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Well posedness conditions for planar conewise linear systems

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Abstract

In this study, we give well-posedness conditions for planar conewise linear systems where the vector field is not necessarily continuous. It is further shown that, for a certain class of planar conewise linear systems, well posedness is independent of the conic partition of $\mathbb{R}^2$. More specifically, the system is well posed for any conic partition of $\mathbb{R}^2$.

Keywords

Switched systems, existence and uniqueness of solution, conewise linear systems, nonsmooth systems, carathedory solution, well posedness, planar systems

Introduction

Conewise linear systems (CLS) are multimodal switched linear systems where the entire space $\mathbb{R}^n$ is divided into polyhedral cones. This division is done by bounding matrices C_i's. It is known by Farkas-Minkowski-Weyl Theorem (Schrijver, 1986) that polyhedral cones can also be represented in terms of the generators $\{v_1, v_2, \ldots v_l\}$ that bound the cones. These are called extreme rays in Polyhedral Combinatorics literature (see Section III). As an alternative point of view, CLS can be regarded as a special class of linear hybrid automata (Lygeros et al., 2003; Shen et al., 2009) or more generally a piecewise linear system.

CLS has been investigated from many different aspects in recent years, like well posedness (Imura and Van der Schaft, 2000; Şahan and Eldem, 2015; Thuan and Çamlibel, 2014; Xia, 2002); stability (Araposthatis and Broucke, 2007; Eldem and Oner, 2015; Eldem and Şahan, 2014, 2016; Pachter and Jacobson, 1981; Shen et al., 2009; Zhendong and Shuzhi, 2011); control (Çamlibel et al., 2008; Heemels et al., 2010) and observability (Çamlibel et al., 2006; Shen, 2010). Many of these works assume that the system has a continuous vector field on the switching boundary, which resolves the issue of well posedness.

Well posedness simply means the existence and uniqueness of the solutions of CLS. For the interior of each cone, well posedness is obvious. However, if we have a discontinuous vector field on the switching boundary, then the existence and uniqueness of the solutions are not so obvious. There are many solution structures presented for the solutions of CLS: Carathédory, Filippov, Krasovskii, Euler, and so forth, to deal with the discontinuity problem on the switching boundary. Among these, the most popular ones are Carathédory and Filippov solutions; see Cortes (2008), Filippov (1998), Pogromsky et.al. (2003) and Bacciotti (2003) and the references therein. We use Carathedory solutions in this work and check the directions that are pointed by the vector fields of each mode sharing the switching boundary (the reader may refer to Van der Schaft and Schumacher [2000: 57] for details).

For the discontinuity problem of planar conewise linear systems (PCLS), Pachter and Jacobson (1981) proposed "flow continuation condition" for $\mathbb{R}^2$. Araposthatis and Broucke (2007) used this concept and gave results for stability of PCLS. Imura and Van der Schaft (2000) imposed "the smooth continuation sets" ($\mathbb{S}_i$) and the decomposition of the space into distinct $\mathbb{S}_i$'s for bimodal systems. Xia (2002) improved this result for multiple modes and multiple criteria piecewise linear systems.

In this work, we give easily verifiable conditions for the existence and uniqueness of the solutions of PCLS in Theorem 1 in the paper. For this purpose, we make use of a "flow continuation condition" like statement which will be helpful for the proofs of subsequent results. In this respect, Theorem 1 in this paper is equivalent to well posedness result given in Pachter and Jacobson (1981). The main results of this work are given in Theorem 2 and Theorem 3. Conditions we propose in Theorem 2 are related to some entries and eigenvalues of the system matrices and thus they are easily verifiable. Theorem 3 provides conditions for well posedness that are valid for random choices of seperation matrices (C_i's).

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The paper is organized as follows. Section II defines the class of systems that we considered and gives some details about CLSs. We introduce the well posedness problem for PCLS and give our main results in Section III.

Notation and Terminology: In the paper we refer to polyhedral cones as *modes*. The notation $sgn(x)$ is used to denote the sign of the number x. $\Theta(.,.)$ is the angle between two vectors. Ω^o is used to denote the interior of the set Ω and $Ker(x)$ is used to define the null space of x.

CLSs

A CLS is a switched linear system where the switching is state dependent. In order to define the bounds of the polyhedral cones, the null space of the matrices C_i's are used. Hence, the change of the $sgn(C_i x)$ can be considered as a switching signal. As a result, the trajectory enters that cone or not or stay on the border. As the system already exhibits a nonlinear and complex behavior, almost all of the works consider a continuous crossing between cones (Çamlıbel et al., 2006; Shen et al., 2006). A formal definition of CLS can be given as follows.

