ALGEBRAIC METHODS AND EXACT SOLUTIONS OF QUANTUM PARAMETRIC OSCILLATORS

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ABSTRACT

ALGEBRAIC METHODS AND EXACT SOLUTIONS OF QUANTUM PARAMETRIC OSCILLATORS

In this thesis, we study different approaches for solving the Schrödinger equation for quantum parametric oscillators. The Wei-Norman algebraic approach, the Lewis-Riesenfeld invariant approach, the Malkin-Manko-Trifonov approach are investigated. For each approach, the wave function solutions of the Schrödinger equation, the propagator and dynamical invariants are found and their relations with each other are shown.

In the Wei-Norman Algebraic approach, for constructing wave functions, explicit form of evolution operator is obtained uniquely in terms of two linearly independent classical solutions of the corresponding classical equation of motion. In Lewis-Riesenfeld approach, quadratic invariants are found in terms of the solution of Ermakov-Pinney equation and using the eigenstates of these invariants, wave function solutions are constructed. Setting initial values for Ermakov-Pinney solution, results of Wei-Norman and Lewis-Riesenfeld approaches are compared, then this solution is expressed in terms of same two linearly independent classical solutions. In Malkin-Manko-Trifonov approach, linear invariants which are symmetry operators for the Schrödinger equation, are constructed in terms of complex-valued solutions of the classical equation. Using these invariants, quadratic invariants are constructed and their eigenstates are used to find wave function solutions. Moreover, initial values for complex solutions of classical equation of motion are posed, and comparison of the three approaches is given.

ÖZET

KUANTUM PARAMETRİK OSİLATÖRLER İÇİN CEBİRSEL YÖNTEMLER VE TAM ÇÖZÜMLER

Bu tezde kuantum parametrik osilatörler için Schrödinger denklemini çözmek amacıyla farklı yaklaşımlar çalışılmıştır. Wei-Norman cebri yaklaşımı, Lewis-Riesenfeld değişmez yaklaşımı, Malkin-Manko-Trifonov yaklaşımı incelenmiştir. Her yaklaşım için, Schrödinger denkleminin dalga fonksiyonu çözümleri, ilerletici (propagatör) ve dinamik değişmezleri bulunmuştur ve birbirleriyle ilişkileri gösterilmiştir.

Wei-Norman cebri yaklaşımında, dalga fonksiyonları inşa etmek için evrim operatörünün tam formu, buna karşılık gelen klasik hareket denkleminin klasik iki lineer bağımsız çözümleri cinsinden tek olarak elde edilmiştir. Lewis-Riesenfeld yaklaşımında, ikinci dereceden değişmezler, Ermakov-Pinney denkleminin çözümü cinsinden bulunmuştur ve bu değişmezlerin özdurumları kullanılarak dalga fonksiyonu çözümleri inşa edilmiştir. Ermakov Pinney çözümü için başlangıç değerleri ayarlanarak, Wei-Norman ve Lewis-Riesenfeld çözümleri karşılaştırılmış, daha sonra bu çözüm aynı klasik iki lineer bağımsız çözümler cinsinden ifade edilmiştir. Malkin-Manko-Trifonov yaklaşımında, Schrödinger denklemi için simetri operatörleri olan lineer değişmezler, klasik denklemin karmaşık değerli çözümleri cinsinden inşa edilmiştir. Bu değişmezler kullanılarak, ikinci dereceden değişmezler inşa edilmiştir ve onların özdurumları kullanılarak dalga fonksiyonu çözümleri bulunmuştur. Bundan başka, klasik denklemin karmaşık çözümleri için başlangıç değerleri gösterilmiştir ve üç yaklaşımın karşılaştırılması verilmiştir.

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CHAPTER 1

INTRODUCTION

Oscillations are happening in all around and inside of us world, from the beating of the human hearts, to vibrating atoms. Mathematically, the system is oscillating or vibrating, if variables determining its state are changing not monotonically, but increasing and decreasing alternately. In the simplest case of the mechanical system, the generalized coordinates during oscillating process are increasing and decreasing, so that mechanical points are moving forward and backward one after another. The simplest type of oscillations are harmonic oscillations of pendulum, described by periodic circular functions. The pendulum was invented in XVII century by Galileo Galilei and studied by Christiaan Huygens. From XVIII century, with developing of mathematical analysis and analytical mechanics, oscillating processes started to be studied on more strong mathematical basis, applied to more big variety of oscillations by L. Euler, D'Alembert and Lagrange. In XIX centure K. Weierstrass solved exactly the nonlinear pendulum problem by elliptic functions. More general character of oscillation theory was described in book "The Theory of Sound" by John William Strutt (Lord Rayleigh).

With discovering quantum mechanics and the Schrödinger equation in XX century, an exact solution of quantum harmonic oscillator becomes fundamental of quantum many body theory, quantum field theory, quantum photonics, etc. It was first derived in matrix mechanics by M. Born, P. Jordan, W. Heisenberg and in wave mechanics by E. Schrödinger (Schrödinger, 1926). As was shown by Schrödinger, the Gaussian wave packet as non-stationary solution of the Schrödinger equation is oscillating according to classical harmonic oscillator equation. The pure algebraic way to solve quantum harmonic oscillator by using algebraic properties of creation and annihilation operators was proposed by P. Dirac in his PhD thesis in 1926. This approach together with the Schrödinger factorization method (Schrödinger, 1940) becomes origin of algebraic methods for solving quantum mechanical problems. After book of Herman Weil "Quantum mechanics and group theory" algebraic and group theoretical methods come to be important tools in solving quantum problems.

Between different type of oscillation motions, the special role plays the so called parametric oscillations, or oscillations with parametric excitation. In systems with parametric excitation, external action influences on the system by periodic time changes of one or several parameters. For example, the pendulum with periodically changing length $l(t)$. It can leads to parametric resonance, which includes a wide class of phenomena, from children swings to cosmology, electronics, quantum optics, Casimir forces, Bose condensates, etc.

Mathematically, oscillations with parametric excitation are described by differential equations with explicit time dependent coefficients (which frequently are periodic functions). For one degrees of freedom oscillator with time dependent mass $m = m(t)$ and frequency $\omega = \omega(t)$ we have equation

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \omega^2(t)x(t) = 0(*)$$

where $\gamma(t) = \dot{m}(t)/m(t)$. This general case can be reduced to the one with constant mass $m = constant$, by replacing $t' = \int \frac{dt}{m(t)}, \omega' = m\omega$ (Perelomov and Zel'dovich,1998). However, frequently it is convenient for analysis of the problem's solution to preserve original variables. And quantization of equivalent classical systems, as is known, frequently leads to different results, since different operators order in quantum Hamiltonian. Due to explicit time dependence, quantization of parametric oscillator leads to non-stationary problems in quantum mechanics. Such problems can be solved exactly very seldom. It turns out that harmonic oscillator with time dependent frequency and mass is an important example of quantum problems which can be solved explicitly (Perelomov and Zel'dovich,1998).

There are many works devoted to solution of this problem by different methods. It is impossible to list all of them here, so we just mention several works as the above book of Perelomov and Zeldovich, book of Malkin and Man'ko (Malkin, 1979) and articles (Hartley and Ray,1982), (Dattoli et al., 1997), (Dantas et al., 1992). From these approaches we like to emphasise the Wei-Norman (WN) algebraic method (Wei and Norman, 1963), the Lewis-Riesenfeld (LR) invariant approach (Lewis and Riesenfeld, 1969), and the Malkin-Man'ko-Trifonov (MMT) method (Malkin, 1970). Common point for all these methods is that solution of quantum problem reduces to solution of classical parametric oscillator problem (*). In paper (Büyükaşık et al., 2009) the method was proposed to solve this problem for wide class of parametric functions, such that solution is represented by special functions of mathematical physics from class of hypergeometric functions. This way, the wide class of quantum parametric oscillators was solved.

The aim of this thesis is to provide explicit solutions of the IVP for the Schrödinger equation and compare these results which are found by the WN, LR and MMT-methods. The thesis is organized as follows:

In Chapter 2, we give definitions, propositions and properties of some basic concepts which will be necessary for further studies.

In Chapter 3, we discuss the simple harmonic oscillator. Considering the standard time-independent Hamiltonian $\hat{H}_0$, we introduce time-independent as well as time-dependent Schrödinger equation. For solving both equations, we write Hamiltonian in terms of $su(1,1)$ Lie algebra generators and found the solutions in terms of Hamiltonian eigenstates. Using these eigenstates, we find the evolution operator, wave function and propagator. Later, we obtain dynamical invariants for simple harmonic oscillator and we also find them in Heisenberg picture.

In Chapter 4, we consider the IVP for time-dependent Schrödinger equation with the quadratic Hamiltonian with real-valued parameters. We construct evolution operator and expresses it explicitly in terms of two linearly independent homogeneous solutions of the corresponding classical equation of motion. Using explicit formula of evolution operator, we find the wave function and propagator. Later, we introduce new notation for constructing dynamical invariants.

In Chapter 5, we study the Lewis-Riesenfeld invariant approach, which is based on finding quadratic invariant for the system described by the time-dependent Hamiltonian. After finding the quadratic invariant in terms of $\sigma(t)$, which is solution of the Ermakov-Pinney equation, we obtain its eigenstates. Then, multiplying these states with phase factor, we construct solution of the IVP for the Schrödinger equation. In addition, the propagator is obtained by using eigenstates of quadratic invariant.

In Chapter 6, we find two linearly independent invariants with time-dependent complex-valued parameters, which are symmetry operators for the Schrödinger equation. By using these invariants, we construct quadratic and Hermitian invariants. After that we find the eigenstates of quadratic invariants, which we use to obtain wave function and propagator. Furthermore, we give initial values for complex solutions of classical equation of motion and compare the results obtained by three different approaches.

In Conclusion, we summarize our main results. Details of some calculations, and required definitions are given in Appendix.

CHAPTER 2

PRELIMINARIES

2.1. Time-dependent Schrödinger Equation and the Evolution Operator

The evolution of a quantum system is described by the time-dependent Schrödinger equation

$$i\hbar\frac{\partial\Psi(q,t)}{\partial t} = \hat{H}\Psi(q,t), \tag{2.1}$$

where $\hat{H}$ is a linear Hermitian operator acting in a complex Hilbert space $L_t^2(\mathbb{R})$, called the Hamiltonian or energy operator, and function $\Psi(q,t)$ characterizes the state of the system at time t and position q. When an initial state of the quantum system at time $t = t_0$ is given as

$$\Psi(q,t_0) = \Psi_0(q), \tag{2.2}$$

then solution of the initial value problem (IVP) (2.1), (2.2) can be found using the evolution operator $\hat{U}(t,t_0)$, that is

$$\Psi(q,t) = \hat{U}(t,t_0)\Psi(q,t_0). \tag{2.3}$$

Substituting wave function (2.3) into IVP (2.1), (2.2) one obtains that the evolution operator must satisfy the operator IVP

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t,t_0) = \hat{H}\hat{U}(t,t_0), \tag{2.4}$$

$$\hat{U}(t_0,t_0) = \hat{1}, \tag{2.5}$$

which is usually accepted as definition of the evolution operator. It follows that

$$\Psi(q,t_2) = \hat{U}(t_2,t_1)\Psi(q,t_1) = \hat{U}(t_2,t_1)\hat{U}(t_1,t_0)\Psi(q,t_0) = \hat{U}(t_2,t_0)\Psi(q,t_0)$$

showing that evolution operator satisfies the composition or group property

$$\hat{U}(t,t_2) = \hat{U}(t,t_1)\hat{U}(t_1,t_2), \quad t_2 < t_1 < t.$$

Now, depending on the Hamiltonian of the system, the following two cases may arise.

Case 1:

Suppose that the quantum system is conservative, so that Hamiltonian is explicitly time-independent. Let us denote this Hamiltonian by $\hat{H}_0$. In this case, the evolution operator is of the form

$$\hat{U}(t,t_0) = \exp\left[\frac{-i}{\hbar}(t-t_0)\hat{H}_0\right], \tag{2.6}$$

and using it, derivative of operator $\hat{U}(t,t_0)$ with respect to time t can be defined exactly like the derivative of ordinary function. One can verify directly that the operator (2.6) satisfies IVP (2.4),(2.5). Since Hamiltonian $\hat{H}_0$ is Hermitian, then

$$\hat{U}^{\dagger}(t,t_0) = \exp[\frac{i}{h}(t-t_0)\hat{H}_0] = \hat{U}^{-1}(t,t_0),$$

$$\hat{U}^{\dagger}(t,t_0)\hat{U}(t,t_0) = \hat{U}(t,t_0)\hat{U}^{\dagger}(t,t_0) = \hat{1},$$

which shows that $\hat{U}(t,t_0)$ is a unitary operator.

Case 2:

Suppose that Hamiltonian depends explicitly on time, and denote it by $\hat{H}(t)$. In this case formal integration can be done, but since one should care about time ordering, usually the evolution operator is written as

$$\hat{U}(t,t_0) = \tau \exp\left[\frac{-i}{\hbar}\int_{t_0}^{t} H(t')dt'\right], \tag{2.7}$$

where τ denotes time-ordering operator.

In the present work, to find the evolution operator we shall use its definition given by (2.4), (2.5) which holds also in the case when Hamiltonian depends on time explicitly. Next proposition shows that the evolution operator is unitary even when Hamiltonian depends on time.

Proposition 2.1 *The evolution operator of a quantum system with explicitly time-dependent Hermitian Hamiltonian is unitary.*

Proof By using equation (2.4) we can write the Hermitian conjugate equation:

$$-i\hbar\frac{\partial}{\partial t}\hat{U}^{\dagger}(t,t_0) = \hat{U}^{\dagger}(t,t_0)\hat{H}^{\dagger}(t). \tag{2.8}$$

Multiplying (2.4) from the left by the $\hat{U}^{\dagger}$, and (2.8) from the right by the $\hat{U}$, then subtracting one from an other we have

$$i\hbar\left(\hat{U}^{\dagger}\frac{\partial\hat{U}}{\partial t} + \frac{\partial\hat{U}^{\dagger}}{\partial t}\hat{U}\right) = (\hat{U}^{\dagger}\hat{H}(t)\hat{U} - \hat{U}^{\dagger}\hat{H}^{\dagger}(t)\hat{U}). \tag{2.9}$$

Since Hamiltonian is Hermitian, the right hand side of (2.9) is zero, so that

$$i\hbar\left(\hat{U}^{\dagger}\frac{\partial\hat{U}}{\partial t} + \frac{\partial\hat{U}^{\dagger}}{\partial t}\hat{U}\right) = i\hbar\frac{d}{dt}\left(\hat{U}^{\dagger}\hat{U}\right) = 0, \tag{2.10}$$

showing that the operator $\hat{U}^\dagger(t,t_0)\hat{U}(t,t_0)$ does not depend on time. Since $\hat{U}(t_0,t_0) = \hat{1}$, this product is identity operator at $t = t_0$ and it follows that $\hat{U}^\dagger(t,t_0)\hat{U}(t,t_0) = \hat{1}$ at any time. □

2.2. Symmetries and dynamical invariants of Schrödinger Equation

In this section we discuss the concept of symmetry, integrals of the motion for quantum systems, and their relations. The symmetry of a physical system is a broad concept both in physics and mathematics, and sometimes different meanings are assigned to it in different contexts. In this thesis we adopt the definitions given in the work of V.I. Man'ko, (Man'ko, 1987). These definitions and some of their elementary properties will be formulated mostly in a way suitable for the present study of the Schrödinger equation (2.1), and the associated Schrödinger operator

$$\hat{S}(t) \equiv i\hbar\frac{\partial}{\partial t} - \hat{H}(t). \tag{2.11}$$

Symmetry of Physical Systems

Definition 2.1 *The dynamical symmetry of a quantum system is a collection of operators which form a Lie algebra and take a solution of the Schrödinger equation to other solutions of the same equation (Man'ko, 1987).*

According to this, the collection of symmetry operators generates the dynamical symmetry group.

Proposition 2.2 *An operator $\hat{K}(t)$ is a dynamical symmetry operator for SE (2.1) if it satisfies*

$$[\hat{S}(t), \hat{K}(t)]\Psi = 0, \tag{2.12}$$

for any Ψ being arbitrary solution of the SE.

Proof Assume $[\hat{S}(t), \hat{K}(t)]\Psi = 0$, that is

$$[\hat{S}(t), \hat{K}(t)]\Psi = \hat{S}(t)\hat{K}(t)\Psi - \hat{K}(t)\hat{S}(t)\Psi = 0.$$

Since $\hat{S}(t)\Psi = 0$, the above equation becomes

$$[\hat{S}(t), \hat{K}(t)]\Psi = \hat{S}(t)\left(\hat{K}(t)\Psi\right) = 0,$$

which shows that $\hat{K}(t)$ is a symmetry operator for the Schrödinger equation. □

Note that an operator $\hat{K}(t)$ is a symmetry for SE (2.1) if it satisfies more strong condition

$$[\hat{S}(t), \hat{K}(t)] = 0, \qquad (2.13)$$

which means the commutator is identically zero.

Proposition 2.3 *Let $\hat{K}(t)$ be a symmetry operator for SE (2.1) and function Ψ_0 satisfies SE, that is $\hat{S}(t)\Psi_0 = 0$. Then,*

a) $\hat{K}^n(t)$ is a symmetry operator for each $n = 1, 2, 3, ...$ and the functions defined as $\Psi_n = (\hat{K}(t))^n\Psi_0$ are solutions of the SE.

b) In general, for any analytic function f, the operator $f(\hat{K}(t))$ is a symmetry and the function $\Psi = f(\hat{K}(t))\Psi_0$ is also solution of the SE.

Proof a) Using mathematical induction, we will show that $\hat{S}(t)\left(\hat{K}^n(t)\Psi_0\right) = 0$. For the case $n = 1$, it is clear that $\hat{S}(t)\left(\hat{K}(t)\Psi_0\right) = 0$ since $\hat{K}(t)$ is a symmetry operator. Now assume that $\hat{S}(t)\left(\hat{K}^{n-1}(t)\Psi_0\right) = 0$. Then,

$$\hat{S}(t)\left(\hat{K}^n(t)\Psi_0\right) = \hat{S}(t)\hat{K}(t)\left(\hat{K}^{n-1}(t)\Psi_0\right) = \hat{K}(t)\hat{S}(t)\left(\hat{K}^{n-1}(t)\Psi_0\right) = 0,$$

which shows that $\left(\hat{K}^n(t)\Psi_0\right) = \Psi_n$ for n=1,2,3,... are solutions of the Schrödinger equation.

b) Using part (a) and the fact that any analytic function has a power series expansion, we have

$$\hat{S}(t)\Big(f(\hat{K}(t))\Psi_0\Big) = \hat{S}(t)\left(\sum_{n=1}^{\infty} c_n \hat{K}^n(t)\Psi_0\right) = c_n\left(\sum_{n=1}^{\infty} \hat{S}(t)\hat{K}^n(t)\Psi_0\right) = 0.$$

Since we find $\hat{S}(t)\Big(f(\hat{K}(t))\Psi_0\Big) = 0$, this shows that it is also solution of the Schrödinger equation. □

Quantum Dynamical Invariants (Quantum Integrals of the Motion)

Definition 2.2 *An operator $\hat{I}(t)$ acting on a state $|\Psi(t)\rangle$ of the quantum system is called a quantum dynamical invariant (quantum integral of the motion), if its expectation value at this state does not change with time, that is*

$$\frac{d}{dt}\Big\langle \Psi(t)\Big|\hat{I}(t)\Big|\Psi(t)\Big\rangle = 0. \tag{2.14}$$

Proposition 2.4 *$\hat{I}(t)$ is a dynamical invariant for SE (2.1) if and only if*

$$\left(i\hbar\frac{\partial \hat{I}(t)}{\partial t} - [\hat{H}(t), \hat{I}(t)]\right)|\Psi\rangle = 0, \tag{2.15}$$

for any Ψ being arbitrary solution of the SE.

Proof Taking the derivative of the equation (2.14), we have

$$\begin{aligned}
\frac{d}{dt}\langle \hat{I}(t)\rangle_\Psi &= \frac{\partial}{\partial t}\langle\Psi|\hat{I}(t)|\Psi\rangle = \langle\frac{\partial\Psi}{\partial t}|\hat{I}(t)|\Psi\rangle + \langle\Psi|\frac{\partial \hat{I}(t)}{\partial t}|\Psi\rangle + \langle\Psi|\hat{I}(t)|\frac{\partial\Psi}{\partial t}\rangle,\\
&= \langle\frac{1}{i\hbar}\hat{H}(t)\Psi|\hat{I}(t)|\Psi\rangle + \langle\Psi|\frac{\partial \hat{I}(t)}{\partial t}|\Psi\rangle + \langle\Psi|\hat{I}(t)|\frac{1}{i\hbar}\hat{H}(t)\Psi\rangle,\\
&= \langle\Psi|\frac{\partial \hat{I}(t)}{\partial t}|\Psi\rangle + \frac{1}{i\hbar}\langle\Psi|\hat{I}(t)\hat{H}(t) - \hat{H}(t)\hat{I}(t)|\Psi\rangle,\\
&= \left\langle\frac{\partial \hat{I}(t)}{\partial t} + \frac{1}{i\hbar}[\hat{I}(t), \hat{H}(t)]\right\rangle_\Psi = \frac{1}{i\hbar}\left\langle i\hbar\frac{\partial \hat{I}(t)}{\partial t} - [\hat{H}(t), \hat{I}(t)]\right\rangle_\Psi.
\end{aligned}$$

If $\hat{I}(t)$ is a dynamical invariant, its expectation value does not change with time, i.e.,

$\frac{d}{dt}\left\langle\Psi(t)\middle|\hat{I}(t)\middle|\Psi(t)\right\rangle = 0$, so we have (2.15). Conversely if (2.15) is valid, it is clear to see $d\langle\hat{I}(t)\rangle_\Psi/dt = 0$ which yields that $\hat{I}(t)$ is a dynamical invariant for SE (2.1). □

We note that, an operator $\hat{I}(t)$ is a dynamical invariant for SE (2.1) if it satisfies

$$i\hbar\frac{\partial\hat{I}(t)}{\partial t} - [\hat{H}(t), \hat{I}(t)] = 0, \tag{2.16}$$

and this condition (2.16) is stronger than condition (2.15).

If a system has different invariants, say $\hat{I}_1$ and $\hat{I}_2$, then arbitrary functions of them, and in particular their commutators $[\hat{I}_1, \hat{I}_2]$, and in general for n, m positive integers $[\hat{I}_1^n, \hat{I}_2^m]$, or anti-commutators $\{\hat{I}_1, \hat{I}_2\}$ are also invariants.

