Dynamics of squeezed states of a generalized quantum parametric oscillator

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Abstract. Time-evolution of squeezed coherent states of a generalized Caldirola-Kanai type quantum parametric oscillator is found explicitly using the exact evolution operator obtained by the Wei-Norman algebraic approach. Properties of these states are investigated according to the parameters of the unitary squeeze operator and the time-variable parameters of the generalized quadratic Hamiltonian. As an application, we consider exactly solvable quantum models with specific frequency modification for which the corresponding classical oscillator is in underdamping case and driving forces are of sinusoidal type. For each model we explicitly provide the evolution of the squeezed coherent states and discuss their behavior.

1. Introduction

Squeezed states of quantum harmonic oscillator have applications in different areas such as nonlinear optical processes, optical communications, the detection of gravitational waves, electromagnetism, etc. [1, 2, 3, 4].

Squeezed coherent states are mostly known as generalization of coherent states, which obey the minimum uncertainty principle, but have less uncertainty in one quadrature at the expense of increased uncertainty in the other. Their main properties were derived by Stoler [5] and Yuen [1], and then studied by many other authors. As known, squeezed states can be defined in different ways, such as certain linear superposition of the number states, as eigenstates of an operator being a linear combination of the raising and lowering operators, or as a result of applying the squeezing operator, [6].

On the other side, it was shown that one can generate squeezed states of harmonic oscillator by adding to standard Hamiltonian a mixed term with squeezing (two-photon) parameter [1], or simply by adding at some moment of time a quadratic term in position. However, when the oscillator has time-dependent mass or/and frequency squeezing effects appear naturally due to the time-variable parameters and the evolution operator behaves like some kind of generalized squeezing operator.

In our recent work [7], we considered time-evolved coherent states of the generalized Caldirola-Kanai oscillator, and investigated their squeezing properties according to the time-dependent parameters of the Hamiltonian. In the present work, using the evolution operator $\hat{U}(t, t_0)$ obtained by Wei-Norman Lie algebraic approach and the squeeze operator $\hat{S}(z)$, we obtain the exact time-evolution of squeezed coherent states in coordinate representation. Then, we investigate their dependence on the complex parameter $z = re^{i\theta}$ of the squeeze operator and on the time-dependent parameters of the Hamiltonian $\hat{H}(t)$. As an application, we consider exactly solvable quantum models with specific frequency modification for which the corresponding classical oscillator is in underdamping case and driving forces are of sinusoidal type. For each model we explicitly provide the evolution of the squeezed coherent states and discuss their squeezing and displacement properties. We see that, parameter $r > 0$ of the squeeze operator $\hat{S}(r, \theta)$ can be used to control only the amplitude of the oscillating widths of the wave packets, while squeezing parameter $B(t)$ can be used to control not only their amplitude and phase, but also their frequency.

2. Time evolution of squeezed coherent states of a Generalized Caldirola-Kanai Oscillator

We consider the evolution problem for a generalized Caldirola-Kanai oscillator

$$i\hbar\frac{\partial}{\partial t}\Psi(q,t) = \hat{H}(t)\Psi(q,t), \quad t > 0, \tag{1}$$

$$\Psi(q,t_0) = \Psi_0(q), \quad -\infty < q < \infty, \tag{2}$$

with Hamiltonian

$$\hat{H}(t) = \frac{-\hbar^2}{2}e^{-\gamma t}\frac{\partial^2}{\partial q^2} + \frac{\omega_0^2}{2}e^{\gamma t}q^2 - i\hbar\frac{B(t)}{2}\left(q\frac{\partial}{\partial q} + \frac{\partial}{\partial q}q\right) - D_0 e^{\gamma t}\cos(\omega t)q, \tag{3}$$

where ω_0 is a constant frequency, $\mu(t) = e^{\gamma t}, \gamma > 0$, is the exponentially increasing mass, $B(t)$ is a real-valued parameter depending on time, and driving forces are taken to be of sinusoidal kind. Since the Hamiltonian is a linear combination of Heisenberg-Weyl and $su(1,1)$ Lie algebra generators,

$$\hat{E}_1 = iq, \quad \hat{E}_2 = \frac{\partial}{\partial q}, \quad \hat{E}_3 = i\hat{I} \tag{4}$$

$$\hat{K}_- = -\frac{i}{2}\frac{\partial^2}{\partial q^2}, \qquad \hat{K}_+ = \frac{i}{2}q^2, \qquad \hat{K}_0 = \frac{1}{2}(q\frac{\partial}{\partial q} + \frac{1}{2}), \tag{5}$$

the evolution operator of the problem can be found using Wei-Norman algebraic approach, see [14]. Explicitly we have

$$\hat{U}(t,t_0) = \exp\left(\frac{i}{\hbar}\int_{t_0}^{t}\left[\frac{-e^{-\gamma s}}{2}p_p^2(s) + \frac{\omega^2}{2}e^{\gamma s}x_p^2(s)\right]ds\right)\exp\left(ip_p(t)q\right)\exp\left(-x_p(t)\frac{\partial}{\partial q}\right)$$
$$\exp\left(\frac{i}{2\hbar}e^{\gamma t}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right)q^2\right)\exp\left(\ln\left|\frac{x_1(t_0)}{x_1(t)}\right|\left(q\frac{\partial}{\partial q} + \frac{1}{2}\right)\right)\exp\left(\frac{i}{2}\hbar x_1^2(t_0)\left(\frac{x_2(t)}{x_1(t)}\right)\frac{\partial^2}{\partial q^2}\right), \tag{6}$$

