0)x_2(t)\right)\alpha\right]^2\right\}, \tag{35}$$

where the squeezing coefficient for the generalized coherent states is

$$\sigma(t) = \sqrt{\frac{x_1^2(t) + (\omega_0 {x_0}^2 x_2(t))^2}{x_0^2}}. \tag{36}$$

The corresponding probability densities become

$$\rho_\alpha(q,t) = \sqrt{\frac{\omega_0}{\pi\hbar}} \times \frac{1}{\sigma(t)} \times \exp\left[-\left(\frac{\omega_0}{\hbar}\right)\left(\frac{q-\langle\hat{q}\rangle_\alpha(t)}{\sigma(t)}\right)^2\right], \tag{37}$$

where the expectation values are

$$\langle\hat{q}\rangle_\alpha(t) = \sqrt{\frac{2\hbar}{\omega_0}}\left(\frac{\alpha_1}{x_0}x_1(t) + \alpha_2(\omega_0 x_0)x_2(t)\right) + x_p(t), \tag{38}$$

$$\langle\hat{p}\rangle_\alpha(t) = \sqrt{\frac{2\hbar}{\omega_0}}\left(\frac{\alpha_1}{x_0}p_1(t) + \alpha_2(\omega_0 x_0)p_2(t)\right) + p_p(t), \tag{39}$$

showing that the center of the wave packets follows the classical trajectory. Then, uncertainties in terms of $\sigma(t)$ are of the form

$$(\Delta\hat{q})_\alpha(t) = \sqrt{\frac{\hbar}{2\omega_0}}\sigma(t),$$

$$(\Delta\hat{p})_\alpha(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{\sigma(t)}\sqrt{1 + \frac{\mu^2(t)\sigma^4(t)}{(\omega_0)^2}\left(\frac{\dot{\sigma}(t)}{\sigma(t)} - B(t)\right)^2},$$

$$(\Delta\hat{q})_\alpha(\Delta\hat{p})_\alpha(t) = \frac{\hbar}{2}\sqrt{1 + \frac{\mu^2(t)\sigma^4(t)}{(\omega_0)^2}\left(\frac{\dot{\sigma}(t)}{\sigma(t)} - B(t)\right)^2}.$$

We note that if $x_1(t)$, $x_2(t)$ are solutions of the simple harmonic oscillator, like in Sec. II for SHO, we have $\sigma(t) = 1$ so there is no squeezing of the wave packets. However, in general, $\sigma(t)$ depends on time, which shows that the time evolution of coherent states do not preserve the minimum uncertainty, and the squeezing properties depend on parameters $\mu(t)$, $\omega^2(t)$, and $B(t)$ of the Hamiltonian.

B. Time evolution of squeezed coherent states

In this part, we provide the main results. First, we obtain time evolution of the squeezed ground state under the generalized evolution operator $\hat{U}_g(t, t_0)$, that is,

$$\chi_{r,\theta}(q,t) = \hat{U}_g(t,t_0)\hat{S}(r,\theta)\varphi_0(q) = \hat{U}_g(t,t_0)\chi^0_{r,\theta}(q), \tag{40}$$

which explicitly becomes

$$\begin{aligned}\chi_{r,\theta}(q,t) = {} & \left(\frac{\omega_0}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{Q_{r,\theta}(t)}} \times \exp\left[-\frac{i}{2}\int_0^r \frac{\sin\theta dr}{(S^0_{r,\theta})^2}\right] \times \exp\left[-\frac{i}{2}\int_{t_0}^t \frac{\omega_0 dt}{\mu(s)(Q_{r,\theta}(s))^2}\right] \\ & \times \exp\left[\frac{i}{\hbar}\int_{t_0}^t\left(-\frac{p_p^2(s)}{2\mu(s)} - E(s)p_p(s) + \frac{\mu(s)\omega^2(s)}{2}x_p^2(s) - F(s)\right)ds\right] \times \exp\left[\frac{i}{\hbar}p_p(t)q\right] \\ & \times \exp\left[\frac{i\omega_0}{2\hbar}\left[\sin\theta\sinh(2r) + \left(\frac{x_0^2\omega_0\left(1+\sin^2\theta\sinh^2(2r)\right)}{(S^0_{r,\theta})^2}\right)\frac{x_2(t)}{x_1(t)}\right]\left(\frac{(q-x_p(t))}{Q_{r,\theta}(t)}\right)^2\right] \\ & \times \exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right)(q-x_p(t))^2\right] \times \exp\left[-\frac{\omega_0}{2\hbar}\left(\frac{(q-x_p(t))}{Q_{r,\theta}(t)}\right)^2\right].\end{aligned} \tag{41}$$

Next, the time evolution of squeezed coherent states under the generalized evolution operator $\hat{U}_g(t, t_0)$ is found according to

$$\chi_{\alpha,r,\theta}(q,t) = \hat{U}_g(t,t_0)\hat{D}(\alpha)\hat{S}(r,\theta)\varphi_0(q) = \hat{U}_g(t,t_0)\chi^0_{\alpha,r,\theta}(q), \tag{42}$$

and this gives the generalized time-dependent squeezed coherent states in the form

$$\chi_{\alpha,r,\theta}(q,t) = \left(\frac{\omega_0}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{Q_{r,\theta}(t)}} \times \exp\left[-\frac{i}{2}\int_0^r \frac{\sin\theta dr}{(S^0_{r,\theta})^2}\right] \times \exp\left[-\frac{i}{2}\int_{t_0}^t \frac{\omega_0 dt}{\mu(s)(Q_{r,\theta}(s))^2}\right]$$
$$\times \exp\left[\frac{i}{\hbar}\int_{t_0}^t\left(-\frac{p_p^2(s)}{2\mu(s)} - E(s)p_p(s) + \frac{\mu(s)\omega^2(s)}{2}x_p^2(s) - F(s)\right)ds\right]$$
$$\times \exp\left[\frac{i}{\hbar}p_p(t)q\right] \times \exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right)(q - x_p(t))^2\right]$$
$$\times \exp\left\{\frac{i\omega_0}{2\hbar}\left[\sin\theta\sinh(2r) + \left(\frac{x_0^2\omega_0\left(1+\sin^2\theta\sinh^2(2r)\right)}{(S^0_{r,\theta})^2}\right)\frac{x_2(t)}{x_1(t)}\right]\right.$$
$$\left.\times\left(\frac{(q-x_p(t)) - \sqrt{2\hbar/\omega_0}x_0^{-1}x_1(t)\lambda(\alpha,r,\theta)}{Q_{r,\theta}(t)}\right)^2\right\}$$
$$\times \exp\left[-\frac{\omega_0}{2\hbar}\left(\frac{(q-x_p(t)) - \sqrt{2\hbar/\omega_0}x_0^{-1}x_1(t)\lambda(\alpha,r,\theta)}{Q_{r,\theta}(t)}\right)^2\right], \tag{43}$$

where $\lambda(\alpha, r, \theta) = \lambda_1(\alpha, r, \theta) + i\lambda_2(\alpha, r, \theta)$ is defined in (26).