Proposition 2.5 *Eigenvalues of Hermitian dynamical invariant $\hat{I}(t)$ are real and do not depend on time.*

Proof Let $\hat{I}(t)|\varphi_n\rangle = \lambda_n|\varphi_n\rangle$, where λ_n is an eigenvalue of $\hat{I}(t)$, with corresponding eigenvectors $|\varphi_n\rangle$. Consider the following equations:

$$\begin{aligned}
\langle\hat{I}(t)\varphi_n|\varphi_n\rangle &= \langle\varphi_n|\hat{I}^\dagger(t)\varphi_n\rangle = \langle\varphi_n|\hat{I}(t)\varphi_n\rangle, \\
\langle\lambda_n\varphi_n|\varphi_n\rangle &= \langle\varphi_n|\lambda_n\varphi_n\rangle, \\
\bar{\lambda}_n\langle\varphi_n|\varphi_n\rangle &= \lambda_n\langle\varphi_n|\varphi_n\rangle, \\
\left(\bar{\lambda}_n - \lambda_n\right)\langle\varphi_n|\varphi_n\rangle &= 0,
\end{aligned}$$

where $\langle\varphi_n|\varphi_m\rangle = \int_{-\infty}^{\infty}\varphi_n^*(q)\varphi_m(q)dq$. Since $\langle\varphi_n, \varphi_n\rangle \neq 0$, we have $\left(\bar{\lambda}_n - \lambda_n\right) = 0$ which shows that λ_n is real. For showing that eigenvalues are time-independent, we will take time-derivative of both sides of the $\hat{I}(t)|\varphi_n\rangle = \lambda_n(t)|\varphi_n\rangle$ which gives

$$\frac{\partial\hat{I}(t)}{\partial t}|\varphi_n\rangle + \hat{I}(t)\frac{\partial|\varphi_n\rangle}{\partial t} = \dot{\lambda}_n(t)|\varphi_n\rangle + \lambda_n(t)\frac{\partial|\varphi_n\rangle}{\partial t}. \tag{2.17}$$

Using equation (2.16) we get

$$\frac{d\lambda_n(t)}{dt}|\varphi_n\rangle = (\hat{I}(t) - \lambda_n(t))\frac{\partial|\varphi_n\rangle}{\partial t} + \frac{i}{\hbar}(\hat{I}\hat{H}|\varphi_n\rangle - \hat{H}\hat{I}|\varphi_n\rangle), \tag{2.18}$$

and arranging the terms gives

$$\frac{d\lambda_n(t)}{dt}|\varphi_n\rangle = (\hat{I}(t) - \lambda_n(t))\Big(\frac{\partial|\varphi_n\rangle}{\partial t} + \frac{i}{\hbar}\hat{H}|\varphi_n\rangle\Big), \quad \forall n. \tag{2.19}$$

Taking the inner product with $\langle\varphi_n|$ gives

$$\frac{d\lambda_n(t)}{dt}\Big\langle\varphi_n|\varphi_n\Big\rangle = \Big\langle\varphi_n|(\hat{I}(t) - \lambda_n(t))\Big(\frac{\partial\varphi_n}{\partial t} + \frac{i}{\hbar}\hat{H}\varphi_n\Big)\Big\rangle, \quad \forall n. \tag{2.20}$$

Since $\hat{I}(t) - \lambda(t)$ is a self-adjoint, we can write

$$\frac{d\lambda_n(t)}{dt}\big\|\varphi_n\big\|^2 = \Big\langle(\hat{I}(t) - \lambda_n(t))\varphi_n|\frac{\partial\varphi_n}{\partial t} + \frac{i}{\hbar}\hat{H}\varphi_n\Big\rangle, \tag{2.21}$$

which clearly implies $\dot{\lambda}_n(t) = 0$, so that $\lambda_n(t) = \lambda_n$ is a constant. □

Proposition 2.6 *Let $\hat{S}(t) = i\hbar(\partial/\partial t) - \hat{H}(t)$ be the Schrödinger operator (2.11). Then,*

$$[\hat{S}(t), \hat{K}(t)] = 0 \quad \Leftrightarrow \quad i\hbar\frac{\partial\hat{K}(t)}{\partial t} - [\hat{H}(t), \hat{K}(t)] = 0. \tag{2.22}$$

Proof Consider the commutation of $\hat{S}(t)$ and $\hat{K}(t)$, i.e.,

$$[\hat{S}(t), \hat{K}(t)] \;=\; \hat{S}(t)\hat{K}(t) - \hat{K}(t)\hat{S}(t) = (i\hbar\frac{\partial}{\partial t} - \hat{H}(t))\hat{K}(t) - \hat{K}(t)(i\hbar\frac{\partial}{\partial t} - \hat{H}(t)).$$

Applying this equality to arbitrary function f, we have

$$\begin{aligned}
\Big(\hat{S}(t)\hat{K}(t) - \hat{K}(t)\hat{S}(t)\Big)f &= (i\hbar\frac{\partial}{\partial t} - \hat{H}(t))\hat{K}(t)f - \hat{K}(t)(i\hbar\frac{\partial}{\partial t} - \hat{H}(t))f, \\
&= i\hbar\hat{K}(t)\frac{\partial f}{\partial t} + i\hbar\frac{\partial\hat{K}(t)}{\partial t}f - i\hbar\hat{K}(t)\frac{\partial f}{\partial t} + [\hat{K}(t), H(t)]f, \\
&= \Big(i\hbar\frac{\partial\hat{K}(t)}{\partial t} - [H(t), \hat{K}(t)]\Big)f.
\end{aligned}$$

which implies

$$[\hat{K}(t), \hat{S}(t)] = \left(i\hbar\frac{\partial\hat{K}(t)}{\partial t} - [H(t), \hat{K}(t)]\right).$$

It follows that if $\hat{K}(t)$ is a symmetry operator satisfying condition (2.13), then it is also a dynamical invariant. Conversely, if $\hat{K}(t)$ is a dynamical invariant satisfying condition (2.16), then it is a symmetry operator. □

Clearly, one can consider also the weaker form of above proposition, that's

$$[\hat{S}(t), \hat{K}(t)]\Psi = 0 \quad \Leftrightarrow \quad i\hbar\frac{\partial\hat{K}(t)}{\partial t}\Psi - [\hat{H}(t), \hat{K}(t)]\Psi = 0, \tag{2.23}$$

where Ψ satisfies the SE. It shows that, in the present context a symmetry operator is equivalent to a dynamical invariant.

2.2.1. Quantum integrals of the motion and Evolution operator formalism

We have seen that solution of the IVP for Schrödinger equation is completely determined by the evolution operator $\hat{U}(t, t_0)$, which carries the initial state $\Psi(q, t_0)$ into the state $\Psi(q, t)$ at later time t. Here, we will show the connection between the integral of the motions and the evolution operator.

Proposition 2.7 *a) Any operator $\hat{I}(t)$ of the form*

$$\hat{I}(t) \quad = \quad \hat{U}(t, t_0)\hat{I}(t_0)\hat{U}^{-1}(t, t_0). \tag{2.24}$$

is an integral of the motion.

b) Conversely, any integral of the motion $\hat{I}(t)$ always has the form (2.24).

Proof a) For the proof it is sufficient to show that the expectation value of $\hat{I}(t)$ does not depend on time. Indeed, using (2.24) and $\Psi(t) = \hat{U}(t, t_0)\Psi(t_0)$, we have the following

$$\begin{aligned}\langle\Psi(t)|\hat{I}(t)|\Psi(t)\rangle &= \langle\hat{U}(t, t_0)\Psi(t_0)|\hat{U}(t, t_0)\hat{I}(t_0)\hat{U}^{-1}(t, t_0)|\hat{U}(t, t_0)\Psi(t_0)\rangle \\ &= \langle\Psi(t_0)|\hat{U}^\dagger\hat{U}\hat{I}(t_0)\hat{U}^{-1}\hat{U}|\Psi(t_0)\rangle,\end{aligned}$$

and since $\hat{U}$ is unitary, it follows that

$$\langle\Psi(t)|\hat{I}(t)|\Psi(t)\rangle = \langle\Psi(t_0)|\hat{I}(t_0)|\Psi(t_0)\rangle, \tag{2.25}$$

which shows that the expectation value does not depend on time. According to the definition of an integral of motion, we can conclude that operator $\hat{I}(t)$ is an integral of motion.

b) Suppose that $\hat{I}(t)$ is an integral of the motion. Then, by definition we know that the equality (2.25) is valid. Replacing state $|\Psi(t)\rangle$ by $|\Psi(t)\rangle = \hat{U}(t, t_0)|\Psi(t_0)\rangle$ in equation (2.25), we get

$$\langle\Psi(t_0)|\hat{U}^\dagger(t, t_0)\hat{I}(t)\hat{U}(t, t_0)|\Psi(t_0)\rangle = \langle\Psi(t_0)|\hat{I}(t_0)|\Psi(t_0)\rangle. \tag{2.26}$$

From the equality (2.26) of matrix elements we get equality for the operators since it holds for any vector $|\Psi(t_0)\rangle$. Therefore we have $\hat{U}^\dagger(t, t_0)\hat{I}(t)\hat{U}(t, t_0) = \hat{I}(t_0)$ or the equality we desire $\hat{I}(t) = \hat{U}(t, t_0)\hat{I}(t_0)\hat{U}^{-1}(t, t_0)$. □

Proposition 2.8 *The eigenvalues of integrals of the motion $\hat{I}(t)$ do not depend on time.*

Proof : Let the operator $\hat{I}(t)$ be an integral of the motion, and assume it has eigenvectors $\varphi_\lambda(t)$ with corresponding eigenvalues $\lambda(t)$, that is

$$\hat{I}(t)\varphi_\lambda(t) = \lambda(t)\varphi_\lambda(t). \tag{2.27}$$

Then, at time $t = t_0$ we have

$$\hat{I}(t_0)\varphi_\lambda(t_0) = \lambda(t_0)\varphi_\lambda(t_0), \tag{2.28}$$

Using $\hat{I}(t_0) = \hat{U}^{-1}(t, t_0)\hat{I}(t)\hat{U}(t, t_0)$, last equality implies

$$\begin{aligned} \hat{U}^{-1}(t, t_0)\hat{I}(t)\hat{U}(t, t_0)\varphi_\lambda(t_0) &= \lambda\varphi_\lambda(t_0), \\ \hat{I}(t)\hat{U}(t, t_0)\varphi_\lambda(t_0) &= \lambda\hat{U}(t, t_0)\varphi_\lambda(t_0), \\ \hat{I}(t)\varphi_\lambda(t) &= \lambda\varphi_\lambda(t), \end{aligned}$$

showing that the eigenvalues λ of the operator $\hat{I}(t)$ do not depend on time by construction. Since the integral of the motion $\hat{I}(t)$ and operator $\hat{I}(t_0)$ are connected by a unitary evolution operator, they have the same spectrum and therefore there are no other eigenvalues of $\hat{I}(t)$ different from λ. □

Proposition 2.9 *If $\hat{I}(t)$ is an integral of the motion, then $\hat{I}^n(t)$ for each $n = 1, 2, 3, ...$ and $f(\hat{I}(t))$ for any analytic function f are also integrals of the motion.*

Proof : According to the previous proposition we have equality (2.24). Using this equality and Proposition 2.7, it follows

$$\hat{I}^2(t) = \hat{U}(t)\hat{I}(t_0)\hat{U}^{-1}(t)\hat{U}(t)\hat{I}(t_0)\hat{U}^{-1}(t) = \hat{U}(t)\hat{I}^2(t_0)\hat{U}^{-1}(t). \tag{2.29}$$

By induction one can easily show that any power of an integral of the motion is again an integral of motion. Since any analytic function f can be represented as a power series, then $f(\hat{I}(t))$ is also an integral of the motion. □

Also the product of the distinct integrals of the motion with the same Hamiltonian is an integral of the motion. It can be proved in the same way we did for the square of the operator-integral of the motion.

Proposition 2.10 *Integral of the motion takes a solution of the Schrödinger equation into a solution of the same equation.*

Proof : Let $\Psi(t) = \hat{U}(t, t_0)\Psi(t_0)$ be solution of the Schrödinger equation. We want to show that the new function defined as $\phi(t) = \hat{I}(t)\Psi(t)$ is also a solution of the SE. Indeed, we have

$$
\begin{aligned}
\phi(t) &= \hat{I}(t)\Psi(t) = \hat{I}(t)\hat{U}(t, t_0)\Psi(t_0) \\
&= \hat{U}(t, t_0)\hat{I}(t_0)\hat{U}^{-1}(t, t_0)\hat{U}(t, t_0)\Psi(t_0) \\
&= \hat{U}(t, t_0)\hat{I}(t_0)\Psi(t_0) = \hat{U}(t, t_0)\phi(t_0),
\end{aligned}
$$

where we define $\phi(t_0) = \hat{I}(t_0)\Psi(t_0)$. Thus, the new function is of the form
$\phi(t) = \hat{U}(t, t_0)\phi(t_0)$, showing that $\phi(t)$ also becomes a solution of the Schrödinger equation.

□

CHAPTER 3

SIMPLE HARMONIC OSCILLATOR

Consider the following Hamiltonian:

$$\hat{H}_0 = \frac{1}{2m}\hat{p}^2 + \frac{m\omega_0^2}{2}\hat{q}^2, \tag{3.1}$$

where $\hat{p}$ and $\hat{q}$ are momentum and position operators, respectively, m is the particle mass, ω_0 is the angular frequency and both of them are time independent. Introducing the operators

$$\hat{a}^\dagger = \sqrt{\frac{m\omega_0}{2\hbar}}\left(\hat{q} - i\frac{1}{m\omega_0}\hat{p}\right), \tag{3.2}$$

$$\hat{a} = \sqrt{\frac{m\omega_0}{2\hbar}}\left(\hat{q} + i\frac{1}{m\omega_0}\hat{p}\right), \tag{3.3}$$

$$\hat{N} = \hat{a}^\dagger\hat{a}, \tag{3.4}$$

we have

$$\hat{H}_0 = \hbar\omega_0(\hat{a}\hat{a}^\dagger - \frac{1}{2}) = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2}) = \hbar\omega_0(\hat{N} + \frac{1}{2}), \tag{3.5}$$

By using above, we get following equalities:

$$\hat{p} = i\sqrt{\frac{m\omega_0\hbar}{2}}\left(\hat{a}^\dagger - \hat{a}\right),\ \hat{q} = \sqrt{\frac{\hbar}{2m\omega_0}}\left(\hat{a}^\dagger + \hat{a}\right). \tag{3.6}$$

The operators $\hat{a}$, $\hat{a^\dagger}$, and $\hat{N}$ satisfy the following commutation relations

$$[\hat{a}, \hat{a^\dagger}] = 1,\ [N, \hat{a}^\dagger] = \hat{a}^\dagger,\ [N, \hat{a}] = -\hat{a}. \tag{3.7}$$

Therefore we have spectrum generating algebra $\{1, \hat{a}^\dagger, \hat{a}, N\}$.

3.1. Eigenvalues and Eigenstates of the Hamiltonian $\hat{H}_0$

The eigenvalue problem for $\hat{H}_0$ also known as time-independent Schrödinger equation is

$$\hat{H}_0\varphi_n(q) = E_n\varphi_n(q), \tag{3.8}$$

where E_n are eigenvalues of Hamiltonian operator $\hat{H}_0$ and $\varphi_n(q)$ are the corresponding eigenstates. Eigenvalues and eigenstates of $\hat{H}_0$ can be found by an algebraic approach (Dirac, 1982) :

Consider the number operator $\hat{N}$ which is a Hermitian operator. Since it is a Hermitian, its eigenvalues must be real. Let us denote eigenvalues of $\hat{N}$ by λ_n and corresponding normalized eigenstates by $|\lambda_n\rangle$, i.e.,

$$\hat{N}|\lambda_n\rangle = \lambda_n|\lambda_n\rangle. \tag{3.9}$$

Firstly, we will show that eigenvalues of $\hat{N}$ are non-negative, that's $\lambda_n \geq 0$,

$$0 \leq ||\hat{a}|\lambda_n\rangle||^2 = \langle\lambda_n|\hat{a}^\dagger\hat{a}|\lambda_n\rangle = \langle\lambda_n|\hat{N}|\lambda_n\rangle = \lambda_n\langle\lambda_n|\lambda_n\rangle = \lambda_n \tag{3.10}$$

Next, we will see that $\hat{a}|n\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue $\lambda_n - 1$, $\hat{a}^2|\lambda_n\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue $\lambda_n - 2$ and so on.

$$\hat{N}\hat{a}|\lambda_n\rangle = \hat{a}(\hat{N} - 1)|\lambda_n\rangle = (\lambda_n - 1)\hat{a}|\lambda_n\rangle, \tag{3.11}$$

$$\hat{N}\hat{a}^2|\lambda_n\rangle = \hat{a}(\hat{N} - 1)\hat{a}|\lambda_n\rangle = (\lambda_n - 2)\hat{a}|\lambda_n\rangle. \tag{3.12}$$

Now we show that eigenvalues λ_n must be integers and the only possible eigenstates are $|0\rangle, |1\rangle, |2\rangle, ...$ Indeed, if λ_n is not an integer, and we apply lowering operator consecutively, eventually we would come to negative eigenvalues. But since we found that the eigenvalues of $\hat{N}$ can not be negative, it is not allowed to obtain negative eigenvalues. So lowest possible eigenstate is $|0\rangle$ with eigenvalue 0 and thus, λ_n can take only integer values, i.e., $\lambda_n = n$.

Since $\hat{a}|n\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue $n-1$, it must be proportional to $|n-1\rangle$, i.e., $\hat{a}|n\rangle = c_n|n-1\rangle$ where c_n is a constant.

$$n = \langle n|\hat{a}^\dagger \hat{a}|n\rangle = |c_n|^2\langle n-1|n-1\rangle = |c_n|^2 \tag{3.13}$$

Thus we found $c_n = \sqrt{n}$. Using this result we have $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and we see that $\hat{a}$ is a lowering operator. If we consider operator $\hat{a}^\dagger$, we can find that $\hat{a}^\dagger|n\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue $n+1$, $(\hat{a}^\dagger)^2|n\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue $n+2$ and since it is proportional to $|n+1\rangle$ we can find $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. Thus $\hat{a}^\dagger$ will be a raising operator.

Since we can express Hamiltonian $\hat{H}_0$ as given in equation (3.5), the eigenstates of $\hat{H}_0$ are the eigenstates of $\hat{N}$, and we have

$$\hat{H}_0|n\rangle \;=\; \hbar\omega_0(N + \frac{1}{2})|n\rangle = \hbar\omega_0(n + \frac{1}{2})|n\rangle. \tag{3.14}$$

which shows that the eigenvalues of $\hat{H}_0$ are $E_n = \hbar\omega_0(n + \frac{1}{2})$ for n=0,1,2,3,...

Eigenstates in Coordinate Representation

Since correspoding eigenstates of E_n are φ_n, $\langle q|n\rangle = \varphi_n(q)$ the equation (3.8) will be

$$\hat{H}_0\varphi_n(q) = \hbar\omega_0(n + \frac{1}{2})\varphi_n(q). \tag{3.15}$$

For finding the eigenstates in coordinate representation we shall use $\hat{q} = q$ and $\hat{p} = -i\hbar\frac{d}{dq}$

in Hamiltonian (3.1). So Hamiltonian becomes

$$\hat{H}_0 = \frac{-\hbar^2}{2m}\frac{d^2}{dq^2} + \frac{m\omega_0^2}{2}q^2.$$

Then $\hat{a}\varphi_0(q) = 0$, that's

$$\hat{a}\varphi_0(q) = \left(\sqrt{\frac{m\omega_0}{2\hbar}}q + \sqrt{\frac{\hbar}{2m\omega_0}}\frac{d}{dq}\right)\varphi_0(q) = 0. \tag{3.16}$$

Above equation is a first order differential equation. Solution of this equation is $\varphi_0(q) = N_0 e^{\frac{-m\omega_0 q^2}{2\hbar}}$. After normalization, we have

$$\varphi_0(q) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{\frac{-m\omega_0 q^2}{2\hbar}}.$$

Other eigenstates can be found by applying $\hat{a}^\dagger$ to the ground state $\varphi_0(q)$. Therefore for finding $\varphi_n(q)$, one needs to apply n times $\hat{a}^\dagger$ to $\varphi_0(q)$, that is

$$\varphi_n(q) = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\varphi_0(q) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{n!}}\left(\sqrt{\frac{m\omega_0}{2\hbar}}q - \sqrt{\frac{\hbar}{2m\omega_0}}\frac{d}{dq}\right)^n e^{\frac{-m\omega_0 q^2}{2\hbar}}.$$

Let $\xi = \sqrt{(m\omega_0)/\hbar}q$, then we obtain $\varphi_n = N_n e^{-\xi^2/2} e^{\xi^2/2}(\xi - d/d\xi)^n e^{-\xi^2/2}$, where $e^{\xi^2/2}(\xi - d/d\xi)^n e^{-\xi^2/2} = H_n(\xi)$ represents the n-th order Hermite polynomial and
N_n is the normalization constant that can be found as $N_n = (2^n n!)^{-1/2}(m\omega_0/\pi\hbar)^{1/4}$. As a result, the normalized eigenstates of the $\hat{H}_0$ are

$$\varphi_n(q) = N_n e^{\frac{-m\omega_0 q^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}q\right), \qquad n = 0, 1, 2, ... \tag{3.17}$$

where corresponding eigenvalues are $E_n = \hbar\omega_0(n + 1/2)$. The system $\{\varphi_n(q)\}_{n=0}^{\infty}$ forms an orthonormal basis for $L^2(\mathbb{R})$. Therefore any $\Psi(q) \in L^2(\mathbb{R})$ has a unique representation.

From $|\Psi\rangle = \sum_{n=0}^{\infty} |n\rangle \langle n| \, |\Psi\rangle$, we have

$$\langle q|\Psi\rangle = \sum_{n=0}^{\infty} \langle q|n\rangle \langle n|\Psi\rangle \,, \tag{3.18}$$

$$\Rightarrow \Psi(q) = \sum_{n=0}^{\infty} c_n \varphi_n(q) = \sum_{n=0}^{\infty} \langle n|\Psi\rangle \, \varphi_n(q), \tag{3.19}$$

where $c_n = \langle n|\Psi\rangle = \langle n|q\rangle \langle q|\Psi\rangle = \int_{-\infty}^{\infty} \varphi_n^*(q)\Psi(q)dq.$

3.2. The time-dependent Schrödinger Equation

Consider the IVP for the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \Psi(q,t)}{\partial t} = \hat{H}_0 \Psi(q,t), \quad -\infty < q < \infty \tag{3.20}$$

$$\Psi(q,t_0) = \Psi_0(q), \tag{3.21}$$

where $\Psi_0(q) \in L^2(\mathbb{R})$ and $\hat{H}_0$ is given by equation (3.1).

3.2.1. Solution of the IVP for the Schrödinger Equation: Standart approach

In this approach we use the evolution operator of the Schrödinger equation (3.20) in the form

$$\hat{U}_0(t,t_0) = e^{-\frac{i}{\hbar}(t-t_0)\hat{H}_0} \tag{3.22}$$

and that the eigenstates $\{\varphi_n(q)\}_{n=0}^{\infty}$ of $\hat{H}_0$ form an orthonormal basis for $L^2(\mathbb{R})$. Then, any initial condition $\Psi_0(q) \in L^2(\mathbb{R})$ can be written as

$$\Psi_0(q) = \sum_{n=0}^{\infty} \langle n|\Psi_0\rangle \varphi_n(q), \tag{3.23}$$

and solution of the IVP for the Schrödinger equation can be found by applying evolution operator (3.22) to $\Psi_0(q)$,

$$\Psi(q,t) = \hat{U}_0(t,t_0)\Psi_0(q) = e^{-\frac{i}{\hbar}(t-t_0)\hat{H}_0}\Bigg(\sum_{n=0}^{\infty}\langle n|\Psi_0\rangle\varphi_n(q)\Bigg), \tag{3.24}$$

$$= \sum_{n=0}^{\infty}\langle n|\Psi_0\rangle e^{-\frac{i}{\hbar}(t-t_0)E_n}\varphi_n(q), \tag{3.25}$$

where we used also the spectral mapping theorem:

$$\hat{H}_0\varphi_n = E_n\varphi_n \Rightarrow f(\hat{H}_0)\varphi_n = f(E_n)\varphi_n, \tag{3.26}$$

for any analytic function f.

