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ARTICLE

On the number of bound states of semirelativistic Hamiltonian with Dirac delta potentials in one dimension

Fatih Erman

Abstract: We study the bound state problem for semirelativistic N attractive Dirac δ-potentials in one dimension. We give a sufficient condition for the Hamiltonian to have N bound states and give an explicit criterion for it.

Key words: number of bound states, Dirac delta potentials, resolvent, Salpeter Hamiltonians.

Résumé : Nous étudions le problème des états liés pour N potentiels δ de Dirac, attractifs et semi-relativistes en une dimension. Nous déterminons une condition suffisante pour que le Hamiltonien ait N états liés et donnons un critère explicite pour sa réalisation. [Traduit par la Rédaction]

Mots-clés : potentiel delta de Dirac, hamiltonien de Saltpeter, nombre d'états liés, renormalisation, noyau de chaleur.

1. Introduction

Schrödinger Hamiltonians with Dirac delta potentials or point interactions are one class of exactly solvable models that describe the short-range interactions in nuclear, atomic, and solid state physics. There is a vast amount of literature in this field from different points of view (see, for example, refs. 1 and 2). Dirac delta potentials in one dimension are even introduced as a pedagogical tool for analytically solvable potentials in standard quantum mechanics textbooks [3]. Two attractive Dirac delta potentials in one dimension are also used as an elementary model of a one-dimensional diatomic ion H_2^+. The Schrödinger equation for this model is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \lambda\delta(x)\psi(x) - \lambda\delta(x-a)\psi(x) = E\psi(x) \tag{1.1}$$

where a detailed bound state analysis has been discussed in refs. 3 and 4. The bound states can be determined by solving the above Schrödinger equation in each region separated by the location of delta potentials and imposing the continuity and jump conditions for ψ at the location of delta potentials, namely,

$$\begin{aligned}
&\psi(0^+) = \psi(0^-) \qquad \psi(a^+) = \psi(a^-)\\
&\psi'(0^+) - \psi'(0^-) = -\frac{2m\lambda}{\hbar^2}\psi(0)\\
&\psi'(a^+) - \psi'(a^-) = -\frac{2m\lambda}{\hbar^2}\psi(a)
\end{aligned} \tag{1.2}$$

These conditions give us the following transcendental equation:

$$e^{-\rho a} = \pm\left(1 - \frac{2\rho}{\mu}\right) \tag{1.3}$$

where $\mu = 2m\lambda/\hbar^2$ and $\rho = \sqrt{-(2mE/\hbar^2)}$. Here we naturally assume that $E < 0$ for the bound states (there cannot be square integrable solutions for $E \geq 0$). When one has "–" in (1.3), then $-1 + (2\rho/\mu)$ is a monotonically increasing function whereas $e^{-\rho a}$ is decreasing in ρ. As a result of this, there always exists one root of (1.3) for this case. On the other hand, when one has "+" in (1.3), $1 - (2\rho/\mu)$ is decreasing and it passes through the point (0, 1). If we assume that $(\partial/\partial\rho)(e^{-\rho a}) < (\partial/\partial\rho)[1 - (2\rho/\mu)]$ at $\rho = 0$, then (1.3) has one root for $\rho > 0$, that is, if we impose

$$a > \frac{\hbar^2}{m\lambda} \tag{1.4}$$

we have two bound states. However, if we consider an arbitrary finite number of centers (say N), it becomes a rather hard problem to determine the sufficiency condition for the equality between the number of bound states (say, $n(H)$) and the number of delta interactions (N). This is a very natural problem when we deal with a particle moving in a one-dimensional crystal modelled by Dirac delta potentials. One solution to the above problem was given by Albeverio and Nizhnik, and they found the necessary and sufficient conditions for $n(H) = N$ when all intensities are negative [5] (i.e., $\lambda_k > 0$ where $k = 1, 2, \ldots, N$). More recently, Ogurisu studied the conditions for $n(H) = N$ with negative intensities [6]. This has been illustrated by reformulating the problem as an eigenvalue problem of a finite dimensional matrix and studying the general behavior of the eigenvalues of this matrix.