For both states (41) and (43), the initial squeezing coefficient $S^0_{r,\theta}$ is given by (23), and

$$Q_{r,\theta}(t) = \sqrt{\left(\frac{S^0_{r,\theta}}{x_0}x_1(t) + \frac{x_0\omega_0\sin\theta\sinh(2r)}{S^0_{r,\theta}}x_2(t)\right)^2 + \left(\frac{x_0\omega_0}{S^0_{r,\theta}}x_2(t)\right)^2}, \tag{44}$$

is the generalized squeezing coefficient, for which we note the following properties:

(i) It depends on the squeezing parameters $r \geq 0$ and $\theta \in [0, 2\pi)$ of the squeeze operator.
(ii) It depends on the solutions $x_1(t)$ and $x_2(t)$ of the classical equation of motion, which, in turn, depend on the time-variable parameters $\mu(t)$, $\omega^2(t)$ and $B(t)$ of the Hamiltonian.
(iii) When the classical solutions are $x_1(t) = x_0\cos(\omega_0 t)$ and $x_2(t) = (1/x_0\omega_0)\sin(\omega_0 t)$, then $Q_{r,\theta}(t)$ is equal to the time-dependent squeezing $Q^0_{r,\theta}(t)$ of the SHO.
(iv) At initial time $t = t_0$, the generalized squeezing reduces to the initial squeezing, that is, $Q_{r,\theta}(t_0) = S^0_{r,\theta}$.
(v) For $r = 0$, we have $S^0_{r,\theta}|_{r=0} = 1$ and $Q_{r,\theta}(t)|_{r=0} = \sigma(t)$. That is, when the squeezing parameter r is zero, the generalized squeezing coefficient $Q_{r,\theta}(t)$ reduces to the squeezing coefficient $\sigma(t)$ for the coherent states. □

Now, knowing the states $\chi_{\alpha,r,\theta}(q, t)$, probability densities $\rho_{\alpha,r,\theta}(q, t) = |\chi_{\alpha,r,\theta}(q, t)|^2$ become

$$\rho_{\alpha,r,\theta}(q,t) = \sqrt{\frac{\omega_0}{\pi\hbar}} \times \frac{1}{Q_{r,\theta}(t)} \times \exp\left\{-\left[\sqrt{\frac{\omega_0}{\hbar}}\left(\frac{q - \langle\hat{q}\rangle_\alpha(t)}{Q_{r,\theta}(t)}\right)\right]^2\right\}, \tag{45}$$

where the expectation values of position and momentum are the same as for the coherent states and given by (38) and (39), but uncertainties and the uncertainty product become

$$(\Delta\hat{q})_{r,\theta}(t) = \sqrt{\frac{\hbar}{2\omega_0}}Q_{r,\theta}(t),$$
$$(\Delta\hat{p})_{r,\theta}(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{Q_{r,\theta}(t)}\sqrt{1 + \frac{(\mu(t)Q^2_{r,\theta}(t))^2}{\omega_0^2}\left(\frac{\dot{Q}_{r,\theta}(t)}{Q_{r,\theta}(t)} - B(t)\right)^2}, \tag{46}$$

$$(\Delta\hat{q}\Delta\hat{p})_{r,\theta}(t) = \frac{\hbar}{2}\sqrt{1 + \frac{(\mu(t)Q^2_{r,\theta}(t))^2}{\omega_0^2}\left(\frac{\dot{Q}_{r,\theta}(t)}{Q_{r,\theta}(t)} - B(t)\right)^2}. \tag{47}$$

For some choices of the phase θ, we have

(1) If $\theta = 0$ and $\theta = \pi$ $(z = \pm r)$, then the generalized squeezing coefficient is

$$Q_{r,\theta}(t) = \sqrt{\left(\frac{e^{\pm r}x_1(t)}{x_0}\right)^2 + \left(x_0\omega_0 e^{\mp r}x_2(t)\right)^2}.$$

(2) If $\theta = \pi/2$ and $\theta = 3\pi/2$ $(z = \pm ir)$, then

$$Q_{r,\theta}(t) = \sqrt{\cosh(2r)\left[\left(\frac{1}{x_0}x_1(t) \pm x_0\omega_0 \tanh(2r)x_2(t)\right)^2 + \left(\frac{x_0\omega_0}{\cosh(2r)}x_2(t)\right)^2\right]}.$$

IV. EXACTLY SOLVABLE MODELS

In this section, we apply our results to find and analyze the behavior of the squeezed coherent states of exactly solvable models, which in this context appear as the simplest generalizations of the SHO. That is, we consider the time-evolution problem for the quantum harmonic oscillator with self-adjoint Hamiltonian of the form

$$\hat{H}(t) = \frac{-\hbar^2}{2}\frac{\partial^2}{\partial q^2} + \frac{\omega_0^2}{2}q^2 - i\hbar\frac{B(t)}{2}\left(q\frac{\partial}{\partial q} + \frac{\partial}{\partial q}q\right) - F\cos(\omega t)q, \tag{48}$$

where $B(t)$ is a real-valued parameter depending on time. Our goal is to investigate the influence of the parameter $z = re^{i\theta}$, together with the influence of some special time-dependent parameters $B(t)$, on the squeezing properties of the wave packets. Since the corresponding classical equation is

$$\ddot{x} + \left(\omega_0^2 - \left(\dot{B}(t) + B^2(t)\right)\right)x = F\cos(\omega t), \tag{49}$$

by requiring that $\dot{B}(t) + B^2(t)$ is a real constant less than ω_0^2, we see that all $B(t) \neq 0$ preserving the harmonic oscillator structure in position space are as follows:

(i) $B(t) = B_0$-constant, where $0 < B_0^2 < \omega_0^2$.
(ii) $B(t) = \Lambda_1 \tanh(\Lambda_1 t + \beta_1)$, where $0 < \Lambda_1^2 < \omega_0^2$ and β_1-arbitrary.
(iii) $B(t) = (t+b)^{-1}$, where b is the arbitrary nonzero constant $(\Lambda_2^2 = 0)$.
(iv) $B(t) = -\Lambda_3 \tan(\Lambda_3 t + \beta_2)$, where $\Lambda_3^2 > 0$ and β_2– arbitrary.

For these choices of $B(t)$, Eq. (49) takes the form $\ddot{x}(t) + \Omega_m^2 x(t) = F\cos(\omega t)$, with modified constant frequency $\Omega_m > 0$, whose exact value depends on $B(t)$.

On the other hand, here driving forces are taken to be $D(t) = -F\cos(\omega t)$, where $\omega > 0$ is the driving frequency and F is the real constant. Since $B(t)$ modifies the frequency, clearly it will have influence also on the displacement of the wave packets. Precisely, $x_p(t)$ will depend on the values of Ω_m and ω, as follows:

(A) If $\omega \neq \Omega_m$, the particular solution satisfying conditions $x_p(0) = 0$ and $\dot{x}_p(0) = 0$ is

$$x_p(t) = F_p\left[\cos(\omega t) - \cos(\Omega_m t)\right] = 2F_p \sin\left(\frac{(\Omega_m - \omega)t}{2}\right)\sin\left(\frac{(\Omega_m + \omega)t}{2}\right),$$

where $F_p = F/(\Omega_m^2 - \omega^2)$ gives the maximum amplitude of the bounded oscillations. Special case of interest occurs, when driving frequency ω is relatively close to Ω_m so that $|\Omega_m - \omega|$ is very small compared with $(\Omega_m + \omega)$ and one can observe formation of beats.

