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A General Expression for the Stagnant Thermal Conductivity of Stochastic and Periodic Structures

A general expression has been obtained to estimate thermal conductivities of both stochastic and periodic structures with high-solid thermal conductivity. An air layer partially occupied by slanted circular rods of high-thermal conductivity was considered to derive the general expression. The thermal conductivity based on this general expression was compared against that obtained from detailed three-dimensional numerical calculations. A good agreement between two sets of results substantiates the validity of the general expression for evaluating the stagnant thermal conductivity of the periodic structures. Subsequently, this expression was averaged over a hemispherical solid angle to estimate the stagnant thermal conductivity for stochastic structures such as a metal foam. The resulting expression was found identical to the one obtained by Hsu et al., Krishnan et al., and Yang and Nakayama. Thus, the general expression can be used for both stochastic and periodic structures. [DOI: 10.1115/1.4038449]

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Introduction

Porous structures have been proposed for many thermal management applications, such as cooling of electronics [1] and metal foam heat exchanger [2]. One of the most fundamental properties required for such applications is the effective stagnant thermal conductivity. Naturally, the effective stagnant thermal conductivity depends on its particular porous structure. A comparatively simple porous structure may be described as a unit cell, for which direct numerical calculations can be conducted to obtain the effective thermal conductivity, exploiting periodic and adiabatic boundary conditions [3,4]. However, such detailed three-dimensional calculations are quite formidable even for simple geometrical configurations. Due to recent advance in manufacturing technologies, a number of geometrically sophisticated cellular materials and lattice frame structures are now commercially available. These stochastic and periodic structures have been considered for possible heat transfer enhancement structures of high specific surface for some time [5–7]. Thus, a general formula is definitely needed for estimating effective stagnant thermal conductivities of such stochastic and periodic structures.

In this study, we shall propose a simple analytical model to obtain a general expression for an effective stagnant thermal conductivity of a three-dimensional solid structure of high-solid thermal conductivity. A horizontal air layer partially occupied by slanted rods of high-thermal conductivity is considered, in which the upper and lower boundaries of the air layer are maintained at low and high temperatures, respectively. This air layer model partially filled with slanted rods, whose details will shortly be discussed in the Analytical Consideration section, seems to be rather specific, but represents most sandwich structures with lattice frame core used in heat exchanger applications. One of the sandwich structures with lattice frame core is illustrated in Fig. 1. In fact, one of our strong motivations to initiate this study was to establish a general formula valid for estimating the effective stagnant thermal conductivity of such sandwich structure with truss core.

The stagnant thermal conductivity is obtained analytically by estimating heat transferring vertically across the air phase and also through the slanted solid rods partially filled within the air layer. Three-dimensional numerical computations will also be conducted under the same geometrical and thermal boundary conditions, so as to verify the present general expression. Subsequently, the general expression will be extended to the case of stochastic structure by averaging it over a hemispherical solid angle.

Analytical Consideration

An air layer partially occupied by slanted circular rods is considered, as illustrated in Fig. 2, in which the bottom boundary is

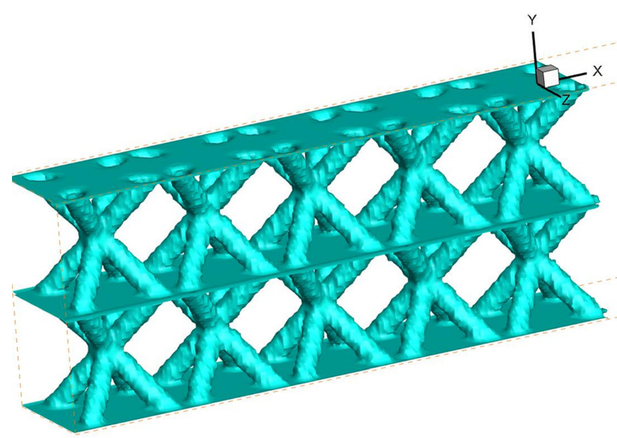

Fig. 1 Typical sandwich structure with truss core

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hotter than the top boundary. The top and bottom boundaries of distance H are kept isothermal with a fixed temperature difference ΔT. We shall consider an effective stagnant thermal conductivity of this layer estimating heat conducting vertically through the air and slanted rods. We assume that convection is absent, and hence, heat transfer between the air and peripheral surface of the rod is negligible.