Another reason why the Dirac delta potentials attract a great deal of interest from the physical point of view is that the Dirac delta potentials in two and three dimensions require the so-called renormalization procedure in quantum field theory [7]. That is why they help us to understand many concepts originally introduced in quantum field theory, such as dimensional transmutation, regularization, asymptotic freedom, etc. Recently, a semirelativistic version of the Schrödinger equation, known as the Salpeter equation, for one and two Dirac delta potentials has been studied in detail [8–10]. What we mean by the semirelativistic Schrödinger

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F. Erman. Department of Mathematics, İzmir Institute of Technology, Urla 35430, İzmir, Turkey.
Email for correspondence: fatih.erman@gmail.com.

equation or Salpeter equation is that the free Hamiltonian is given by the nonlocal operator [11, 12]

$$H_0 = \sqrt{-\frac{d^2}{dx^2} + m^2} \tag{1.5}$$

where we have used units such that $\hbar = c = 1$. The square root $\sqrt{p^2 + m^2}$ is the symbol of the pseudo-differential operator $\sqrt{-(d^2/dx^2) + m^2}$ defined by

$$\sqrt{-\frac{d^2}{dx^2} + m^2}\psi(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\sqrt{p^2 + m^2}e^{ipx}\hat{\psi}(p) \tag{1.6}$$

where $\hat{\psi}(p)$ is the Fourier transform of $\psi(x)$. If we substitute back $\hat{\psi}(p)$, we can express the action of H_0 on the function in coordinate representation as

$$H_0\psi(x) = -\frac{m}{\pi}\int_{-\infty}^{\infty} dy \frac{K_1(x - y)}{|x - y|}\psi(y) \tag{1.7}$$

where K_1 is the modified Bessel function of the first kind. This Hamiltonian can be considered as a good approximation to relativistic systems in which particle creations and annihilations are not allowed. The Salpeter Hamiltonians (1.5) with several potentials have been studied and considered as very successful models in phenomenological meson physics [13–15]. We have recently considered a single semirelativistic quantum mechanical particle of mass m moving in one dimension in the presence of finitely many δ-interactions located at isolated points a_k. The Hamiltonian is formally given by [16]

$$H = \sqrt{-\frac{d^2}{dx^2} + m^2} - \sum_{k=1}^{N}\lambda_k\delta(x - a_k) \tag{1.8}$$

where λ_k are assumed to be positive and called the coupling constants (or strengths), and a_k are the locations of the Dirac-δ centers in $\mathbb{R}$. We assume that $a_k \neq a_j$ for $k \neq j$. This problem requires renormalization in contrast to its one-dimensional non-relativistic version. The renormalization of the problem is summarized in the following section to make the paper self-contained. In particular, we have considered two centers and discussed the bound state spectrum of the problem (see also ref. 10). Therein, we have also found a condition sufficient for $n(H) = 2$ studying numerically the bound state solutions. This problem is analytically hard to solve since the expressions that we have are not analytically closed expressions.

For some class of regular potentials in the non-relativistic case, the various upper and lower bounds (e.g., the Bargmann's bound, the Calogero bound, the Birmann-Schwinger bound, the Lieb-Thirring bound) on the number of bound states for a given potential are given in the literature [17]. This subject is also partially summarized in a more elementary way in ref. 18. All these bounds are valid only for regular potentials when the renormalization is not required. Hence, it is not obvious to find the upper and lower bounds on the number of bound states for singular potentials and find the conditions for $n(H) = N$ (the conditions that saturate the upper bound of the number of bound states). For the non-relativistic version of the problem with N Dirac delta centers for various dimensions,

$$H_{nr} = -\frac{\hbar^2}{2m}\Delta - \sum_{k=1}^{N}\lambda_k\delta(x - a_k) \tag{1.9}$$

the conditions $n(H_{nr}) = N$ have been rigorously studied in great detail from several points of view [5, 6, 19–23]. The main aim of this work is to give a sufficient condition for $n(H) = N$. Our proof is basically the extension of the work in ref. 6 to the semirelativistic case, where the renormalization is required and explicit analytical expressions are absent.

The paper is organized as follows. We first summarize the renormalization of the Salpeter Hamiltonian with N attractive Dirac delta potentials in Sect. 2. Then, the bound state spectrum is shortly described in Sect. 3. Finally, we give the main result of this paper in Sect. 4.

2. Renormalization of one-dimensional Salpeter Hamiltonian with Dirac delta potentials

Before discussing the number of bound states of the problem, we first summarize the renormalization of the model through the heat kernel that has been recently considered in ref. 16. The Salpeter equation for the formal Hamiltonian (1.8) is given by

$$\begin{aligned}\langle x|H|\psi\rangle &= \langle x|H_0|\psi\rangle - \sum_{k=1}^{N}\lambda_k\delta(x - a_k)\psi(x)\\ &= \langle x|\left(H_0 - \sum_{k=1}^{N}\lambda_k|a_k\rangle\langle a_k|\right)|\psi\rangle = E\psi(x)\end{aligned} \tag{2.1}$$

where the kets $|a_k\rangle$ are the generalized eigenkets of the position operator with the generalized eigenvalue a_k. These generalized Dirac's bras and kets live in the so-called Rigged Hilbert spaces [24]. As in the case of two- and three-dimensional non-relativistic versions of the problem, the above formal Hamiltonian is not a well-defined self-adjoint operator in the above formal form. Here we first give a heuristic construction through the heat kernel regularization of the above formal Hamiltonian. In other words, we start with the regularized Salpeter Hamiltonian