(B) If $\omega = \Omega_m$, that is, the driving frequency is equal to the modified natural frequency, then the particular solution satisfying $x_p(0) = 0$ and $\dot{x}_p(0) = 0$ becomes

$$x_p(t) = \frac{F}{2\omega}t\sin(\omega t),$$

which describes oscillations whose amplitude grows linearly with time t and leads to resonance phenomena. □

In what follows, first we study the quantum oscillator model with $B(t) = B_0$– constant, where $B_0 = 0$ corresponds to SHO. Then, we consider three other models, with the different time-variable parameters $B(t)$ introduced above.

A. Model B_0 constant

First, we consider a generalized harmonic oscillator with Hamiltonian (48), where the squeezing parameter is constant, that is, $B(t) = B_0$ such that $0 \leq B_0^2 < \omega_0^2$. The corresponding classical equation of motion is $\ddot{x}(t) + \Omega_0^2 x(t) = F\cos(\omega t)$, where the modified frequency $\Omega_0 = \sqrt{\omega_0^2 - B_0^2}$ becomes smaller or equal to the natural frequency ω_0, i.e., $0 < \Omega_0 \le \omega_0$, and required initial conditions $x_1(0) = x_0 \neq 0$, $\dot{x}_1(0) = x_0 B_0$ and $x_2(0) = 0$, $\dot{x}_2(0) = 1/x_0$ give the solutions

$$x_1(t) = \frac{x_0\omega_0}{\Omega_0}\cos(\Omega_0 t - \beta), \quad x_2(t) = \frac{1}{x_0\Omega_0}\sin(\Omega_0 t), \quad \beta = \arctan(B_0/\Omega_0).$$

Then, the equation for momentum $\ddot{p}(t) + \Omega_0^2 p(t) = F(\omega\sin(\omega t) - B_0\cos(\omega t))$ has the same modified frequency and its homogeneous solutions are

$$p_1(t) = -x_0\frac{\omega_0^2}{\Omega_0}\sin(\Omega_0 t), \quad p_2(t) = \frac{1}{x_0}\left(\cos(\Omega_0 t) - \frac{B_0}{\Omega_0}\sin(\Omega_0 t)\right).$$

In this model, the generalized squeezing coefficient for the time-evolved squeezed coherent states $\chi_{\alpha,r,\theta}(q,t)$ given by (43) becomes

$$Q_{r,\theta}(t) = \frac{\omega_0}{\Omega_0}\sqrt{\left(S^0_{r,\theta}\cos(\Omega_0 t - \beta) + \frac{\sin\theta\sinh(2r)}{S^0_{r,\theta}}\sin(\Omega_0 t)\right)^2 + \left(\frac{\sin(\Omega_0 t)}{S^0_{r,\theta}}\right)^2}, \tag{50}$$

where $S^0_{r,\theta}$ is given by (23). Thus, it depends on parameters $r \ge 0$, $\theta \in [0, 2\pi)$ and parameter B_0 which determines the modified frequency $\Omega_0 \in (0, \omega_0]$ and phase β. The case $r = 0$ gives the squeezing coefficient for the coherent states $\Phi_\alpha(q,t)$, that is,

$$Q_{r,\theta}(t)|_{r=0} \equiv \sigma(t) = \frac{\omega_0}{\Omega_0}\sqrt{\cos^2(\Omega_0 t - \beta) + \sin^2(\Omega_0 t)}.$$

For the special choices $\theta = 0$ and $\theta = \pi$ ($z = \pm r$), the squeezing coefficient (50) is

$$Q_{r,\theta}(t) = \frac{\omega_0}{\Omega_0}\sqrt{e^{\pm 2r}\cos^2(\Omega_0 t - \beta) + e^{\mp 2r}\sin^2(\Omega_0 t)},$$

and uncertainties become $(\Delta\hat{q})_{r,\theta}(t) = \sqrt{\hbar/2\omega_0}\,Q_{r,\theta}(t)$,

$$(\Delta\hat{p})_{r,\theta}(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{Q_{r,\theta}(t)}\sqrt{1 + \frac{Q^4_{r,\theta}}{\omega_0^2}\left(\frac{\dot{Q}_{r,\theta}(t)}{Q_{r,\theta}(t)} - B_0\right)^2},$$

$$(\Delta\hat{q}\Delta\hat{p})_{r,\theta}(t) = \frac{\hbar}{2}\left\{1 + \frac{\omega_0^2}{4\Omega_0^2}\left[\frac{2B_0}{\Omega_0}\left(e^{\pm 2r}\cos^2(\Omega_0 t - \beta) + e^{\mp 2r}\sin^2(\Omega_0 t)\right)\right.\right.$$
$$\left.\left. + e^{\pm 2r}\sin(2(\Omega_0 t - \beta)) + e^{\mp 2r}\sin(2\Omega_0 t)\right]^2\right\}^{1/2}.$$

Thus, we see that parameter $r > 0$ influences only the amplitude of the oscillating width of the wave packets; however, B_0 has influence not only on the amplitude but also on their frequency and phase. For fixed $r > 0$, when $|B_0|$ approaches $\omega_0 > 0$, frequency Ω_0 decreases, but the amplitude of oscillations increases, as shown in Fig. 1. As expected, when $B_0 = 0$, one has $\Omega_0 = \omega_0$, $\beta = 0$ so that the squeezing

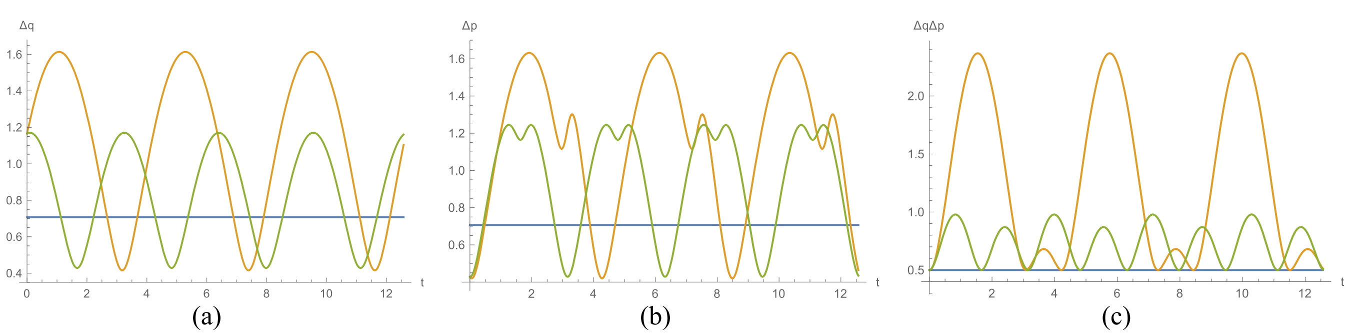


FIG. 1. Model $B(t) = B_0$ with $B_0 = 1/12, 2/3$, $r = 1/2$, $\theta = 0$, and $\omega_0 = \hbar = 1$. (a) Uncertainty $(\Delta\hat{q})_{r,\theta}(t)$, (b) uncertainty $(\Delta\hat{p})_{r,\theta}(t)$, and (c) uncertainty product $(\Delta\hat{q}\Delta\hat{p})_{r,\theta}$.