Referring to Fig. 2, heat transfer rate from the hot bottom to cold top boundaries by conduction through any particular slanted rod of the thermal conductivity k_{s_j}, projected cross-sectional area A_{sj}, and directional angle β_j may be estimated as follows:

$$(A_{s_j}\mathbf{j})\cdot\mathbf{q}_{s_j}\cong(A_{s_j}\mathbf{j})\cdot\left(k_{sj}\frac{\Delta T}{(H/\cos\beta_j)}\mathbf{s}_j\right)=A_{sj}k_{sj}\frac{\Delta T}{H}\cos^2\beta_j \quad (1)$$

where $j = 1, 2, 3, N$.

The temperature gradient along the rod axis is estimated to be $-\Delta T/(H/\cos\beta_j)$. Furthermore, note that $\mathbf{s}_j$ is the unit vector along the rod j such that $\mathbf{j}\cdot\mathbf{s}_j=\cos\beta_j$. Hence, the effective stagnant thermal conductivity of the air layer with N slanted rods within the space $A\times H$ can be estimated as

$$k_{\text{eff}}=\frac{\sum_{j=1}^{N}A_{sj}k_{sj}\frac{\Delta T}{H}\cos^2\beta_j+\left(A-\sum_{j=1}^{N}A_{s_j}\right)k_f\frac{\Delta T}{H}}{A\frac{\Delta T}{H}}$$

$$=\sum_{j=1}^{N}\frac{A_{sj}}{A}k_{sj}\cos^2\beta_j+\varepsilon k_f \quad (2)$$

where k_f is the thermal conductivity of the air, whereas $\varepsilon = 1-\sum_{j=1}^{N}A_{s_j}/A$ is the porosity.

Hence, for the case as illustrated in Fig. 2, where

$$k_{s_1}=k_{s_2}=\cdots=k_{s_N}\equiv k_s \text{ and } \beta_1=\beta_2=\cdots=\beta_N\equiv\beta \quad (3)$$

we have the following compact expression:

$$k_{\text{eff}}=(1-\varepsilon)k_s\cos^2\beta+\varepsilon k_f \quad (4)$$

There are many effective medium models for estimating the stagnant thermal conductivity of composite material, such as the one proposed by Nan et al. [8] on the basis of the multiple-scattering approach. These models assume a continuous variation of local thermal conductivity which can be expressed by a homogeneous part and an arbitrary fluctuation part. On the other hand, the present model based on a fluid layer with slanted rods allows a discrete variation of local thermal conductivity in which each rod can have its own location, size, orientation, and thermal conductivity.

In the conventional effective medium models, small particles are assumed to be dispersed in the matrix, whereas in the present air layer model, the solid phase is not dispersed in the fluid phase (matrix) but is described individually by the collection of slanted rods with different location, size, orientation, and thermal conductivity. Thus, the present model, unlike the conventional continuous model, is capable of estimating the effective thermal conductivity of sandwich structures with truss core.

It is interesting to note that the model proposed by Nan et al. on the basis of the multiple-scattering approach, despite its difference in variation of local thermal conductivity, reduce to the present expression (4), for the limiting case of aligned continuous fibers (see Eq. (18*b*) in their paper).

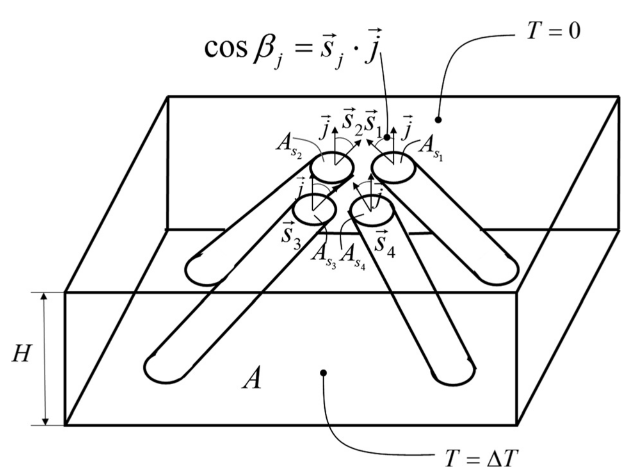

Fig. 2 Air layer with slanted rods

Numerical Consideration

Full three-dimensional calculations were carried out using open FOAM 4.1 for the case as illustrated in Fig. 3. All four vertical boundaries are assumed to be adiabatic, whereas the top and bottom boundaries are set to be isothermal, with the temperature difference, ΔT. Computations were carried out for the case in which