$$\langle x|H_\epsilon|\psi\rangle = \langle x|H_0|\psi\rangle - \sum_{k=1}^{N}\lambda_k(\epsilon)\langle x|a_k^\epsilon\rangle\langle a_k^\epsilon|\psi\rangle \tag{2.2}$$

where ϵ is the short "time" cutoff and the coupling constants are considered to depend on ϵ and $\langle x|a_k^\epsilon\rangle \to \langle x|a_k\rangle = \delta(x - a_k)$ as $\epsilon \to 0^+$ in the distributional sense. For this reason, one natural choice of the regularization scheme is to use the heat kernel (i.e., $\langle x|a_k^\epsilon\rangle := K_{\epsilon/2}(x, a_k)$; note that $K_{\epsilon/2}(x, a_k) \to \delta(x - a_k)$ in the distributional sense). We recall that the heat kernel is defined as the fundamental solution to the heat equation [25]

$$H_0K_t(x, y) = -\frac{\partial K_t(x, y)}{\partial t} \tag{2.3}$$

Actually, what we mean by (2.2) is

$$H_\epsilon = H_0 - \sum_{k=1}^{N}\lambda_k(\epsilon)(K_{\epsilon/2}(\cdot, a_k), \cdot)K_{\epsilon/2}(x, a_k) \tag{2.4}$$

where $(\cdot, \cdot)$ is the L^2 inner product (i.e., $(K_{\epsilon/2}(\cdot, a_k), \psi) = \int_{-\infty}^{\infty}dxK_{\epsilon/2}(x, a_k)\overline{\psi(x)}$). Then, by solving the following inhomogenous equation:

$$H_0\psi(x) - \sum_{k=1}^{N}\lambda_k(\epsilon)K_{\epsilon/2}(x,a_k)\int_{-\infty}^{\infty}dyK_{\epsilon/2}(y,a_k)\overline{\psi}(y) - E\psi(x) = \rho(x) \tag{2.5}$$

we can find the regularized resolvent $R_\epsilon(E) = (H_\epsilon - E)^{-1}$ using the integral representation of the free resolvent $(H_0 - E)^{-1} = \int_0^\infty dt e^{-t(H_0-E)}$,

$$R_\epsilon(E) = R_0(E) + \sum_{k,j=1}^{N}\left[\int_0^\infty dte^{tE}K_{t+\epsilon/2}(x,a_k)\right][\Phi^{-1}(\epsilon,E)]_{kj}\left(\int_0^\infty dte^{tE}K_{t+\epsilon/2}(a_j,\cdot),\cdot\right) \tag{2.6}$$

where

$$\Phi_{kj}(\epsilon,E) = \begin{cases} \dfrac{1}{\lambda_k(\epsilon)} - \displaystyle\int_0^\infty dtK_{t+\epsilon}(a_k,a_k)e^{tE} & \text{if } k=j \\ -\displaystyle\int_0^\infty dtK_{t+\epsilon}(a_k,a_j)e^{tE} & \text{if } k\neq j \end{cases} \tag{2.7}$$

If we choose the coupling constants

$$\frac{1}{\lambda_k(\epsilon)} = \int_0^\infty dtK_{t+\epsilon}(a_k,a_k)e^{tE_B^k} \tag{2.8}$$

where E_B^k is the bound state energy of the system for the kth center in the absence of the rest of the centers, and take the formal limit as $\epsilon \to 0^+$, we obtain a non-trivial finite expression for the resolvent

$$R(E) = R_0(E) + \sum_{k,j=1}^{N}\left[\int_0^\infty dte^{tE}K_t(x,a_k)\right][\Phi^{-1}(E)]_{kj}\left(\int_0^\infty dte^{tE}K_t(a_j,\cdot),\cdot\right) \tag{2.9}$$

where

$$\Phi_{kj}(E) = \begin{cases} \displaystyle\int_0^\infty dtK_t(a_k,a_k)(e^{tE_B^k} - e^{tE}) & \text{if } k=j \\ -\displaystyle\int_0^\infty dtK_t(a_k,a_j)e^{tE} & \text{if } k\neq j \end{cases} \tag{2.10}$$

where $\Re(E) < m$. We call the matrix $\Phi(E) = [\Phi_{kj}(E)]_{k,j=1}^N$ the principal matrix. We assume that (2.10) can be extended onto the largest possible subset of the complex plane by analytic continuation. The principal matrix satisfies $\Phi^\dagger(E) = \Phi(\overline{E})$. The resolvent formula (2.9) is a kind of Krein's formula [26] and it is expressed in terms of the heat kernel. The explicit expression of the heat kernel associated with the operator $\sqrt{-(d^2/dx^2)+m^2}$ is given in [25] by the following formula:

$$K_t(x,y) = \frac{mt}{\pi\sqrt{(x-y)^2+t^2}}K_1\left[m\sqrt{(x-y)^2+t^2}\right] \tag{2.11}$$

for any $x, y \in \mathbb{R}$ and $t > 0$. In particular, the formula contains the massless case $m = 0$. In this case, the heat kernel associated with $H_0 = \sqrt{-(d^2/dx^2)}$ is given by [25]

$$K_t(x,y) = \frac{1}{\pi}\left[\frac{t}{t^2+(x-y)^2}\right] \tag{2.12}$$

The above condition $\Re(E) < m$ is imposed to guarantee the convergence of the integrals due to the exponential decaying behavior of the heat kernel for large values of t [16]

$$K_t(a_k,a_k) \sim \frac{m}{\pi}\sqrt{\frac{m}{2mt}}e^{-tm} \tag{2.13}$$

The principal matrix (2.10) can also be expressed in the momentum space by using the generalized completeness relation $\int_{-\infty}^{\infty}(dp/2\pi)|p\rangle\langle p| = 1$ [24],

$$\Phi_{kj}(E) = \begin{cases} \displaystyle\int_{-\infty}^{\infty}\frac{dp}{2\pi}\left(\frac{1}{\sqrt{p^2+m^2}-E_B^k} - \frac{1}{\sqrt{p^2+m^2}-E}\right) & \text{if } k=j \\ -\displaystyle\int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{\exp[ip(a_k-a_j)]}{\sqrt{p^2+m^2}-E} & \text{if } k\neq j \end{cases} \tag{2.14}$$

where $\Re(E) < m$. The integral in the diagonal terms can be directly evaluated

$$\Phi_{kk}(E) = \frac{E_B^k}{\pi\sqrt{m^2-(E_B^k)^2}}\left[\frac{\pi}{2} + \arctan\frac{E_B^k}{\sqrt{m^2-(E_B^k)^2}}\right] - \frac{E}{\pi\sqrt{m^2-E^2}}\left(\frac{\pi}{2} + \arctan\frac{E}{\sqrt{m^2-E^2}}\right) \tag{2.15}$$

where we have used $\arctan z = (1/2i)\log[(i-z)/(i+z)]$ and chosen the principal branch of $\arctan z$ such that $\arctan 0 = 0$. The off-diagonal elements can also be expressed in the following form by using the residue theorem in refs. 8 and 9]:

$$\Phi_{kj}(E) = \begin{cases} -\dfrac{1}{\pi}\displaystyle\int_m^\infty d\mu\,\exp(-\mu|a_k-a_j|)\frac{\sqrt{\mu^2-m^2}}{\mu^2-m^2+E^2} & \text{if } \Re(E) < m \\ -i\dfrac{\exp\left(i\sqrt{E^2-m^2}|a_k-a_j|\right)}{\sqrt{1-(m^2/E^2)}} - \dfrac{1}{\pi}\displaystyle\int_m^\infty d\mu\,\exp(-\mu|a_k-a_j|)\frac{\sqrt{\mu^2-m^2}}{\mu^2-m^2+E^2} & \text{if } \Re(E) > m \end{cases} \tag{2.16}$$

Here the integral over the variable μ is due to the branch cut along $[im, i\infty)$. Expressing the integral in the off-diagonal part of principal matrix (2.14) by (2.16) is very useful when we study the scattering spectrum of the problem [16].

We will now show that $R_\epsilon(E)$ converges to $R(E)$ as $\epsilon \to 0^+$ in the Hilbert–Schmidt norm. For this reason, we need to prove that

$$\lim_{\epsilon\to 0^+}\sum_{l=1}^{\infty}\left\|\left(\int_0^\infty dt e^{tE}K_{t+\epsilon/2}(a_j,\cdot),f_l(\cdot)\right)\left[\int_0^\infty dt e^{tE}K_{t+\epsilon/2}(a_j,x)\right]-\left(\int_0^\infty dt e^{tE}K_t(a_j,\cdot),f_l(\cdot)\right)\left[\int_0^\infty dt e^{tE}K_t(a_j,x)\right]\right\| = 0 \tag{2.17}$$

where $\{f_l\}_{l\in I}$ is an orthonormal basis for the Hilbert space $L^2(\mathbb{R})$. By adding and subtracting a term

$$\left(\int_0^\infty dt e^{tE}K_t(a_j,\cdot),f_l(\cdot)\right)\left[\int_0^\infty dt e^{tE}K_{t+\epsilon/2}(a_j,x)\right] \tag{2.18}$$

and combining the common factors we end up with the terms including the following factor:

$$\int_{-\infty}^{\infty}dx\int_0^\infty dt e^{tE}[K_{t+\epsilon/2}(a_j,x)-K_t(a_j,x)]^2 \tag{2.19}$$

where we have used the triangle and Cauchy–Schwarz inequalities. Using $\int_{-\infty}^{\infty}dx K_{t_1}(a_i,x)K_{t_2}(x,a_j)=K_{t_1+t_2}(a_i,a_j)$ and the upper bound of the modified Bessel function of the first kind [27]

$$K_1(x) < e^{-x/2}\left(\frac{1}{x}+\frac{1}{2}\right) \tag{2.20}$$

it is easy to show that the limit is zero.