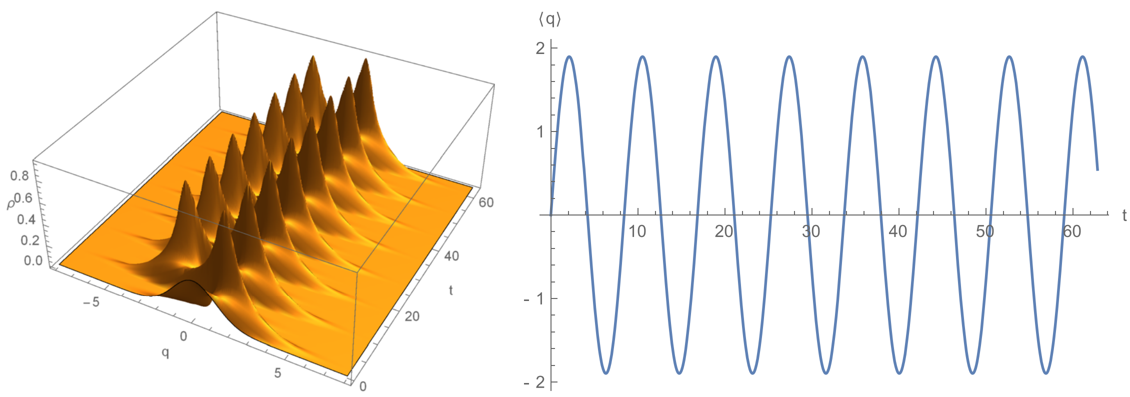


FIG. 2. Model-B_0: (a) probability density $\rho_{\alpha,r,\theta}(q,t)$ with $\alpha = i$, $r = 1/2$, $\theta = 0$, $B_0 = 2/3$, $\omega_0 = \hbar = 1$, and $x_p(t) = 0$; (b) expectation of position $\langle\hat{q}\rangle_\alpha(t)$.

coefficient reduces to that of the SHO. For example, in Fig. 2(a), we show the probability density $\rho_{\alpha,r,\theta}(q,t)$ for $r = 1/2$, $\theta = 0$, and $B_0 = 2/3$, when $D(t) = 0$. The width of the wave packet oscillates with frequency $2\Omega_0 = 2\sqrt{5}/3$, and its center following the classical trajectory $\langle\hat{q}\rangle_\alpha(t)$ given in Fig. 2(b) oscillates with frequency $\Omega_0 = \sqrt{5}/3$.

B. Model 1

Now, we consider the quantum parametric oscillator with Hamiltonian (48) and squeezing parameter $B(t) = \Lambda_1\tanh(\Lambda_1 t)$, $0 < \Lambda_1^2 < \omega_0^2$. The corresponding classical equation of motion in position space is of the form $\ddot{x}(t) + \Omega_1^2 x(t) = F\cos(\omega t)$, $0 < \Omega_1 < \omega_0$, with modified frequency $\Omega_1 = \sqrt{\omega_0^2 - \Lambda_1^2}$, which is less than the natural frequency ω_0, as in the previous model. However, the equation in momentum space becomes

$$\ddot{p}(t) + (\omega_0^2 + \Upsilon_1^2(t))p = F(\omega\sin(\omega t) - \Lambda_1\tanh(\Lambda_1 t)\cos(\omega t)),$$

with frequency modification $\Upsilon_1^2(t) = \Lambda_1^2(1 - 2\tanh^2(\Lambda_1(t))$ depending on time. Therefore, the expectation values at states $\chi_{\alpha,r,\theta}(q,t)$ found using (38) and (39) are

$$\langle\hat{q}\rangle_\alpha(t) = \sqrt{\frac{2\hbar}{\omega_0}}\left(\alpha_1\cos(\Omega_1 t) + \alpha_2\frac{\omega_0}{\Omega_1}\sin(\Omega_1 t)\right) + x_p(t),$$

$$\langle\hat{p}\rangle_\alpha(t) = \sqrt{\frac{2\hbar}{\omega_0}}\Big[-\alpha_1\big(\Omega_1\sin(\Omega_1 t) + \Lambda_1\tanh(\Lambda_1 t)\cos(\Omega_1 t)\big) + \alpha_2\omega_0\Big(\cos(\Omega_1 t) - \frac{\Lambda_1}{\Omega_1}\tanh(\Lambda_1 t)\sin(\Omega_1 t)\Big)\Big] + p_p(t),$$

showing that expectation in position is periodic, but expectation in momentum is no longer periodic due to the influence of $B(t)$. For this model, a generalized squeezing coefficient for the time-evolved squeezed coherent states $\chi_{\alpha,r,\theta}(q,t)$ given by (43) is found as

$$Q_{r,\theta}(t) = \sqrt{\left(S^0_{r,\theta}\cos(\Omega_1 t) + \frac{\omega_0\sin\theta\sinh(2r)}{\Omega_1 S^0_{r,\theta}}\sin(\Omega_1 t)\right)^2 + \left(\frac{\omega_0}{\Omega_1 S^0_{r,\theta}}\sin(\Omega_1 t)\right)^2},$$

which depends on r, θ, and the modified frequency Ω_1. In particular, when $r = 0$, one gets the squeezing coefficient for the time-evolved coherent states $\Phi_\alpha(q,t)$, that is,

$$Q_{r,\theta}(t)|_{r=0} \equiv \sigma(t) = \sqrt{\cos^2(\Omega_1 t) + \frac{\omega_0^2}{\Omega_1^2}\sin^2(\Omega_1 t)}, \quad 0 < \Omega_1 < \omega_0.$$

For the special choices of the phase $\theta = 0$ and $\theta = \pi$, we have the squeezing coefficient