where $x_1(t)$, $x_2(t)$ are linearly independent homogeneous solutions of the classical equation of motion

$$\ddot{x} + \gamma\dot{x} + \left(\omega_0^2 - \left(\dot{B} + B^2 + \gamma B\right)\right)x = D_0\cos(\omega t), \tag{7}$$

satisfying the initial conditions $x_1(0) = x_0 \neq 0, \dot{x}_1(0) = x_0 B(0)$; $x_2(0) = 0, \dot{x}_2(0) = 1/x_0$, and $x_p(t)$ is a particular solution of (7) satisfying $x_p(0) = 0, \dot{x}_p(0) = 0$. The corresponding equation of motion for classical momentum is

$$\ddot{p} - \gamma\dot{p} + (\omega_0^2 + (\dot{B} - B^2 - \gamma B))p = -D_0 e^{\gamma t}(\omega\sin(\omega t) + B(t)\cos(\omega t)), \tag{8}$$

with homogeneous solutions $p_1(t) = e^{\gamma t}\Big(\dot{x}_1(t) - B(t)x_1(t)\Big)$, $p_2(t) = e^{\gamma t}\Big(\dot{x}_2(t) - B(t)x_2(t)\Big)$, and particular solution $p_p(t) = e^{\gamma t}\Big(\dot{x}_p(t) - B(t)x_p(t)\Big)$.

As well-known, coherent states of standard harmonic oscillator can be defined by the action of the displacement operator

$$\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}), \quad \alpha = \alpha_1 + i\alpha_2,\ \alpha_1,\ \alpha_2 \in R, \tag{9}$$

on the ground state. Here, $\hat{a}$ and $\hat{a}^\dagger$ are the usual annihilation and creation operators, which in coordinate representation take form

$$\hat{a} = \sqrt{\frac{\omega_0}{2\hbar}}q + \sqrt{\frac{\hbar}{2\omega_0}}\frac{\partial}{\partial q}, \quad \hat{a}^\dagger = \sqrt{\frac{\omega_0}{2\hbar}}q - \sqrt{\frac{\hbar}{2\omega_0}}\frac{\partial}{\partial q}, \tag{10}$$

and using the ground state $\varphi_0(q) = (\omega_0/\pi\hbar)^{1/4}e^{-\frac{\omega_0}{2\hbar}q^2}$, one gets the standard time-independent coherent states

$$\phi^0_\alpha(q) = \hat{D}(\alpha)\varphi_0(q) = \left(\frac{\omega_0}{\pi\hbar}\right)^{1/4} e^{-\frac{i}{2\hbar}\langle\hat{q}\rangle_\alpha\langle\hat{p}\rangle_\alpha} e^{\frac{i}{\hbar}\langle\hat{p}\rangle_\alpha q} e^{-\frac{\omega_0}{2\hbar}(q-\langle\hat{q}\rangle_\alpha)^2}, \tag{11}$$

where $\langle\hat{q}\rangle_\alpha = \sqrt{2\hbar/(\omega_0)}\,\alpha_1$, $\langle\hat{p}\rangle_\alpha = \sqrt{2\omega_0\hbar}\,\alpha_2$, $\alpha = \alpha_1 + i\alpha_2, \alpha_1, \alpha_2-$ real constants.

On the other side, squeezed states can be found by applying the unitary squeeze operator

$$\hat{S}(z) = \exp\left[\frac{1}{2}(z\hat{a}^{\dagger 2} - z^*\hat{a}^2)\right], \quad z = z_1 + iz_2, z_1, z_2 \in R, \tag{12}$$

as introduced in [11] and discussed for example in [12]. Using polar coordinate representation $z = re^{i\theta}$, with $r \geq 0$, $\theta \in [0, 2\pi)$, squeeze operator becomes

$$\hat{S}(r,\theta) = \exp\left[r\left(i\frac{\omega_0}{2\hbar}(\sin\theta)q^2 - (\cos\theta)(q\frac{\partial}{\partial q} + \frac{1}{2}) + i\frac{\hbar}{2\omega_0}\sin\theta\frac{\partial^2}{\partial q^2}\right)\right], \tag{13}$$

and it can be disentangled as product of exponential operators in the form

$$\hat{S}(r,\theta) = \frac{1}{\sqrt{\cosh r + \cos\theta\sinh r}} \times \exp\left[\frac{i\omega_0}{2\hbar}\left(\frac{\sin\theta\sinh r}{\cosh r + \cos\theta\sinh r}\right)q^2\right]$$
$$\times \exp\left[-\ln(\cosh r + \cos\theta\sinh r)q\frac{\partial}{\partial q}\right] \times \exp\left[\frac{i\hbar}{2\omega_0}\left(\frac{\sin\theta\sinh r}{\cosh r + \cos\theta\sinh r}\right)\frac{\partial^2}{\partial q^2}\right]. \tag{14}$$