3. Bound states

Because the bound state energies are the poles of resolvent (2.9), they must be the points of the real E axis such that the principal matrix is not invertible. Equivalently, E must be the solution to

$$\det\Phi(E) = 0 \tag{3.1}$$

This is a transcendental equation so it is rather hard to solve for an arbitrary number of centers. It is important to emphasize that free resolvent has no bound state spectrum, and it has only a continuous spectrum starting from m on the real E axis. In ref. 9, a terminology for the bound states depending on the values of the energy on the real axis has been introduced, namely, weak, strong, and ultrastrong bound states. However, we are not going to study the number of bound states according to this classification and we do not distinguish them in this paper.

To study the bound state spectrum for an arbitrary number of centers and make some general conclusions about the number of bound states, we consider the eigenvalue equation of the principal matrix

$$\Phi(E)A(E) = \omega(E)A(E) \tag{3.2}$$

Then, the solutions of (3.1) are actually zeros of the eigenvalues of the principal matrix. Moreover, the principal matrix is a symmetric analytic family in the sense of Kato [28] so that its eigenvalues and eigenprojections are analytic on the real axis due to theorem 6.1 in ref. 28. For real E, the principal matrix is Hermitian because $K_t(a_k, a_j) = K_t(a_j, a_k)$. Hence, all the eigenvalues ω are real. Because it is legitimate to interchange the order of momentum integration in the matrix elements (2.14) and the derivation with respect to E, we can find $(\partial\Phi_{kj}/\partial E)$. Then, from the Feynman–Hellmann theorem [29, 30]

$$\frac{\partial\omega_r}{\partial E} = \sum_{k,j=1}^{N}\overline{A}_k\frac{\partial\Phi_{kj}}{\partial E}A_j \tag{3.3}$$

we find that the eigenvalues ω are decreasing functions of E [16]

$$\frac{\partial\omega_r(E)}{\partial E} < 0 \tag{3.4}$$

Here the index r stands for the index of the eigenvalues. The monotonic behavior of the eigenvalues implies that there are at most N bound states (including the weak, strong, and ultrastrong ones). From the explicit expression of the principal matrix (2.16), all the off-diagonal terms vanish as $|a_k - a_j| \to \infty$ for all k, j. Hence, $\omega_r \to \Phi_{rr}$. If all E_B^k are the same, then we have N degenerate bound states.

For simplicity, we have considered in our previous work [16] the special case, where we have two ($E_B^1 = E_B^2 = E_B$) centers located at $a_1 = 0$ and $a_2 = a$. In this case, we numerically illustrated that there are exactly two bound states if