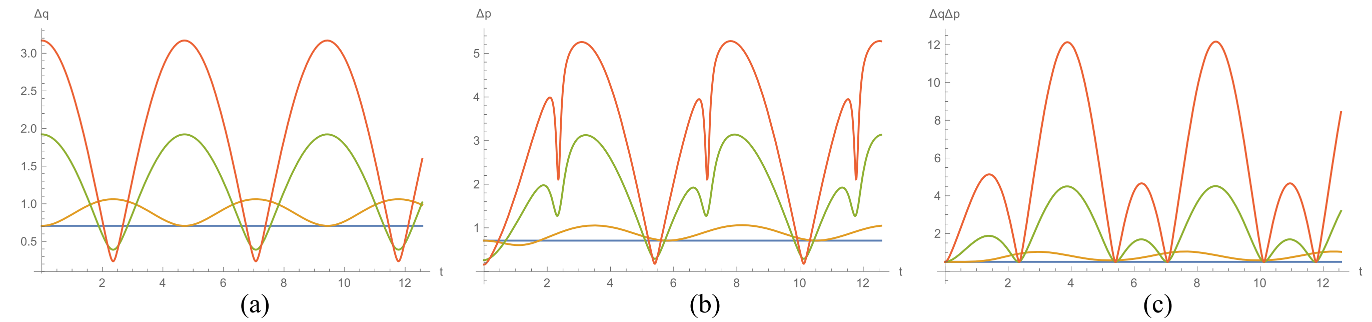


FIG. 3. Model 1: $\omega_0 = \hbar = 1$, $\Omega_1 = 2/3$, $r = 0, 1, 3/2$, $\theta = 0$. (a) Uncertainty $(\Delta\hat{q})_{r,\theta}(t)$, (b)uncertainty $(\Delta\hat{p})_{r,\theta}(t)$, and (c) uncertainty product $(\Delta\hat{q}\Delta\hat{p})_{r,\theta}$.

$$Q_{r,\theta}(t) = \sqrt{e^{\pm 2r}\cos^2(\Omega_1 t) + \frac{\omega_0^2}{\Omega_1^2} e^{\mp 2r}\sin^2(\Omega_1 t)}, \tag{51}$$

and it follows that uncertainties and the uncertainty product at states $\chi_{\alpha,r,\theta}(q,t)$ are $(\Delta\hat{q})_{r,\theta}(t) = \sqrt{\hbar/2\omega_0}Q_{r,\theta}(t)$,

$$(\Delta\hat{p})_{r,\theta}(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{Q_{r,\theta}(t)}\sqrt{1 + \frac{Q_{r,\theta}^4}{\omega_0^2}\left(\frac{\dot{Q}_{r,\theta}(t)}{Q_{r,\theta}(t)} - \Lambda_1\tanh(\Lambda_1 t)\right)^2},$$

$$(\Delta\hat{q}\Delta\hat{p})_{r,\theta}(t) = \frac{\hbar}{2}\left\{1 + \frac{1}{4\omega_0^2}\left[\left(\frac{\omega_0^2}{\Omega_1}e^{\pm 2r} - \Omega_1 e^{\mp 2r}\right)\sin(2\Omega_1 t) - 2\Lambda_1\tanh(\Lambda_1 t)\left(e^{\pm 2r}\cos^2(\Omega_1 t) + \frac{\omega_0^2}{\Omega_1^2}e^{\mp 2r}\sin^2(\Omega_1 t)\right)\right]^2\right\}^{1/2}.$$

In Fig. 3(a), we show that for given Ω_1, the amplitude of oscillations of $(\Delta\hat{q})_{r,\theta}(t)$ increases as r increases. Also, when there are no external forces $[D(t) = 0]$, expectation of position is periodic with classical period $T_1 = 2\pi/\Omega_1$ and uncertainty in position has period $T_1/2$. This shows that the center of the probability density $\rho_{\alpha,r,\theta}(q,t) = |\chi_{\alpha,r,\theta}(q,t)|^2$ given by (45) will oscillate with frequency Ω_1, and its width will oscillate with frequency $2\Omega_1$, where $\Omega_1 \in (0, \omega_0)$. Thus, the wave packet will periodically exhibit exact full revival. When there is a periodic force, behavior will depend on driving frequency ω. For example, in Fig. 4(a), we plot the probability density $\rho_{\alpha,r,\theta}(q,t)$ of the ground

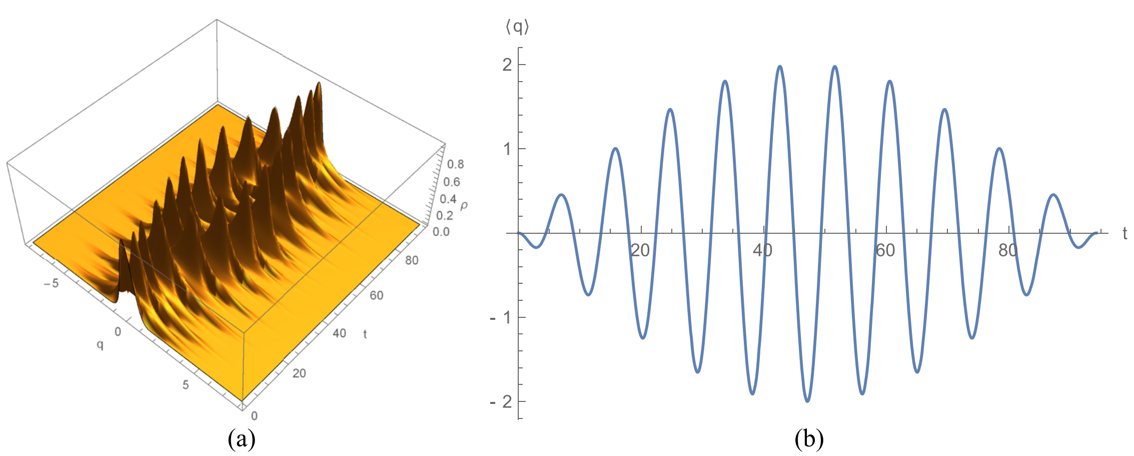


FIG. 4. Model 1: (a) probability density $\rho_{\alpha,r,\theta}(q, t)$ with $\alpha = 0$, $r = 1/2$, $\theta = \pi$, $\Omega_1 = 2/3$, $\omega_0 = \hbar = 1$, and $x_p(t)$ for driving frequency $\omega = 22/30$; (b) expectation of position $\langle\hat{q}\rangle_\alpha(t) = x_p(t)$.

state ($\alpha = 0$), where $\Omega_1 = 2/3$, with $x_p(t)$ at driving frequency $\omega = 22/30$. Since Ω_1 is relatively close to ω, the center of the wave packet follows the periodic beatlike trajectory shown in Fig. 4(b).