An equivalent procedure for solving the IVP (3.20),(3.21) is first to apply the evolution operator (3.22) to $\varphi_n(q)$ and find solutions of the Schrödinger equation (3.20) as

$$\begin{aligned}
\Psi_n(q,t) &= \hat{U}_0(t,t_0)\varphi_n(q) = \exp\left[-\frac{i}{\hbar}(t-t_0)E_n\right]\varphi_n(q), \\
&= N_n \times \exp\left[-\frac{i}{\hbar}(t-t_0)E_n\right] \times \exp\left[\frac{-m\omega_0 q^2}{2\hbar}\right] \times H_n\Bigg(\sqrt{\frac{m\omega_0}{\hbar}}q\Bigg) \\
&= N_n \times \exp\left[-i\omega_0(t-t_0)(n+\frac{1}{2})\right] \times \exp\left[\frac{-m\omega_0 q^2}{2\hbar}\right] \times H_n\Bigg(\sqrt{\frac{m\omega_0}{\hbar}}q\Bigg),
\end{aligned} \tag{3.27}$$

for each $n = 0, 1, 2, 3...$ Since the set $\{\Psi_n(q,t)\}_{n=0}^{\infty}$ is an orthonormal basis for the solution space of the Schrödinger equation, then any other solution of (3.20) is of the form

$$\Psi(q,t) = \sum_{n=0}^{\infty} c_n \Psi_n(q,t). \tag{3.28}$$

Here the coefficients c_n are fixed by the given initial condition (3.21), $\Psi(q,t_0) = \Psi_0(q)$ as $c_n = \langle n|\Psi_0\rangle$. Then solution of the IVP is

$$\Psi(q,t) = \sum_{n=0}^{\infty} \langle n|\Psi_0\rangle \Psi_n(q,t), \tag{3.29}$$

which is same with solution (3.25).

3.2.2. Lie Algebra

In this part, we give definition and properties of a Lie algebra which will be necessary for further calculations.

Definition 3.1 *A Lie algebra is a vector space over a field* $\mathbb{F}$ *with a multiplication on the vector space defined as Lie bracket and denoted by* $[.,.]$, *i.e.,* $[.,.] : L \times L \to L$ *such that the following properties are satisfied:*

1) A Lie algebra is bilinear, i.e.,

$$\begin{aligned} \left[aX + bY, Z\right] &= a\left[X,Z\right] + b\left[Y,Z\right], \\ \left[Z, aX + bY\right] &= a\left[Z,X\right] + b\left[Z,Y\right]. \end{aligned}$$

2) It satisfies Jacobi Identity which is

$$\left[X,\left[Y,Z\right]\right] + \left[Y,\left[Z,X\right]\right] + \left[Z,\left[X,Y\right]\right] = 0.$$

3) It is skew-symmetric, i.e.,

$$\left[X, Y\right] = -\left[Y, X\right].$$

for all $a, b, c \in \mathbb{F}$ and $X, Y, Z \in L$.

Example 3.1 *Consider the operators*

$$\hat{K}_- = \frac{-i}{2}\frac{\partial^2}{\partial q^2}, \qquad \hat{K}_+ = \frac{i}{2}q^2, \qquad \hat{K}_0 = \frac{1}{2}(q\frac{\partial}{\partial q} + \frac{1}{2}). \tag{3.30}$$

These operators are generators of $su(1,1)$ Lie algebra (Dattoli et al., 1997) and satisfy the following commutation relations:

$$\left[\hat{K}_-, \hat{K}_+\right] = 2\hat{K}_0,\ \left[\hat{K}_0, \hat{K}_+\right] = \hat{K}_+,\ \left[\hat{K}_0, \hat{K}_-\right] = -\hat{K}_-. \tag{3.31}$$

For the proofs, see the Appendix A.

3.2.3. Solution of the IVP for the Schrödinger Equation using the Wei-Norman algebraic Approach

Now, we will solve SE using Wei-Norman Algebraic approach. For this, note that $\hat{H}_0$ can be written as a linear combination of $su(1,1)$ Lie algebra generators, i.e.,

$$\hat{H}_0 = +i\left(\frac{-\hbar^2}{m}\hat{K}_- - m\omega_0^2\hat{K}_+\right). \tag{3.32}$$

where $\hat{K}_-$, $\hat{K}_+$, $\hat{K}_0$ are as defined in (3.30). It follows that we can write the evolution operator $\hat{U}_0(t, t_0)$ as a product of exponential operators, which are generators of $SU(1,1)$

Lie algebra, as $\hat{U}_0(t,t_0) = e^{f_0(t)\hat{K}_+} e^{2h_0(t)\hat{K}_0} e^{g_0(t)\hat{K}_-}$, that's

$$\hat{U}_0(t,t_0) \;=\; \exp\left[\frac{i}{2}f_0(t)q^2\right]\exp\left[h_0(t)\left(q\frac{\partial}{\partial q}+\frac{1}{2}\right)\right]\exp\left[-\frac{i}{2}g_0(t)\frac{\partial^2}{\partial q^2}\right], \tag{3.33}$$

where $f_0(t)$, $g_0(t)$, $h_0(t)$ are real-valued functions to be determined. Using the equation (2.4), one needs to find $\frac{\partial \hat{U}_0}{\partial t}$ and $\hat{H}_0\hat{U}_0$:

$$\begin{aligned}\frac{\partial \hat{U}_0}{\partial t} \;&=\; \dot{f}_o(t)\hat{K}_+\hat{U}_0 + \dot{g}_0(t)e^{f_0(t)\hat{K}_+}\left(e^{2h_0(t)\hat{K}_0}\hat{K}_- e^{-2h_0(t)\hat{K}_0}\right)e^{2h_0(t)\hat{K}_0}e^{g_0(t)\hat{K}_-}\hat{U}_0. \\ &+2\dot{h}_0(t)\left(e^{f_0(t)\hat{K}_+}\hat{K}_0 e^{-f_0(t)\hat{K}_+}\right)\hat{U}_0\end{aligned} \tag{3.34}$$

Rewriting $\frac{\partial \hat{U}_0}{\partial t}$ after using Baker-Hausdorff identity

$$e^{\xi\hat{A}}\hat{B}e^{-\xi\hat{A}} = \hat{B} + \xi[\hat{A},\hat{B}] + \frac{\xi^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{\xi^3}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + .., \tag{3.35}$$

we get

$$\begin{aligned}\frac{\partial \hat{U}_0}{\partial t} \;&=\; \left(\left[\dot{f}_0(t) - 2\dot{h}_0 f_0(t) + e^{-2h_0(t)}\dot{g}_0(t)f_0^2(t)\right]\hat{K}_+\right. \\ &+\; \left.\left[e^{-2h_0(t)}\dot{g}_0(t)\right]\hat{K}_- + \left[2\dot{h}_0(t) - 2f_0(t)\dot{g}_0(t)e^{-2h_0(t)}\right]\hat{K}_0\right)\hat{U}_0.\end{aligned}$$

Since $i\hbar\frac{\partial \hat{U}_0}{\partial t}$ must be equal to $\hat{H}_0\hat{U}_0(t,t_0)$ and $\hat{H}_0$ is given by equation (3.32), then the following relation must hold:

$$\begin{aligned}i\left(\frac{-\hbar^2}{m}\hat{K}_- - m\omega_0^2\hat{K}_+\right) \;&=\; i\hbar\left(\left[\dot{f}_0(t) - 2\dot{h}_0 f_0(t) + e^{-2h_0(t)}\dot{g}_0(t)f_0^2(t)\right]\hat{K}_+ + \left[e^{-2h_0(t)}\dot{g}_0(t)\right]\hat{K}_-\right. \\ &+\; \left.\left[2\dot{h}_0(t) - 2f_0(t)\dot{g}_0(t)e^{-2h_0(t)}\right]\hat{K}_0\right).\end{aligned}$$

For this equality, we come up with IVP for a nonlinear system of three first-order ordinary differential equations for $f_0(t)$, $g_0(t)$, $h_0(t)$, that's

$$\begin{aligned} \dot{f}_0 + \frac{\hbar f_0^2}{m} + \frac{m\omega_0^2}{\hbar} &= 0, \quad f_0(t_0) = 0, &(3.36)\\ \dot{h}_0 + \frac{\hbar f_0}{m} &= 0, \quad h_0(t_0) = 0, &(3.37)\\ \dot{g}_0 + \frac{\hbar e^{2h_0}}{m} &= 0, \quad g_0(t_0) = 0. &(3.38) \end{aligned}$$

Notice that the equation (3.36) is a Ricatti equation and by substitution $f_0(t) = m\dot{x}/\hbar x$, it can be linearized as

$$\ddot{x}(t) + \omega_0^2 x(t) = 0. \tag{3.39}$$

Therefore, solution $f_0(t)$ of equation (3.36) can be expressed in terms of the linear independent solutions of the equation (3.39) namely $x_1(t)$ and $x_2(t)$, which satisfy the initial conditions:

$$\begin{aligned} x_1(t_0) &= x_0 \neq 0 \quad , \quad \dot{x}_1(t_0) = 0, &(3.40)\\ x_2(t_0) &= 0 \quad , \quad \dot{x}_2(t_0) = \frac{1}{mx_0}. &(3.41) \end{aligned}$$

According to this, solutions $x_1(t)$ and $x_2(t)$ will be explicitly

$$\begin{aligned} x_1(t) &= x_0 \cos\Big(\omega_0(t - t_0)\Big), &(3.42)\\ x_2(t) &= \frac{1}{m\omega_0 x_0} \sin\Big(\omega_0(t - t_0)\Big). &(3.43) \end{aligned}$$

By solving IVP for the nonlinear system of three first-order ordinary differential equations we find the following system for $f_0(t)$, $g_0(t)$, $h_0(t)$ in terms of $x_1(t)$ and $x_2(t)$,

$$f_0(t) = \frac{m\dot{x}_1(t)}{\hbar x_1(t)} = \frac{-m\omega_0}{\hbar}\tan\Big(\omega_0(t-t_0)\Big), \tag{3.44}$$

$$g_0(t) = -\hbar x_0^2\Big(\frac{x_2(t)}{x_1(t)}\Big) = \frac{-\hbar}{m\omega_0}\tan\Big(\omega_0(t-t_0)\Big), \tag{3.45}$$

$$h_0(t) = -\ln\left|\frac{x_1(t)}{x_0}\right| = -\ln\Big|\cos(\omega_0(t-t_0))\Big|. \tag{3.46}$$

Substituting $f_0(t)$, $g_0(t)$, $h_0(t)$ into the equation (3.33) we have evolution operator as

$$\hat{U}_0(t,t_0) = \exp\left[\frac{im}{2\hbar}\Big(\frac{\dot{x}_1(t)}{x_1(t)}\Big)q^2\right]\times\exp\left[-\ln\left|\frac{x_1(t)}{x_1(t_0)}\right|(q\frac{\partial}{\partial q}+\frac{1}{2})\right]$$
$$\times\exp\left[\frac{i\hbar x_1^2(t_0)}{2}\Big(\frac{x_2(t)}{x_1(t)}\Big)\frac{\partial^2}{\partial q^2}\right],$$

or

$$\hat{U}_0(t,t_0) = \exp\left[\frac{-im\omega_0}{2\hbar}\tan\Big(\omega_0(t-t_0)\Big)q^2\right]\times\exp\left[-\ln\Big|\cos\left(\omega_0(t-t_0)\right)\Big|\Big(q\frac{\partial}{\partial q}+\frac{1}{2}\Big)\right]$$
$$\times\exp\left[\frac{i\hbar}{2m\omega_0}\tan\Big(\omega_0(t-t_0)\Big)\frac{\partial^2}{\partial q^2}\right].$$

Now, we apply this evolution operator to $\varphi_n(q)$, i.e.,

$$\Psi_n(q,t) = \hat{U}_0(t,t_0)\varphi_n(q) = \exp\left[\frac{i}{2}f_0(t)q^2\right]\exp\left[h_0(t)\Big(q\frac{\partial}{\partial q}+\frac{1}{2}\Big)\right]$$
$$\times\exp\left[-\frac{i}{2}g_0(t)\frac{\partial^2}{\partial q^2}\right]\Big(N_n\exp\big[\frac{-m\omega_0 q^2}{2\hbar}\big]H_n\big(\sqrt{\frac{m\omega_0}{\hbar}}q\big)\Big).$$

By using the result given in Appendix C, we have

$$
\begin{aligned}
\Psi_n(q,t) &= N_n \exp\left[\frac{i}{2}f_0(t)q^2\right]\exp\left[h_0(t)\left(q\frac{\partial}{\partial q}+\frac{1}{2}\right)\right]\times\frac{1}{(1+(\frac{m\omega_0}{\hbar}g_0)^2)^{1/4}} \\
&\times\exp\left[-i\left(\frac{\frac{m\omega_0}{\hbar}g_0}{1+(\frac{m\omega_0}{\hbar}g_0)^2}\right)\frac{m\omega_0}{2\hbar}q^2\right]\times\exp\left[i\omega_0\left(n+\frac{1}{2}\right)\arctan(\frac{m\omega_0}{\hbar}g_0)\right] \\
&\times\exp\left[-\left(\frac{1}{1+(\frac{m\omega_0}{\hbar}g_0)^2}\right)\frac{m\omega_0}{2\hbar}q^2\right]\times H_n\left(\left(\frac{1}{(1+(\frac{m\omega_0}{\hbar}g_0)^2)^{1/2}}\right)\sqrt{\frac{m\omega_0}{\hbar}}q\right)
\end{aligned}
\tag{3.47}
$$

Now, we rewrite this expression by using dilatation operator (D.2) and results obtained for $f_0(t)$, $g_0(t)$, and $h_0(t)$,

$$
\Psi_n(q,t) = N_n\times\exp\left[-i\omega_0(n+\frac{1}{2})(t-t_0)\right]\times\exp\left[\frac{-m\omega_0}{2\hbar}q^2\right]\times H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}q\right). \tag{3.48}
$$

This shows that $\Psi_n(q,t)$ obtained by Wei-Norman algebraic approach gives the same result with the one obtained before (3.27).

3.3. Propagator For Standart Harmonic Oscillator

Consider the IVP for the Schrödinger equation given by equations (3.20) and (3.21). Solution of this IVP can be found as a result of applying an integral operator to the initial wave function. Precisely, solution can be written in the form

$$
\Psi(q,t) = \int_{-\infty}^{\infty}\mathcal{K}_0(q,t;q',t_o)\Psi_0(q')dq'. \tag{3.49}
$$

where the kernel $\mathcal{K}_0(q,t;q',t_0)$ of the integral operator is known as the "propagator" or Green's function for the IVP. The propagator $\mathcal{K}_0(q,t;q',t_0)$ satisfies the Schrödinger equation in variables q and t, with q' and t_0 fixed, and at initial time $t\to t_0$ is equal to the

Dirac-delta function localized at $q = q'$, that's

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\mathcal{K}_0(q,t;q',t_0) &= \hat{H}_0\mathcal{K}_0(q,t;q',t_0), && (3.50)\\
\mathcal{K}_0(q,t;q',t_0)|_{t=t_0} &= \lim_{t\to t_0}\mathcal{K}_0(q,t;q',t_0) = \delta(q-q'). && (3.51)
\end{aligned}
$$

Because of these properties, the propagator $\mathcal{K}_0(q,t;q',t_0)$ as a function of q, can be seen as the wave function at time t of a particle that was localized at point q' at the initial time t_0 (Sakurai and Napolitano, 2010).

From another side, we know that the solution can be found by using the evolution operator. Writing the initial state as

$$
\Psi_0(q) = \int_{-\infty}^{\infty}\delta(q-q')\Psi_0(q')dq', \tag{3.52}
$$

where $\delta(q-q')$ is the Dirac-delta distribution, and substituting equation (3.52) in equation (2.3) we get

$$
\Psi(q,t) = \hat{U}_0(t,t_0)\int_{-\infty}^{\infty}\delta(q-q')\Psi_0(q')dq' = \int_{-\infty}^{\infty}\hat{U}_0(t,t_0)\delta(q-q')\Psi_0(q')dq' \tag{3.53}
$$

Comparing equations (3.49) and (3.53) we obtain the relation between the propagator and the evolution operator as

$$
\mathcal{K}_0(q,t;q',t_0) = \hat{U}_0(t,t_0)\delta(q-q'), \tag{3.54}
$$

where t_0 is initial time such that $\hat{U}_0(t_0,t_0) = \hat{1}$.

Below, firstly we shall find explicitly the propagator $\mathcal{K}_0(q,t;q',t_0)$ by using the eigenstate representation of Dirac-Delta function. Secondly, we shall find the propagator by using the representation of $\hat{U}_0(t,t_0)$ given by (3.33) and applying it to Dirac-Delta function. Details are as follows.

3.3.1. Finding the propagator: first approach

First, we will find the propagator by using the eigenstates of $\hat{H}_0$ which are $\varphi_n(q)$. For this aim, it is required to find the representation of Dirac-delta in terms of the orthonormal basis. Remember that in Section 3.1, we found that $\left\{\varphi_n(q)\right\}_{n=0}^{\infty}$ forms an orthonormal basis for $L^2(\mathbb{R})$. Therefore we have the following

$$\begin{aligned}\Psi_0(q) &= \sum_{n=0}^{\infty}\langle n|\Psi_0\rangle\varphi_n(q) = \sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\varphi_n^*(q')\Psi_0(q')dq'\varphi_n(q), \\ &= \int_{-\infty}^{\infty}\left(\sum_{n=0}^{\infty}\varphi_n^*(q')\varphi_n(q)\right)\Psi_0(q')dq'. \end{aligned} \tag{3.55}$$

It follows that

$$\delta(q-q') = \sum_{n=0}^{\infty}\varphi_n^*(q')\varphi_n(q). \tag{3.56}$$

Therefore, using $\hat{U}_0(t,t_0)$ given by (3.22) and Dirac-Delta representation given by (3.56) we have

$$\begin{aligned}\mathcal{K}_0(q,t;q',t_0) &= \hat{U}_0(t,t_0)\delta(q-q') = e^{-\frac{i}{\hbar}(t-t_0)\hat{H}_0}\sum_{n=0}^{\infty}\varphi_n^*(q')\varphi_n(q), \\ &= \sum_{n=0}^{\infty}\varphi_n^*(q')e^{-\frac{i}{\hbar}(t-t_0)\hat{H}_0}\varphi_n(q), \\ &= \sum_{n=0}^{\infty}\varphi_n^*(q')e^{-\frac{i}{\hbar}(t-t_0)\hat{E}_n}\varphi_n(q), \quad (\textit{by spectral mapping theorem}) \\ \mathcal{K}_0(q,t;q',t_0) &= \sum_{n=0}^{\infty}\left(e^{-\frac{i}{\hbar}(t_0)\hat{E}_n}\varphi_n(q')\right)^*\left(e^{-\frac{i}{\hbar}(t)\hat{E}_n}\varphi_n(q)\right) = \sum_{n=0}^{\infty}\Psi_n^*(q',t_0)\Psi_n(q,t). \end{aligned} \tag{3.57}$$

Substituting $\varphi_n(q)$ given by equation (3.17) and eigenvalue E_n into the equation (3.57), we have

$$\begin{aligned} \mathcal{K}_0(q,t;q',t_0) &= \sqrt{\frac{m\omega_0}{\pi\hbar}} \exp\left[\frac{-m\omega_0}{2\hbar}(q^2+q'^2)\right] \times \exp\left[\frac{i\omega_0(t_0-t)}{2}\right] \\ &\times \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{e^{i\omega_0(t_0-t)}}{2}\right)^n H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}q'\right) H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}q\right). \end{aligned} \tag{3.58}$$

For the expression under summation, we use Mehler's formula (Zhukov, 1999), that is

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{2}\tau\right)^n H_n(x)H_n(y) = \frac{1}{\sqrt{1-\tau^2}} \exp\left[\frac{2xy\tau-(x^2+y^2)\tau^2}{1-\tau^2}\right]. \tag{3.59}$$

Then $\mathcal{K}_0(q,t;q',t_0)$ will be

$$\begin{aligned} \mathcal{K}_0(q,t;q',t_0) &= \sqrt{\frac{m\omega_0}{\pi\hbar}} \exp\left[\frac{-m\omega_0}{2\hbar}(q^2+q'^2)\right] \times \exp\left[\frac{i\omega_0(t_0-t)}{2}\right] \\ &\times \frac{1}{\sqrt{1-e^{2i\omega_0(t_0-t)}}} \exp\left[\frac{2\left(\frac{m\omega_0}{\hbar}\right)qq'e^{i\omega_0(t_0-t)} - \left(\frac{m\omega_0}{\hbar}\right)(q^2+q'^2)e^{2i\omega_0(t_0-t)}}{1-e^{2i\omega_0(t_0-t)}}\right]. \end{aligned}$$

After necessary calculations and arrangements, above equation becomes

$$\begin{aligned} \mathcal{K}_0(q,t;q',t_0) &= \sqrt{\frac{m\omega_0}{2\pi i\hbar\sin\left(\omega_0(t-t_0)\right)}} \exp\left[\frac{im\omega_0}{2\hbar\sin\left(\omega_0(t-t_0)\right)}\right. \\ &\left.\times\Big((q^2+q'^2)\cos\left(\omega_0(t-t_0)\right)-2qq'\Big)\right]. \end{aligned} \tag{3.60}$$

The propagator $\mathcal{K}_0(q,t;q',t_0)$ (3.60) coincides with the propagator found in (Sakurai and Napolitano, 2010).

3.3.2. Finding the propagator: second approach

In this part, we will use the evolution operator $\hat{U}(t,t_0)$ which is given by equation (3.33) to find explicitly the propagator $\mathcal{K}_0(q,t;q',t_0)$. For this we will apply $\hat{U}(t,t_0)$ to

$\delta(q-q')$, i.e.,

$$\begin{aligned}\mathcal{K}_0(q,t;q',t_0) &= \hat{U}_0(t,t_0)\delta(q-q'),\\ &= \exp\left[\frac{i}{2}f_0(t)q^2\right]\exp\left[h_0(t)\left(q\frac{\partial}{\partial q}+\frac{1}{2}\right)\right]\exp\left[-\frac{i}{2}g_0(t)\frac{\partial^2}{\partial q^2}\right]\delta(q-q').\end{aligned}$$

Using the result given in Appendix C, we have

$$\mathcal{K}_0(q,t;q',t_0) = \frac{1}{\sqrt{2\pi}}\exp\left[\frac{i}{2}f_0(t)q^2\right]\exp\left[h_0(t)\left(q\frac{\partial}{\partial q}+\frac{1}{2}\right)\right]\sqrt{\frac{i}{g_0(t)}}\exp\left[\frac{-i}{2g_0(t)}(q-\acute{q})^2\right].$$

Now we need to use properties (D.1),(D.2) and substitute $f_0(t)$, $g_0(t)$ and $h_0(t)$. After necessary calculations, $\mathcal{K}_0(q,t;q',t_0)$ becomes

$$\begin{aligned}\mathcal{K}_0(q,t;q',t_0) &= \sqrt{\frac{m\omega_0}{2\pi i\hbar\sin(\omega_0(t-t_0))}}\exp\left[\frac{im\omega_0}{2\hbar\sin(\omega_0(t-t_0))}\right.\\ &\quad\left.\times\Big((q^2+q'^2)\cos(\omega_0(t-t_0))-2qq'\Big)\right]. \qquad (3.61)\end{aligned}$$

Observe that the result is exactly the same with equation (3.60). Therefore we can conclude that, we obtained the same result for propagator as expected by using two different approaches.

3.4. Standart Harmonic Oscillator in Heisenberg Picture

Definition 3.2 *If $\hat{A}$ is an operator in Schrödinger picture, then the corresponding operator in Heisenberg picture is defined as*

$$\hat{A}_H(t) = \hat{U}_0^\dagger(t,t_0)\hat{A}_s\hat{U}_0(t,t_0),$$

where $\hat{U}_0(t,t_0)$ is the evolution operator of the physical system.