$$\Delta T=100\text{K},\ H=0.005\,\text{m},\ k_{s_1}=k_{s_2}=k_s=2.57\,\text{W/mK},$$
$$k_f=0.0257\,\text{W/mK},$$
$$A=144\,\text{mm}^2,\ A_{s_1}=A_{s_2}=3.32\,\text{mm}^2(\text{i.e.}, \varepsilon=0.954)\ \beta=\pi/3$$

The three-dimensional temperature field in the air layer with two slanted rods of high thermal conductivity predicted by the numerical computation is presented in Figs. 4(*a*)–4(*c*). As expected, most heat conducts through the two slanted rods of high thermal conductivity, while the rest of heat conducts vertically through the air. Using these numerical results, the effective stagnant thermal conductivity was calculated from

$$k_{\text{eff}}=\frac{\int_{A_s}-k_s\frac{\partial T}{\partial y}\Big|_{y=0}dA_s+\int_{A_f}-k_f\frac{\partial T}{\partial y}\Big|_{y=0}dA_f}{A\frac{\Delta T}{H}} \quad (5)$$

which is found to be

$$k_{\text{eff}}=0.0580\,\text{W/mK}$$

from the three-dimensional numerical computation. This value is very close to the one estimated from the general expression (4) as follows:

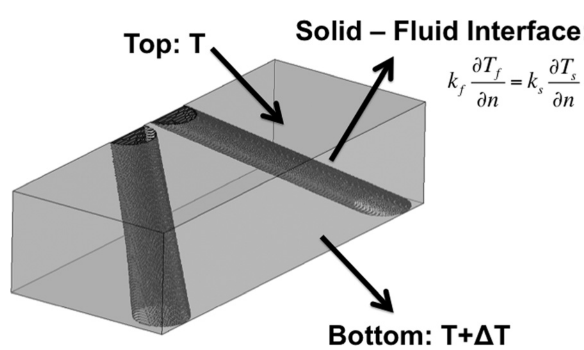

Fig. 3 Physical model for detailed three-dimensional numerical computation

$$k_{\mathrm{eff}} = (1-\varepsilon)k_s \cos^2\beta + \varepsilon k_f$$
$$= (1-0.954)\times 2.57 \times \cos^2(\pi/3)$$
$$+ 0.954 \times 0.0257 = 0.0540\,\mathrm{W/mK} \qquad (6)$$

A good agreement between the suggested general expression and numerical results can be observed, which clearly indicates the validity of Eq. (4).

A series of numerical computations were carried out for different directional angles to investigate possible thermal interference among rods. The numerical results, however, proved that the effect of such interference among rods on the effective thermal conductivity is negligibly small, and that the general expression (4) is still valid for the fluid and solid combination in which the thermal conductivity of the solid is much higher than that of the fluid.

Extension to Stochastic Structures

Analytical expressions for estimating stagnant thermal conductivities of porous structures have been proposed by various researchers. Hsu et al. [9] presented a general lumped parameter model for stagnant thermal conductivity of periodic porous media. Paek et al. [10] used the Dul'nev's model [11] for periodic structures. Both models transform an open-cell structure of polygonal geometry to an equilateral cubic model and reduce to

$$k_{\mathrm{eff}} = \left(\left(\frac{d}{H_c}\right)^2 + \frac{2\left(\frac{d}{H_c}\right)\left(1-\frac{d}{H_c}\right)}{\left(\frac{d}{H_c}\right) + \frac{k_s}{k_f}\left(1-\frac{d}{H_c}\right)} \right) k_s + \left(1-\frac{d}{H_c}\right)^2 k_f \qquad (7)$$

where

$$\varepsilon = 1 - 3\left(\frac{d}{H_c}\right)^2 + 2\left(\frac{d}{H_c}\right)^3 \qquad (8)$$