C. Model 2

Now, we study the oscillator with Hamiltonian (48) and squeezing parameter $B(t) = 1/(t + b)$, where b is the arbitrary nonzero real constant. We note that for time $t > 0$, if $b > 0$, then $B(t)$ is smooth, but if $b < 0$, it will have a singularity at positive time $t = |b|$. For this model, the classical equation of motion becomes $\ddot{x}(t) + \omega_0^2 x(t) = F\cos(\omega t)$, showing that this choice of $B(t)$ does not influence the natural frequency $\omega_0 > 0$. However, the initial conditions change according to $b \neq 0$ as $x_1(0) = x_0 \neq 0$, $\dot{x}_1(0) = x_0/b$; $x_2(0) = 0$, $\dot{x}_2(0) = 1/x_0$, and this is reflected in the amplitude and phase of the solutions as follows:

$$x_1(t) = x_0\sqrt{1 + \frac{1}{\omega_0^2 b^2}}\cos(\omega_0 t - \delta), \quad x_2(t) = \frac{1}{\omega_0 x_0}\sin(\omega_0 t), \quad \delta = \arctan(1/\omega_0 b).$$

On the other hand, the classical equation for momentum is

$$\ddot{p}(t) + \left(\omega_0^2 + \frac{2}{(t+b)^2}\right)p(t) = F\left(\omega\sin(\omega t) - \left(\frac{1}{t+b}\right)\cos(\omega t)\right),$$

with frequency modification $\Upsilon_2^2(t) = -2/(t+b)^2$ depending explicitly on time. Its homogeneous solutions become

$$p_1(t) = -x_0\sqrt{1 + \frac{1}{\omega_0^2 b^2}}\left(\omega_0\sin(\omega_0 t - \delta) + \frac{\cos(\omega_0 t - \delta)}{t+b}\right), \quad p_2(t) = \frac{1}{\omega_0 x_0}\left(\omega_0\cos(\omega_0 t) - \frac{\sin(\omega_0 t)}{t+b}\right),$$

which are no longer periodic, and if $B(t)$ has singularity, it will be reflected in momentum space. Consequently, the probability densities of the time-evolved squeezed coherent states in position space are smooth and periodic with the generalized squeezing coefficient

$$Q_{r,\theta}(t) = \sqrt{\left(\sqrt{1 + \frac{1}{\omega_0^2 b^2}}S_0(r,\theta)\cos(\omega_0 t - \delta) + \frac{\sin\theta\sinh(2r)}{S_0(r,\theta)}\sin(\omega_0 t)\right)^2 + \left(\frac{\sin(\omega_0 t)}{S_0(r,\theta)}\right)^2},$$

showing that the amplitude of oscillations can be controlled by $r > 0$ and $B(t)$, but in that model without changing the frequency $\omega_0 > 0$. When $r = 0$, squeezing of coherent states $\Phi_\alpha(q, t)$ is determined by

$$Q_{r,\theta}(t)|_{r=0} \equiv \sigma(t) = \sqrt{\left(1 + \frac{1}{\omega_0^2 b^2}\right)\cos^2(\omega_0 t - \delta) + \sin^2(\omega_0 t)}.$$

For the special choices $\theta = 0$ and $\theta = \pi$ ($z = \pm r$), the squeezing coefficient of states $\chi_{\alpha,r,\theta}(q, t)$ given by (43) becomes

$$Q_{r,\theta}(t) = \sqrt{e^{\pm 2r}\left(1 + \frac{1}{\omega_0^2 b^2}\right)\cos^2(\omega_0 t - \delta) + e^{\mp 2r}\sin^2(\omega_0 t)},$$

and uncertainties are found as follows: $(\Delta\hat{q})_{r,\theta}(t) = \sqrt{\hbar/2\omega_0}Q_{r,\theta}(t)$,

$$(\Delta\hat{p})_{r,\theta}(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{Q_{r,\theta}(t)}\sqrt{1 + \frac{Q_{r,\theta}^4(t)}{\omega_0^2}\left(\frac{\dot{Q}_{r,\theta}(t)}{Q_{r,\theta}(t)} - \frac{1}{t+b}\right)^2},$$

$$(\Delta\hat{q})_{r,\theta}(\Delta\hat{p})_{r,\theta}(t) = \frac{\hbar}{2}\left\{1 + \frac{1}{4\omega_0^2}\left[\left(\frac{2}{t+b}\right)\left(e^{\pm 2r}\left(1 + \frac{1}{\omega_0^2 b^2}\right)\cos^2(\omega_0 t - \delta) + e^{\mp 2r}\sin^2(\omega_0 t)\right)\right.\right.$$
$$\left.\left. + e^{\pm 2r}\left(\omega_0 + \frac{1}{\omega_0 b^2}\right)\sin(2(\omega_0 t - \delta)) - \omega_0 e^{\mp 2r}\sin(2\omega_0 t)\right]^2\right\}^{1/2}.$$

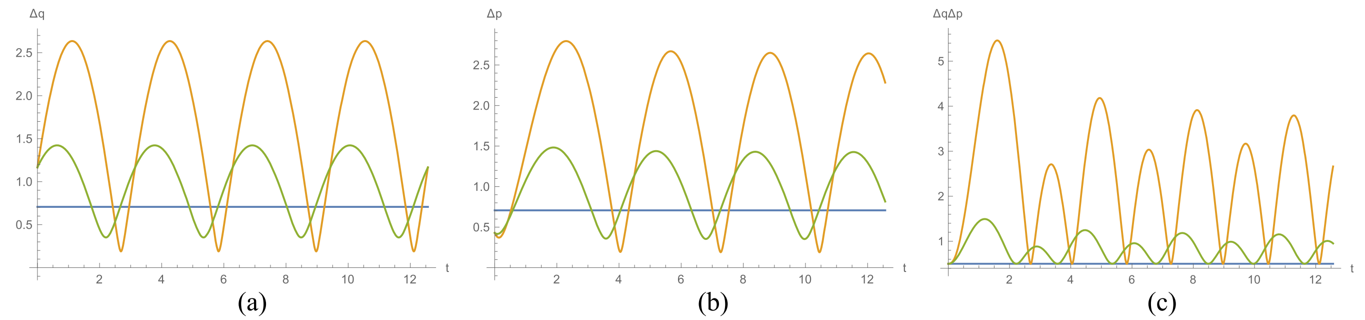


FIG. 5. Model 2: $\omega_0 = \hbar = 1$, $b = 1/2, 3/2$, $r = 1/2$, $\theta = 0$. (a) Uncertainty $(\Delta\hat{q})_{r,\theta}(t)$, (b) uncertainty $(\Delta\hat{p})_{r,\theta}(t)$, and (c) uncertainty product $(\Delta\hat{q}\Delta\hat{p})_{r,\theta}(t)$.

We note that for fixed values of $r > 0$, when $|b|$ increases, the amplitude of oscillations of $(\Delta\hat{q})_{r,\theta}(t)$ decreases and $\delta \to 0$ so that the uncertainty product $(\Delta\hat{q})_{r,\theta}(\Delta\hat{p})_{r,\theta}(t)$ approaches that for the SHO. When $|b| \to 0$, the amplitude of oscillations increases, but the frequency does not change, as shown in Fig. 5. In Fig. 6(a), we plot the probability density $\rho_{\alpha,r,\theta}(q, t)$ with $x_p(t)$ at driving frequency $\omega = 3$, which is different than the natural frequency $\omega_0 = 1$, and observe the squeezing properties of the wave packet, whose center follows the trajectory shown in Fig. 6(b).