According to this, using the evolution operator found in equation (3.22) and the expressions of $\hat{q}$ and $\hat{p}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$, which are given in equation (3.6), we can find the position and momentum operators in Heisenberg picture. For this aim and further calculations, we need to use the Baker-Hausdorff identity (3.35). For using this identity, we need commuatation relations which are given in equation (3.7). This way, we can find the Heisenberg operators corresponding to the simple harmonic oscillator as

$$
\begin{aligned}
\hat{a}_H(t) &= e^{\frac{i}{\hbar}(t-t_0)\hat{H}_0}\hat{a}e^{-\frac{i}{\hbar}(t-t_0)\hat{H}_0} = e^{-i\omega_0(t-t_0)}\hat{a}, \\
&= \sqrt{\frac{m\omega_0}{2\hbar}}e^{-i\omega_0(t-t_0)}\hat{q} + i\frac{1}{\sqrt{2\hbar m\omega_0}}e^{-i\omega_0(t-t_0)}\hat{p}, \qquad (3.62) \\
\hat{a}^\dagger_H(t) &= e^{\frac{i}{\hbar}(t-t_0)}\hat{a}^\dagger e^{-\frac{i}{\hbar}(t-t_0)} = e^{i\omega_0(t-t_0)}\hat{a}^\dagger, \\
&= \sqrt{\frac{m\omega_0}{2\hbar}}e^{i\omega_0(t-t_0)}\hat{q} - i\frac{1}{\sqrt{2\hbar m\omega_0}}e^{i\omega_0(t-t_0)}\hat{p}. \qquad (3.63)
\end{aligned}
$$

Using equations (3.62) and (3.63), we find the position and momentum operators in Heisenberg picture as

$$
\begin{aligned}
\hat{q}_H(t) &= \cos\Big(\omega_0(t-t_0)\Big)\hat{q} + \frac{1}{m\omega_0}\sin\Big(\omega_0(t-t_0)\Big)\hat{p}, \qquad (3.64) \\
\hat{p}_H(t) &= -m\omega_0\sin\Big(\omega_0(t-t_0)\Big)\hat{q} + \cos\Big(\omega_0(t-t_0)\Big)\hat{p}, \qquad (3.65)
\end{aligned}
$$

where $\hat{q} \equiv \hat{q}_H(t_0)$ and $\hat{p} \equiv \hat{p}_H(t_0)$. For $\hat{q}$ and $\hat{p}$, are known in the Heisenberg picture, $\hat{q}^2$ and $\hat{p}^2$ can also be calculated easily in the Heisenberg picure, as well as $\hat{q}^n$ and $\hat{p}^n$, where n is a positive integer. This way, $\hat{q}^2$ and $\hat{p}^2$ will be

$$
\begin{aligned}
\hat{q}^2_H(t) &= \hat{U}^\dagger_0(t,t_0)\hat{q}^2\hat{U}_0(t,t_0) = \Big(\hat{U}^\dagger_0(t,t_0)\hat{q}\hat{U}_0(t,t_0)\Big)^2, \\
\hat{p}^2_H(t) &= \hat{U}^\dagger_0(t,t_0)\hat{p}^2\hat{U}_0(t,t_0) = \Big(\hat{U}^\dagger_0(t,t_0)\hat{p}\hat{U}_0(t,t_0)\Big)^2.
\end{aligned}
$$

Using equations (3.64),(3.65), $\hat{q}_H^2(t)$ and $\hat{p}_H^2(t)$ then become

$$
\begin{aligned}
\hat{q}_H^2(t) &= \cos^2\Big(\omega_0(t-t_0)\Big)\hat{q}^2 + \frac{1}{m^2\omega_0^2}\sin^2\Big(\omega_0(t-t_0)\Big)\hat{p}^2 \\
&+ \frac{1}{m\omega_0}\cos\Big(\omega_0(t-t_0)\Big)\sin\Big(\omega_0(t-t_0)\Big)\Big(\hat{q}\hat{p}+\hat{p}\hat{q}\Big), \qquad (3.66) \\
\hat{p}_H^2(t) &= m^2\omega_0^2\sin^2\Big(\omega_0(t-t_0)\Big)\hat{q}^2 + \cos^2\Big(\omega_0(t-t_0)\Big)\hat{p}^2 \\
&- m\omega_0\sin\Big(\omega_0(t-t_0)\Big)\cos\Big(\omega_0(t-t_0)\Big)\Big(\hat{q}\hat{p}+\hat{p}\hat{q}\Big). \qquad (3.67)
\end{aligned}
$$

By this result we can see that Hamiltonian in both of the Schrödinger picture and the Heisenberg picture are the same,

$$
\hat{H}_H(t) = \frac{1}{2m}\hat{p}_H^2(t) + \frac{m\omega_0^2}{2}\hat{q}_H^2(t) = \frac{1}{2m}\hat{p}^2 + \frac{m\omega_0^2}{2}\hat{q}^2. \qquad (3.68)
$$

One can easily check that position and momentum operators which are given by equations (3.64) and (3.65), are the solutions of classical equations,

$$
\begin{aligned}
\frac{d^2}{dt^2}\hat{q}_H(t) + \omega_0^2\hat{q}_H(t) &= 0, \\
\frac{d^2}{dt^2}\hat{p}_H(t) + \omega_0^2\hat{p}_H(t) &= 0,
\end{aligned}
$$

and satisfy Heisenberg equations of motion,

$$
\begin{aligned}
\frac{d}{dt}\hat{q}_H(t) &= \frac{\hat{p}_H(t)}{m}, \\
\frac{d}{dt}\hat{p}_H(t) &= -m\omega_0^2(t)\hat{q}_H(t).
\end{aligned}
$$

3.5. Dynamical Invariants for Standart Harmonic Oscillator

Since we know the evolution operator, we can find the invariants corresponding to position and momentum operators like we did in previous section by using their representations in terms of $\hat{a}$ and $\hat{a}^\dagger$. Firstly we find $\hat{a}_0(t)$ and $\hat{a}_0^\dagger(t)$ as following:

$$\begin{aligned}
\hat{a}(t) &= \hat{U}_0(t,t_0)\hat{a}\hat{U}_0^\dagger(t,t_0) = e^{-\frac{i}{\hbar}(t-t_0)\hat{H}_0}\hat{a}e^{\frac{i}{\hbar}(t-t_0)\hat{H}_0} = e^{i\omega_0(t-t_0)}\hat{a}, \\
&= \sqrt{\frac{m\omega_0}{2\hbar}}e^{i\omega_0(t-t_0)}\hat{q} + i\frac{1}{\sqrt{2\hbar m\omega_0}}e^{i\omega_0(t-t_0)}\hat{p}, \qquad (3.69)\\
\hat{a}^\dagger(t) &= \hat{U}_0(t,t_0)\hat{a}^\dagger\hat{U}_0^\dagger(t,t_0) = e^{-\frac{i}{\hbar}(t-t_0)\hat{H}_0}\hat{a}^\dagger e^{\frac{i}{\hbar}(t-t_0)\hat{H}_0} = e^{-i\omega_0(t-t_0)}\hat{a}^\dagger, \\
&= \sqrt{\frac{m\omega_0}{2\hbar}}e^{-i\omega_0(t-t_0)}\hat{q} - i\frac{1}{\sqrt{2\hbar m\omega_0}}e^{-i\omega_0(t-t_0)}\hat{p}. \qquad (3.70)
\end{aligned}$$

Then using equations (3.69) and (3.70), the invariants corresponding to position and momentum operators will be

$$\begin{aligned}
\hat{q}(t) &= \cos\Big(\omega_0(t-t_0)\Big)\hat{q} - \frac{1}{m\omega_0}\sin\Big(\omega_0(t-t_0)\Big)\hat{p}, \\
\hat{p}(t) &= m\omega_0\sin\Big(\omega_0(t-t_0)\Big)\hat{q} + \cos\Big(\omega_0(t-t_0)\Big)\hat{p}.
\end{aligned}$$

Now consider the Hamiltonian $\hat{H}_0 = \frac{1}{2m}\hat{p}^2 + \frac{m\omega_0^2}{2}\hat{q}^2$, then $\hat{I}(t) = \hat{U}_0(t,t_0)\hat{H}_0\hat{U}_0^\dagger(t,t_0) = \hat{H}_0$ is an invariant. It is clear since $\hat{H}_0$ has no time dependency, $\left[\hat{H}_0, \hat{U}_0\right] = 0$.

CHAPTER 4

QUANTUM PARAMETRIC OSCILLATOR: WEI-NORMAN ALGEBRAIC METHOD

Consider the IVP

$$i\hbar\frac{\partial\Psi(q,t)}{\partial t} = \hat{H}(t)\Psi(q,t), \tag{4.1}$$

$$\Psi(q,t_0) = \Psi_0(q), \tag{4.2}$$

where $q \in \mathbb{R}$ and $t \geq 0$. For time dependent Hamiltonian of quantum parametric oscillator

$$\hat{H}(t) = \frac{1}{2\mu(t)}\hat{p}^2 + \frac{\mu(t)\omega^2(t)}{2}q^2, \tag{4.3}$$

where $\mu(t)$, $\omega(t)$ are real-valued functions of time, so that $\hat{H}^\dagger(t) = \hat{H}(t)$.

The Wei-Norman Algebraic method (Evolution Operator method) for solving an IVP for Schrödinger equation (4.1) was introduced in (Wei and Norman, 1963) and later used in many works such as (Büyükaşık et al., 2009), (Dattoli et al., 1997). This method is based on the Lie algebraic properties of the Hamiltonian (4.3), which is quadratic in $\hat{p}$ and $\hat{q}$ and therefore can be written as a linear superposition of generators of su(1,1) Lie algebra. Then, the evolution operator $\hat{U}(t,t_0)$ can be written as product of exponential operators that are elements of the corresponding SU(1,1) group. This allows us to find explicitly the evolution operator and solution of the IVP for the Schrödinger equation. We give the details in next sections.

4.1. Construction of the evolution operator

The Hamiltonian given by equation (4.3) can be written as a linear superposition of generators of $su(1,1)$ Lie algebra as follows

$$\hat{H}(t) = i\left(\frac{-\hbar^2}{\mu(t)}\hat{K}_- - \mu(t)\omega^2(t)\hat{K}_+\right). \tag{4.4}$$

Then, the evolution operator is of the form

$$\hat{U}(t,t_0) = e^{f(t)\hat{K}_+}e^{2h(t)\hat{K}_0}e^{g(t)\hat{K}_-}, \tag{4.5}$$

where $f(t)$, $g(t)$, $h(t)$ are real-valued functions to be determined. Using definition of evolution operator (2.4),(2.5), we need to find $\frac{\partial \hat{U}}{\partial t}$ and $\hat{H}\hat{U}$:

$$\begin{aligned}\frac{\partial \hat{U}}{\partial t} &= \dot{f}(t)\hat{K}_+\hat{U} + e^{f(t)\hat{K}_+}2\dot{h}(t)\hat{K}_0 e^{2h(t)\hat{K}_0}e^{g(t)\hat{K}_-} + e^{f(t)\hat{K}_+}e^{2h(t)\hat{K}_0}\dot{g}(t)\hat{K}_- e^{g(t)\hat{K}_-},\\ &= \dot{f}(t)\hat{K}_+\hat{U} + 2\dot{h}(t)\left(e^{f(t)\hat{K}_+}\hat{K}_0 e^{-f(t)\hat{K}_+}\right)\hat{U} + \dot{g}(t)e^{f(t)\hat{K}_+}\left(e^{2h(t)\hat{K}_0}\hat{K}_- e^{-2h(t)\hat{K}_0}\right)e^{2h(t)\hat{K}_0}e^{g(t)\hat{K}_-}\hat{U}.\end{aligned}$$

Rewriting $\frac{\partial \hat{U}}{\partial t}$ after using Baker-Hausdorff identity (3.35) and multiplying it with $i\hbar$, we get

$$\begin{aligned}i\hbar\frac{\partial \hat{U}}{\partial t} &= i\hbar\Bigg(\left[\dot{f}(t) - 2\dot{h}f(t) + e^{-2h(t)}\dot{g}(t)f^2(t)\right]\hat{K}_+\\ &+ \left[e^{-2h(t)}\dot{g}(t)\right]\hat{K}_- + \left[2\dot{h}(t) - 2f(t)\dot{g}(t)e^{-2h(t)}\right]\hat{K}_0\Bigg)\hat{U}.\end{aligned}$$

Since $i\hbar\frac{\partial\hat{U}}{\partial t}$ must be equal to $\hat{H}(t)\hat{U}(t,t_0)$ and $\hat{H}(t)$ is given by equation(4.4), then the following relation must hold:

$$\begin{aligned} i\Big(\frac{-\hbar^2}{\mu(t)}\hat{K}_- - \mu(t)\omega^2(t)\hat{K}_+\Big) &= i\hbar\Bigg(\Big[\dot{f}(t) - 2\dot{h}f(t) + e^{-2h(t)}\dot{g}(t)f^2(t)\Big]\hat{K}_+ + \Big[e^{-2h(t)}\dot{g}(t)\Big]\hat{K}_- \\ &+ \Big[2\dot{h}(t) - 2f(t)\dot{g}(t)e^{-2h(t)}\Big]\hat{K}_0\Bigg). \end{aligned}$$

For this equality, we get IVP for a nonlinear system of three first-order ordinary differential equations for the unknown real-valued functions $f(t), g(t)$ and $h(t)$(Initial conditions are chosen as "0" because $\hat{U}(t_0,t_0)$ must satisfy (2.5)).

$$\dot{f} + \frac{\hbar f^2}{\mu(t)} + \frac{\mu(t)\omega^2(t)}{\hbar} = 0 \quad , \quad f(t_0) = 0, \tag{4.6}$$

$$\dot{h} + \frac{\hbar f}{\mu(t)} = 0 \quad , \quad h(t_0) = 0, \tag{4.7}$$

$$\dot{g} + \frac{\hbar e^{2h}}{\mu(t)} = 0 \quad , \quad g(t_0) = 0. \tag{4.8}$$

Since equation (4.6) is a Ricatti equation, by substitution $f(t) = \mu(t)(\dot{x}/x)/\hbar$, it can be linearized in the form of a classical damped parametric oscillator with time dependent damping $\frac{\dot{\mu}}{\mu}$ and frequency $\omega(t)$:

$$\ddot{x} + \frac{\dot{\mu}}{\mu}\dot{x} + \omega^2(t)x = 0. \tag{4.9}$$

Let $x_1(t)$ and $x_2(t)$ be two linearly independent real solutions of equation (4.9), satisfying the initial conditions

$$x_1(t_0) = x_0 \neq 0, \quad \dot{x}_1(t_0) = 0, \tag{4.10}$$

$$x_2(t_0) = 0, \quad \dot{x}_2(t_0) = \frac{1}{\mu(t_0)x_0}. \tag{4.11}$$

Note that, the Wronskian of $x_1(t)$ and $x_2(t)$ at t_0 is

$$W(x_1, x_2)(t_0) = \begin{vmatrix} x_0 & 0 \\ 0 & \frac{1}{\mu(t_0)x_0} \end{vmatrix} = \frac{1}{\mu(t_0)},$$

and using the Wronskian formula

$$W(x_1, x_2)(t) = W(x_1, x_2)(t_0) \exp\Big[-\int_{t_0}^{t} \frac{\dot{\mu}(s)}{\mu(s)} ds\Big],$$

we get

$$W(x_1, x_2)(t) = \frac{1}{\mu(t_0)} \exp\Big[-\ln \frac{\mu(t)}{\mu(t_0)}\Big] = \frac{1}{\mu(t)}. \quad (4.12)$$

Now, solving three first-order ordinary differential equations (4.6), (4.7) and (4.8), we find

$$f(t) = \frac{\mu(t)}{\hbar}\Big(\frac{\dot{x}_1(t)}{x_1(t)}\Big), \quad (4.13)$$

$$g(t) = -\hbar x_1^2(t_0)\Big(\frac{x_2(t)}{x_1(t)}\Big), \quad (4.14)$$

$$h(t) = -\ln\Big|\frac{x_1(t)}{x_1(t_0)}\Big|. \quad (4.15)$$

Substituting these functions into equation (4.5), we get the evolution operator

$$\hat{U}(t, t_0) = \exp\Big[\frac{i}{2} f(t) q^2\Big] \exp\Big[h(t)\Big(q\frac{\partial}{\partial q} + \frac{1}{2}\Big)\Big] \exp\Big[-\frac{i}{2} g(t) \frac{\partial^2}{\partial q^2}\Big]. \quad (4.16)$$

It can be also expressed in terms of $x_1(t)$ and $x_2(t)$ as follows

$$\hat{U}(t, t_0) = \exp\Big[\frac{i\mu(t)}{2\hbar}\Big(\frac{\dot{x}_1(t)}{x_1(t)}\Big)q^2\Big] \times \exp\Big[-\ln\Big|\frac{x_1(t)}{x_1(t_0)}\Big|(q\frac{\partial}{\partial q} + \frac{1}{2})\Big]$$
$$\times \exp\Big[\frac{i\hbar x_1^2(t_0)}{2}\Big(\frac{x_2(t)}{x_1(t)}\Big)\frac{\partial^2}{\partial q^2}\Big]. \quad (4.17)$$

4.2. The wave functions

To obtain an explicit form for evolving in time states $\Psi(q,t)$, we will use $\Psi(q,t) = \hat{U}(t,t_0)\Psi_0(q)$. For using this equation, we need to know $\hat{U}(t,t_0)$ and $\Psi_0(q)$. We found $\hat{U}(t,t_0)$ in equation (4.16), and for $\Psi_0(q)$, we will use the most general form of initial wave function in $L_t^2(\mathbb{R})$, whose expansion is

$$\Psi_0(q) = \sum_{n=0}^{\infty}\langle n|\Psi_0\rangle\varphi_n(q),$$

where $\varphi_n(q)$ are the normalized eigenstates of $\hat{H}_0$ (3.1) which are expressed in equation (3.17), with corresponding to eigenvalues $E_n = \hbar\omega_0(n+\frac{1}{2})$. Substituting $\Psi_0(q)$ into $\Psi(q,t) = \hat{U}(t,t_0)\Psi_0(q)$ will give,

$$\Psi(q,t) = \hat{U}(t,t_0)\Psi_0(q) = \sum_{n=0}^{\infty}\langle n|\Psi_0\rangle\hat{U}(t,t_0)\varphi_n(q).$$

For finding $\Psi_n(q,t) = \hat{U}(t,t_0)\varphi_n(q)$, we use results which are given in Appendix C, the identities (D.1),(D.2), so that we have

$$\begin{aligned}
\Psi_n(q,t) &= \hat{U}(t,t_0)\varphi_n(q),\\
&= \exp\left[\frac{i}{2}f(t)q^2\right]\exp\left[h(t)\left(q\frac{\partial}{\partial q}+\frac{1}{2}\right)\right]\exp\left[-\frac{i}{2}g(t)\frac{\partial^2}{\partial q^2}\right]\varphi_n(q),\\
&= N_n\exp\left[\frac{h(t)}{2}\right]\exp\left[\frac{i}{2}f(t)q^2\right]\frac{1}{(1+(\frac{m\omega_0}{\hbar}g(t))^2)^{1/4}}\\
&\times\ H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}\left(\frac{1}{(1+(\frac{m\omega_0}{\hbar}g(t))^2)^{1/2}}\right)e^{h(t)}q\right)\\
&\times\ \exp\left[\frac{-i\frac{m\omega_0}{\hbar}}{2}\left(\frac{\frac{m\omega_0}{\hbar}g(t)}{1+(\frac{m\omega_0}{\hbar}g(t))^2}\right)\left(e^{h(t)}q\right)^2\right]\exp\left[i\left(n+\frac{1}{2}\right)\arctan\left(\frac{m\omega_0}{\hbar}g(t)\right)\right]\\
&\times\ \exp\left[-\frac{m\omega_0}{2\hbar}\left(\frac{1}{1+(\frac{m\omega_0}{\hbar}g(t))^2}\right)\left(e^{h(t)}q\right)^2\right].
\end{aligned}\tag{4.18}$$

where $N_n = (2^n n!)^{-1/2}(m\omega_0/\pi\hbar)^{1/4}$. Substituting $f(t)$, $g(t)$, and $h(t)$ into the equation (4.18) and using the Wronskian $W(x_1(t), x_2(t)) = 1/\mu(t)$, the wave functions in terms of the classical solutions $x_1(t)$ and $x_2(t)$ of equation (4.9) become

$$\begin{aligned}\Psi_n(q,t) &= \left(\frac{1}{\pi\hbar 2^n n!}\right)^{1/2}\frac{1}{\sqrt{\sigma_0(t)}}\times\exp\left[\frac{i\mu(t)\dot{x}_1(t)}{2\hbar x_1(t)}q^2\right]\times\exp\left[\frac{i}{2\hbar}m\omega_0\hbar x_1^2(t_0)\frac{x_2(t)}{x_1(t)}\frac{1}{\sigma_0^2(t)}q^2\right]\\ &\times\ \exp\left[i\left(n+\frac{1}{2}\right)\arctan\left(-m\omega_0 x_1^2(t_0)\frac{x_2(t)}{x_1(t)}\right)\right]\\ &\times\ \exp\left[-\frac{1}{2\hbar\sigma_0^2(t)}q^2\right]\times H_n\left(\frac{q}{\sqrt{\hbar}\sigma_0(t)}\right)\end{aligned} \tag{4.19}$$

where

$$\sigma_0(t) = \frac{1}{\sqrt{m\omega_0}}\left(\frac{x_1^2(t)+m^2\omega_0^2 x_1^4(t_0)x_2^2(t)}{x_1^2(t_0)}\right)^{1/2}. \tag{4.20}$$

By using $\rho_n(q,t) = |\Psi_n(q,t)|^2$, probability densities can also be calculated as

$$\rho_n(q,t) = \left(\frac{1}{\pi\hbar 2^n n!}\right)\times\frac{1}{\sigma_0(t)}\times\exp\left[-\left(\frac{1}{\sqrt{\hbar}\sigma_0(t)}q\right)^2\right]\times H_n^2\left(\frac{q}{\sqrt{\hbar}\sigma_0(t)}\right). \tag{4.21}$$

Functions $\Psi_n(q,t)$ and $\rho_n(q,t)$, which are found above coincide with the results in (Büyükaşık et al., 2009), and references given there.

Motion of Zeros

Let $\tau_n^{(l)}$ for $l = 1, 2, ...n$, be the zeros of Hermite polynomial $\hat{H}_n(q)$, i.e., $\hat{H}_n(\tau_n^{(l)}) = 0$. From solution (4.19), one can see that the zeros of wave function $\Psi_n(q,t)$ are the zeros of $H_n\left(q/\sqrt{\hbar}\sigma_0(t)\right)$. Then, motion of zeros of the wave function (4.19) is described by the function

$$q_n^{(l)}(t) = \sqrt{\hbar}\tau_n^{(l)}\sigma_0(t),$$

where $\sigma_0(t)$ is expressed in (4.20).