Therefore, time-independent squeezed coherent states of standard harmonic oscillator, which we denote by $\chi^0_{\alpha,r,\theta}(q)$ are obtained using

$$\chi^0_{\alpha,r,\theta}(q) = \hat{D}(\alpha)\hat{S}(r,\theta)\varphi_0(q),$$

and explicitly we get

$$\chi^0_{\alpha,r,\theta}(q) = \sqrt{\frac{\omega_0}{\pi\hbar}} \times \frac{1}{\sqrt{S^0_{r,\theta}}} \times \exp[-i\alpha_1\alpha_2] \times \exp\left[-\frac{i}{2}\int_0^r \frac{\sin\theta}{(S^0_{r,\theta})^2}dr\right] \times \exp\left[i\alpha_2\sqrt{\frac{2\omega_0}{\hbar}}q\right]$$
$$\times \exp\left[\frac{i\omega_0}{2\hbar}\sin\theta\sinh(2r)\left(\frac{q - \alpha_1\sqrt{2\hbar/\omega_0}}{S^0_{r,\theta}}\right)^2\right] \times \exp\left[-\frac{\omega_0}{2\hbar}\left(\frac{q - \alpha_1\sqrt{2\hbar/\omega_0}}{S^0_{r,\theta}}\right)^2\right], \tag{15}$$

where

$$S^0_{r,\theta} = \sqrt{\cosh^2 r + \cos\theta\sinh 2r + \sinh^2 r}, \tag{16}$$

denotes the initial squeezing, that is $S^0_{r,\theta}$ is the squeezing coefficient due to the action of the squeeze operator $\hat{S}(r,\theta)$ on the ground state. In this work, we shall consider the cases when $\theta = 0$ and $\theta = \pi$, which lead to squeezing of the form $S^0_{r,\theta} = e^{\pm r}$.

Then, time-evolution of squeezed coherent states is found according to

$$\chi_{\alpha,r,\theta}(q,t) = \hat{U}(t,t_0)\chi^0_{\alpha,r,\theta}(q), \tag{17}$$

and explicitly we get

$$\begin{aligned}
\chi_{\alpha,r,\theta}(q,t) &= \Big(\frac{\omega_0}{\pi\hbar}\Big)^{\frac{1}{4}} \frac{1}{\sqrt{Q_{r,\theta}(t)}} \times \exp\Big[-\frac{i}{2}\int_{t_0}^{t} \frac{\omega_0 e^{-\gamma s} ds}{Q^2_{r,\theta}(s)}\Big] \\
&\times \exp\Big[\frac{i}{\hbar}\int_{t_0}^{t}\Big(-\frac{e^{\gamma s}}{2}(\dot{x}_p(s) - B(s)x_p(s))^2 + \frac{\omega_0^2 e^{\gamma s}}{2}x_p^2(s)\Big)ds\Big] \\
&\times \exp\Big[\frac{i}{\hbar}(\dot{x}_p(t) - B(t)x_p(t))q\Big] \times \exp\Big[\frac{ie^{\gamma t}}{2\hbar}\Big(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\Big)(q - x_p(t))^2\Big] \\
&\times \exp\Big\{\frac{i}{2\hbar}\Big(\frac{\omega_0 x_0}{e^{\pm r}}\Big)^2 \frac{x_2(t)}{x_1(t)}\Big(\frac{(q - x_p(t)) - \sqrt{2\hbar/\omega_0}x_0^{-1}x_1(t)\alpha_1 + ie^{\pm r}\alpha_2}{Q_{r,\theta}(t)}\Big)^2\Big\} \\
&\times \exp\Big[-\frac{\omega_0}{2\hbar}\Big(\frac{(q - x_p(t)) - \sqrt{2\hbar/\omega_0}x_0^{-1}x_1(t)\alpha_1 + ie^{\pm r}\alpha_2}{Q_{r,\theta}(t)}\Big)^2\Big],
\end{aligned} \tag{18}$$

where

$$Q_{r,\theta}(t) = \sqrt{\Big(\frac{e^{\pm r}x_1(t)}{x_0}\Big)^2 + \Big(x_0\omega_0 e^{\mp r}x_2(t)\Big)^2}. \tag{19}$$

is the generalized squeezing coefficient for $\theta = 0$ and $\theta = \pi$. The corresponding probability density $\rho_{\alpha,r,\theta}(q,t) = |\chi_{\alpha,r,\theta}(q,t)|^2$ becomes

$$\rho_{\alpha,r,\theta}(q,t) = \sqrt{\frac{\omega_0}{\pi\hbar}} \times \frac{1}{Q_{r,\theta}(t)} \times \exp\Big\{-\Big[\sqrt{\frac{\omega_0}{\hbar}}\Big(\frac{q - \langle\hat{q}\rangle_\alpha(t)}{Q_{r,\theta}(t)}\Big)\Big]^2\Big\}, \tag{20}$$

where expectation values of position and momentum are

$$\langle\hat{q}\rangle_\alpha(t) = \sqrt{\frac{2\hbar}{\omega_0}}\Big(\frac{\alpha_1}{x_0}x_1(t) + \alpha_2(\omega_0 x_0)x_2(t)\Big) + x_p(t), \tag{21}$$

$$\langle\hat{p}\rangle_\alpha(t) = \sqrt{\frac{2\hbar}{\omega_0}}\Big(\frac{\alpha_1}{x_0}p_1(t) + \alpha_2(\omega_0 x_0)p_2(t)\Big) + p_p(t), \tag{22}$$