Definition 1: Consider a finite set of real $n \times n$ constant matrices $A_1, A_2, \ldots A_l$ and a finite set of convex polyhedral cones $\{\chi_1, \chi_2, \ldots \chi_l\}$ with $\cup_{i=1}^{l} \chi_i = \mathbb{R}^n$ and $\chi_i^0 \cap \chi_j^0 = \varnothing$ for $i \neq j$. A conewise linear system is defined as follows.

$$\dot{x} = f(x) = A_i x, \; x \in \chi_i \quad (1)$$

where

$$\chi_i = \{x : C_i x \geq 0\}, \; i = 1, 2, \ldots, l \text{ and } C_i \in \mathbb{R}^{l_i x n} \quad (2)$$

and the notation $C_i x \geq 0$ implies componentwise inequality. Conventionally, it is assumed that l_i, the number of the inequalities in the description, is minimum; in other words, the description is not redundant.

Well posedness for CLSs

Unlike the complex structured higher dimensional CLS; PCLS has a simple formation where the cones are bounded by two lines, so $l_i = 2$ for $\forall i$. We define the bounding matrices C_i's as follows

$$C_i = \begin{bmatrix} n_i \\ -n_{i+1} \end{bmatrix}, i = 1, 2, \ldots l, 0 \neq n_i \in \mathbb{R}^{1x2} \quad (3)$$

where $n_{l+1} \equiv n_1$. This completes the loop and makes a formal conic partition for $\mathbb{R}^2$. Using this partition, the bounds of the polyhedral cones are rays $\{v_1, v_2, \ldots, v_l\}$ directed counter clockwise that lies in $Kern_i's$. Although a formal definition as (3) is not given in the literature, many of the examples given obey this rule (see example 2 of Shen [2010] and example 3.47 of Zhendong and Shuzhi [2011]). At this point **we assume that** $\boldsymbol{n_i^T v_{i+1} \geq 0}$ **and complete the loop as** $\boldsymbol{v_{l+1} \equiv v_1}$. Here, the condition $n_i^T v_{i+1} \geq 0$ is equivalent to saying that $\Theta(v_i, v_{i+1}) \leq \pi$ for $i = 1, 2, \ldots l$. As a result, the modes are convex cones and can be represented with a single criteria, that is, with only one C_i for each mode. By this partition, the bounds of the polyhedral cones can also be defined as $v_i = \chi_i \cap \chi_{i-1}$ where $\chi_{l+1} \equiv \chi_1$. Hence, both of the neighborhood modes are active on the bound v_i and as a result existence and uniqueness of the solution is not so trivial. Therefore, as mentioned in the Introduction, it is generally assumed in the literature that the **vector field is continuous** on each bound to handle the issue of well posedness (Çamlıbel et al., 2006; Shen, 2010; Shen et al., 2009), or eqivalently

$$x \in \chi_i \cap \chi_j \Rightarrow A_i x = A_j x.$$

In order to resolve multiple conic subdivision problem for CLS, (Shen, 2010) and (Iwatani and Hara, 2006) give the definitions for simple and memoryless system for continuous and discontinuous CLSs, respectively. In order to give the definition of a memoryless system, "the smooth continuation sets" ($\mathbb{S}_i's$) of Imura and Van der Schaft (2000) are used. Recall that $\mathbb{S}_i's$ are the set of initial conditions from which the solutions starts and continues into the i^{th} cone. The system $(1)-(2)$ is said to be **memoryless**, if all the following conditions hold:

- $\mathbb{S}_i \neq \mathbb{R}^n$, $\forall i$,
- $(\mathbb{S}_i)^o \neq \varnothing$, $\forall i$,
- $\cup_{i=1}^{l} \mathbb{S}_i = \mathbb{R}^n$,
- $(\mathbb{S}_i \cap \mathbb{S}_j)^o = \varnothing, \forall (i,j), \; (i \neq j)$

As emphasized in Iwatani and Hara (2006), these are natural and not restrictive assumptions. **Unless otherwise stated, we assume that the conditions above hold**. As the modes are already linear, the theory of differential equations imply that there exists a unique solution for initial conditions in χ_i^o. But when we consider the initial conditions on the switching boundaries which has a discontinuous right hand side, we may have unique or non unique solutions or no solutions. We define the well posedness formally as follows (Şahan and Eldem, 2015).

Definition 2: Consider a CLS (1)–(2). The system is **well posed** if and only if smooth continuation is possible in only one of the two modes from every initial state $x_0 \in \mathbb{R}^n$.