4.3. Finding the Propagator using the evolution operator

From section 3.3, we know that propagator is

$$K(q,t;q',t_0) \quad = \quad \hat{U}(t,t_0)\delta(q-q'), \tag{4.22}$$

where t_0 is the initial time such that $\hat{U}(t_0,t_0) = \hat{1}$. Therefore, using the evolution operator obtained in equation (4.16) and applying it to $\delta(q-q')$ will give us the propagator explicitly. Using the result given in Appendix C, we have

$$\begin{aligned} K(q,t;q',t_0) &= \hat{U}(t,t_0)\delta(q-q'), \\ &= \exp\left[\frac{i}{2}f(t)q^2\right]\exp\left[h(t)\left(q\frac{\partial}{\partial q}+\frac{1}{2}\right)\right]\exp\left[-\frac{i}{2}g(t)\frac{\partial^2}{\partial q^2}\right]\delta(q-q'), \\ &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{i}{g(t)}}\exp\left[\frac{h(t)}{2}\right]\exp\left[\frac{i}{2}f(t)q^2\right]\exp\left[\frac{-i}{2g(t)}(e^{h(t)}q-q')^2\right], \end{aligned} \tag{4.23}$$

or

$$\begin{aligned} K(q,t;q',t_0) &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{-i}{\hbar x_1(t_0)x_2(t)}}\exp\left[\frac{i}{2}\frac{\mu(t)}{\hbar}\frac{\dot{x}_1(t)}{x_1(t)}q^2\right] \\ &\times \exp\left[\frac{ix_1(t)}{2\hbar x_1^2(t_0)x_2(t)}\left(\frac{x_1(t_0)}{x_1(t)}q-q'\right)^2\right]. \end{aligned} \tag{4.24}$$

Now, to describe the evolution of a state from an arbitrary time t_1 to t_2, where $t_1 < t_2$, we can use the evolution operator $\hat{U}(t_2,t_1)$ or the propagator $K(q,t_2;q',t_1)$,

$$\Psi(q,t_2) \quad = \quad \hat{U}(t_2,t_1)\Psi(q,t_1) = \int_{-\infty}^{\infty} K(q,t_2;q',t_1)\Psi(q',t_1)dq'.$$

To find $\hat{U}(t_2,t_1)$ explicitly we use the composition rule, $\hat{U}(t_2,t_1)\hat{U}(t_1,t_0) = \hat{U}(t_2,t_0)$ which gives $\hat{U}(t_2,t_1) = \hat{U}(t_2,t_0)\hat{U}^\dagger(t_1,t_0)$. By using $\hat{U}(t,t_0)$ found in equation (4.16), $\hat{U}(t_2,t_0)$

and $\hat{U}^{\dagger}(t_1, t_0)$ can be written respectively as follows

$$
\begin{aligned}
\hat{U}(t_2, t_0) &= \exp\left[\frac{i}{2}f(t_2)q^2\right]\exp\left[h(t_2)\left(q\frac{\partial}{\partial q}+\frac{1}{2}\right)\right]\exp\left[-\frac{i}{2}g(t_2)\frac{\partial^2}{\partial q^2}\right],\\
\hat{U}^{\dagger}(t_1, t_0) &= \exp\left[\frac{i}{2}g(t_1)\frac{\partial^2}{\partial q^2}\right]\exp\left[-h(t_1)\left(q\frac{\partial}{\partial q}+\frac{1}{2}\right)\right]\exp\left[\frac{-i}{2}f(t_1)q^2\right].
\end{aligned}
$$

Therefore, using the result given in Appendix C and identities (D.1),(D.2), we have

$$
\begin{aligned}
K(q, t_2; q', t_1) &= \hat{U}(t_2, t_1)\delta(q-q') = \hat{U}(t_2, t_0)\hat{U}^{\dagger}(t_1, t_0)\delta(q-q'),\\
&= \exp\left[\frac{h(t_2)-h(t_1)}{2}\right]\exp\left[\frac{i}{2}f(t_2)q^2\right]\exp\left[h(t_2)\left(q\frac{\partial}{\partial q}\right)\right]\exp\left[-h(t_1)\left(q\frac{\partial}{\partial q}\right)\right]\\
&\quad\times\exp\left[\frac{i}{2}\left(g(t_1)-g(t_2)\right)\frac{\partial^2}{\partial q^2}\right]\times\exp\left[\frac{-i}{2}f(t_1)q^2\right]\delta(q-q'),\\
&= \exp\left[\frac{h(t_2)-h(t_1)}{2}\right]\exp\left[\frac{i}{2}f(t_2)q^2\right]\exp\left[h(t_2)\left(q\frac{\partial}{\partial q}\right)\right]\\
&\quad\times\exp\left[\frac{i}{2}\left(g(t_1)-g(t_2)\right)\frac{\partial^2}{\partial q^2}\right]\exp\left[\frac{-i}{2}f(t_1)e^{-2h(t_1)}q^2\right]\delta(e^{-h(t_1)}q-q'),
\end{aligned}
$$

or

$$
\begin{aligned}
K(q, t_2; q', t_1) &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{i}{g(t_2)-g(t_1)}}\exp\left[\frac{h(t_2)-h(t_1)}{2}\right]\exp\left[\frac{i}{2}\left(f(t_2)q^2-f(t_1)q'^2\right)\right]\\
&\quad\times\exp\left[\frac{-i}{2\left(g(t_2)-g(t_1)\right)}(e^{h(t_2)}q-e^{h(t_1)}q')^2\right].
\end{aligned}
$$

4.4. Heisenberg Picture

According to Definition 3.2, by using the evolution operator (4.16), we can find the position and momentum operators in Heisenberg picture in terms of $x_1(t)$ and $x_2(t)$ as

$$\hat{Q}_H(t) = \hat{U}^{\dagger}(t, t_0)\hat{q}\hat{U}(t, t_0) = \frac{1}{x_0}x_1(t)\hat{q} + x_0x_2(t)\hat{p}, \tag{4.25}$$

$$\hat{P}_H(t) = \hat{U}^{\dagger}(t, t_0)\hat{p}\hat{U}(t, t_0) = \frac{1}{x_0}\mu(t)\dot{x}_1(t)\hat{q} + x_0\mu(t)\dot{x}_2(t)\hat{p}, \tag{4.26}$$

where we note that $\hat{q} = \hat{Q}_H(t_0)$ and $\hat{p} = \hat{P}_H(t_0)$. They can also be expressed in terms of $\sigma_0(t)$,

$$
\begin{aligned}
\hat{Q}_H(t) &= \sqrt{m\omega_0}\cos\theta_0(t)\sigma_0(t)\hat{q} + \frac{1}{\sqrt{m\omega_0}}\sin\theta_0(t)\sigma_0(t)\hat{p}, \\
\hat{P}_H(t) &= \mu(t)\left(\frac{\sqrt{m\omega_0}\sin\theta_0(t)}{\mu(t)\sigma_0(t)} + \sqrt{m\omega_0}\cos\theta_0(t)\dot{\sigma}_0(t)\right)\hat{q} \\
&\quad + \frac{\mu(t)}{m\omega_0}\left(\frac{\sqrt{m\omega_0}\cos\theta_0(t)}{\mu(t)\sigma_0(t)} + \sqrt{m\omega_0}\sin\theta_0(t)\dot{\sigma}_0(t)\right)\hat{p},
\end{aligned}
$$

where $\theta_0(t) = \int_{t_0}^{t}\frac{1}{\mu(s)\sigma_0^2(s)}ds$. Knowing $\hat{Q}$ and $\hat{P}$ in the Heisenberg picture allows us to compute easily $\hat{Q}^2$ and $\hat{P}^2$ in the Heisenberg picure, and also $\hat{Q}^n$ and $\hat{P}^n$, where n is a positive integer. For $\hat{Q}^2$ and $\hat{P}^2$ we have

$$
\begin{aligned}
\hat{Q}_H^2(t) &= \hat{U}^\dagger(t,t_0)\hat{q}^2\hat{U}(t,t_0) = \left(\hat{U}^\dagger(t,t_0)\hat{q}\hat{U}(t,t_0)\right)^2, \\
\hat{P}_H^2(t) &= \hat{U}^\dagger(t,t_0)\hat{p}^2\hat{U}(t,t_0) = \left(\hat{U}^\dagger(t,t_0)\hat{p}\hat{U}(t,t_0)\right)^2.
\end{aligned}
$$

Rewriting $\hat{Q}_H^2(t)$ and $\hat{P}_H^2(t)$ by using equations (4.25) and (4.26), they become

$$
\begin{aligned}
\hat{Q}_H^2(t) &= \frac{1}{x_0^2}x_1^2(t)\hat{q}^2 + x_0^2x_2^2(t)\hat{p} + x_1(t)x_2(t)\{\hat{q},\hat{p}\}, \\
\hat{P}_H^2(t) &= \mu^2(t)\left(\frac{1}{x_0^2}\dot{x}_1{}^2(t)\hat{q}^2 + x_0^2\dot{x}_2^2(t)\hat{p} + \dot{x}_1(t)\dot{x}_2(t)\{\hat{q},\hat{p}\}\right),
\end{aligned}
$$

where $\{\hat{q},\hat{p}\} = \left(\hat{q}\hat{p} + \hat{p}\hat{q}\right)$ is the anticommutator of $\hat{q}$ and $\hat{p}$. In addition, mixed terms $\hat{Q}_H(t)\hat{P}_H(t)$ and $\hat{P}_H(t)\hat{Q}_H(t)$ are calculated as

$$
\begin{aligned}
\hat{Q}_H(t)\hat{P}_H(t) &= \frac{\mu(t)x_1(t)\dot{x}_1(t)}{x_0^2}\hat{q}^2 + \mu(t)x_1(t)\dot{x}_2(t)\hat{q}\hat{p} + \mu(t)\dot{x}_1(t)x_2(t)\hat{p}\hat{q} \\
&\quad + \mu(t)x_0^2x_2\dot{x}_2(t)\hat{p}^2, \\
\hat{P}_H(t)\hat{Q}_H(t) &= \frac{\mu(t)x_1(t)\dot{x}_1(t)}{x_0^2}\hat{q}^2 + \mu(t)\dot{x}_1(t)x_2(t)\hat{q}\hat{p} + \mu(t)x_1(t)\dot{x}_2(t)\hat{p}\hat{q} \\
&\quad + \mu(t)x_0^2x_2\dot{x}_2(t)\hat{p}^2.
\end{aligned}
$$

Next, we find the Heisenberg operators corresponding to damped parametric oscillator (4.3) as

$$\begin{aligned}
\hat{A}_H(t) &= \hat{U}^\dagger(t,t_0)\hat{a}\hat{U}(t,t_0) = \hat{U}^\dagger(t,t_0)\sqrt{\frac{m\omega_0}{2\hbar}}\left(\hat{q} + i\frac{1}{m\omega_0}\hat{p}\right)\hat{U}(t,t_0), \\
&= \left(\sqrt{\frac{m\omega_0}{2\hbar}}\frac{1}{x_0}x_1(t) + i\frac{1}{\sqrt{2\hbar m\omega_0}}\frac{1}{x_0}\mu(t)\dot{x}_1(t)\right)\hat{q} \\
&+ \left(\sqrt{\frac{m\omega_0}{2\hbar}}x_0x_2(t) + i\frac{1}{\sqrt{2\hbar m\omega_0}}x_0\mu(t)\dot{x}_2(t)\right)\hat{p}.
\end{aligned}$$

Similarly $\hat{A}_H^\dagger(t)$ can be calculated

$$\begin{aligned}
\hat{A}_H^\dagger(t) &= \left(\sqrt{\frac{m\omega_0}{2\hbar}}\frac{1}{x_0}x_1(t) - i\frac{1}{\sqrt{2\hbar m\omega_0}}\frac{1}{x_0}\mu(t)\dot{x}_1(t)\right)\hat{q} \\
&+ \left(\sqrt{\frac{m\omega_0}{2\hbar}}x_0x_2(t) - i\frac{1}{\sqrt{2\hbar m\omega_0}}x_0\mu(t)\dot{x}_2(t)\right)\hat{p}.
\end{aligned}$$

One can easily check that position and momentum operators which are given by equations (4.25) and (4.26), are the solutions of classical equations,

$$\begin{aligned}
\frac{d^2}{dt^2}\hat{Q}_H(t) + \frac{\dot{\mu}(t)}{\mu(t)}\frac{d}{dt}\hat{Q}_H(t) + \omega^2(t)\hat{Q}_H(t) &= 0, \\
\frac{d^2}{dt^2}\hat{P}_H(t) - \frac{(\dot{\mu\omega^2})}{\mu\omega^2}\frac{d}{dt}\hat{P}_H(t) + \omega^2(t)\hat{P}_H(t) &= 0,
\end{aligned}$$

and satisfy Heisenberg equations of motion,

$$\begin{aligned}
\frac{d}{dt}\hat{Q}_H(t) &= \frac{\hat{P}_H(t)}{\mu(t)}, \\
\frac{d}{dt}\hat{P}_H(t) &= -\mu(t)\omega^2(t)\hat{Q}_H(t).
\end{aligned}$$

4.5. Complex Function Representation of Dynamical Invariants

It is instructive, instead of two real independent solutions $x_1(t)$ and $x_2(t)$ of equation (4.9), to introduce one complex functions solution $\varepsilon_0(t)$ of the same equation:

$$\begin{aligned}\varepsilon_0(t) &= \frac{x_1(t)}{\sqrt{m\omega_0}x_0} + i\sqrt{m\omega_0}x_0 x_2(t), & (4.27)\\ &= \sigma_0(t)e^{i\theta_0(t)} = \sigma_0(t)\Big(\cos(\theta_0(t)) + i\sin(\theta_0(t))\Big), & (4.28)\end{aligned}$$

where

$$\begin{aligned}|\varepsilon_0(t)| &= \sigma_0(t) = \frac{1}{\sqrt{m\omega_0}}\sqrt{\frac{x_1^2(t)}{x_0^2} + (m\omega_0 x_0 x_2(t))^2}, & (4.29)\\ \theta_0(t) &= \int_{t_0}^{t}\frac{dt'}{\mu(t')|\varepsilon_0(t')|^2}. & (4.30)\end{aligned}$$

The complex function $\varepsilon_0(t)$ defined here satisfies the equation (4.9) and the initial conditions,

$$\varepsilon_0(t_0) = \frac{1}{\sqrt{m\omega_0}}, \quad \dot{\varepsilon}_0(t_0) = \frac{i\sqrt{m\omega_0}}{\mu(t_0)}. \tag{4.31}$$

Moreover,

$$\varepsilon_0^*(t_0) = \frac{1}{\sqrt{m\omega_0}}, \quad \dot{\varepsilon}_0^*(t_0) = \frac{-i\sqrt{m\omega_0}}{\mu(t_0)}, \tag{4.32}$$

which implies that the Wronskian is

$$W(\varepsilon_0(t), \varepsilon_0^*(t)) \equiv \varepsilon_0(t)\dot{\varepsilon}_0^*(t) - \dot{\varepsilon}_0(t)\varepsilon_0^*(t) = -\frac{2i}{\mu(t)}. \tag{4.33}$$

Using the new notation, the evolution operator (4.17), the wave functions (4.19) and the propagator (4.24) can be written as

$$\begin{aligned}\hat{U}(t,t_0) &= \exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{\dot{\varepsilon}_0(t)+\dot{\varepsilon}_0^*(t)}{\varepsilon_0(t)+\varepsilon_0^*(t)}\right)q^2\right]\exp\left[\ln\left|\frac{\sqrt{m\omega_0}(\varepsilon_0(t)+\varepsilon_0^*(t))}{2}\right|^{-1}\left(q\frac{\partial}{\partial q}+\frac{1}{2}\right)\right]\\ &\times\exp\left[\frac{\hbar}{2m\omega_0}\left(\frac{\varepsilon_0(t)-\varepsilon_0^*(t)}{\varepsilon_0(t)+\varepsilon_0^*(t)}\right)\frac{\partial^2}{\partial q^2}\right], \qquad (4.34)\end{aligned}$$

$$\Psi(q,t) = \frac{1}{(n!\sqrt{\hbar\pi})^{\frac{1}{2}}}\frac{1}{\sqrt{\varepsilon_0(t)}}\left(\frac{\varepsilon_0^*(t)}{2\varepsilon_0(t)}\right)^{n/2}\exp\left[\frac{i\mu(t)}{2\hbar}\frac{\dot{\varepsilon}_0(t)}{\varepsilon_0(t)}q^2\right]H_n\left(\frac{q}{\sqrt{\hbar}|\varepsilon_0(t)|}\right), \qquad (4.35)$$

$$\begin{aligned}K(q,t;q',t_0) &= \frac{1}{\sqrt{2\pi\hbar i|\varepsilon_0(t)||\varepsilon_0(t_0)|\sin(\theta_0(t))}}\times\exp\left[\frac{-iqq'}{\hbar\sin(\theta_0(t))|\varepsilon_0(t)||\varepsilon_0(t_0)|}\right]\\ &\times\exp\left[\frac{i}{2\hbar}\cot(\theta_0(t))\left(\frac{q^2}{|\varepsilon_0(t)|^2}+\frac{q'^2}{|\varepsilon_0(t_0)|^2}\right)\right]\\ &\times\exp\left[\frac{i}{4\hbar}\left(\frac{\mu(t)q^2}{|\varepsilon_0(t)|^2}\frac{d}{dt}|\varepsilon_0(t)|^2\right)\right]. \qquad (4.36)\end{aligned}$$

Since we know the evolution operator, we can find the invariants corresponding to position and momentum operators

$$\begin{aligned}\hat{Q}_0(t) &= \hat{U}(t,t_0)\hat{q}\hat{U}^\dagger(t,t_0) = x_0\mu(t)\dot{x}_2(t)\hat{q} - x_0x_2(t)\hat{p},\\ \hat{P}_0(t) &= \hat{U}(t,t_0)\hat{p}\hat{U}^\dagger(t,t_0) = \frac{-\mu(t)}{x_0}\dot{x}_1(t)\hat{q} + \frac{x_1(t)}{x_0}\hat{p}.\end{aligned}$$

expressed in terms of new complex variable (4.27) as

$$\begin{aligned}\hat{Q}_0(t) &= \frac{\mu(t)}{2i\sqrt{m\omega_0}}(\dot{\varepsilon}_0(t)-\dot{\varepsilon}_0^*(t))\hat{q} - \frac{1}{2i\sqrt{m\omega_0}}(\varepsilon_0(t)-\varepsilon_0^*(t))\hat{p},\\ \hat{P}_0(t) &= \frac{-\sqrt{m\omega_0}\mu(t)(\dot{\varepsilon}_0(t)+\dot{\varepsilon}_0^*(t))}{2}\hat{q} + \frac{\sqrt{m\omega_0}(\varepsilon_0(t)+\varepsilon_0^*(t))}{2}\hat{p}.\end{aligned}$$

Similarly, the invariants corresponding to creation and annihilation operators in terms of $\varepsilon_0(t)$ will be

$$\hat{A}_0(t) = \hat{U}(t,t_0)\hat{a}\hat{U}^\dagger(t,t_0) = \frac{i}{\sqrt{2\hbar}}\Big(\varepsilon_0(t)\hat{p} - \mu(t)\dot{\varepsilon}_0(t)\hat{q}\Big), \tag{4.37}$$

$$\hat{A}_0^\dagger(t) = \hat{U}(t,t_0)\hat{a}^\dagger\hat{U}^\dagger(t,t_0) = \frac{i}{\sqrt{2\hbar}}\Big(-\varepsilon_0^\dagger(t)\hat{p} + \mu(t)\dot{\varepsilon}_0^\dagger(t)\hat{q}\Big). \tag{4.38}$$

If we consider the Hamiltonian $\hat{H}_0 = \frac{1}{2m}\hat{p}^2 + \frac{m\omega_0^2}{2}\hat{q}^2$, then corresponding invariant

$$\begin{aligned}\hat{I}_0(t) &= \hat{U}(t,t_0)\hat{H}_0\hat{U}^\dagger(t,t_0), \quad \hat{I}_0(t_0) = \hat{H}_0,\\ &= \frac{1}{2}\omega_0\Big(|\varepsilon_0(t)|\hat{p} - \mu(t)|\dot{\varepsilon}_0(t)|\hat{q}\Big)^2.\end{aligned} \tag{4.39}$$

CHAPTER 5

QUANTUM PARAMETRIC OSCILLATOR: THE LEWIS RIESENFELD INVARIANT METHOD

Consider again the Schrödinger equation,

$$i\hbar \frac{\partial \Psi(q,t)}{\partial t} = \hat{H}(t)\Psi(q,t), \tag{5.1}$$

where $\hat{H}(t)$ given by equation (4.3) is a self-adjoint operator depending explicitly on time. It was explored by Lewis and Riesenfeld in paper (Lewis and Riesenfeld, 1969), and later it was used in other works such as (Hartley and Ray,1982), (Dantas et al., 1992). The Lewis-Riesenfeld method for solving Schrödinger equation (5.1) is based on finding quadratic invariant for the system described by the Hamiltonian (4.3). More precisely, it is motivated by the basic results given in the following proposition.

Proposition 5.1 *Let the operator $\hat{I}(t)$ be a spatial self-adjoint invariant of Schrödinger equation (5.1) defined on a Hilbert space. Assume $\left\{\Phi_n(q,t)\right\}$ is a complete set of orthonormal eigenstates of $\hat{I}(t)$ corresponding to eigenvalues $\left\{\lambda_n\right\}$. Then,*

(i) The eigenvalues are time-independent, i.e.,

$$\frac{d}{dt}\lambda_n(t) = 0. \tag{5.2}$$

(ii) The wave functions defined as

$$\psi_n^{(L)}(q,t) = e^{i\nu_n(t)}\Phi_n(q,t), \tag{5.3}$$

where $\nu_n(t)$ satisfies

$$\frac{d\nu_n(t)}{dt} = \frac{1}{\hbar}\langle\Phi_n\big|i\hbar\frac{\partial}{\partial t} - \hat{H}(t)\big|\Phi_n\rangle, \tag{5.4}$$

are solutions of Schrödinger equation, provided the eigenvalues λ_n are non-degenerate.

Proof :

(i) By the assumption we have

$$\hat{I}(t)\Phi_n(q,t) = \lambda_n(t)\Phi_n(q,t), \quad \forall n. \tag{5.5}$$

Taking time-derivative of both sides of equation (5.5) gives

$$\frac{\partial\hat{I}(t)}{\partial t}\Phi_n + \hat{I}(t)\frac{\partial\Phi_n}{\partial t} = \dot{\lambda}_n(t)\Phi_n(t) + \lambda_n(t)\frac{\partial\Phi_n}{\partial t}. \tag{5.6}$$

Using equation (2.16) we get

$$\frac{d\lambda_n(t)}{dt}\Phi_n = (\hat{I}(t) - \lambda_n(t))\frac{\partial\Phi_n}{\partial t} + \frac{i}{\hbar}(\hat{I}\hat{H}\Phi_n - \hat{H}\hat{I}\Phi_n), \tag{5.7}$$

and arranging the terms gives

$$\frac{d\lambda_n(t)}{dt}\Phi_n(t) = (\hat{I}(t) - \lambda_n(t))\Big(\frac{\partial\Phi_n}{\partial t} + \frac{i}{\hbar}\hat{H}\Phi_n\Big), \quad \forall n. \tag{5.8}$$

Taking inner product with Φ_n gives

$$\frac{d\lambda_n(t)}{dt}\Big\langle\Phi_n\Big|\Phi_n\Big\rangle = \Big\langle\Phi_n(t)\Big|(\hat{I}(t) - \lambda_n(t))\Big(\frac{\partial\Phi_n}{\partial t} + \frac{i}{\hbar}\hat{H}\Phi_n\Big)\Big\rangle, \quad \forall n. \tag{5.9}$$

Since $\hat{I}(t) - \lambda(t)$ is a self-adjoint, we can write

$$\frac{d\lambda_n(t)}{dt}\left\|\Phi_n\right\|^2 = \left\langle (\hat{I}(t) - \lambda_n(t))\Phi_n \middle| \frac{\partial \Phi_n}{\partial t} + \frac{i}{\hbar}\hat{H}\Phi_n \right\rangle, \tag{5.10}$$

which clearly implies $\dot{\lambda}_n(t) = 0$, so that $\lambda_n(t) = \lambda_n-$ constant.

(ii) Since $\dot{\lambda}_n(t) = 0$, from equation (5.9) it follows that

$$\left\langle \Phi_m \middle| (\hat{I} - \lambda_n)\left(\frac{\partial \Phi_n}{\partial t} + \frac{i}{\hbar}\hat{H}\Phi_n\right) \right\rangle = 0, \quad \forall n, m. \tag{5.11}$$

Since equation (5.11) holds for $\forall n, m$ and $\{\Phi_n(q,t)\}$ is an orthonormal basis, it implies that $(\hat{I} - \lambda_n)\left(\frac{\partial \Phi_n}{\partial t} + \frac{i}{\hbar}\hat{H}\Phi_n\right) = 0$, which means $\frac{\partial \Phi_n}{\partial t} + \frac{i}{\hbar}\hat{H}\Phi_n$ is an eigenstate of $\hat{I}(t)$ corresponding to eigenvalue λ_n. If the eigenspace corresponding to λ_n is one-dimensional, then

$$i\hbar\frac{\partial \Phi_n}{\partial t} - \hat{H}\Phi_n = c_n(t)\Phi_n, \tag{5.12}$$

for some $c_n(t) \neq 0$. Thus $\Phi_n(q,t)$ is a solution of Schrödinger equation (5.1) with modified Hamiltonian

$$\hat{H}_c(t) = \hat{H}(t) + c_n(t). \tag{5.13}$$

This suggests that solution of Schrödinger equation (5.1) is of the form (5.3), i.e.,

$$\psi_n^{(L)}(q,t) = e^{i\nu_n(t)}\Phi_n(q,t),$$

for some real-valued function $\nu_n(t)$. To find $\nu_n(t)$, we substitute $\psi_n^{(L)}(q,t)$ to the Schrödinger equation (5.1), and obtain equation (5.4). Solving this first-order differential equation will determine $\nu_n(t)$ up to an arbitrary constant of integration, which can be fixed by normalization of the state. □

In next sections we show how one can find the dynamical invariant, its eigenstates and phase factors explicitly . Then, we provide the corresponding solutions of the Schrödinger equation (5.1).