$$ma > 0.775 \tag{3.5}$$

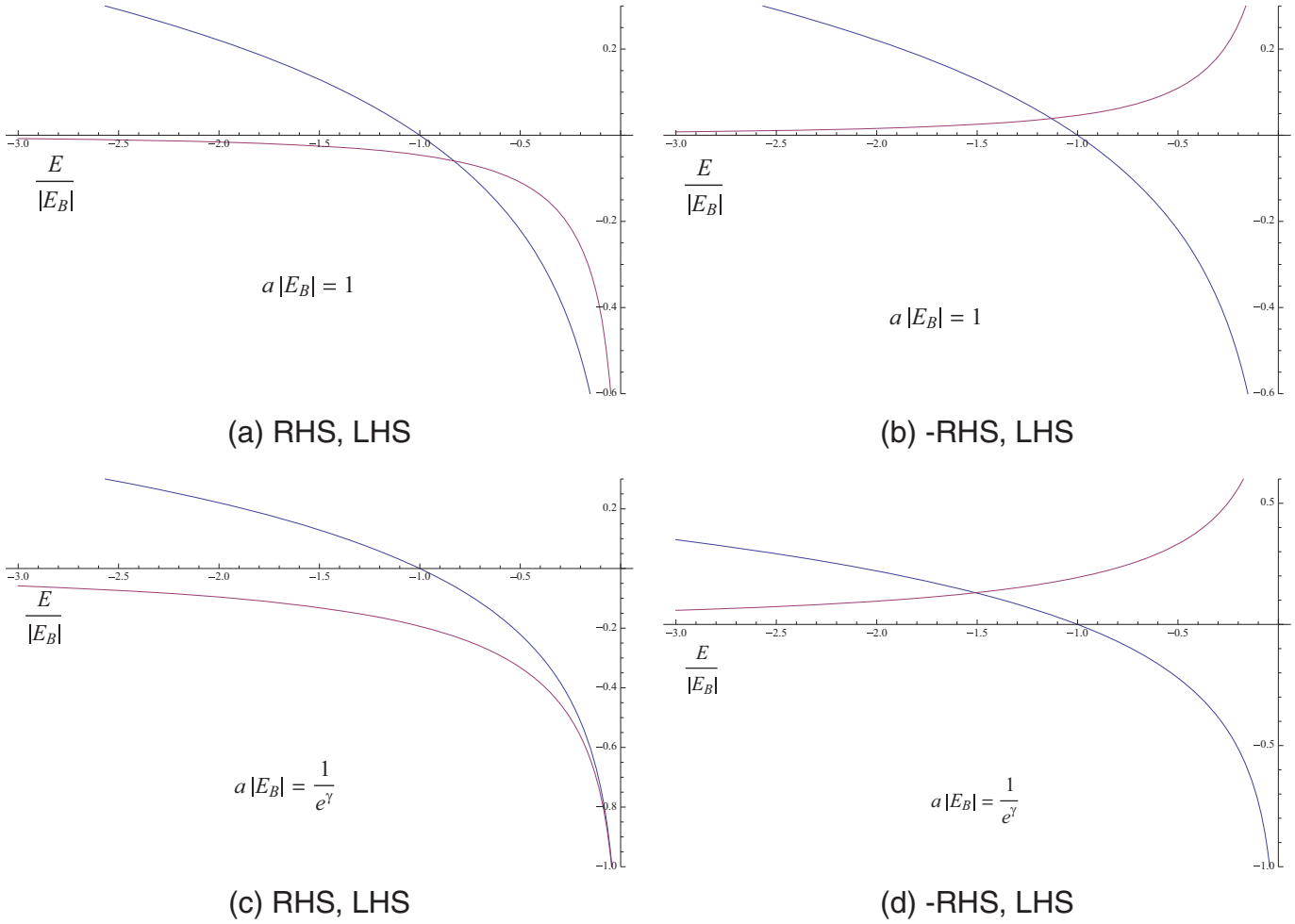
for $E_B/m = 1/2$. In other words, there appears a second bound state beyond the certain critical value of the distance between the centers. However, when we increase the number of centers, it is very hard to make numerical computations and determine whether such a type of condition exists. For a typical example for modeling one-dimensional lattice, N is very large. In the following section, we will determine such conditions for N bound states by following the ideas originally given in ref. 6 for the non-relativistic case. In contrast to the non-relativistic case, the principal matrix in this case is not given in a closed form.

In the massless case, we can explicitly calculate the principal matrix given by

$$\Phi_{kj}(E) = \begin{cases} \dfrac{1}{\pi}\log\left(\dfrac{E}{E_B^k}\right) & \text{if } k = j \\ \dfrac{1}{2\pi}\{2\cos[E(a_k-a_j)]\mathrm{Ci}(-E|a_k-a_j|)+\sin(E|a_k-a_j|)[\pi+2\mathrm{Si}(E|a_k-a_j|)]\} & \text{if } k\neq j \end{cases} \tag{3.6}$$

where $E_B^k < 0$. Here Ci and Si are the sine integral and the cosine integral functions defined by their integral representations [31]

Fig. 1. The graphs of RHS (–RHS) and LHS of (3.8) for the particular values of the parameters.



$$\mathrm{Ci}(x) = -\int_x^\infty dt \frac{\cos t}{t} \qquad \mathrm{Si}(x) = \int_0^x dt \frac{\sin t}{t} \tag{3.7}$$

For the same problem for two centers located symmetrically, the condition sufficient for $n(H) = 2$ can be analytically found [16] by solving the equation $\det\Phi(E) = 0$, or

$$\log\left(\frac{E}{E_B}\right) = \pm\left\{\frac{2\cos(aE)\mathrm{Ci}(-aE) + \sin(aE)[\pi + 2\mathrm{Si}(aE)]}{2}\right\} \tag{3.8}$$

We can plot the left-hand side and right-hand side as a function of $E/|E_B|$ for particular values of $a|E_B| = 1$ and $a|E_B| = 1/e^\gamma$, as shown in Fig. 1. Let RHS be the right hand side of (3.8) with positive sign in front of the parenthesis. Using the arguments similar to the ones in the non-relativistic case, we obtain the sufficient condition for $n(H) = 2$

$$a|E_B| > \frac{1}{e^\gamma} \tag{3.9}$$

where $\gamma \approx 0.5772$ is Euler's constant.