If the existence and uniqueness of the solution is guaranteed for any initial condition, then the system is called well posed. For the clarification of well posedness problem on the border of a piecewise linear system, vector field of each mode sharing the same boundary must point out the same mode (Van der Schaft and Schumacher, 2000). This idea is used to solve well posedness problem for bimodal piecewise affine system in Şahan and Eldem (2015). For CLS where $l \geq 2$ but $n = 2$, Pachter and Jacobson (1981) imposed the "**flow continuation**" condition for CLS with dicontinuous vector field, as follows

$$sgn\left(v'_{i+1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A_i v_{i+1}\right) = sgn\left(v'_{i+1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A_{i+1} v_{i+1}\right)$$
for each $i = 1, 2, \ldots l$.

(4)

The main idea that lies under this condition is as explained in van der Schaft and Schumacher (2000: 58). In this work, we remove the continuity assumption for the vector field and try to find the existence and uniqueness conditions for a solution on the boundary v_i for a not necessarily continuous PCLS. We use a revised version of the "flow continuation" condition and give a new condition for well posedness that will be helpful to prove the main results in this work and enable us to generalize them to n-dimensional space in the future works.

Now, let us see what kind of border problems we may have.

Example 1 : Consider the following PCLS $\dot{x} = A_i x$ with four modes where

$$A_i = \begin{bmatrix} a^i_{11} & a^i_{12} \\ a^i_{21} & a^i_{22} \end{bmatrix} \text{ for } i = 1, 2, 3, 4;$$

$$C_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, C_3 = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \text{ and}$$

$$C_4 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Using our set-up

$$C_1 = \begin{bmatrix} n_1 \\ -n_2 \end{bmatrix}, C_2 = \begin{bmatrix} n_2 \\ -n_3 \end{bmatrix}, C_3 = \begin{bmatrix} n_3 \\ -n_4 \end{bmatrix} \text{ and } C_4 = \begin{bmatrix} n_4 \\ -n_1 \end{bmatrix}.$$

The corresponding geometry is shown in the Figure 1.

For the initial conditions $x_0 \in v_1$, we have $x_0 = \begin{bmatrix} \gamma_1 \\ 0 \end{bmatrix}, \gamma_1 > 0$ and note that here, both of the modes are active, i.e. both $C_1 x$ and $C_4 x \geq 0$. If we choose $A_1 = \begin{bmatrix} -1 & \times \\ 1 & \times \end{bmatrix}$ and $A_4 = \begin{bmatrix} -1 & \times \\ 1 & \times \end{bmatrix}$ then $A_1 x = A_4 x$ for $x \in v_1$, so the system is continuous on v_1. What are the conditions for the existence and uniqueness of the solution if we have $A_1 x \neq A_4 x$ for $x \in \chi_1 \cap \chi_4$?

In case of $A_1 x \neq A_4 x$ for $x \in \chi_1 \cap \chi_4$, both vector fields must point the same direction for well posedness. Choose

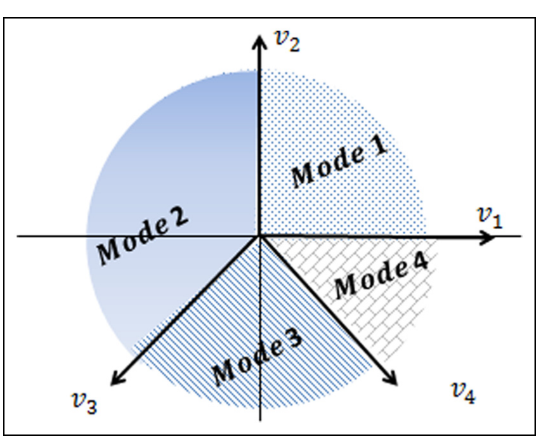

Figure 1. Plane Geometry for Example 1.