and uncertainties and uncertainty product become

$$\begin{aligned}
(\Delta\hat{q})_{r,\theta}(t) &= \sqrt{\frac{\hbar}{2\omega_0}}Q_{r,\theta}(t), \\
(\Delta\hat{p})_{r,\theta}(t) &= \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{Q_{r,\theta}(t)}\sqrt{1 + \frac{(e^{\gamma t}Q^2_{r,\theta}(t))^2}{\omega_0^2}\Big(\frac{\dot{Q}_{r,\theta}(t)}{Q_{r,\theta}(t)} - B(t)\Big)^2},
\end{aligned} \tag{23}$$

$$(\Delta\hat{q}\Delta\hat{p})_{r,\theta}(t) = \frac{\hbar}{2}\sqrt{1 + \frac{(e^{\gamma t}Q^2_{r,\theta}(t))^2}{\omega_0^2}\Big(\frac{\dot{Q}_{r,\theta}(t)}{Q_{r,\theta}(t)} - B(t)\Big)^2}. \tag{24}$$

We note that for $r = 0$, above results reduce to that for the time-evolved coherent states.

3. Exactly solvable models

Squeezing coefficient $Q_{r,\theta}(t)$ given by (19), clearly depends not only on the parameters r and θ of the squeezing operator, but also on the homogeneous solutions $x_1(t)$ and $x_2(t)$ of the classical equation of motion, which in turn depend on the time-variable parameter $B(t)$. In this section, we find and discuss the influence of these parameters on the squeezing properties of the wave packets.

From the classical equation of motion given by Eq.(7), one can see that, in general parameter $B(t)$ can essentially modify the original frequency. In this work, we choose $B(t)$ such that the Caldirola-Kanai type oscillator structure is preserved. Precisely, we take $-(\dot{B}+B^2+\gamma B)=\Lambda_0^2$, where $\Lambda_0^2>\omega_0^2$. Therefore, the classical equation becomes

$$\ddot{x}+\gamma\dot{x}+(\omega_0^2+\Lambda_0^2)x=D_0\cos(\omega t), \tag{25}$$

with constant frequency $\omega_0^2+\Lambda_0^2>0$, Λ_0^2- being the frequency modification in position space, and

$$\Omega_d^2=\omega_0^2+\Lambda_0^2-\gamma^2/4$$

gives the frequency Ω_d of the modified damped oscillator. Depending on the sign of $\Omega_B^2=\Lambda_0^2-\gamma^2/4$, time-dependent functions $B(t)$ satisfying these conditions are:

(i) $B(t)=-(\gamma/2)+\Omega_B'\tanh(\Omega_B't-\beta)$, $\Omega_B'=\sqrt{(\gamma^2/4)-\Lambda_0^2}$, when $-\omega_0^2<\Lambda_0^2<\gamma^2/4$,

(ii) $B(t)=-(\gamma/2)+b/(1+bt)$, when $\Lambda_0^2=\gamma^2/4$,

(iii) $B(t)=-(\gamma/2)-\Omega_B\tan(\Omega_Bt-\beta)$, $\Omega_B=\sqrt{\Lambda_0^2-(\gamma^2/4)}$, when $\Lambda_0^2>\gamma^2/4$.

According to this, we investigate the following three models.

3.1. Model 1

First, we consider a generalized Caldirola-Kanai oscillator with Hamiltonian (3) and squeezing parameter $B(t)=-(\gamma/2)+\Omega_B'\tanh(\Omega_B't)$, where $\Omega_B'=\sqrt{\gamma^2/4-\Lambda_0^2}$ and $-\omega_0^2<\Lambda_0^2<\gamma^2/4$. Then, $\Omega_d^2=\omega_0^2+\Lambda_0^2-\gamma^2/4$ and we have the cases: (i) $\Omega_d^2<0$ (overdamping), (ii) $\Omega_d^2=0$ (critical damping), and (iii) $\Omega_d^2>0$ (underdamping). We give the results only for the underdamping case. That is, let $\Omega_d^2>0$, which means $-\omega_0^2+\gamma^2/4<\Lambda_0^2<\gamma^2/4$. Then, homogenous solutions of the classical equation in position space are

$$x_1(t)=x_0e^{-\gamma t/2}\cos(\Omega_dt),\quad x_2(t)=\frac{1}{x_0\Omega_d}e^{-\gamma t/2}\sin(\Omega_dt),$$

and particular solution is

$$x_p(t)=A_he^{-\gamma t/2}\cos(\Omega_dt-\beta_h)+A_p\cos(\omega t-\delta_p),$$

where A_h and θ_h are constants such that $x_p(t)$ satisfies the initial conditions $x_p(0)=0,\dot{x}_p(0)=0$. The amplitude and phase shift of the steady-state part are

$$A_p=\frac{D_0}{\sqrt{((\omega_0^2+\Lambda_0^2)-\omega^2)^2+\gamma^2\omega^2}},\quad \delta_p=\tan^{-1}\left(\frac{\gamma\omega}{(\omega_0^2+\Lambda_0^2)-\omega^2}\right),$$

and resonance frequency and maximum amplitude are found as

$$\omega_{res}=\sqrt{(\omega_0^2+\Lambda_0^2)-\gamma^2/2},\quad A_p(\omega_{res})=\frac{D_0}{\sqrt{(\omega_0^2+\Lambda_0^2)\gamma^2-\frac{\gamma^4}{4}}},\quad \Lambda_0^2>\gamma^2/4.$$