4. Main results

From the explicit expression of the principal matrix, it is real symmetric and continuously differentiable matrix-valued function on the complex half-plane with $\Re(E) < m$. Let us first now recall the following theorem [28, Theorem II.6.8]:

Theorem 1 *If* $\mathbf{T}(\lambda) = [t_{kj}(\lambda)]_{k,j=1}^N$ *is a real symmetric and continuously differentiable matrix and*

$$\lim_{\lambda\to-\infty} \mathbf{T}(\lambda) = \mathrm{diag}(a_1, a_2, \ldots, a_N) \tag{4.1}$$

then, the following assertions hold:

(*i*) *There exist N continuously differentiable functions* $\tau_j(\lambda)$ *that represent the repeated eigenvalues of the matrix* $\mathbf{T}(\lambda)$.
(*ii*) $\lim_{\lambda\to-\infty}\tau_j(\lambda) = a_j$ *for all* $j = 1, \ldots, N$.

In light of Theorem 1, let us define

$$T_{kj}(E) = \frac{1}{g(E)}\Phi_{kj}(E) \tag{4.2}$$

where

$$g(E) = -\frac{E}{\pi\sqrt{m^2 - E^2}}\left(\frac{\pi}{2} + \arctan\frac{E}{\sqrt{m^2 - E^2}}\right) \tag{4.3}$$

where Φ_{kj} are the elements of the principal matrix for the massive case (we shall consider the massless case separately) and we assume that E is restricted to the real axis and $E < m$. Thanks to the Lebesgue dominated convergence theorem, we can change the order of limit and integration with respect to t in (2.10). Suppose furthermore that $E_B^1 = E_B^2 = \ldots = E_B^N = E_B$ for simplicity. Then, we obtain $\lim_{E\to-\infty}[1/g(E)]\Phi(E) = \mathrm{diag}(1, 1, \ldots, 1)$. Hence, the above matrix $(1/g(E))\Phi(E)$ satisfies all the hypothesis of Theorem 1 and there exist N continuously differentiable functions $\omega_k(E)/g(E)$ that represent the eigenvalues of $(1/g(E))\Phi(E)$, where $\omega_k(E)$ is the kth eigenvalue of the principal matrix $\Phi(E)$. Moreover, $\lim_{E\to-\infty}[\omega_k(E)/g(E)] = 1$ for all k. This means that the eigenvalues ω_k are positive for sufficiently negative values of E for all k.

If $\Phi(E)$ is negative definite with some $E_0 < m$ (i.e., $\omega_k(E_0) < 0$ for all k), then there exist at least N numbers E_k such that $\omega_k(E_k) = 0$ for all k due to the intermediate value theorem. Hence, it implies that $\det\Phi(E_k) = 0$, so that E_k is an eigenvalue. The monotonic behaviour of the eigenvalues ω_k guarantees that there exists exactly N number of E_k such that $\omega_k(E_k) = 0$ for all k. In other words, there are N bound states.

To find an explicit condition for N bound states, we recall the following theorem, called Gerschgorin theorem [32] in matrix analysis:

Theorem 2 *All the eigenvalues of the matrix* $\mathbf{T} = (T_{kj})_{k,j=1}^N$ *are contained in the union of Gerschgorin's disks*

$$G_k = \left\{z \in \mathbb{C}; |z - T_{kk}| \le \sum_{j\neq k}|T_{kj}|\right\} \tag{4.4}$$

for $k = 1, \ldots, N$.

Let us apply Gerschgorin theorem for the principal matrix. Then, all the eigenvalues ω of the principal matrix are located in the union of following disks G_k:

$$G_k(E) = \left[\Phi_{kk}(E) - \sum_{j\neq k}|\Phi_{kj}(E)|, \Phi_{kk}(E) + \sum_{j\neq k}|\Phi_{kj}(E)|\right] \tag{4.5}$$

Thus, all the eigenvalues $\omega_k(E)$ are negative if

$$\Phi_{kk}(E) + \sum_{j\neq k}|\Phi_{kj}(E)| < 0 \tag{4.6}$$

for some $E = E_0 < m$. Let us denote the above sum $\sum_{j\neq k}|\Phi_{kj}(E)|$ by $b_k(E)$.

Let us now find an explicit condition for $b_k(E)$ to be negative. For that purpose, we first notice that

$$\Phi_{kk}(E) + \sum_{j\neq k}|\Phi_{kj}(E)| \le \max_{1\le k\le N}\Phi_{kk}(E) + (N-1)\max_{1\le k\le N}\max_{1\le j\neq k\le N}|\Phi_{kj}(E)| \tag{4.7}$$

Because all the E_B^k are assumed to be the same, we have

$$\Phi_{kk}(E) + \sum_{j\neq k}|\Phi_{kj}(E)| \le \Phi_{kk}(E) + (N-1)\max_{1\le k\le N}\max_{1\le j\neq k\le N}|\Phi_{kj}(E)| \tag{4.8}$$