$A_1 = \begin{bmatrix} -1 & \times \\ 2 & \times \end{bmatrix}$ and $A_4 = \begin{bmatrix} -3 & \times \\ 1 & \times \end{bmatrix}$. Then $A_1 x \neq A_4 x$, but well posedness is guaranteed on this line. Morever, the choice of $A_1 = \begin{bmatrix} 1 & \times \\ 2 & \times \end{bmatrix}$ and $A_4 = \begin{bmatrix} -3 & \times \\ 1 & \times \end{bmatrix}$ also works. But when it comes to v_3 or v_4, even if $A_i v_i$ and $A_{i-1} v_i$ have same sign, they may point out different modes or stay on the v_i and be a sliding mode. For example, let us revise the 4^{th} system matrix as $A_4 = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}$ and also give $A_3 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$. Then, $A_3 v_4 = \begin{bmatrix} -\gamma_1 \\ \gamma_1 \end{bmatrix}$ and $A_4 v_4 = \begin{bmatrix} -3\gamma_1 \\ \gamma_1 \end{bmatrix}$, the 3^{rd} mode corresponding entries have same sign. But while the solution with respect to the 3rd mode stays on v_4, the solution with respect to the 4^{th} mode points out the 3^{rd} mode (see Figure 2). This is true because v_4 is an eigenvector for *mode* 3. But the absolute amount of change of the 1^{st} variable for the solution with respect to the 4^{th} mode is equal to three times the absolute amount of change of the 2^{nd} variable. That is why it flows into the 3^{rd} mode.

As depicted in the example above, if v_i (or v_{i+1}) which bounds *mode i* as an eigenvector of the system matrix A_i, then the trajectory stays on the border and becomes a sliding mode since $A_i v_i = \lambda v_i$. In order to remedy this, we put forward the following assumption.

Assumption 1: Neither v_i nor v_{i+1} is not an eigenvector of the system matrix A_i.

Throughout the paper, we assume that Assumption 1 holds, unless otherwise stated. This implies that the pairs (n_i, A_i) are observable. Now, we also define $\alpha_i^+ : = n_i^T A_i v_i$ and $\alpha_{i-1}^- : = n_i^T A_{i-1} v_i$ as used in Araposthatis and Broucke (2007) and give the following result on well posedness of PCLS. This result is equivalent to the "flow continuation condition" of Pachter and Jacobson (1981). Instead of flow continuation condition, we used α_i^+ and α_i^- both in the statement and the proof of the following result. This approach leads us to the main results of this paper (Theorem 2 and Theorem 3).

Theorem 1: Consider the planar conewise linear system $\dot{x} = f(x) = A_i x, x \in \chi_i = \{x : C_i x \geq 0\}$ where $A_i, C_i \in \mathbb{R}^{2x2}$, $i = 1, 2, \ldots l$ (described by equations (1) and (2)). Assume that the vector field is not necessarily continuous.

Then, PCLS is well posed if and only if $\alpha_i^+ \alpha_{i-1}^- > 0$ for $i = 1, 2, \ldots l$.

Proof : Consider an initial condition $0 \neq x_0 \in v_i$. Both i^{th} and $(i-1)^{th}$ modes are active on this line. Also note that the vectors $A_i v_i$ and $A_{i-1} v_i$ are the derivatives of i^{th} and $(i-1)^{th}$ modes, respectively (see Figure 3). If the inner products $n_i, A_i v_i$ and $n_i, A_{i-1} v_i$ are both positive, then both derivatives points into the i^{th} mode. If the inner products $n_i, A_i v_i$ and $n_i, A_{i-1} v_i$ are both negative then, both derivatives points out into the $(i-1)^{th}$ mode. For the case where $n_i, A_i v_i$ and $n_i, A_{i-1} v_i$ have different signs either there are two solutions (one of them continues smoothly in the i^{th} mode and the other smoothly continuing in the $(i-1)^{th}$ mode), or there

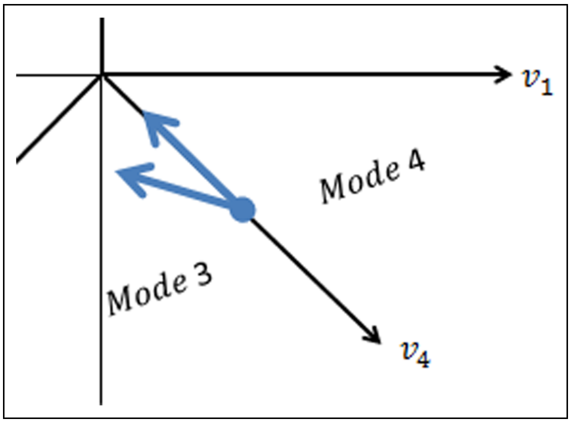

Figure 2. Directions of the trajectories starting on v_4.

are no solutions in the sense of Carathédory. Thus, $\alpha_i^+\alpha_{i-1}^- > 0$ guarantees the well posedness.

The sufficiency part can also be proved in a same way and this completes the proof.