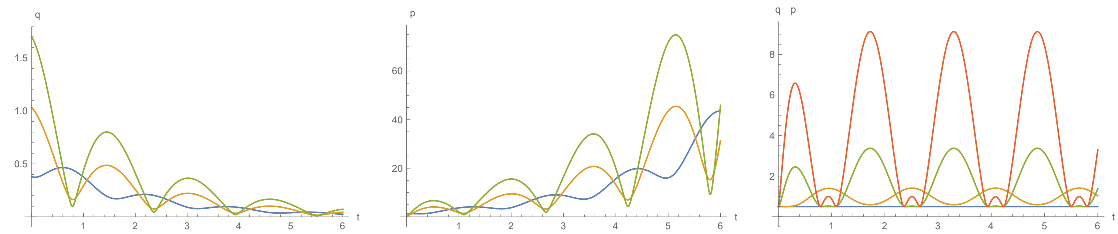
For the special choices $\theta = 0$ and $\theta = \pi (z = \pm r)$, squeezing coefficient is

$$Q_{r,\theta}(t) = e^{-\gamma t/2}\sqrt{e^{\pm 2r}\cos^2(\Omega_d t) + \frac{\omega_0^2}{\Omega_d^2}e^{\mp 2r}\sin^2(\Omega_d t)}, \tag{26}$$

and uncertainties become $(\Delta\hat{q})_{r,\theta}(t) = \sqrt{\hbar/2\omega_0}Q_{r,\theta}(t)$,

$$(\Delta\hat{p})_{r,\theta}(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{Q_{r,\theta}(t)}\sqrt{1 + \frac{e^{2\gamma t}Q_{r,\theta}^4}{\omega_0^2}\left(\frac{\dot{Q}_{r,\theta}(t)}{Q_{r,\theta}(t)} + \frac{\gamma}{2} - \Omega_B'\tanh(\Omega_B' t)\right)^2},$$

$$(\Delta\hat{q}\Delta\hat{p})_{r,\theta}(t) = \frac{\hbar}{2}\Bigg\{1 + \frac{1}{\omega_0^2}\Bigg[\Omega_B'\tanh(\Omega_B' t)(e^{\pm 2r}\cos^2(\Omega_d t) + \frac{\omega_0^2}{\Omega_d^2}e^{\mp 2r}\sin^2(\Omega_d t)) + \frac{1}{2}\left(\Omega_d e^{\pm 2r} - \frac{\omega_0^2}{\Omega_d}e^{\mp 2r}\right)\sin(2\Omega_d t)\Bigg]^2\Bigg\}^{1/2}.$$



(a) (b) (c)

Figure 1. Model 1:For $\omega_0 = \sqrt{12}, \gamma = 1$, $\Lambda_0^2 = -31/4, \Omega_d = 2, \Omega_B' = 2\sqrt{2}$, $r = 0, 1, 3/2, \theta = 0$. (a)Uncertainty $(\Delta\hat{q})_{r,\theta}(t)$, (b)Uncertainty $(\Delta\hat{p})_{r,\theta}(t)$, (c) Uncertainty product $(\Delta\hat{q}\Delta\hat{p})_{r,\theta}$.

In Fig.1(a), we show that for given values γ, ω_0 and Λ_0^2, when r increases, the amplitude of oscillations of $(\Delta\hat{q})_{r,\theta}(t)$ increases. As an example, in Fig.2(a), we plot the probability density $\rho_{\alpha,r,\theta}(q,t)$ of the ground state ($\alpha = 0$) without displacement ($x_p(t) = 0$) and observe oscillatory squeezing of the width. In Fig.2(b), we show $\rho_{\alpha,r,\theta}(q,t)$ under periodic displacement $x_p(t) = \cos(\sqrt{23/2}t - \tan^{-1}(\sqrt{46}))$ at resonance frequency $\omega = \sqrt{23/2}$.