For (4.8) to be negative, it is necessary that $\Phi_{kk}(E) < 0$, which is true for $E > E_B$. Using a lower bound of the modified Bessel function K_1 of the first order [16]

$$K_1(x) > \frac{e^{-x}}{x} \tag{4.9}$$

for all $x > 0$, we can find a upper bound of Φ_{kk} for $E > E_B$

$$\Phi_{kk}(E) = \int_0^\infty dt K_t(a_k, a_k)(e^{tE_B} - e^{tE}) < \frac{1}{\pi}\log\left(\frac{m-E}{m-E_B}\right) \tag{4.10}$$

If we define $d = \min_{1\le k,j\le N}\{|a_k - a_j|; k \neq j\}$ and use the upper bound of the Bessel function (2.20) together with $e^{-(m/2)\sqrt{d^2+t^2}} \le e^{-(m/2)t}$ and $\sqrt{d^2+t^2} \ge t$ for all t, we can find an upper bound of the off-diagonal terms of the principal matrix

$$\begin{aligned}|\Phi_{kj}(E)| &\le \int_0^\infty dt\frac{mt}{\pi\sqrt{d^2+t^2}}e^{tE}\exp\left(-\frac{m}{2}\sqrt{d^2+t^2}\right)\left(\frac{1}{m\sqrt{d^2+t^2}} + \frac{1}{2}\right)\\ &< \frac{1}{\pi d^2}\int_0^\infty dtt\exp\left[-t\left(\frac{m}{2} - E\right)\right] + \frac{m}{2\pi d}\int_0^\infty dtt\exp\left[-t\left(\frac{m}{2} - E\right)\right]\\ &< \frac{1}{(E-m)^2}\left(\frac{1}{\pi d^2} + \frac{m}{2\pi d}\right)\end{aligned} \tag{4.11}$$

where we have assumed $E < m/2$ for the convergence of the integrals. If we now impose

$$\frac{1}{\pi}\log\left(\frac{m-E}{m-E_B}\right) + (N-1)\left[\frac{1}{(E-m)^2}\left(\frac{1}{\pi d^2} + \frac{m}{2\pi d}\right)\right] < 0 \tag{4.12}$$

it implies the condition (4.6). Let $t = m - E$ and $t_B = m - E_B$ and define $f(t) = (1/\pi)\log(t/t_B) + (N-1)[C(d)/t^2]$, where $C(d) = (1/\pi d^2) + (m/2\pi d)$ and $nt > 0$. The unique critical point of $f(t)$ is $t_c = \sqrt{2\pi C(N-1)}$. It is easy to show that $f''(t_c) > 0$, thus, t_c is the point of the global minimum of f. Hence, if we impose that $f(t_c) < 0$ with the condition $t > 0$, or more explicitly

$$N - 1 < \frac{(m - E_B)^2}{2\pi e[(1/\pi d^2) + (m/2\pi d)]} \tag{4.13}$$

then we will get (4.12). Therefore, if condition (4.13) is satisfied, then we have exactly N bound states.

For the massless case, the upper bound of the off-diagonal term of the principal matrix is given by

$$|\Phi_{kj}(E)| \le \frac{1}{\pi d^2 E^2} \tag{4.14}$$

where we have used $d^2 + t^2 \ge d^2$. Then, applying the similar arguments above, we obtain

$$N - 1 < \frac{d^2E_B^2}{2e} \tag{4.15}$$

for the condition of N bound states.

Conditions (4.13) in the massive and (4.15) in the massless case for N bound states are consistent with the numerical computations for $N = 2$, summarized above. However, condition (4.13) for $N = 2$

$$2\pi e\left(\frac{1}{\pi a^2} + \frac{m}{2\pi a}\right) < (m - E_B)^2 \tag{4.16}$$

and condition (4.15) for $N = 2$

$$a|E_B| > e \tag{4.17}$$

are not exactly reduced to conditions (3.5) and (3.9) because we have used the upper bounds of the principal matrix to obtain analytical results.

5. Concluding remarks

In this paper, we elaborated the sufficient conditions (4.13) and (4.15) for Salpeter Hamiltonian with N attractive delta potentials to have N bound states. The condition that we have found overestimates the real bound computed numerically for delta potentials. For $E_B/m = 1/2$, condition (4.13) becomes

$$\frac{1}{a^2m^2} + \frac{1}{2am} < \frac{1}{8e} \tag{5.1}$$

or equivalently $ma > 2e[1 + \sqrt{1 + (2/e)}]$. The bound is definitely less stringent than the one obtained numerically ($ma > 0.775$). The reason for this is that we have used upper bounds instead of the exact values of the principal matrix to find the sufficient condition analytically. Similarly, in the massless case, our bound for $N = 2$ is larger than $1/e^{\gamma}$, where $\gamma \approx 0.5772$ is Euler's constant.

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