H_c is the size of the unit cubic cell, whereas d is the diameter of the ligament. Yang and Nakayama [12], on the other hand, faithfully followed a volume averaging theory and derived the same expression. Krishnan et al. [13] focused on Lemlich's theory [14] and assumed that heat conduction for the case of high porosity occurs only through the ligament of solid along its axis and not through its periphery. Then, they derived a remarkably simple formula as follows:

$$k_{\mathrm{eff}} = \frac{1-\varepsilon}{3} k_s \qquad (9)$$

This corresponds to the case of $d/H_c \ll 1$ and $k_f/k_s \ll 1$, in which Eq. (7) reduces to the form identical to the foregoing Eq. (9).

The general expression based on the air layer partially occupied by slanted rods can be extended to the case of stochastic structure. The stochastic structure as in a metal foam may be approximated as a collection of randomly oriented rods. Thus, the effective stagnant thermal conductivity can be evaluated by summing up all heat conducting through these randomly oriented rods. Hence, Eq. (4) may be averaged by integrating it over a hemispherical solid

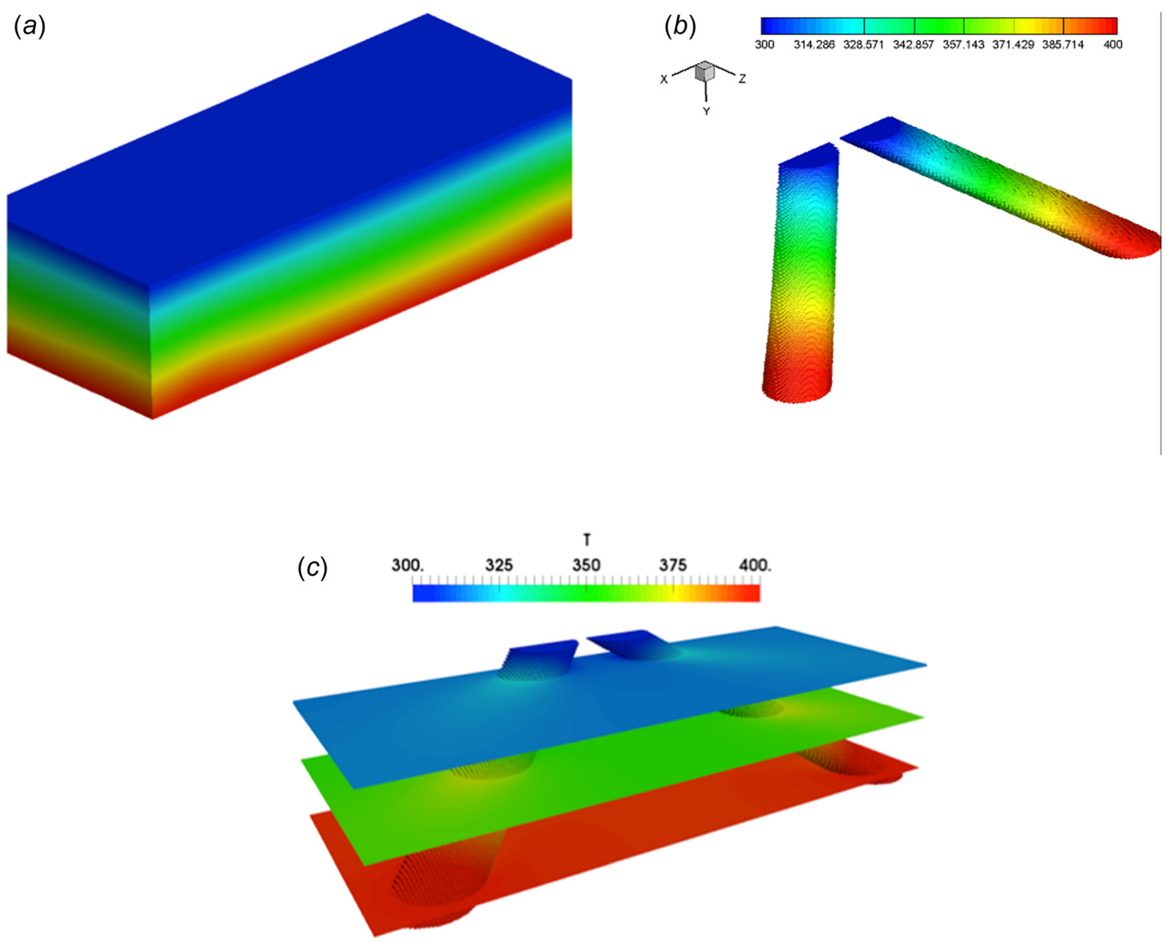

Fig. 4 Temperature distribution obtained from three-dimensional numerical computation: (*a*) air phase, (*b*) solid phase, and (*c*) cross-sectional view

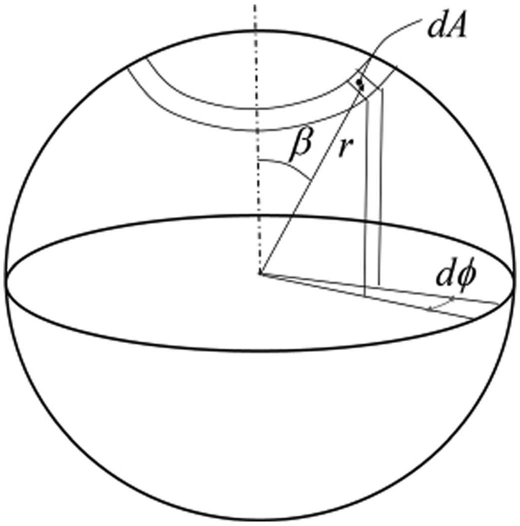

Fig. 5 Polar coordinate and solid angle

angle, 2π, as shown in Fig. 5. The polar angle β stays the same for the entire azimuthal angle over $0 \le \phi \le 2\pi$. Upon nothing that an infinitesimal solid angle is given by $d\omega = dA/r^2 = (rd\beta)(r\sin\beta d\phi)/r^2 = d\phi \sin\beta d\beta$, we have

$$k_{\text{eff}} = \frac{1}{2\pi}\int_0^{2\pi}\left((1-\varepsilon)k_s\cos^2\beta + \varepsilon k_f\right)d\omega = \frac{1}{2\pi}\int_0^{2\pi}d\phi\int_0^{\pi/2}\left((1-\varepsilon)k_s\cos^2\beta + \varepsilon k_f\right)\sin\beta d\beta = \frac{1-\varepsilon}{3}k_s + \varepsilon k_f \quad (10)$$