5.1. Finding the Lewis-Riesenfeld Quadratic Invariant

Lewis-Riesenfeld approach for solving time-dependent SE is based on finding self-adjoint quadratic invariant. The invariant $\hat{I}_{LR}(t)$ is assumed to be a linear superposition of generators of su(1,1) Lie algebra defined in Section 3.2.1, that is

$$\hat{I}_{LR}(t) = -\frac{i}{\hbar}(\alpha(t)\hat{K}_+ + \beta(t)\hat{K}_- + \gamma(t)\hat{K}_0), \tag{5.14}$$

where $\alpha(t), \beta(t), \gamma(t)$ are real-valued so that $\hat{I}_{LR}(t) = \hat{I}^{\dagger}_{LR}(t)$. On the other hand Hamiltonian $\hat{H}(t)$ can also be written in terms of su(1,1) Lie algebra generators as given in equation (4.4). Substituting equation (5.14) and (4.4) into the equation defining the dynamical invariant operator (2.15), that is

$$i\frac{\partial \hat{I}_{LR}(t)}{\partial t} = \frac{-1}{\hbar}[\hat{I}_{LR}(t), \hat{H}(t)],$$

one can determine the unknown functions $\alpha(t), \beta(t), \gamma(t)$. For this first, we calculate $[I_{LR}, H]$ using the commutation relations (3.31).

$$\begin{aligned} [\hat{I}_{LR}, \hat{H}] &= \left[-i\Big(\alpha(t)\hat{K}_+ + \beta(t)\hat{K}_- + \gamma(t)\hat{K}_0\Big), -i\Big(\frac{\hbar^2}{\mu(t)}\hat{K}_- + \mu(t)\omega^2(t)\hat{K}_+\Big)\right], \\ &= \frac{\hbar^2\alpha(t)}{\mu(t)}2\hat{K}_0 - \beta(t)\mu(t)\omega^2(t)2\hat{K}_0 + \frac{\hbar^2\gamma(t)}{\mu(t)}\hat{K}_- - \gamma(t)\mu(t)\omega^2(t)\hat{K}_+. \end{aligned} \tag{5.15}$$

Substituting above result and $\partial \hat{I}_{LR}/\partial t$ into equation (2.15) we have:

$$\begin{aligned}\dot{\alpha}(t)\hat{K}_+ + \dot{\beta}(t)\hat{K}_- + \dot{\gamma}(t)\hat{K}_0 = & -\Bigg(\Big(2\frac{\hbar\alpha(t)}{\mu(t)} - 2\frac{\beta(t)\mu(t)\omega^2(t)}{\hbar}\Big)\hat{K}_0 + \frac{\hbar\gamma(t)}{\mu(t)}\hat{K}_- \\ & - \frac{\gamma(t)\mu(t)\omega^2(t)}{\hbar}\hat{K}_+\Bigg).\end{aligned}$$

From last equation we get a system of three first-order ordinary differential equations:

$$\dot{\alpha}(t) - \frac{\gamma(t)\mu(t)\omega^2(t)}{\hbar} = 0, \tag{5.16}$$

$$\dot{\beta}(t) + \frac{\hbar\gamma(t)}{\mu(t)} = 0, \tag{5.17}$$

$$\dot{\gamma}(t) + 2\hbar\frac{\alpha(t)}{\mu(t)} - \frac{2\beta(t)\mu(t)\omega^2}{\hbar} = 0. \tag{5.18}$$

To solve this system, we introduce the auxiliary real-valued function $\sigma(t)$, such that $\beta(t) = \hbar^2\sigma^2(t)$. Then from equation (5.17) we get $\gamma(t) = -2\hbar\sigma(t)\dot{\sigma}(t)\mu(t)$. Writing $\beta(t)$ and $\gamma(t)$ in (5.18), we find $\alpha(t)$ in terms of $\sigma(t)$ as:

$$\alpha(t) = \sigma\mu^2(\ddot{\sigma} + \frac{\dot{\mu}}{\mu}\sigma + \omega^2\sigma) + \dot{\sigma}^2\mu^2.$$

Now, substituting $\alpha(t)$ and $\gamma(t)$ into equation (5.16) we have:

$$\begin{aligned}\frac{d}{dt}\Big(\sigma\mu^2(\ddot{\sigma} + \frac{\dot{\mu}}{\mu}\sigma + \omega^2\sigma) + \dot{\sigma}^2\mu^2\Big) + \frac{2\sigma\dot{\sigma}\hbar\mu^2\omega^2}{\hbar} &= 0, \\ \frac{d}{dt}\Big(\sigma\mu^2(\ddot{\sigma} + \frac{\dot{\mu}}{\mu}\sigma + \omega^2\sigma)\Big) + 2\dot{\sigma}\mu^2\Big((\ddot{\sigma} + \frac{\dot{\mu}}{\mu}\sigma + \omega^2\sigma\Big) &= 0.\end{aligned} \tag{5.19}$$

Let us define $y(t) = \sigma(t)\mu^2(t)\Big(\ddot{\sigma}(t) + \frac{\dot{\mu}(t)}{\mu(t)}\sigma(t) + \omega^2(t)\sigma(t)\Big)$. Then equation (5.19) becomes:

$$\frac{d}{dt}y(t) + 2\frac{\dot{\sigma}(t)}{\sigma(t)}y(t) = 0. \tag{5.20}$$

Equation (5.20) is the first-order homogeneous linear equation, and $y(t) = 0$ is always a solution. It has one nonzero solution as well. Therefore, we consider two cases:

Case 5.1 *When $y \equiv 0$, that's*

$$\ddot{\sigma}(t) + \frac{\dot{\mu}(t)}{\mu(t)}\dot{\sigma}(t) + \omega^2(t)\sigma(t) = 0, \tag{5.21}$$

where $\sigma(t)$ is real-valued by assumption. Let $x(t)$ denote a solution of equation (5.21). In this case $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ become:

$$\begin{cases} \alpha(t) = \mu^2(t)\dot{x}^2(t), \\ \beta(t) = \hbar^2 x^2(t), \\ \gamma(t) = -2\hbar\mu(t)x(t)\dot{x}(t). \end{cases}$$

Now, by substituting $\alpha(t)$, $\gamma(t)$, $\beta(t)$ into the equation (5.14), we get the special quadratic invariant, which we denote by $\hat{I}_x(t)$

$$\begin{aligned} \hat{I}_x(t) &= -\frac{i}{\hbar}(\mu^2(t)\dot{x}^2(t))\Big(\frac{iq^2}{2}\Big) - \frac{i}{\hbar}(x^2(t)\hbar^2)\Big(\frac{-i}{2}\frac{\partial^2}{\partial q^2}\Big) + \frac{2ix(t)\dot{x}(t)\hbar\mu(t)}{2\hbar}\Big(q\frac{\partial}{\partial q} + \frac{1}{2}\Big), \\ \text{or} \qquad \hat{I}_x(t) &= \frac{1}{2\hbar}\Big(\mu(t)\dot{x}(t)\hat{q} - x(t)\hat{p}\Big)^2. \end{aligned}$$

Case 5.2 *For nonzero $y(t)$ we solve equation (5.20) and find*

$$y(t) = \sigma^{-2}(t) = \frac{c}{\sigma^2(t)},$$

where we can choose $c = 1$. Now, since $y(t) = \Big(\mu^2(t)\sigma(t)\Big(\ddot{\sigma}(t) + \frac{\dot{\mu}(t)}{\mu(t)}\sigma(t) + \omega^2(t)\sigma(t)\Big)\Big)$ and we found that $y(t) = 1/\sigma^2(t)$, we have the following equality, which is known as

Ermakov-Pinney nonlinear differential equation,

$$\ddot{\sigma}(t) + \frac{\dot{\mu}(t)}{\mu(t)}\sigma(t) + \omega^2(t)\sigma(t) \quad = \quad \frac{1}{\mu^2(t)\sigma^3(t)}. \tag{5.22}$$

As a result, $\alpha(t)$, $\beta(t)$ *and* $\gamma(t)$ *can be found in terms of a solution* $\sigma(t)$ *of equation (5.22) as*

$$\begin{cases} \alpha(t) = \frac{1}{\sigma^2(t)} + \mu^2(t)\dot{\sigma}^2(t), \\ \\ \beta(t) = \hbar^2\sigma^2(t), \\ \\ \gamma(t) = -2\hbar\mu(t)\sigma(t)\dot{\sigma}(t). \end{cases}$$

Now, by substituting $\alpha(t)$, $\gamma(t)$, $\beta(t)$ *into equation (5.14), we have the following quadratic invariant:*

$$\hat{I}_{LR}(t) = \frac{1}{2\hbar}\left(\left(\mu(t)\dot{\sigma}(t)\hat{q} - \sigma(t)\hat{p}\right)^2 + \frac{\hat{q}^2}{\sigma^2(t)}\right). \tag{5.23}$$

5.2. Eigenvalues and Eigenstates of the Invariant

The eigenvalues and eigenstates of the quadratic, self-adjoint invariant $\hat{I}_{LR}(t)$ can be found by an operator method that is completely analogous to the method introduced by P. Dirac for diagonalizing the standart harmonic oscillator, as described in Chapter 3. Here, we will show that the self-adjoint quadratic invariant $\hat{I}_{LR}(t)$ has discrete spectrum and complete orthonormal set of eigenstates. First, we will factorize $\hat{I}_{LR}(t)$ given by the equation (5.23) in the form

$$\hat{I}_{LR}(t) = \left(A^\dagger(t)A(t) + \frac{1}{2}\right). \tag{5.24}$$

After necessary calculations we can find $A(t)$ and $A^{\dagger}(t)$ as follows:

$$A^{\dagger}(t) = \frac{1}{\sqrt{2\hbar}}\left(\frac{\hat{q}}{\sigma} - i\Big(\sigma\hat{p} - \dot{\sigma}\mu\hat{q}\Big)\right), \tag{5.25}$$

$$A(t) = \frac{1}{\sqrt{2\hbar}}\left(\frac{\hat{q}}{\sigma} + i\Big(\sigma\hat{p} - \dot{\sigma}\mu\hat{q}\Big)\right). \tag{5.26}$$

The operators $A^{\dagger}(t)$, $A(t)$ and $\hat{I}_{LR}(t)$ satisfy the following commutation relations

$$[A(t), A^{\dagger}(t)] = \hat{1},\ \ [\hat{I}_{LR}(t), A^{\dagger}(t)] = A^{\dagger}(t),\ \ [\hat{I}_{LR}(t), A(t)] = -A(t). \tag{5.27}$$

We note that these commutation relations hold for every solution $\sigma(t)$ of equation (5.22). Thus, we have the spectrum generating algebra $\{\hat{1}, \hat{I}_{LR}(t), A^{\dagger}(t), A(t)\}$. From algebraic point of view, the integral (5.24) with algebra (5.27) is equivalent to harmonic oscillator with Heisenberg-Weyl algebra. This allows us to construct eigenvalues and eigenstates for it in the same way.

5.2.1. Finding the Eigenvalues

Assume that $\hat{I}_{LR}(t)$ has a complete set of orthonormal eigenfunctions $\{|\Phi_n\rangle\}_{n=0}^{\infty}$, that is

$$\hat{I}_{LR}(t)|\Phi_n\rangle = \lambda_n|\Phi_n\rangle, \qquad n = 0, 1, 2, 3, ... \tag{5.28}$$

where λ_n are the discrete real eigenvalues and $\langle\Phi_n, \Phi_m\rangle = \delta_{nm}$. Let us show that eigenvalues λ_n satisfy $1/2 \leq \lambda_n$ which shows that they are non-negative numbers,

$$0 \leq ||A|\Phi_n\rangle||^2 = \langle\Phi_n|A^{\dagger}A|\Phi_n\rangle = \langle\Phi_n|\hat{I}_{LR} - \frac{1}{2}|\Phi_n\rangle = \lambda_n - \frac{1}{2}, \tag{5.29}$$

$$\frac{1}{2} \leq \lambda_n. \tag{5.30}$$

Consider the following equations, which show that $A^\dagger\Phi_n$ and $A\Phi_n$ are eigenstates of the $\hat{I}_{LR}(t)$:

$$\begin{aligned} \hat{I}_{LR}\Big(A^\dagger|\Phi_n\rangle\Big) &= (A^\dagger A + \frac{1}{2})A^\dagger|\Phi_n\rangle = \Big(\lambda_n + 1\Big)A^\dagger|\Phi_n\rangle, && (5.31) \\ \hat{I}_{LR}\Big(A|\Phi_n\rangle\Big) &= (A^\dagger A + \frac{1}{2})A|\Phi_n\rangle = \Big(\lambda_n - 1\Big)A|\Phi_n\rangle. && (5.32) \end{aligned}$$

We can see that operator $\hat{A}(t)$ lowers and the operator $\hat{A}^\dagger(t)$ raises the energy of the system so, $\hat{A}(t)$ is the lowering operator and $\hat{A}^\dagger(t)$ is the raising operator. Since we have $1/2 \leq \lambda_n$, the energy of the system can not be negative. Thus we must have the lower limit for the energy in the state $|\Phi_0\rangle$, such that $A|\Phi_0\rangle = 0$, which implies

$$\begin{aligned} \hat{I}_{LR}|\Phi_0\rangle &= \lambda_0|\Phi_0\rangle, \\ (A^\dagger A + \frac{1}{2})|\Phi_0\rangle &= \lambda_0|\Phi_0\rangle, \\ \lambda_0 &= \frac{1}{2}. \end{aligned}$$

Applying $A^\dagger$ to the ground state $|\Phi_0\rangle$ by using equation (5.31), we have

$$\hat{I}_{LR}A^\dagger|\Phi_0\rangle = (\lambda_0 + 1)A^\dagger|\Phi_0\rangle = \Big(\frac{3}{2}\Big)A^\dagger|\Phi_0\rangle.$$

Then, applying n times $A^\dagger$ to the ground state Φ_0, we can find λ_n as

$$\lambda_n = n + \frac{1}{2}, \qquad n = 0, 1, 2... \tag{5.33}$$

Therefore equation (5.28) becomes

$$\hat{I}_{LR}|\Phi_n\rangle = \Big(n + \frac{1}{2}\Big)|\Phi_n\rangle. \tag{5.34}$$

In addition, according to (5.32), $\hat{A}|\Phi_n\rangle$ is an eigenstate of $\hat{I}_{LR}(t)$ with the eigenvalue $n-\frac{1}{2}$, then it must be proportional to $|\Phi_{n-1}\rangle$, i.e., $\hat{A}|\Phi_n\rangle| = c_n|\Phi_{n-1}\rangle$, where c_n is a constant. Consider the following equations:

$$\begin{aligned}\langle\Phi_n|A^\dagger A|\Phi_n\rangle| &= |c_n|^2 \quad\Rightarrow\quad \langle\Phi_n|\hat{I}_{LR}-\frac{1}{2}|\Phi_n\rangle| = |c_n|^2,\\ \lambda_n-\frac{1}{2} &= |c_n|^2, \quad\Rightarrow\quad n=|c_n|^2.\end{aligned}$$

Thus $c_n = \sqrt{n}$, so the equation $A|\Phi_n\rangle| = c_n|\Phi_{n-1}\rangle$ will be $A|\Phi_n\rangle| = \sqrt{n}|\Phi_{n-1}\rangle$. For the eigenstate $A^\dagger|\Phi_n\rangle$, we can also find that $A^\dagger|\Phi_n\rangle| = \sqrt{n+1}|\Phi_{n+1}\rangle$. Using these results, we can derive the next expression for eigenstate,

$$|\Phi_n\rangle = \frac{\left(A^\dagger\right)^n|\Phi_0\rangle}{\sqrt{n!}}. \tag{5.35}$$

5.2.2. Explicit derivation of the eigenstates in coordinate representation

Now, since $\hat{I}_{LR}$ is the self-adjoint operator, the eigenstates corresponding to distinct eigenvalues are orthogonal, and the set $\{\Phi_n\}_{n=0}^{\infty}$ is an orthogonal basis for $L^2(\mathbb{R})$. To find the eigenstates $\Phi_n(q,t)$ explicitly, we need to find the ground state $\Phi_0(q,t)$ by solving $\hat{A}\Phi_0(q,t)=0$, that's

$$\frac{1}{\sqrt{2\hbar}}\left(\frac{q}{\sigma}+i\left(-i\hbar\sigma\frac{d}{dq}-\dot{\sigma}\mu q\right)\right)\Phi_0 = 0. \tag{5.36}$$

Solving differential equation (5.36), and then doing normalization give us the ground state wave function

$$\Phi_0(q,t) = \left(\frac{1}{\sqrt{\hbar\pi}}\right)^{1/2}\frac{1}{\sqrt{\sigma(t)}}\exp\left[\frac{i}{2\hbar}\mu\left(\frac{\dot{\sigma}}{\sigma}+\frac{i}{\mu\sigma^2}\right)q^2\right]. \tag{5.37}$$

Using equation (5.35) and propositions given in Appendix D, we can find wave functions $\Phi_n(q,t)$ as well. Writing equation (5.35) explicitly, we have

$$\begin{aligned}
\Phi_n = \frac{\left(A^\dagger\right)^n \Phi_0}{\sqrt{n!}} &= \frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2\hbar}}\left(\frac{\hat{q}}{\sigma} - i(\sigma\hat{p} - \dot{\sigma}\mu\hat{q})\right)\right)^n \left(\left(\frac{1}{\sqrt{\hbar\pi}}\right)^{1/2} \frac{1}{\sqrt{\sigma(t)}} \right.\\
&\left. \times \exp\left[\frac{i}{2\hbar}\mu\left(\frac{\dot{\sigma}}{\sigma} + \frac{i}{\mu\sigma^2}\right)q^2\right]\right),\\
&= \left(\frac{1}{2^n n!\sqrt{\hbar\pi}}\right)^{1/2} \frac{1}{\sqrt{\sigma(t)}}\left(\frac{1}{\sqrt{\hbar}}\right)^n \left(\hbar\sigma(t)\right)^n \left(\left(\frac{1}{\hbar\sigma^2(t)} + \frac{i\mu\dot{\sigma}}{\hbar\sigma}\right)q - \frac{d}{dq}\right)^n\\
&\times \exp\left[\frac{i}{2\hbar}\mu\left(\frac{\dot{\sigma}}{\sigma} + \frac{i}{\mu\sigma^2}\right)q^2\right]\Bigg),\\
&= \left(\frac{1}{2^n n!\sqrt{\hbar\pi}}\right)^{1/2} \frac{1}{\sqrt{\sigma(t)}}\left(\frac{1}{\sqrt{\hbar}}\right)^n \left(\hbar\sigma(t)\right)^n \left((-1)^n \exp\left[\left(\frac{1}{\hbar\sigma^2} + \frac{i\mu\dot{\mu}}{\hbar\sigma}\right)\frac{q^2}{2}\right]\right.\\
&\left.\times\left(\frac{d^n}{dq^n}\right)\exp\left[-\left(\frac{1}{\hbar\sigma^2} + \frac{i\mu\dot{\mu}}{\hbar\sigma}\right)\frac{q^2}{2}\right]\right) \times \exp\left[\frac{i\mu}{2\hbar}\left(\frac{\dot{\sigma}}{\sigma} + \frac{i}{\mu\sigma^2}\right)q^2\right],\\
&= (-1)^n\left(\frac{1}{2^n n!\sqrt{\hbar\pi}}\right)^{1/2} \frac{1}{\sqrt{\sigma(t)}}\left(\frac{1}{\sqrt{\hbar}}\right)^n \left(\hbar\sigma(t)\right)^n \exp\left[\left(\frac{1}{\hbar\sigma^2} + \frac{i\mu\dot{\mu}}{\hbar\sigma}\right)\frac{q^2}{2}\right]\\
&\times\left(\left(\frac{d^n}{dq^n}\right)\exp\left[\frac{-q^2}{\hbar\sigma^2}\right]\right).\\
&= (-1)^n\left(\frac{1}{2^n n!\sqrt{\hbar\pi}}\right)^{1/2} \frac{1}{\sqrt{\sigma(t)}}\left(\frac{1}{\sqrt{\hbar}}\right)^n \left(\hbar\sigma(t)\right)^n \exp\left[\left(\frac{1}{\hbar\sigma^2} + \frac{i\mu\dot{\mu}}{\hbar\sigma}\right)\frac{q^2}{2}\right]\\
&\times\left((-1)^n\left(\frac{1}{\sqrt{\hbar\sigma}}\right)^n H_n\left(\frac{q}{\sqrt{\hbar\sigma}}\right)\exp\left[\frac{-q^2}{\hbar\sigma^2}\right]\right).
\end{aligned}$$

As a result, we get

$$\Phi_n(q,t) = \left(\frac{1}{\sqrt{\hbar\pi}2^n n!}\right)^{1/2} \frac{1}{\sqrt{\sigma(t)}} H_n\left(\frac{q}{\sqrt{\hbar}\sigma(t)}\right) \exp\left[\frac{i\mu}{2\hbar}\left(\frac{\dot{\sigma}(t)}{\sigma(t)} + \frac{i}{\mu\sigma^2(t)}\right)q^2\right],$$

for $n = 0, 1, 2, 3, ..,$ where H_n are the Hermite polynomials.