D. Model 3

The last quantum oscillator model which we discuss is given by Hamiltonian (48) where parameter $B(t) = -\Lambda_3 \tan(\Lambda_3 t)$, $\Lambda_3^2 > 0$, is periodic with singularities at times $t = (n - 1/2)\pi/\Lambda_3$, $n = 1, 2, \ldots$. The corresponding classical equation of motion is $\ddot{x}(t) + \Omega_3^2 x(t) = F\cos(\omega t)$, $\Omega_3 > \omega_0$, with modified frequency $\Omega_3 = \sqrt{\omega_0^2 + \Lambda_3^2}$, which in this model is greater than the natural frequency $\omega_0 > 0$. The homogeneous solutions $x_1(t) = x_0 \cos(\Omega_3 t)$, $x_2(t) = (1/\Omega_3 x_0) \sin(\Omega_3 t)$ are smooth and periodic. However, in momentum space, we have

$$\ddot{p} + \left(\omega_0^2 + \Upsilon_3^2(t)\right)p = F(\omega \sin(\omega t) + \Lambda_3 \tan(\Lambda_3 t)\cos(\omega t)),$$

where $\Upsilon_3^2(t) = -\Lambda_3^2(1+2\tan^2(\Lambda_3 t))$ is the time-dependent frequency modification, which reflects the singularities of $B(t)$ in the corresponding homogeneous solutions

$$p_1(t) = x_0\left(\Lambda_3 \tan(\Lambda_3 t)\cos(\Lambda_3 t) - \Omega_3 \sin(\Omega_3 t)\right), \quad p_2(t) = \frac{1}{x_0}\left(\frac{\Lambda_3}{\Omega_3}\tan(\Lambda_3 t)\sin(\Omega_3 t) + \cos(\Omega_3 t)\right).$$

In this model, for squeezed coherent states $\chi_{\alpha,r,\theta}(q, t)$, the generalized squeezing coefficient is

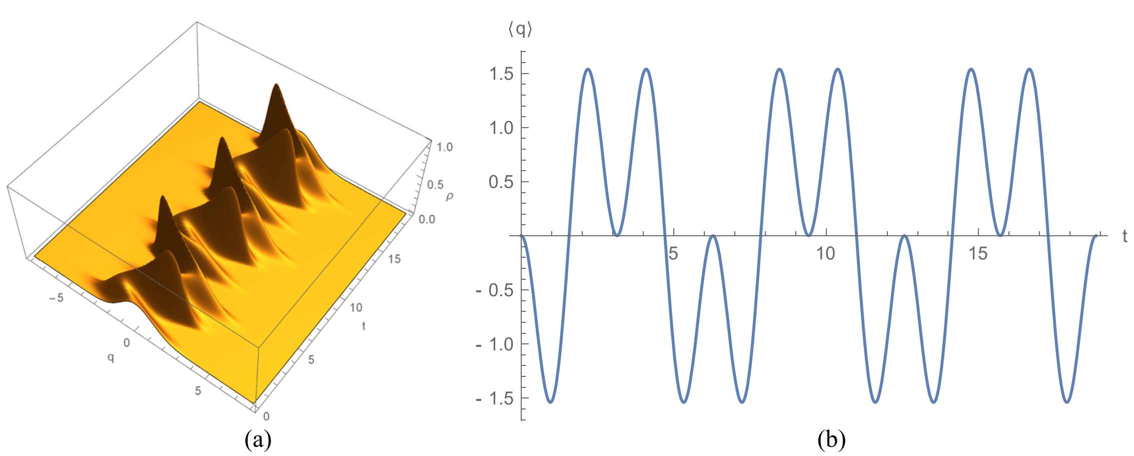
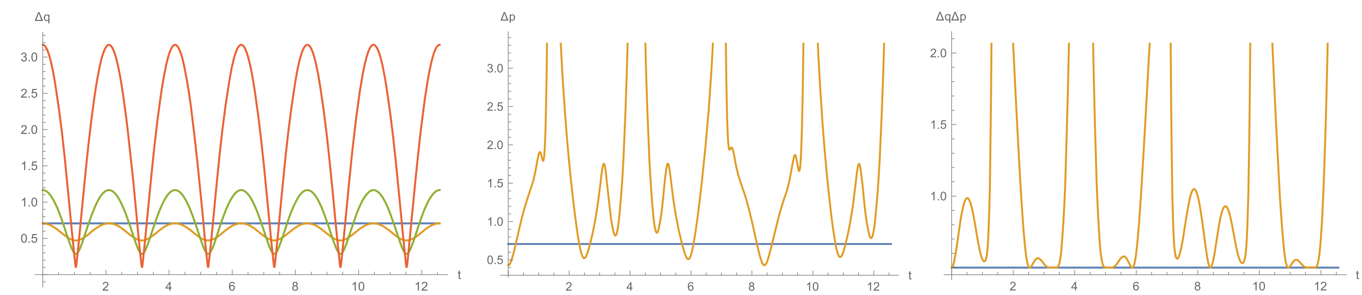


FIG. 6. Model 2: (a) probability density $\rho_{\alpha,r,\theta}(q, t)$ with $\alpha = 0$, $r = 1/2$, $\theta = 0$, $b = 3$, $\omega_0 = \hbar = 1$, and $x_p(t)$ at driving frequency $\omega = 3$; (b) expectation of position $\langle\hat{q}\rangle_0(t) = x_p(t)$.



(a) (b) (c)

FIG. 7. Model 3: $\omega_0 = \hbar = 1$, $\Omega_3 = 3/2$, $\theta = 0$. (a) Uncertainty $(\Delta\hat{q})_{r,\theta}(t)$ for $r = 0, 1/2, 3/2$, $\theta = 0$, (b) uncertainty $(\Delta\hat{p})_{r,\theta}(t)$ for $r = 1/2$, and (c) uncertainty product $(\Delta\hat{q}\Delta\hat{p})_{r,\theta}$ for $r = 1/2$.

$$Q_{r,\theta}(t) = \sqrt{\left(S_0(r,\theta)\cos(\Omega_3 t) + \frac{\omega_0 \sin\theta \sinh(2r)}{\Omega_3 S_0(r,\theta)}\sin(\Omega_3 t)\right)^2 + \left(\frac{\omega_0}{\Omega_3 S_0(r,\theta)}\sin(\Omega_3 t)\right)^2}.$$

When $r = 0$, we obtain the squeezing coefficient of coherent states $\Phi_\alpha(q, t)$ as