Now let us consider an extreme ray v_i, corresponding vector n_i, two active modes i and $i-1$ and their matrices A_i. and A_{i-1}.

$$\alpha_i^+ = n_i A_i v_i = [x_1 \quad x_2]\begin{bmatrix} a_{11}^i & a_{12}^i \\ a_{21}^i & a_{22}^i \end{bmatrix}\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = a_{21}^i x_2^2 + (a_{11}^i - a_{22}^i)x_1 x_2 - a_{12}^i x_1^2.$$

If $x_1 = 0$, then $\alpha_i^+ = a_{21}^i x_2^2$. So the sign of α_i^+ is determined by the sign of a_{21}^i. Assuming that $x_1 \neq 0$,

$$\alpha_i^+ = x_1^2\left[a_{21}^i\left(\frac{x_2}{x_1}\right)^2 + (a_{11}^i - a_{22}^i)\frac{x_2}{x_1} - a_{12}^i\right]. \quad (5)$$

In a similar fashion

$$\alpha_{i-1}^- = x_1^2\left[a_{21}^{i-1}\left(\frac{x_2}{x_1}\right)^2 + (a_{11}^{i-1} - a_{22}^{i-1})\frac{x_2}{x_1} - a_{12}^{i-1}\right].$$

Note that this is a quadratic equation in terms of the ratio $\frac{x_2}{x_1}$. Then, $\alpha_i^+ > 0(<0)$ for every arbitrary choice of $n_i = [x_1 \quad x_2]$ if and only if

$$\left.\begin{matrix} \Delta_i = (a_{11}^i - a_{22}^i)^2 + 4a_{12}^i a_{21}^i < 0 \\ a_{21}^i > 0(<0) \end{matrix}\right\} \quad (6)$$

This is also true for α_{i-1}^- and for the entries of A_{i-1} as well (in case of $x_1 = 0$, we need the sign of a_{21}^i again). It is interesting that although the coefficients of α_i^+ and the coefficients of the characteristic equation $p_{A_i}(\lambda)$ of the system matrix A_i are different, Δ_i (which plays a key role for determining the eigenvalues A_i) is the same. More precisely, if Δ_i in (6) is negative, then the eigenvalues of the matrix A_i are also complex as the roots of α_i^+.

Also note that if we assume that the eigenvalues of A_i are complex and a_{21}^i's have the same sign for all modes, then it follows that the sign of a_{12}^i's are the same for all modes, because $\Delta_i < 0$ implies that $a_{12}^i a_{21}^i < 0$ as well. Now, we give the following result.

Theorem 2: Consider a PCLS defined by equations (1)–(3). Suppose that the system matrices A_i have either complex or real, multiple eigenvalues. Then, the PCLS is well posed if and only if

(i) $|a_{12}^i| + |a_{21}^i| \neq 0$,
(ii) $(a_{12}^i)(a_{21}^i) \leq 0$ for $1 \leq i \leq l$ and $sgn(a_{12}^i)$ is the same for all matrices A_i if $a_{12}^i \neq 0$.

Proof: Suppose that the hypothesis hold and the system is well posed. Then, by Theorem 1, $\alpha_i^+\alpha_{i-1}^- > 0$ for any subsequent modes $i-1$ and i.

Let $x_1 = 0$. Then the sign of $\alpha_i^+ = a_{21}^i x_2^2$ is determined by the sign of a_{21}^i. If also $a_{21}^i = 0$, then $\alpha_i^+ = 0$ and it means that $v_i := \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$ is the eigenvector of *mode i* that contradicts Assumption 1. Then, a_{21}^i must be nonzero, which implies that conditions (i) and (ii) of the Theorem hold.

Let $x_1 \neq 0$. Then α_i^+ can be written as $\alpha_i^+ = x_1^2\left[a_{21}^i\left(\frac{x_2}{x_1}\right)^2 + (a_{11}^i - a_{22}^i)\frac{x_2}{x_1} - a_{12}^i\right]$. If also $a_{21}^i \neq 0$, then the roots of the equation $\alpha_i^+ = 0$ are $m_{1,2} = \frac{-(a_{11}^i - a_{22}^i) \mp \sqrt{(a_{11}^i - a_{22}^i)^2 + 4a_{12}^i a_{21}^i}}{2a_{21}^i}$. Now, consider the characteristic polynomial of the system matrix A_i. The roots are defined by $\lambda_{1,2} = \frac{-(a_{11}^i + a_{22}^i) \mp \sqrt{(a_{11}^i - a_{22}^i)^2 + 4a_{12}^i a_{21}^i}}{2}$.