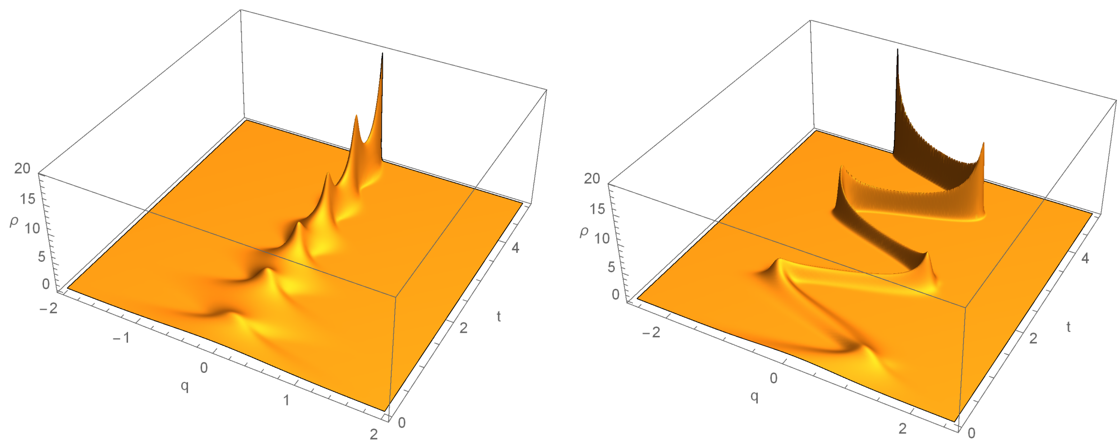
3.2. Model 2

Next, we consider the quantum Hamiltonian given by (3) with squeezing parameter $B(t) = -(\gamma/2) + b/(1 + bt)$, where b is an arbitrary constant. Here, since $\Lambda_0^2 = \gamma^2/4$ for any real constant b, we have $\Omega_d = \omega_0$. Thus, no matter what is the sign of $\Omega_0^2 = \omega_0^2 - \gamma^2/4$ for the original oscillator, if one adds to the system $B(t)$ defined here, the new oscillator always becomes in the underdamping case. The corresponding homogenous solutions are of the form

$$x_1(t) = \frac{x_0}{\omega_0}\sqrt{\omega_0^2 + b^2}e^{-\gamma t/2}\cos(\omega_0 t - \delta), \quad x_2(t) = \frac{1}{\omega_0 x_0}e^{-\gamma t/2}\sin(\omega_0 t), \quad \delta = \tan^{-1}(b/\omega_0),$$

and particular solution $x_p(t) = x_h(t) + A_p\cos(\omega t - \delta_p)$, where $x_h(t)$ is the transient part such that $x_p(t)$ satisfies initial conditions $x_p(0) = 0, \dot{x}_p(0) = 0$. The amplitude and phase shift of the steady-state part of $x_p(t)$ are

$$A_p = \frac{D_0}{\sqrt{((\omega_0^2 + \gamma^2/4) - \omega^2)^2 + \gamma^2\omega^2}}, \quad \delta_p = \arctan\left(\frac{\gamma\omega}{(\omega_0^2 + \gamma^2/4) - \omega^2}\right).$$



(a) (b)

Figure 2. Model 1: Probability density $\rho_{\alpha,r,\theta}(q,t)$ with $\gamma=1, \omega_0=\sqrt{12}, \Lambda_0^2=0, \hbar=1, \alpha=0, r=1/2, \theta=0$ (a)$\rho_{\alpha,r,\theta}(q,t)$ without displacement and (b) $\rho_{\alpha,r,\theta}(q,t)$ displaced by $x_p(t)$ at resonance frequency $\omega=\sqrt{23/2}, D_0=\sqrt{47}/2$.

For $\theta=0$ and $\theta=\pi$, we have squeezing coefficient

$$Q_{r,\theta}(t)|_{r=0} = e^{-\gamma t/2}\sqrt{\left(\frac{\omega_0^2+b^2}{\omega_0^2}\right)e^{\pm 2r}\cos^2(\omega_0 t-\delta)+e^{\mp 2r}\sin^2(\omega_0 t)},$$

which is smooth for any constant b, and uncertainties are found as follows:

$$(\Delta\hat{q})_{r,\theta}(t) = \sqrt{\hbar/2\omega_0 Q_{r,\theta}(t)},$$

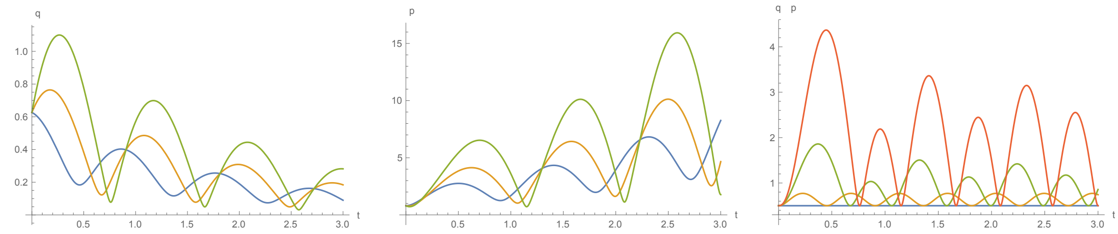
$$(\Delta\hat{p})_{r,\theta}(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{Q_{r,\theta}(t)}\sqrt{1+\frac{e^{2\gamma t}Q_{r,\theta}^4}{\omega_0^2}\left(\frac{\dot{Q}_{r,\theta}(t)}{Q_{r,\theta}(t)}+\frac{\gamma}{2}-\frac{b}{1+bt}\right)^2},$$

$$(\Delta\hat{q})_{r,\theta}(\Delta\hat{p})_{r,\theta}(t) = \frac{\hbar}{2}\left\{1+\frac{1}{4}\left[\left(\frac{2b/\omega_0}{1+bt}\right)\left(\left(\frac{\omega_0^2+b^2}{\omega_0^2}\right)e^{\pm 2r}\cos^2(\omega_0 t-\delta)+e^{\mp 2r}\sin^2(\omega_0 t)\right)\right.\right.$$
$$\left.\left.+\left(\frac{\omega_0^2+b^2}{\omega_0^2}\right)e^{\pm 2r}\sin(2(\omega_0 t-\delta))-e^{\mp 2r}\sin(2\omega_0 t)\right]^2\right\}^{1/2}.$$

In this model, for fixed values of $r>0$ and the frequency ω_0, the amplitude of oscillations of $(\Delta\hat{q})_{r,\theta}(t)$ can be increased by increasing the value of $|b|$. And when $|b|\to 0$, the amplitude of oscillations decreases, as one can see in Fig.3(a). In Fig.4, we show the probability density $\rho_{\alpha,r,\theta}(q,\theta)$ when $b=6$, (a) for $x_p(t)=0$ and (b) for $x_p(t)\cos(\sqrt{47}/2t-\tan^{-1}(\sqrt{47}))$ at resonance frequency $\omega=\sqrt{47}/2$ and observe the oscillatory squeezing in $(\Delta\hat{q})_{r,\theta}(t)$.