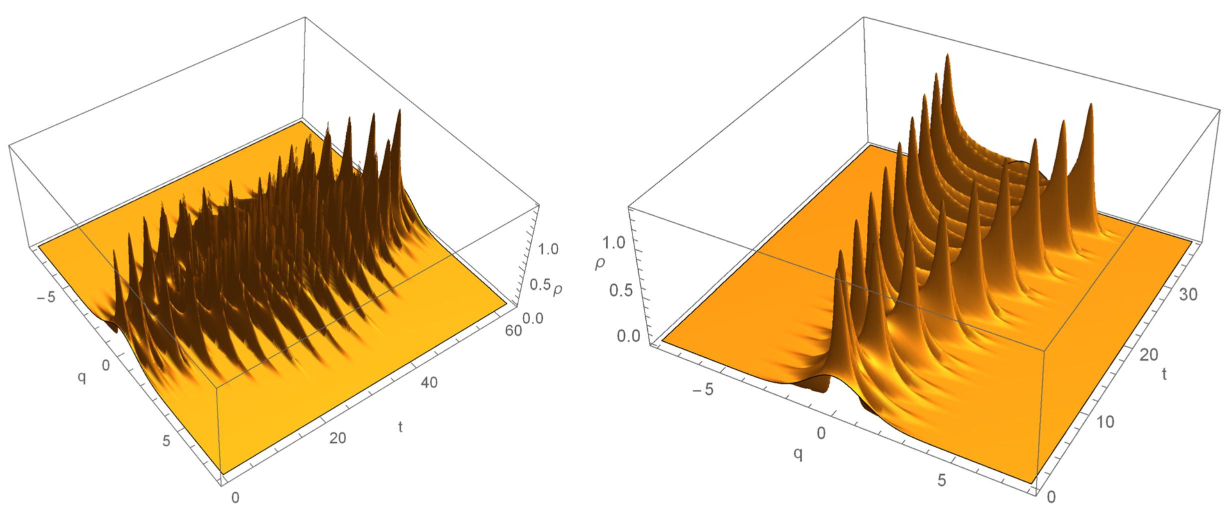
$$Q_{r,\theta}(t)|_{r=0} = \sigma(t) = \sqrt{\cos^2(\Omega_3 t) + \frac{\omega_0^2}{\Omega_3^2}\sin^2(\Omega_3 t)}, \quad \Omega_3 > \omega_0,$$

and we see that both amplitude and frequency of its oscillations can be increased by increasing the value of $\Lambda_3^2 > 0$.

When $\theta = 0$ and $\theta = \pi$, squeezing of the states $\chi_{\alpha,r,\theta}(q, t)$ is determined by

$$Q_{r,\theta}(t) = \sqrt{e^{\pm 2r}\cos^2(\Omega_3 t) + \frac{\omega_0^2}{\Omega_3^2}e^{\mp 2r}\sin^2(\Omega_3 t)},$$

according to which uncertainties become $(\Delta\hat{q})_{r,\theta}(t) = \sqrt{\hbar/2\omega_0}\,Q_{r,\theta}(t)$,



(a) (b)

FIG. 8. Model 3: probability density $\rho_{\alpha,r,\theta}(q, t)$ for $\alpha = 0$, $\Omega_3 = 3/2$, $r = 1/2$, $\theta = 0$, and $\omega_0 = \hbar = 1$ (a) with $x_p(t)$ at driving frequency $\omega = 8/5$ and (b) with $x_p(t)$ at driving frequency $\omega = 3/2$.

$$(\Delta\hat{p})_{r,\theta}(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{Q_{r,\theta}(t)}\sqrt{1+\frac{Q_{r,\theta}^4}{\omega_0^2}\left(\frac{\dot{Q}_{r,\theta}(t)}{Q_{r,\theta}(t)} - \Lambda_3\tan(\Lambda_3 t)\right)^2},$$

$$(\Delta\hat{q}\Delta\hat{p})_{r,\theta}(t) = \frac{\hbar}{2}\Bigg\{1+\frac{1}{4\omega_0^2}\Bigg[\left(\frac{\omega_0^2}{\Omega_3}e^{\pm 2r} - \Omega_3 e^{\mp 2r}\right)\sin(2\Omega_3 t) + 2\Lambda_3\tan(\Lambda_3 t)\left(e^{\pm 2r}\cos^2(\Omega_3 t) + \frac{\omega_0^2}{\Omega_3^2}e^{\mp 2r}\sin^2(\Omega_3 t)\right)\Bigg]^2\Bigg\}^{1/2}.$$

In this model, uncertainty in position is smooth and periodic, with frequency Ω_3 greater than the natural frequency ω_0. On the other hand, uncertainty in momentum and the uncertainty relation have singularities due to $B(t)$ (see Fig. 7). Moreover, for given values of ω_0 and Ω_3, when $r > 0$ increases, the amplitude of oscillations of $(\Delta\hat{q})_{r,\theta}(t)$ also increases. For example, taking $\omega_0 = 1$, $\Omega_3 = 3/2$, the behavior of $(\Delta\hat{q})_{r,\theta}(t)$ for different values $r = 0, 1/2, 3/2$ is shown in Fig. 7(a). Clearly, in general, the width of the wave packets in position space can be increased by increasing both r and Ω_3, but frequency can be increased only by increasing Ω_3. For example, in Fig. 8(a), we plot the probability density $\rho_{\alpha,r,\theta}(q,t)$ with $\alpha = 0$ and $x_p(t)$ at driving frequency ω relatively close to Ω_3 so that the center of the wave packet follows the beatlike trajectory. In Fig. 8(b), the same probability density is shown, but with displacement $x_p(t)$ at resonance driving frequency $\omega = \Omega_3$.

V. CONCLUSION

We considered the most general quantum parametric oscillator, whose Hamiltonian $\hat{H}_g(t)$ can be written as a linear combination of generators of the finite dimensional su(1, 1) and Heisenberg-Weyl Lie algebras, that is, su(1, 1) $\oplus$ W. Consequently, the displacement operator $\hat{D}(\alpha)$, squeeze operator $\hat{S}(z)$, and the evolution operator $\hat{U}_g(t, t_0)$, all being unitary, can be represented as finite products of exponential operators, which are generators of the corresponding Lie groups. Based on these representations, we found the exact time evolution of the squeezed coherent states, their probability densities, expectations, and uncertainties. This allowed us to determine explicitly how displacement of the wave packets depends on the complex parameter α and on all parameters of the Hamiltonian, and how squeezing properties depend on the complex parameter $z = re^{i\theta}$ and the time-dependent parameters $\mu(t)$, $\omega(t)$ and $B(t)$.

As an application of these results, we introduced a generalization of the standard quantum harmonic oscillator by adding to it a mixed term $B(t)(\hat{q}\hat{p} + \hat{p}\hat{q})/2$ and a linear term $D(t)\hat{q}$. Then, we found all parameters $B(t)$ for which the structure of the corresponding classical harmonic oscillator in position space is preserved and separately discussed the quantum models with these different squeezing parameters $B(t)$. It happens that for given frequency $\omega_0 > 0$, the squeezing properties of the wave packets depend both on $r > 0$ and $B(t)$. However, parameters $r > 0$ and $\theta \in [0, 2\pi)$ of the squeeze operator $\hat{S}(r,\theta)$ can be used to control only the amplitude and phase of the oscillating widths of the wave packets, while squeezing parameter $B(t)$ can be used to control not only their amplitude and phase but also their frequency. In fact, model $B(t) = B_0 \neq 0$ and Model 1 show that frequency can be modified by $B(t)$ in such a way that it becomes smaller than the natural frequency ω_0. Model 2 shows that $B(t)$ can be used to change the amplitude of oscillations, without modifying the original frequency ω_0. Finally, in Model 3, we see that $B(t)$ can be used to increase simultaneously the frequency and amplitude of the oscillating widths of the wave packets in position space, and this is achieved by allowing $B(t)$ to have singularities at finite times. These singularities do not appear in position space estimates, but are reflected in momentum uncertainties and the uncertainty relation.

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