5.3. Determining the phase factor $\nu_n(t)$

For the phase factor $\nu_n(t)$ from Proposition 5.1, we have equation (5.4),

$$\frac{d\nu_n(t)}{dt} = i\langle\Phi_n|\frac{\partial}{\partial t}|\Phi_n\rangle - \frac{1}{\hbar}\langle\Phi_n|\hat{H}(t)|\Phi_n\rangle.$$

To evaluate $\nu_n(t)$, we need to compute $\langle\Phi_n|H(t)|\Phi_n\rangle$ and $\langle\Phi_n|\frac{\partial}{\partial t}|\Phi_n\rangle$. Before starting to calculate these diagonal elements we need to write $\hat{p}$ and $\hat{q}$ in terms of $\hat{A}$ and $\hat{A}^\dagger$ using equations (5.25) and (5.26), i.e.,

$$\hat{q} = \sqrt{\frac{\hbar}{2}}\sigma(t)(\hat{A} + \hat{A}^\dagger), \quad \hat{p} = \frac{\sqrt{\hbar}}{\sqrt{2}i}\frac{(\hat{A} - \hat{A}^\dagger)}{\sigma(t)} + \sqrt{\frac{\hbar}{2}}\mu\dot{\sigma}(t)(\hat{A} + \hat{A}^\dagger). \tag{5.38}$$

Now, we calculate $\langle\Phi_n|H(t)|\Phi_n\rangle$ as follows:

$$\begin{aligned}\langle\Phi_n|H(t)|\Phi_n\rangle &= \langle\Phi_n|\frac{1}{2\mu}\hat{p}^2 + \frac{\mu\omega^2}{2}\hat{q}^2|\Phi_n\rangle = \frac{1}{2\mu}\langle\Phi_n|\left(\frac{\sqrt{\hbar}(\hat{A} - \hat{A}^\dagger)}{\sigma\sqrt{2}i} + \mu\dot{\sigma}\frac{\sqrt{\hbar}(\hat{A} + \hat{A}^\dagger)}{\sqrt{2}}\right)^2|\Phi_n\rangle, \\ &+ \frac{\mu\omega^2}{2}\langle\Phi_n|\left(\frac{\sqrt{\hbar}\sigma(\hat{A} + \hat{A}^\dagger)}{\sqrt{2}}\right)^2|\Phi_n\rangle = \frac{\hbar}{2}(n + \frac{1}{2})\left[\frac{1}{\mu\sigma^2} + \dot{\sigma}^2\mu + \mu\omega^2\sigma^2\right]. \end{aligned} \tag{5.39}$$

Then, we calculate $\langle\Phi_n|\frac{\partial}{\partial t}|\Phi_n\rangle$ by using the equality $A|\Phi_n\rangle = \sqrt{n}|\Phi_{n-1}\rangle$, i.e.,

$$\begin{aligned} \frac{\partial}{\partial t}\Big(A|\Phi_n\rangle\Big) &= \frac{\partial}{\partial t}\Big(\sqrt{n}|\Phi_{n-1}\rangle\Big), \\ \text{or} \qquad \frac{\partial A}{\partial t}|\Phi_n\rangle + A\frac{\partial}{\partial t}|\Phi_n\rangle &= \sqrt{n}\frac{\partial}{\partial t}|\Phi_{n-1}\rangle. \end{aligned}$$

Taking inner product of last equation with $\langle\Phi_{n-1}|$, gives

$$
\begin{aligned}
\langle\Phi_{n-1}|A\frac{\partial}{\partial t}|\Phi_n\rangle &= \sqrt{n}\langle\Phi_{n-1}|\frac{\partial}{\partial t}|\Phi_{n-1}\rangle - \langle\Phi_{n-1}|\frac{\partial A}{\partial t}|\Phi_n\rangle, \\
\langle A^\dagger\Phi_{n-1}|\frac{\partial}{\partial t}|\Phi_n\rangle &= \sqrt{n}\langle\Phi_{n-1}|\frac{\partial}{\partial t}|\Phi_{n-1}\rangle - \langle\Phi_{n-1}|\frac{\partial A}{\partial t}|\Phi_n\rangle, \\
\sqrt{n}\langle\Phi_n|\frac{\partial}{\partial t}|\Phi_n\rangle &= \sqrt{n}\langle\Phi_{n-1}|\frac{\partial}{\partial t}|\Phi_{n-1}\rangle - \langle\Phi_{n-1}|\frac{\partial A}{\partial t}|\Phi_n\rangle, \\
\langle\Phi_n|\frac{\partial}{\partial t}|\Phi_n\rangle &= \langle\Phi_{n-1}|\frac{\partial}{\partial t}|\Phi_{n-1}\rangle + \frac{1}{\sqrt{n}}\langle\Phi_n|\frac{\partial A^\dagger}{\partial t}|\Phi_{n-1}\rangle. \qquad (5.40)
\end{aligned}
$$

For the calculation of $\langle\Phi_n|\frac{\partial A^\dagger}{\partial t}|\Phi_{n-1}\rangle$, it is needed to find $\frac{\partial A^\dagger}{\partial t}$. Differentiating equation (5.25), we have

$$
\frac{\partial A^\dagger}{\partial t} = \frac{1}{\sqrt{2\hbar}}\Bigg(\frac{-\dot{\sigma}\hat{q}}{\sigma^2} - i\Big(\dot{\sigma}\hat{p} - \dot{\sigma}\dot{\mu}\hat{q} - \ddot{\sigma}\mu\hat{q}\Big)\Bigg) = \frac{1}{\sqrt{2\hbar}}\Bigg(\Big(\frac{-\dot{\sigma}}{\sigma^2} + i\dot{\sigma}\dot{\mu} + i\ddot{\sigma}\mu\Big)\hat{q} - i\dot{\sigma}\hat{p}\Bigg).
$$

Using equations (5.38), we rewrite it in terms of operators $\hat{A}$ and $\hat{A}^\dagger$,

$$
\frac{\partial A^\dagger}{\partial t} = \frac{1}{2}\Bigg[\Big(\frac{-2\dot{\sigma}}{\sigma} + i(\dot{\mu}\dot{\sigma}\sigma + \mu\sigma\ddot{\sigma} - \mu\dot{\sigma}^2)\Big)\hat{A} + i\Big(\dot{\mu}\dot{\sigma}\sigma + \mu\sigma\ddot{\sigma} - \mu\dot{\sigma}^2\Big)\hat{A}^\dagger\Bigg].
$$

Now we are ready to calculate $\langle\Phi_n|\frac{\partial A^\dagger}{\partial t}|\Phi_{n-1}\rangle$.

$$
\begin{aligned}
\langle\Phi_n|\frac{\partial A^\dagger}{\partial t}|\Phi_{n-1}\rangle &= \langle\Phi_n|\frac{1}{2}\Bigg[\Big(\frac{-2\dot{\sigma}}{\sigma} + i(\dot{\mu}\dot{\sigma}\sigma + \mu\sigma\ddot{\sigma} - \mu\dot{\sigma}^2)\Big)\hat{A} \\
&+ i\Big(\dot{\mu}\dot{\sigma}\sigma + \mu\sigma\ddot{\sigma} - \mu\dot{\sigma}^2\Big)\hat{A}^\dagger\Bigg]|\Phi_{n-1}\rangle, \\
&= \frac{i}{2}\Big(\dot{\mu}\dot{\sigma}\sigma + \mu\sigma\ddot{\sigma} - \mu\dot{\sigma}^2\Big)\sqrt{n}. \qquad (5.41)
\end{aligned}
$$

By using equation (5.41), equation (5.40) takes the following form:

$$
\langle\Phi_n|\frac{\partial}{\partial t}|\Phi_n\rangle = \langle\Phi_{n-1}|\frac{\partial}{\partial t}|\Phi_{n-1}\rangle + \frac{i}{2}\Big(\dot{\mu}\dot{\sigma}\sigma + \mu\sigma\ddot{\sigma} - \mu\dot{\sigma}^2\Big). \qquad (5.42)
$$

We observe that equation (5.42) is a recursion formula. So it can be used for evolution of $\langle\Phi_{n-1}|\frac{\partial}{\partial t}|\Phi_{n-1}\rangle$ in terms of $\langle\Phi_{n-2}|\frac{\partial}{\partial t}|\Phi_{n-2}\rangle$, so that

$$\langle\Phi_n|\frac{\partial}{\partial t}|\Phi_n\rangle = \langle\Phi_{n-2}|\frac{\partial}{\partial t}|\Phi_{n-2}\rangle + \frac{i}{2}\Big(\dot{\mu}\dot{\sigma}\sigma + \mu\sigma\ddot{\sigma} - \mu\dot{\sigma}^2\Big) + \frac{i}{2}\Big(\dot{\mu}\dot{\sigma}\sigma + \mu\sigma\ddot{\sigma} - \mu\dot{\sigma}^2\Big).$$

The same recursion formula can be applied for the $\langle\Phi_{n-2}|\frac{\partial}{\partial t}|\Phi_{n-2}\rangle$. Continuing this recursion n times we get:

$$\langle\Phi_n|\frac{\partial}{\partial t}|\Phi_n\rangle = \langle\Phi_0|\frac{\partial}{\partial t}|\Phi_0\rangle + n\frac{i}{2}\Big(\dot{\mu}\dot{\sigma}\sigma + \mu\sigma\ddot{\sigma} - \mu\dot{\sigma}^2\Big). \tag{5.43}$$

Since Φ_0 is known and given by (5.37), we calculate $\langle\Phi_0|\frac{\partial}{\partial t}|\Phi_0\rangle$ as

$$\langle\Phi_0|\frac{\partial}{\partial t}|\Phi_0\rangle = \frac{i}{4}\Big(\dot{\mu}\dot{\sigma}\sigma + \mu\sigma\ddot{\sigma} - \mu\dot{\sigma}^2\Big).$$

Then, equation (5.43) gives:

$$\begin{aligned}\langle\Phi_n|\frac{\partial}{\partial t}|\Phi_n\rangle &= \frac{i}{4}\Big(\dot{\mu}\dot{\sigma}\sigma + \mu\sigma\ddot{\sigma} - \mu\dot{\sigma}^2\Big) + n\frac{i}{2}\Big(\dot{\mu}\dot{\sigma}\sigma + \mu\sigma\ddot{\sigma} - \mu\dot{\sigma}^2\Big),\\ &= \frac{i}{2}(n+\frac{1}{2})\Big(\dot{\mu}\dot{\sigma}\sigma + \mu\sigma\ddot{\sigma} - \mu\dot{\sigma}^2\Big).\end{aligned}$$

The calculated $\langle\Phi_n|\frac{\partial}{\partial t}|\Phi_n\rangle$ and $\langle\Phi_n|H(t)|\Phi_n\rangle$, we can substitute into equation (5.4), giving

$$\frac{d}{dt}\nu_n(t) = \frac{-1}{2}\Big(n + \frac{1}{2}\Big)\Big[\mu\sigma(\ddot{\sigma} + \frac{\dot{\mu}}{\mu}\dot{\sigma} + w^2\sigma) + \frac{1}{\mu\sigma^2}\Big].$$

Using the Ermakov-Pinney equation (5.22), we can simplify the right hand side of above equation as

$$\frac{d}{dt}\nu_n(t) = -\Big(n + \frac{1}{2}\Big)\Big[\frac{1}{\mu\sigma^2}\Big]. \tag{5.44}$$

Taking integral of this equation we have

$$\nu_n(t) = -\Big(n+\frac{1}{2}\Big)\int_{t_0}^{t}\frac{d\epsilon}{\mu(\epsilon)\sigma^2(\epsilon)},$$

where $\nu_n(t_0) = 0$. As a result, we have shown that the Schrödinger equation (5.1) has the set of orthonormal solutions, given explicitly in the form,

$$\begin{aligned}\psi_n^{(L)}(q,t) &= \Big(\frac{1}{\sqrt{\hbar\pi}2^n n!}\Big)^{1/2}\times\frac{1}{\sqrt{\sigma(t)}}\times\exp\Big[-i(n+\frac{1}{2})\int_{t_0}^{t}\frac{d\epsilon}{\mu(\epsilon)\sigma^2(\epsilon)}\Big]\times H_n\Big(\frac{q}{\sqrt{\hbar}\sigma}\Big)\\ &\times\ \exp\Big[\frac{i\mu(t)\dot{\sigma}(t)}{2\hbar\sigma}q^2\Big]\times\exp\Big[-\frac{1}{2\hbar\sigma^2}q^2\Big]. \qquad (5.45)\end{aligned}$$

where the upper script "L" denotes solutions found by the Lewis-Riesenfeld approach. Let $\theta_0(t) \equiv \int_{t_0}^{t}\frac{d\epsilon}{\mu(\epsilon)\sigma^2(\epsilon)}$, $\theta_0(t_0) = 0$. Then $\nu_n(t) = -\Big(n+\frac{1}{2}\Big)\theta_0(t)$ and equation (5.45) can be rewritten as

$$\begin{aligned}\psi_n^{(L)}(q,t) &= \Big(\frac{1}{\sqrt{\hbar\pi}2^n n!}\Big)^{1/2}\times\frac{1}{\sqrt{\sigma(t)}}\times\exp\Big[-i(n+\frac{1}{2})\theta_0(t)\Big]\times H_n\Big(\frac{q}{\sqrt{\hbar}\sigma}\Big)\\ &\times\ \exp\Big[\frac{i\mu(t)\dot{\sigma}(t)}{2\hbar\sigma}q^2\Big]\times\exp\Big[-\frac{1}{2\hbar\sigma^2}q^2\Big]. \qquad (5.46)\end{aligned}$$

The probability densities corresponding to solutions (5.46) are

$$\rho_n^{(L)}(q,t) = \Big(\frac{1}{\sqrt{\hbar\pi}2^n n!}\Big)\times\frac{1}{\sigma(t)}\times\exp\Big[-\Big(\frac{q}{\sqrt{\hbar}\sigma(t)}\Big)^2\Big]\times H_n^2\Big(\frac{q}{\sqrt{\hbar}\sigma(t)}\Big). \qquad (5.47)$$

Now, we compare this result (5.46) with the wave function (4.19) which is found by the Evolution Operator Method. By setting $\theta_0(t) = \arctan\Big(m\omega_0 x_1^2(t_0)\frac{x_2(t)}{x_1(t)}\Big)$ and using the following proposition it can be shown that the orthonormal solutions of Schrödinger equation which are found in both Lewis Riesenfeld and Evolution Operator Method are actually the same.

Proposition 5.2 *If $x_1(t)$ and $x_2(t)$ are two linearly independent real solutions of equation (4.9) satisfying initial conditions (4.10) and (4.11) as defined in Wei-Norman method,* *then*

$$\sigma(t) = \frac{1}{\sqrt{m\omega_0}}\left(\frac{x_1^2(t) + m^2\omega_0^2 x_1^4(t_0) x_2^2(t)}{x_1^2(t_0)}\right)^{1/2} = \sigma_0(t),$$

satisfies the Ermakov-Pinney equation (5.22) with following initial conditions:

$$\sigma(t_0) = \frac{1}{\sqrt{m\omega_0}}, \quad \dot{\sigma}(t_0) = 0. \tag{5.48}$$

5.4. Finding the propagator using eigenstates of the quadratic invariant

Since the orthonormal solutions of Schrödinger equation $\{\psi_n^{(L)}(q,t)\}$ form an orthonormal basis for $L_t^2(\mathbb{R})$, then any solution of IVP for SE (5.1) is of the form

$$\Psi(q,t) = \sum_{n=0}^{\infty} c_n \psi_n^{(L)}(q,t). \tag{5.49}$$

Let us see what will be the exact form of c_n. In equation (5.1) the initial value $\psi_n^L(q,t_0)$ is given. In addition, we found the exact form of $\psi_n^{(L)}(q,t)$ by the Lewis-Riesenfeld Invariant Method. So let $t = t_0$ in equation (5.49),

$$\Psi(q,t_0) = \sum_{n=0}^{\infty} c_n \psi_n^{(L)}(q,t_0). \tag{5.50}$$

Then, taking inner product of both side with $\psi_n^{(L)}(q,t_0)$, the c_n will be

$$c_n = \langle \psi_n^{(L)}(q,t_0) | \Psi(q,t_0) \rangle = \int_{-\infty}^{\infty} \Psi(q',t_0) {\psi^{(L)}}_n^*(q',t_0) dq'. \tag{5.51}$$

Therefore $\Psi(q,t)$ is

$$\begin{aligned}\Psi(q,t) &= \sum_{n=0}^{\infty}\Big[\int_{-\infty}^{\infty}\Psi(q',t_0)\psi^{(L)*}_{n}(q',t_0)dq'\Big]\psi_n^{(L)}(q,t),\\ &= \int_{-\infty}^{\infty}\Big[\sum_{n=0}^{\infty}\psi_n^{(L)}(q,t)\psi^{(L)*}_{n}(q',t_0)\Big]\Psi(q',t_0)dq'.\end{aligned}$$

The expression under the summation is the propagator, that is

$$\sum_{n=0}^{\infty}\psi_n^{(L)}(q,t)\psi^{(L)*}_{n}(q',t_0) = K(q,t;q',t_o). \tag{5.52}$$

So $\Psi(q,t)$ become

$$\Psi(q,t) = \int_{-\infty}^{\infty}K(q,t;q',t_o)\Psi(q',t_0)dq'. \tag{5.53}$$

Next, we find the closed form for propagator $K(q,t;q',t_o)$. Recall the solutions $\psi_n^{(L)}(q,t)$ given by (5.46). Then $\left(\psi_n^{(L)}\right)^*(q',t_0)$ will be

$$\begin{aligned}\left(\psi_n^{(L)}\right)^*(q',t_0) &= \widetilde{N}_n\frac{1}{\sqrt{\sigma(t_0)}}\exp\Big[i(n+\frac{1}{2})\theta(t_0)\Big]\exp\Big[\frac{-i\mu(t_0)\dot{\sigma}(t_0)}{2\hbar\sigma(t_0)}q'^2\Big]\\ &\times\ H_n\Big(\frac{q'}{\sqrt{\hbar}\sigma(t_0)}\Big)\exp\Big[-\frac{1}{2\hbar\sigma^2(t_0)}q'^2\Big].\end{aligned}$$

Substituting $\psi_n^{(L)}(q,t)$ and $\left(\psi_n^{(L)}\right)^*(q',t_0)$ into the equation (5.52) we have

$$\begin{aligned}K_{LR}(q,t;q',t_o) &= \sum_{n=0}^{\infty}\frac{1}{\sqrt{\hbar\pi}2^n n!}\times\exp\Big[-i(n+\frac{1}{2})\theta_0(t)\Big]\times\exp\Big[i(n+\frac{1}{2})\theta_0(t_0)\Big]\\ &\times H_n\Big(\frac{q}{\sqrt{\hbar}\sigma(t)}\Big)\times H_n\Big(\frac{q'}{\sqrt{\hbar}\sigma(t_0)}\Big)\frac{1}{\sqrt{\sigma(t)\sigma(t_0)}}\times\exp\Big[\frac{i\mu(t)\dot{\sigma}(t)}{2\hbar\sigma}q^2\Big]\\ &\times\exp\Big[\frac{-i\mu(t_0)\dot{\sigma}(t_0)}{2\hbar\sigma(t_0)}q'^2\Big]\times\exp\Big[-\frac{1}{2\hbar\sigma^2(t)}q^2\Big]\times\exp\Big[-\frac{1}{2\hbar\sigma^2(t_0)}q'^2\Big],\end{aligned}$$

$$
\begin{aligned}
K_{LR}(q,t;q',t_o) &= \frac{1}{\sqrt{\hbar\pi}}\frac{1}{\sqrt{\sigma(t)\sigma(t_0)}}\times\exp\left[\frac{i\mu(t)\dot{\sigma}(t)}{2\hbar\sigma}q^2\right]\times\exp\left[\frac{-i\mu(t_0)\dot{\sigma}(t_0)}{2\hbar\sigma(t_0)}q'^2\right]\\
&\times\exp\left[-\frac{1}{2\hbar\sigma^2(t)}q^2\right]\times\exp\left[-\frac{1}{2\hbar\sigma^2(t_0)}q'^2\right]\times\exp\left[\frac{i}{2}(\theta_0(t_0)-\theta_0(t))\right]\\
&\times\left(\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}\exp\left[i(\theta_0(t_0)-\theta_0(t))\right]\right)^n H_n\left(\frac{q}{\sqrt{\hbar}\sigma(t)}\right)H_n\left(\frac{q'}{\sqrt{\hbar}\sigma(t_0)}\right)\right).
\end{aligned}
$$

After making arrangements and using Mehler's formula (3.59) for the expression under the summation, we get,

$$
\begin{aligned}
K_{LR}(q,t;q',t_o) &= \frac{1}{\sqrt{2\pi i\hbar\sin(\theta_0(t)-\theta_0(t_0))\sigma(t)\sigma(t_0)}}\\
&\times\exp\left[\frac{i}{2\hbar}\left(\frac{\mu(t)\dot{\sigma}(t)}{\sigma(t)}+\frac{\cot(\theta_0(t)-\theta_0(t_0))}{\sigma^2(t)}\right)q^2\right]\\
&\times\exp\left[\frac{i}{2\hbar}\left(\frac{-\mu(t_0)\dot{\sigma}(t_0)}{\sigma(t_0)}+\frac{\cot(\theta_0(t)-\theta_0(t_0))}{\sigma^2(t_0)}\right)q'^2\right]\\
&\times\exp\left[\frac{-i}{\hbar\sigma(t)\sigma(t_0)}\frac{qq'}{\sin(\theta_0(t)-\theta_0(t_0))}\right]. \qquad (5.54)
\end{aligned}
$$

This formula for the propagator, $K_{LR}(q,t;q',t_o)$ coincides with the one found in (Yeon et al., 1993).

Particular Case:

For the initial values $\theta_0(t_0)=0$, $\sigma(t_0)=\frac{1}{\sqrt{m\omega_0}}$ and $\dot{\sigma}(t_0)=0$ given in Proposition 5.2 , the last equation becomes

$$
\begin{aligned}
K_{LR}(q,t;q',t_o) &= \frac{1}{\sqrt{\frac{2\pi i\hbar}{m\omega_0}\sin\theta_0(t)\sigma(t)}}\times\exp\left[\frac{i}{2\hbar}\left(\frac{\mu(t)\dot{\sigma}(t)}{\sigma(t)}+\frac{\cot\theta_0(t)}{\sigma^2(t)}\right)q^2\right]\\
&\times\ \exp\left[\frac{im\omega_0}{2\hbar}\cot\theta_0(t)q'^2\right]\times\exp\left[\frac{-i}{\hbar}\frac{\sqrt{m\omega_0}}{\sigma(t)}\frac{qq'}{\sin\theta_0(t)}\right].
\end{aligned}
$$

By substituting $\sigma(t)$ and $\dot{\sigma}(t)$, the propagator is

$$\begin{aligned} K_{LR}(q,t;q',t_o) &= \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\ln\left|\frac{x_1(t)}{x_1(t_0)}\right|\right]\sqrt{\frac{i}{-\hbar x_1^2(t_0)\left(\frac{x_2(t)}{x_1(t)}\right)}}\exp\left[\frac{ix_1}{2\hbar x_0^2 x_2}q'^2\right] \\ &\times\exp\left[\frac{-iqq'}{\hbar x_0 x_2}\right]\exp\left[\frac{i\mu x_1\dot{x}_1 q^2}{2\hbar(x_1^2+m^2\omega_0^2x_0^4x_2^2)}\right]\exp\left[\frac{im^2\omega_0^2x_0^4x_2q^2}{2\hbar x_1(x_1^2+m^2\omega_0^2x_0^4x_2^2)}\right] \\ &\times\exp\left[\frac{i\mu m^2\omega_0^2x_0^4\dot{x}_1x_2^2}{2\hbar x_1(x_1^2+m^2\omega_0^2x_0^4x_2^2)}q^2\right]\times\exp\left[\frac{ix_1}{2\hbar x_2(x_1^2+m^2\omega_0^2x_0^4x_2^2)}q^2\right]. \end{aligned}$$

By using functions $f(t)$, $g(t)$, and $h(t)$ which are given by equations (4.13), (4.14) and (4.15) we have

$$K_{LR}(q,t;q',t_o) = \frac{1}{\sqrt{2\pi}}\exp\left[\frac{h(t)}{2}\right]\sqrt{\frac{i}{g(t)}}\exp\left[\frac{if(t)}{2}q^2\right]\exp\left[\frac{-i}{2g(t)}\left(e^{h(t)}q-q'\right)^2\right]. \tag{5.55}$$

We observe that the last equation is exactly the same as equation (4.23). Since here we used Lewis Riesenfield's approach to find propagator $K_{LR}(q,t;q',t_o)$ and we found equality of it with equation (4.23), the results obtained by the Evolution Operator method and the Lewis Riesenfeld method coincides for the particular case with initial conditions $\sigma(t_0) = 1/\sqrt{m\omega_0}$ and $\dot{\sigma}(t_0) = 0$, imposed on the function $\sigma(t)$.

CHAPTER 6

QUANTUM PARAMETRIC OSCILLATOR: MALKIN-MANKO-TRIFONOV APPROACH

Here we consider again the one-dimensional time-dependent Schrödinger equation (SE)

$$\hat{S}(t)\Psi(q,t) \equiv (i\hbar\frac{\partial}{\partial t} - \hat{H}(t))\Psi(q,t) = 0, \tag{6.1}$$

where $\hat{S}(t)$ denotes the Schrödinger operator and the Hamiltonian is given by (4.3), that is $\hat{H}(t) = (1/2\mu(t))\hat{p}^2 + (\mu(t)\omega^2(t)/2)\hat{q}^2$.

In this Chapter we describe another approach, the Malkin-Manko-Trifonov (MMT) method for finding solutions of SE (6.1). This method was introduced in (Malkin, 1970), and later used in other works such as (Malkin, 1971), (Dodonov and Man'ko,1979). It is based on finding symmetries of SE (6.1), which by Definition 2.1 are operators that map a solution of SE to an other solution of the same equation. According to Proposition 2.6, symmetry operators of SE are also dynamical invariants (integrals of motion), and because of this MMT-method is also known as an approach based on finding dynamical invariants linear in momentum $\hat{p}$ and coordinate $\hat{q}$.