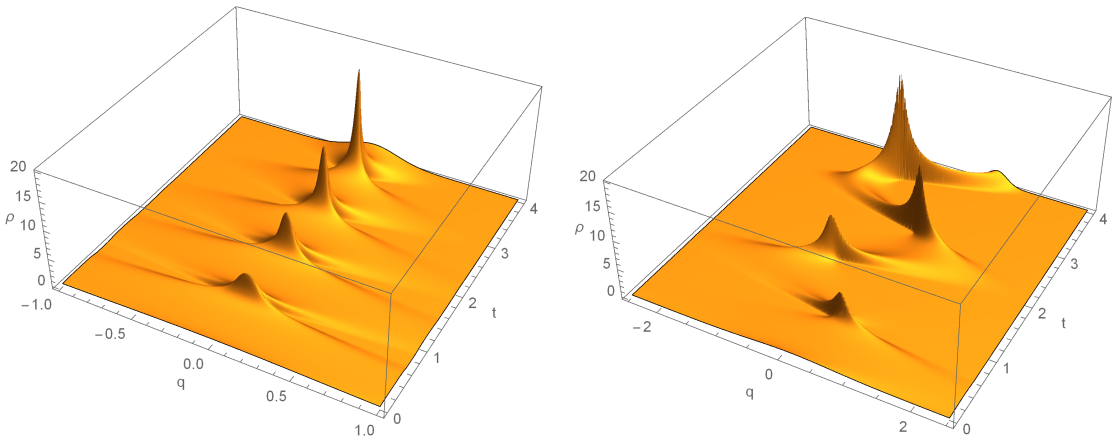
3.3. Model 3

Now, consider the Hamiltonian (3) with $B(t)=-(\gamma/2)-\Omega_B\tan(\Omega_B t)$, where $\Omega_B=\sqrt{\Lambda_0^2-\gamma^2/4}$ and $\Lambda_0^2>\gamma^2/4$. Here, $B(t)$ is periodic with singularities at times $t=(n-1/2)\pi/\Omega_B, n=1,2,....$



(a) (b) (c)

Figure 3. Model 2:For $\omega_0 = \sqrt{12}, \gamma = 1$, and $b = 0, 3, 6$, $r = 1/2, \theta = 0$. (a)Uncertainty $(\Delta\hat{q})_{r,\theta}(t)$, (b)Uncertainty $(\Delta\hat{p})_{r,\theta}(t)$, (c) Uncertainty product $(\Delta\hat{q}\Delta\hat{p})_{r,\theta}$.



(a) (b)

Figure 4. Model 2: Probability density $\rho_{\alpha,r,\theta}(q,t)$ with $\gamma = 1, \omega_0 = \sqrt{12}, b = 6, \hbar = 1, \alpha = 0, r = 1/2, \theta = 0$ (a)$\rho_{\alpha,r,\theta}(q,t)$ without displacement and (b) $\rho_{\alpha,r,\theta}(q,t)$ displaced by $x_p(t)$ at resonance frequency $\omega = \sqrt{47}/2, D_0 = \sqrt{12}$.

In this model, since $\Lambda_0^2 - \gamma^2/4 > 0$, we have $\Omega_d^2 = \omega_0^2 + \Lambda_0^2 - \gamma^2/4 > 0$ for any γ and ω_0, which means the modified oscillator is always in the special underdamping case. Solutions of the classical oscillator have the same form as in Model 1, with only difference the range of the allowed values of Λ_0^2. Therefore, for $\theta = 0$ and $\theta = \pi\,(z = \pm r)$, squeezing coefficient becomes

$$Q_{r,\theta}(t) = e^{-\gamma t/2}\sqrt{e^{\pm 2r}\cos^2(\Omega_d t) + \frac{\omega_0^2}{\Omega_d^2}e^{\mp 2r}\sin^2(\Omega_d t)}, \tag{27}$$

and uncertainties are of the form $(\Delta\hat{q})_{r,\theta}(t) = \sqrt{\hbar/2\omega_0}\,Q_{r,\theta}(t)$,

$$(\Delta\hat{p})_{r,\theta}(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{Q_{r,\theta}(t)}\sqrt{1 + \frac{e^{2\gamma t}Q_{r,\theta}^4}{\omega_0^2}\left(\frac{\dot{Q}_{r,\theta}(t)}{Q_{r,\theta}(t)} + \frac{\gamma}{2} + \Omega_B\tan(\Omega_B t)\right)^2},$$

$$(\Delta\hat{q}\Delta\hat{p})_{r,\theta}(t) = \frac{\hbar}{2}\left\{1 + \frac{1}{\omega_0^2}\left[\Omega_B\tan(\Omega_B t)(e^{\pm 2r}\cos^2(\Omega_d t) + \frac{\omega_0^2}{\Omega_d^2}e^{\mp 2r}\sin^2(\Omega_d t)) + \frac{1}{2}\left(\Omega_d e^{\pm 2r} - \frac{\omega_0^2}{\Omega_d}e^{\mp 2r}\right)\sin(2\Omega_d t)\right]^2\right\}^{1/2}.$$