Note that $\Delta_i = (a_{11}^i - a_{22}^i)^2 + 4a_{12}^i a_{21}^i$ is the same for each equation. As a result, if A_i have complex eigenvalues, then $\Delta_i < 0$. Thus, the $sgn(\alpha_i^+)$ is fixed and and determined by the $sgn(a_{21}^i)$ for any number $\frac{x_2}{x_1}$ and for any choice of v_i (It is true even for the limit case $x_1 \to 0$). Moreover, $a_{12}^i a_{21}^i < 0$ as $\Delta_i < 0$. These imply that conditions (i) and (ii) of the Theorem hold.

Now suppose that $\Delta_i = 0$. Then the eigenvalues of A_i are real, multiple and obviously $a_{12}^i a_{21}^i \leq 0$. Let us first assume that $a_{12}^i a_{21}^i < 0$. The equation $\alpha_i^+ = 0$ also has real, multiple roots: $m_1 = m_2 = -\frac{a_{11}^i - a_{22}^i}{2a_{21}^i}$. If

- $\frac{x_2}{x_1} < -\frac{a_{11}^i - a_{22}^i}{2a_{21}^i}$ or $\frac{x_2}{x_1} > -\frac{a_{11}^i - a_{22}^i}{2a_{21}^i}$ (including $x_1 \to 0$) $\Rightarrow sgn(\alpha_i^+) = sgn(a_{21}^i)$
- $\frac{x_2}{x_1} = -\frac{a_{11}^i - a_{22}^i}{2a_{21}^i} \Rightarrow \alpha_i^+ = 0$. But in this case $v_i = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$ is an eigenvector of A_i that contradicts Assumption 1.

Consequently, we have $sgn(\alpha_i^+) = sgn(a_{21}^i)$. Since $\alpha_i^+\alpha_{i-1}^- > 0$ it follows that a_{21}^i and a_{21}^{i-1} have same sign as well.

Also note that if $a_{12}^i a_{21}^i \neq 0$ and $\Delta_i \leq 0$ then $(a_{11}^i - a_{22}^i)^2 \leq -4a_{12}^i a_{21}^i$, which means $sgn(a_{12}^i) = -sgn(a_{21}^i)$. As a result, $sgn(a_{21}^i) = sgn(a_{21}^{i-1})$ requires that $sgn(a_{12}^i) = sgn(a_{12}^{i-1})$. Therefore, conditions (i) and (ii) of the Theorem hold again.

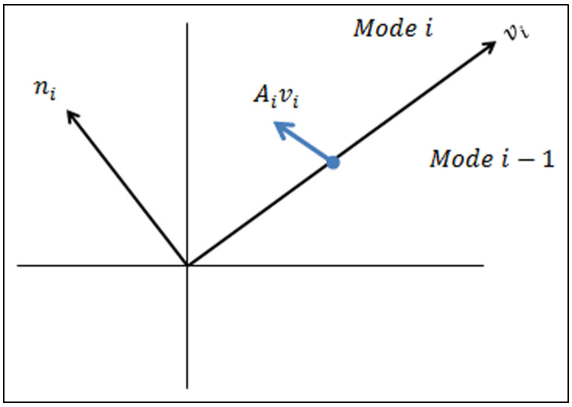

Figure 3. The derivative A_iv_i on the bound v_i.

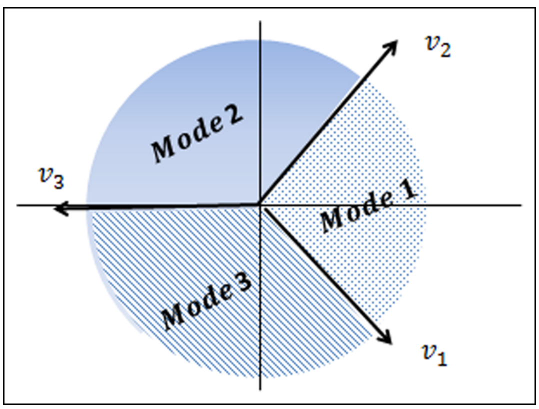

Figure 4. Geometry of the Plane for Example 2.

The final case is $x_1 \neq 0$ and $a^i_{12}a^i_{21} = 0$. Since the eigenvalues of A_i are multiple, it follows that $\Delta_i = 0$. Consequently, $a^i_{11} = a^i_{22}$. For $a^i_{12}a^i_{21} = 0$, we have three options:

(1) If both $a^i_{12}, a^i_{21} = 0$, then the geometric multiplicity of $\lambda_1 = \lambda_2$ is 2 and v_i is also an eigenvector that contradicts Assumption 1. This implies that $|a^i_{12}| + |a^i_{21}| \neq 0$.