The foregoing equation (10) naturally reduces to the expression (9) proposed by Krishnan et al. [13] for the case of $k_f/k_s \ll 1$. Thus, the present general expression is valid for both periodic and stochastic porous structures.

Conclusions

An air layer partially occupied by slanted rods was proposed as an analytical model to estimate effective stagnant thermal conductivities of periodic porous structures. Heat conduction through the slanted rods was accounted along with heat conduction through the air, whereas heat transfer between the air and peripheral surface of the rod was neglected. Detailed three-dimensional numerical computations were also conducted under the same geometrical and thermal boundary conditions. The numerical results are found in good accord with those estimated from the general expression, revealing the validity of the general expression for evaluating the stagnant thermal conductivity of the periodic structures. Moreover, the expression was averaged over a hemispherical solid angle to estimate the stagnant thermal conductivity for stochastic structures. The present expression is found quite useful to both stochastic and periodic structures.

Nomenclature

- A = top and bottom boundary surfaces (m^2)
- d = diameter of the solid rod(m)
- H = height of the air layer (m)
- H_c = size of the unit cubic cell (m)
- $\mathbf{j}$ = vertical unit vector
- k = thermal conductivity (W/mK)
- k_{eff} = effective stagnant thermal conductivity (W/mK)
- $\mathbf{q}$ = heat flux vector (W/m^2)
- r = radial coordinate
- $\mathbf{s}$ = unit vector along the rod axis
- T = temperature (K)
- ΔT = temperature difference between the bottom and top boundaries (K)
- β = directional angle, polar angle (rad)
- ε = porosity
- ϕ = polar angle (rad)
- ω = solid angle (sr)

Subscripts

- j = index for the rod
- f = fluid
- s = solid

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