For this model, uncertainty of position $(\Delta\hat{q})_{r,\theta}(t)$ is smooth and oscillatory, and approaches zero

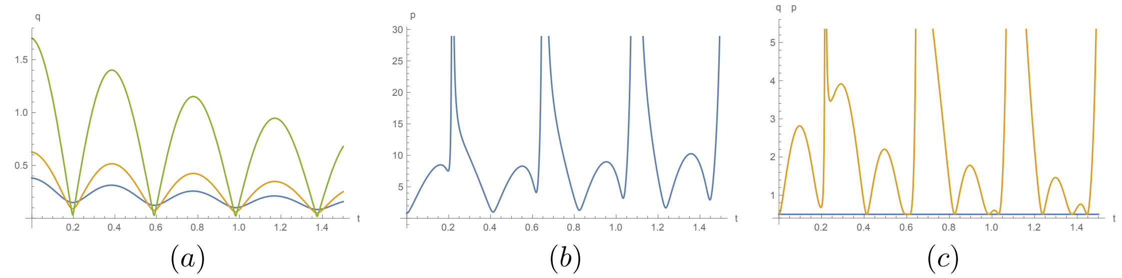


Figure 5. Model 3: $\omega_0 = \sqrt{12}, \gamma = 1$, and $\Lambda_0 = \sqrt{209}/2, \theta = 0$. (a)Uncertainty $(\Delta\hat{q})_{r,\theta}(t)$ for $r = 0, 1/2, 3/2$, (b)Uncertainty $(\Delta\hat{p})_{r,\theta}(t)$ for $r = 1/2$, (c) Uncertainty product $(\Delta\hat{q}\Delta\hat{p})_{r,\theta}$ for $r = 1/2$.

as increasing time, see Fig.5(a). But uncertainty of momentum and uncertainty product have singularities at the points where $B(t)$ is singular, see Fig.5(b) and 5(c). Then, in Fig.6(a), we plot the probability density $\rho_{\alpha,r,\theta}(q,t)$, without displacement ($x_p(t) = 0$) and observe the oscillatory squeezing of the width. Then, in Fig.6(b), we show $\rho_{\alpha,r,\theta}(q,t)$ under periodic displacement $x_p(t) = \cos(\sqrt{255}/2t - \tan^{-1}(\sqrt{255}))$ at resonance frequency $\omega = \sqrt{255}/2$.

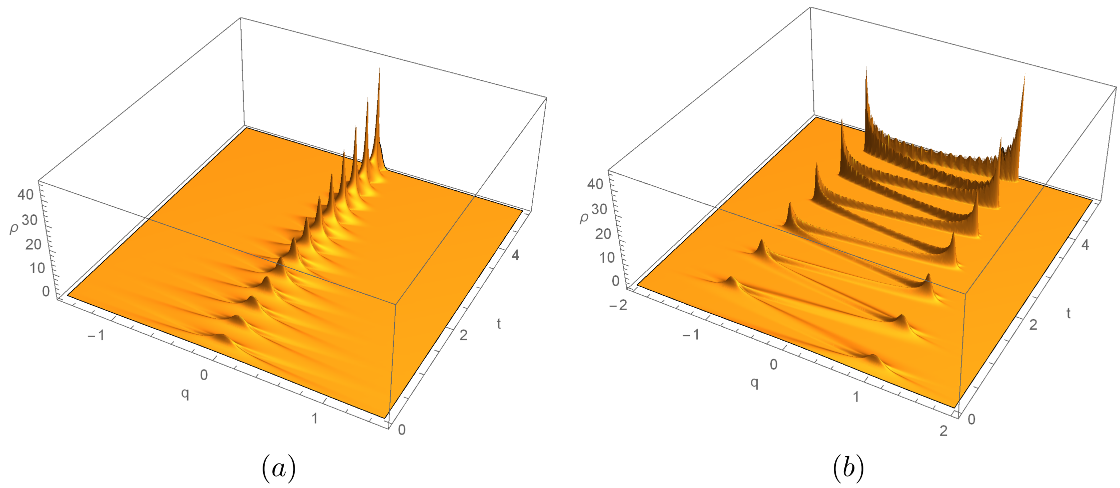


Figure 6. Model 3: Probability density $\rho_{\alpha,r,\theta}(q,t)$ with $\gamma = 1, \omega_0 = \sqrt{12}, \Lambda_0 = \sqrt{209}/2, \hbar = 1, \alpha = 0, r = 1/2, \theta = 0$ (a)$\rho_{\alpha,r,\theta}(q,t)$ without displacement and (b) $\rho_{\alpha,r,\theta}(q,t)$ displaced by $x_p(t)$ at resonance frequency $\omega = \sqrt{255}/2, D_0 = 8$.

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