(2) If $a^i_{12} = 0$, but $a^i_{21} \neq 0$, then $\alpha^+_i = a^i_{21}x^2_2$ and so the sign is determined by a^i_{21}. Since well posedness is equivalent to saying that $\alpha^+_i \alpha^-_{i-1} > 0$, it follows that $sgn(a^i_{21}) = sgn(a^{i-1}_{21})$. What if $a^{i-1}_{12} \neq 0$? In this case, $sgn(a^{i-1}_{12}) = -sgn(a^{i-1}_{21}) = -sgn(a^i_{21})$. The same is also true for $(i-2)^{th}$ mode. That is, $-sgn(a^{i-1}_{21}) = -sgn(a^{i-2}_{21}) = sgn(a^{i-2}_{12}) \Rightarrow sgn(a^{i-1}_{12}) = sgn(a^{i-2}_{12})$.

(3) If $a^i_{21} = 0$, but $a^i_{12} \neq 0$, then $\alpha^+_i = -a^i_{12}x^2_1$ and so the sign is determined by a^i_{12}. If also $x_1 = 0$, this contradicts Assumption 1 again as $v_i = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$ is an eigenvector in this case. If $a^{i-1}_{21} \neq 0$, the analysis done in the previous case is again so conditions (i) and (ii) of the Theorem 2 hold.

The proof for necessity follows the same lines.

It is important how we divide the whole space into polyhedral cones as it is used in the stability analysis of CLS (Iervolino et al., 2017). We see by the proof of Theorem 2 that the borders can be choosen arbitrarily when the eigenvalues are complex. Next the result gives the conditions for arbitrary polyhedral partition.

Theorem 3: Consider a PCLS defined by equations (1) – (3). The system is well posed for any polyhedral conic subdivision of $\mathbb{R}^2$, that is, for any choice of C_i's, if and only if

(i) The eigenvalues of A_i are complex for $1 \le i \le l$.
(ii) The entries a^i_{21}'s (or a^i_{12}'s) have same sign for all system matrices A_i, $1 \le i \le l$.

Proof: Consider a PCLS whose i^{th} mode is bordered by v_i and v_{i+1}. If PCLS is well posed then for an initial condition on v_i the sign of α^+_i and α^-_{i-1} must be the same for any random choice of v_i. If the i^{th} subsystem have real eigenvalues then v_i can be chosen as an eigenvector of this real eigenvalue. This contradicts Assumption 1. Thus, well posedness may not be achieved when the choice of the borders are random. On the other hand, suppose that both subsystems that share v_i have complex eigenvalues, but the signs of a^i_{21} and a^{i-1}_{21} are different. Then, one of the trajectories turns clockwise and the other turns counter clockwise. This implies that either there are two solutions starting from v_i that smoothly continue into seperate modes or there is a border collision on v_i (equivalently there are no solutions in the sense of Carathédory). Since well posedness is not possible in both cases, it follows that a^i_{21}'s must have the same sign for $1 \le i \le l$. Finally, since all A_i's have complex eigenvalues, it follows that $a^i_{12}a^i_{21} < 0$ for $1 \le i \le l$. This implies that a^i_{12}'s must have the same sign for $1 \le i \le l$, which conludes the proof of necessity. The proof for sufficiency follows along the similar lines.

Note that in case we have real and distinct eigenvalues, the corresponding mode may include the eigenvectors. In this case the sign of α^+_i changes in mode when $\frac{x_2}{x_1}$ is equal to the slope of the eigenvector. Therefore, the eigenvectors of A_i must be outside of the cone bounded by v_i and v_{i+1}.

Example 2 : Consider $\dot{x} = A_ix$ with 3-modes where

$$A_1 = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & -6 \\ 1 & 0 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} -4 & -4 \\ 1 & 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, C_2 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } C_3 = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$$

The regions are as in Figure 4.

The system 1 and 2 have complex eigenvalues while the 3rd has real-multiple eigenvalues. The system is not well posed since a^i_{21}'s have different sign: $a^1_{21} < 0$ but a^2_{21} and $a^3_{21} > 0$. It can be also checked that $x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow$ For the solution with respect to the 1st mode, we have $A_1x = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$ (points out the 3rd mode). But with respect to the 3rd mode, we have $A_3x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (points out the 1st mode).

It is originated from the difference signs of a_{21}^1 and a_{21}^3 . The eigenvalues of the system matrix A_1 is $1\pm 2i$. If we change the entries as $A_1 = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, then the spectrum of A_1 is still $1\pm 2i$. But as the sign of a_{21}^i is same for all $i = 1, 2, 3$, the system is well posed.

Now, let us change n_1, to emphasize the importance of Assumption 1. Assume that $C_1 = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$, C_2 is the same and $C_3 = \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix}$; $A_1 = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, and A_2, A_3 are the same.

Note that $v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is the eigenvector of A_3. If $x_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, then for the solution with respect to the 1st mode, we have $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ (points out the 1stmode). But with respect to the 3rd mode, we have $A_3 x = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$ (it is a sliding mode). It shows that we need Assumption 1.

Example 3 : Consider $\dot{x} = A_i x$ with 3-modes where

$$A_1 = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & -8 \\ 1 & 0 \end{bmatrix} \text{and}\, A_3 = \begin{bmatrix} 1 & -1 \\ 1 & 5 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, C_2 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \text{and}\, C_3 = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}.$$

It has same regions as example 2. The sign of a_{21}^i is the same for all $i = 1, 2, 3$ and the sign of a_{12}^i is same for all $i = 1, 2, 3$ but the system is not well posed.

If $x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ then for the solution with respect to the 1st mode, we have $A_1 x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (points out the 1st mode). But with respect to the 3rd mode, we have $A_3 x = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ (points the 3rd mode). It is originated as A_3 has real and distinct eigenvalues and one of the eigenvectors lies in χ_3.

Remark 1: Note that the eigenvectors in the cone changes the sign of the $\alpha_i^{+/-}$ and causes ill posedness. Well posed PCLS may be achieved by changing the borders of the cone even if one of the modes has real and distinct eigenvalues. So there are no eigenvectors inside of the mode or sometimes both are in it. But it is not possible to infer a general conclusion as we did in Theorem 2 or 3.

Example 4: Consider $\dot{x} = A_i x$ with 3-modes where

$$A_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & 0 \\ 1 & -2 \end{bmatrix} \text{and}\, A_3 = \begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, C_2 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \text{and}\, C_3 = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}.$$

It has same regions with example 2 again. Although a_{21}^1 and a_{12}^2 are zero, $sgn(a_{21}^2) = sgn(a_{21}^3)$ and $sgn(a_{12}^1) = sgn(a_{12}^3)$ are the same. Consequently, the system is well posed.

For example, if $x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ then for the solution with respect to the 1st mode, we have $A_1 x = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ (points out the 1st mode). Similarly with respect to 3rd mode, we have $A_3 x = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ (points out the 1st mode too). Same computation can be done for the other borders.

Conclusions and future works

This study includes some results about existence and uniqueness of solutions for planar multimodal systems. We give the properties of the system matrices of a well posed PCLS. One of the results is given for arbitrary conic subdivision as well. Well posedness of PCLS is the first step to prove stability, controllability or some other issues. It is quite important to investigate the behaviour of the trajectories at the borders of the polyhedral cones so to obtain a crossing condition. Thus, while some of the works consider only well posedness conditions of PCLS, some others investigate in the context of stability, controllability, and so forth. Recent works also show the necessity of well posedness conditions for the construction of the appropriate discontinuous Lyapunov functions, (Iervolino et al., 2017). Therefore, the results presented in this work are the first step towards the investigation of PCLS in the context of stability, or controllability.

The work Şahan and Eldem (2015) uses these $\mathbb{S}_i$'s and gives the well posedness conditions for bimodal piecewise linear systems in $\mathbb{R}^n$. Şahan and Eldem (2015) explain the structure of the system matrices that a well posed bimodal system must have. It defines which part of the seperating hyperplane of the modes belongs to $\mathbb{S}_1$ or $\mathbb{S}_2$ using the subspaces where the 1st, 2nd, …nth order derivatives are equal to the zero. Because of the geometry of $\mathbb{R}^2$, we do not encounter these kind of subspaces in our work. If we reduce our main result to $l = 2$ and consider the main result of Şahan and Eldem (2015) for $n = 2$, we all have $a_{21}^i > 0$. Thus, Theorem 2 and 3 generalizes this result to multimodal case.

For the generalization of our work to higher dimensional cases, the subspaces that are mentioned in Şahan and Eldem (2015) must be placed properly on seperating hyperplanes, that is, on different dimensional faces, The placement of these subspaces on different dimensional faces gives many different variants. Thus, to give a necessary and sufficient condition for the structure of a well posed system is quite an hard issue. In our opinion, it will be easier to give just a necessary condition first or just for some special systems for these cases.

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