6.1. Linear Dynamical Invariants

It is known that (Man'ko, 1987) all invariants of an one-dimensional system with quadratic Hamiltonian (4.3) can be constructed from two independent linear $\hat{p}$ and $\hat{q}$ invariants of the form,

$$\mathcal{A}(t) = a(t)\hat{q} + b(t)\hat{p} + c(t), \tag{6.2}$$

where $a(t), b(t)$ and $c(t)$ are time dependent complex-valued functions. For finding invariants of the form (6.2) explicitly, we will use condition $[\hat{S}(t), \mathcal{A}(t)] = 0$. Calculations give

$$\begin{aligned}
[\hat{S}(t), \mathcal{A}(t)] &= i\hbar\Big[\frac{\partial}{\partial t}, a(t)\hat{q} + b(t)\hat{p} + c(t)\Big] - \Big[\hat{H}(t), a(t)\hat{q} + b(t)\hat{p} + c(t)\Big] = 0, \\
&\Rightarrow \; i\hbar\Big[\frac{\partial}{\partial t}, a(t)\hat{q} + b(t)\hat{p} + c(t)\Big] - \Big[\frac{\hat{p}^2}{2\mu(t)}, a(t)\hat{q} + b(t)\hat{p} + c(t)\Big] \\
&\quad -\Big[\frac{\mu(t)\omega^2(t)\hat{q}^2}{2}, a(t)\hat{q} + b(t)\hat{p} + c(t)\Big] = 0, \\
&\Rightarrow \; i\hbar(\dot{a}\hat{q} + \dot{b}\hat{p} + \dot{c}) + \frac{i\hbar a\hat{p}}{\mu(t)} - i\hbar\mu(t)\omega^2(t)b\hat{q} = 0.
\end{aligned}$$

Combining terms in the last expression, we see that the following identity must hold

$$\Big(\dot{b} + \frac{a}{\mu(t)}\Big)\hat{p} + \Big(\dot{a} - \mu(t)\omega^2(t)b\Big)\hat{q} + \dot{c} = 0, \tag{6.3}$$

leading to the system of first-order differential equations:

$$\begin{cases}
\dot{a}(t) = \mu(t)\omega^2(t)b(t), \\
\dot{b}(t) = -\frac{1}{\mu(t)}a(t), \\
\dot{c}(t) = 0.
\end{cases}$$

Taking derivative of second equation and then using the first one, we get

$$\ddot{b} + \frac{\dot{\mu}}{\mu}\dot{b} + \omega^2 b = 0. \tag{6.4}$$

For convenience in comparison of (6.2) with operators $\hat{a}$ and $\hat{a}^\dagger$, we will use the following notation $b(t) \equiv (i/\sqrt{2\hbar})\varepsilon(t)$. Then $a(t), b(t)$ and $c(t)$ become

$$a(t) = \frac{-i}{\sqrt{2\hbar}}\mu(t)\dot{\varepsilon}(t), \quad b(t) = \frac{i}{\sqrt{2\hbar}}\varepsilon(t), \quad c(t) = c_0 \tag{6.5}$$

and without loss of generality we take $c_0 = 0$. As a result we obtain two independent non-Hermitian linear invariants

$$\mathcal{A}(t) = \frac{i}{\sqrt{2\hbar}}\Big(\varepsilon(t)\hat{p} - \mu(t)\dot{\varepsilon}(t)\hat{q}\Big), \tag{6.6}$$
$$\mathcal{A}^{\dagger}(t) = \frac{-i}{\sqrt{2\hbar}}\Big(\varepsilon^{*}(t)\hat{p} - \mu(t)\dot{\varepsilon}^{*}(t)\hat{q}\Big), \tag{6.7}$$

where $\varepsilon(t)$ is a complex-valued solution of the linear differential equation

$$\ddot{\varepsilon}(t) + \frac{\dot{\mu}}{\mu}\dot{\varepsilon}(t) + \omega^2(t)\varepsilon(t) = 0. \tag{6.8}$$

For the commutator of the linear invariants it's convenient to impose condition $[\mathcal{A}(t), \mathcal{A}^{\dagger}(t)] = 1$, which is equivalent to:

$$\varepsilon(t)\dot{\varepsilon}^{*}(t) - \dot{\varepsilon}(t)\varepsilon^{*}(t) = \frac{-2i}{\mu(t)}. \tag{6.9}$$

We note that this condition does not fix the initial data $\varepsilon(t_0)$ and $\dot{\varepsilon}(t_0)$, but imposes only a relation between them. Then, any particular complex solution of equation (6.8), will give linear invariants of the form (6.6) and (6.7).

Using the linear invariants (6.6) and (6.7), one can find time-dependent coherent states by applying the displacement operator $\hat{D}(\alpha) = \exp(\alpha\hat{\mathcal{A}}^{\dagger}(t) - \alpha^{*}\hat{\mathcal{A}}(t))$ to a state $\psi_0^{(M)}(q,t)$ satisfying $\mathcal{A}(t)\psi_0^{(M)}(q,t) = 0$ and $\hat{S}(t)\psi_0^{(M)}(q,t) = 0$.

In this work, we discuss the construction of wave function solutions of the parametric oscillator, which can be seen as generalizations of the Fock states. As we will see in next section, in MMT-approach these solutions of SE appear as the eigenstates of a Hermitian quadratic invariant.

6.2. Quadratic invariants and time-dependent wave function solutions

Using the linear invariants (6.6) and (6.7), one can easily construct quadratic Hermitian invariants. Indeed, we note that for the quantum system described by $\hat{S}(t)$, the operators $\mathcal{A}(t)\mathcal{A}^\dagger(t)$ and $\mathcal{A}^\dagger(t)\mathcal{A}(t)$ are Hermitian quadratic invariants. That is, they satisfy

$$\left(\mathcal{A}\mathcal{A}^\dagger\right)^\dagger = \mathcal{A}\mathcal{A}^\dagger, \quad \left(\mathcal{A}^\dagger\mathcal{A}\right)^\dagger = \mathcal{A}^\dagger\mathcal{A},$$

showing that operators are Hermitian. Since $[S(t), \mathcal{A}(t)] = 0,\ [S(t), \mathcal{A}^\dagger(t)] = 0$ it is not difficult to show that these operators are quadratic invariants

$$\left[\hat{S}(t), \mathcal{A}^\dagger(t)\mathcal{A}(t)\right] = 0, \quad \left[\hat{S}(t), \mathcal{A}(t)\mathcal{A}^\dagger(t)\right] = 0. \tag{6.10}$$

Note that $\mathcal{A}(t)\mathcal{A}^\dagger(t) \neq \mathcal{A}^\dagger(t)\mathcal{A}(t)$, and more general quadratic Hermitian invariant will be

$$\hat{I}_M(t) = \frac{\mathcal{A}^\dagger(t)\mathcal{A}(t) + \mathcal{A}(t)\mathcal{A}^\dagger(t)}{2} = \mathcal{A}^\dagger(t)\mathcal{A}(t) + \frac{1}{2} = \mathcal{N}(t) + \frac{1}{2}, \tag{6.11}$$

where $\mathcal{N}(t) = \mathcal{A}^\dagger(t)\mathcal{A}(t)$. As a result, we have found three operators $\mathcal{A}(t)$, $\mathcal{A}^\dagger(t)$, $\mathcal{N}(t)$ satisfying the commutation relations

$$\left[\mathcal{A}(t), \mathcal{A}^\dagger(t)\right] = 1, \quad \left[\mathcal{N}(t), \mathcal{A}(t)\right] = -\mathcal{A}(t), \quad \left[\mathcal{N}(t), \mathcal{A}^\dagger(t)\right] = \mathcal{A}^\dagger(t), \tag{6.12}$$

of spectrum generating algebra for the operator $\mathcal{N}(t)$, and also for the quadratic invariant $\hat{I}_M(t)$. Since $\mathcal{N}(t)$ is a Hermitian operator invariant, it has a real, time-independent eigenvalues, and due to above commutation relations it acts on the states as a number operator. On the other side, the operators $\mathcal{A}^\dagger(t)$ and $\mathcal{A}(t)$ are the raising and lowering operators, respectively.

Now, the eigenstates of $\mathcal{N}(t)$ can be constructed by a standard procedure. Let $\psi_0^{(M)}(q, t)$ be such that $\mathcal{A}(t)\psi_0^{(M)}(q, t) = 0$, where the upper script $'M'$ will denote states obtained by the MMT-method. Then, we have

$$\mathcal{N}(t)\psi_n^{(M)}(q,t) = n\psi_n^{(M)}(q,t), \quad n = 0, 1, 2, ..., \tag{6.13}$$

and the eigenstates are

$$\psi_n^{(M)}(q,t) = \frac{\left(\mathcal{A}^\dagger(t)\right)^n}{\sqrt{n!}}\psi_0^{(M)}(q,t). \tag{6.14}$$

Clearly, $\psi_n^{(M)}(q,t)$ are also eigenstates of the dynamical invariant $\hat{I}_M(t)$, since

$$\hat{I}_M(t)\psi_n^{(M)}(q,t) = (n + \frac{1}{2})\psi_n^{(M)}(q,t). \tag{6.15}$$

In general, the eigenstates (6.14) do not need to satisfy the Schrödinger equation (6.1). However, if the function $\psi_0^{(M)}(q,t)$ satisfies both equations

$$\mathcal{A}(t)\psi_0^{(M)}(q,t) = 0 \quad and \quad \hat{S}(t)\psi_0^{(M)}(q,t) = 0,$$

then eigenstates $\psi_n^{(M)}(q,t)$ defined by equation (6.14) will be also solutions of the SE, that is $\hat{S}(t)\psi_n^{(M)}(q,t) = 0$. This is because $\mathcal{A}^\dagger(t)$ is a symmetry operator for $\hat{S}(t)$, and therefore $\left(\mathcal{A}^\dagger(t)\right)^n$, for all $n = 1, 2, 3....$ are also symmetry operators, in other words they commute with $\hat{S}(t)$. This shows that, by Malkin-Manko-Trifonov approach one can directly find solutions of the Schrödinger equation, using the operators $\mathcal{A}(t)$ and $\mathcal{A}^\dagger(t)$, since they are symmetry operators by construction. We recall that in the Lewis-Riesenfeld approach the lowering and raising operators $A(t)$ and $A^\dagger(t)$ defined by equations (5.25) and (5.26), are not invariants, and this explains some of the technical differences such as finding the phase factor in LR-approach. In next section, we give the details of finding solutions of Schrödinger equation by using the MMT-method.

6.3. Solutions of Schrödinger equation in coordinate representation

In this section, we construct a function $\psi_0^{(M)}(q,t)$, which satisfies both equations $\mathcal{A}(t)\psi_0^{(M)}(q,t) = 0$ and $\hat{S}(t)\psi_0^{(M)}(q,t) = 0$. For this, writing the equation $\mathcal{A}(t)\psi_0^{(M)}(q,t) = 0$ in coordinate representation

$$\frac{i}{\sqrt{2\hbar}}\Big(\varepsilon(t)(-i\hbar\frac{\partial}{\partial q} - \mu(t)\dot{\varepsilon}(t)q\Big)\psi_0^{(M)}(q,t) = 0, \tag{6.16}$$

and solving it, we get $\psi_0^{(M)}(q,t) = \mathrm{N}_0(t)\exp\left[\frac{i\mu(t)}{2\hbar}\frac{\dot{\varepsilon}(t)}{\varepsilon(t)}q^2\right]$, where $\mathrm{N}_0(t) = \exp[c_1(t) + ic_2(t)]$ and $c_1(t)$ and $c_2(t)$ are arbitrary real-valued time-dependent functions. Doing normalization we find that $\exp[c_1(t)] = 1/((\hbar\pi)^{\frac{1}{4}}\sqrt{|\varepsilon(t)|})$, so that

$$\psi_0^{(M)}(q,t) = \frac{1}{(\hbar\pi)^{\frac{1}{4}}}\frac{1}{\sqrt{|\varepsilon(t)|}}\exp\Big[ic_2(t)\Big]\exp\left[\frac{i\mu(t)}{2\hbar}\frac{\dot{\varepsilon}(t)}{\varepsilon(t)}q^2\right]. \tag{6.17}$$

Now, to fix $c_2(t)$ we use condition $\hat{S}(t)\psi_0(q,t) = 0$, and find that $c_2(t) = \frac{-i}{2}\ln\frac{|\varepsilon(t)|}{\varepsilon(t)}$. Therefore, function (6.17) becomes

$$\psi_0^{(M)}(q,t) = \frac{1}{(\hbar\pi)^{\frac{1}{4}}}\frac{1}{\sqrt{\varepsilon(t)}}\exp\left[\frac{i\mu(t)}{2\hbar}\frac{\dot{\varepsilon}(t)}{\varepsilon(t)}q^2\right]. \tag{6.18}$$

Applying n-times the raising operator $\mathcal{A}^{\dagger}(t)$ to the ground state (6.18) and by using propositions given in Appendix D, we find solutions of SE defined by (6.14) in coordinate representation:

$$
\begin{aligned}
\psi_n^{(M)}(q,t) &= \frac{\left(\mathcal{A}^\dagger(t)\right)^n}{\sqrt{n!}}\psi_0^{(M)}(q,t),\\
&= \frac{1}{\sqrt{n!}}\left(\frac{-i}{\sqrt{2\hbar}}\left(\varepsilon^*(t)\hat{p}-\mu(t)\dot{\varepsilon}^*(t)\hat{q}\right)\right)^n\left(\frac{1}{(\hbar\pi)^{\frac{1}{4}}}\frac{1}{\sqrt{\varepsilon(t)}}\exp\left[\frac{i\mu(t)}{2\hbar}\frac{\dot{\varepsilon}(t)}{\varepsilon(t)}q^2\right]\right),\\
&= \frac{1}{(n!\sqrt{\hbar\pi})^{\frac{1}{2}}}\frac{1}{\sqrt{\varepsilon(t)}}\left(\frac{\sqrt{\hbar}\varepsilon^*(t)}{\sqrt{2}}\right)^n\left(\frac{i\mu\dot{\varepsilon}^*}{\hbar\varepsilon^*}q-\frac{d}{dq}\right)^n\exp\left[\frac{i\mu(t)}{2\hbar}\frac{\dot{\varepsilon}(t)}{\varepsilon(t)}q^2\right],\\
&= \frac{(-1)^n}{(n!\sqrt{\hbar\pi})^{\frac{1}{2}}}\frac{1}{\sqrt{\varepsilon(t)}}\left(\frac{\sqrt{\hbar}\varepsilon^*(t)}{\sqrt{2}}\right)^n\left(\exp\left[\frac{i\mu(t)}{2\hbar}\frac{\dot{\varepsilon}^*(t)}{\varepsilon^*(t)}q^2\right]\frac{d^n}{dq^n}\exp\left[\frac{-i\mu(t)}{2\hbar}\frac{\dot{\varepsilon}^*(t)}{\varepsilon^*(t)}q^2\right]\right)\\
&\times \exp\left[\frac{i\mu(t)}{2\hbar}\frac{\dot{\varepsilon}^*(t)}{\varepsilon^*(t)}q^2\right],\\
&= \frac{(-1)^n}{(n!\sqrt{\hbar\pi})^{\frac{1}{2}}}\frac{1}{\sqrt{\varepsilon(t)}}\left(\frac{\sqrt{\hbar}\varepsilon^*(t)}{\sqrt{2}}\right)^n\exp\left[\frac{i\mu(t)}{2\hbar}\frac{\dot{\varepsilon}^*(t)}{\varepsilon^*(t)}q^2\right]\frac{d^n}{dq^n}\exp\left[\frac{-q^2}{\hbar|\varepsilon(t)|^2}\right],\\
&= \frac{(-1)^n}{(n!\sqrt{\hbar\pi})^{\frac{1}{2}}}\frac{1}{\sqrt{\varepsilon(t)}}\left(\frac{\sqrt{\hbar}\varepsilon^*(t)}{\sqrt{2}}\right)^n\exp\left[\frac{i\mu(t)}{2\hbar}\frac{\dot{\varepsilon}^*(t)}{\varepsilon^*(t)}q^2\right]\left((-1)^n\left(\frac{1}{\sqrt{\hbar}|\varepsilon(t)|}\right)^n\right.\\
&\times \left.H_n\left(\frac{q}{\sqrt{\hbar}|\varepsilon(t)|}\right)\exp\left[\frac{-q^2}{\hbar|\varepsilon(t)|^2}\right]\right),\\
&= \frac{1}{(n!\sqrt{\hbar\pi})^{\frac{1}{2}}}\frac{1}{\sqrt{\varepsilon(t)}}\left(\frac{\varepsilon^*(t)}{2\varepsilon(t)}\right)^{n/2}\exp\left[\frac{i\mu(t)}{2\hbar}\frac{\dot{\varepsilon}^*(t)}{\varepsilon^*(t)}q^2\right]\exp\left[\frac{-q^2}{\hbar|\varepsilon(t)|^2}\right]H_n\left(\frac{q}{\sqrt{\hbar}|\varepsilon(t)|}\right).
\end{aligned}
$$

Finally, we get

$$
\psi_n^{(M)}(q,t) = \frac{1}{(n!\sqrt{\hbar\pi})^{\frac{1}{2}}}\frac{1}{\sqrt{\varepsilon(t)}}\left(\frac{\varepsilon^*(t)}{2\varepsilon(t)}\right)^{n/2}\exp\left[\frac{i\mu(t)}{2\hbar}\frac{\dot{\varepsilon}(t)}{\varepsilon(t)}q^2\right]H_n\left(\frac{q}{\sqrt{\hbar}|\varepsilon(t)|}\right). \tag{6.19}
$$

with corresponding probability density

$$
\rho_n^{(M)}(q,t) = \frac{1}{2^n n!\sqrt{\pi\hbar}}\frac{1}{|\varepsilon(t)|}H_n^2\left(\frac{q}{\sqrt{\hbar}|\varepsilon(t)|}\right). \tag{6.20}
$$

These solutions are orthonormal and any solution of SE can be written in the form

$$
\psi(q,t) = \sum_{n=0}^{\infty}\left\langle\psi_n^{(M)}(q,t_0)\middle|\psi(q,t_0)\right\rangle\psi_n^{(M)}(q,t) = \int K^{(M)}(q,t;q',t_0)\psi(q',t_0)dq'
$$

where the propagator is $K^{(M)}(q,t;q',t_0) = \sum_{n=0}^{\infty} \psi^{(M)}{}_n(q,t)\psi^{(M)*}_n(q',t_0)$. Here, we can find a closed form of the propagator using $\varepsilon(t)$ in the polar coordinates

$$\varepsilon(t) = |\varepsilon(t)|e^{i\theta(t)}, \quad \theta(t) = \int_{t_0}^{t} \frac{dt'}{\mu(t')|\varepsilon(t')|^2}. \tag{6.21}$$

The calculations are as follows

$$\begin{aligned}
K^{(M)}(q,t;q',t_0) &= \sum_{n=0}^{\infty} \psi^{(M)}{}_n(q,t)\psi^{(M)*}_n(q',t_0) \\
&= \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\varepsilon(t)}} \frac{1}{\sqrt{\varepsilon^*(t_0)}} \exp\left[\frac{i}{2}\left(\frac{\dot{\varepsilon}(t)}{\varepsilon(t)}q^2 - \frac{\dot{\varepsilon}^*(t_0)}{\varepsilon^*(t_0)}q'^2\right)\right] \\
&\times \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{2}\frac{|\varepsilon(t)||\varepsilon(t_0)|}{\varepsilon(t)\varepsilon^*(t_0)}\right)^n H_n\left(\frac{q}{|\varepsilon(t)|}\right) H_n\left(\frac{q'}{|\varepsilon^*(t_0)|}\right).
\end{aligned}$$

Using Mehler's formula (3.59) for the expression under the summation and making proper arrangements, we find the propagator

$$\begin{aligned}
K^{(M)}(q,t;q',t_0) &= \frac{1}{\sqrt{2\pi\hbar i|\varepsilon(t)||\varepsilon(t_0)|\sin(\theta(t)-\theta(t_0))}} \times \exp\left[\frac{-iqq'}{\hbar\sin(\theta(t)-\theta(t_0))|\varepsilon(t)||\varepsilon(t_0)|}\right] \\
&\times \exp\left[\frac{i}{2\hbar}\cot(\theta(t)-\theta(t_0))\left(\frac{q^2}{|\varepsilon(t)|^2} + \frac{q'^2}{|\varepsilon(t_0)|^2}\right)\right] \\
&\times \exp\left[\frac{i}{4\hbar}\left(\left(\frac{\mu(t)}{|\varepsilon(t)|^2}\frac{d}{dt}|\varepsilon(t)|^2\right)q^2 - \left(\frac{\mu(t_0)}{|\varepsilon(t_0)|^2}\frac{d}{dt_0}|\varepsilon(t_0)|^2\right)q'^2\right)\right],
\end{aligned} \tag{6.22}$$

where $|\varepsilon(t)|$ satisfies the Ermakov-Pinney equation, that is

$$\frac{d^2}{dt^2}|\epsilon(t)| + \frac{\dot{\mu}(t)}{\mu(t)}\frac{d}{dt}|\epsilon(t)| + \omega^2(t)|\epsilon(t)| = \frac{1}{\mu^2(t)|\epsilon(t)|^3}.$$

6.4. Comparison of Malkin-Manko-Trifonov results by those obtained in the previous approaches

In coordinate and momentum representation by using equations (6.6) and (6.7), the dynamical invariant of MMT-method becomes

$$\hat{I}_M(t) = \frac{1}{2\hbar}\Big(\varepsilon(t)\varepsilon^*(t)\hat{p}^2 + \mu^2\dot{\varepsilon}(t)\dot{\varepsilon}^*(t)\hat{q}^2 - \mu\dot{\varepsilon}(t)\varepsilon^*(t)\hat{p}\hat{q} - \mu\varepsilon(t)\dot{\varepsilon}^*(t)\hat{q}\hat{p}\Big) + \frac{1}{2}. \qquad (6.23)$$

If one takes $\varepsilon(t)$ to be solution of (6.8), satisfying the initial conditions

$$\varepsilon(t_0) = \frac{1}{\sqrt{m\omega_0}}, \qquad \dot{\varepsilon}(t_0) = \frac{i\sqrt{m\omega_0}}{\mu(t_0)}, \qquad (6.24)$$

then $\varepsilon(t)$ is the same as $\varepsilon_0(t)$ defined by (4.27) in WN-method. One can see that this invariant coincides with the one obtained by WN-method. Comparing also the raising and lowering operators in MMT and WN-methods which are equations (6.6),(6.7) and (4.37),(4.38) respectively, it can be seen that they are exactly the same.

Similarly, if $\varepsilon(t)$ satisfies the conditions (6.24) and $\sigma(t)$ satisfies the conditions (5.48), one can see that invariant $\hat{I}_M(t)$ coincides with the dynamical invariant $\hat{I}_{LR}(t)$ obtained by LR-method and given by (5.23). Also, in this case raising and lowering operators in LR and MMT-methods differ by a phase factor, that is

$$\begin{aligned}\mathcal{A}(t) &= e^{i\theta(t)}A(t),\\ \mathcal{A}^\dagger(t) &= e^{-i\theta(t)}A^\dagger(t).\end{aligned}$$

Clearly, wave functions $\psi_n^{(M)}(q,t)$ given by (6.19) will depend on the particular choice of the complex-valued solutions $\varepsilon(t)$ of the classical equation (6.8). That is, different choices of $\varepsilon(t)$ will give wave function solutions of SE in the form (6.19) corresponding to different initial states. In particular, if $\varepsilon(t)$ satisfies initial conditions (6.24), at time $t = t_0$ we have $\psi_0^{(M)}(q,t_0) = \Psi_0(q,t_0) = \varphi_0(q)$. In this case solutions $\psi_n^{(M)}(q,t)$ obtained by the MMT-method coincide with solutions $\Psi_n(q,t)$ obtained by the Wei-Norman method, which are given in terms of $\varepsilon_0(t)$ by equation (4.35). Also, the propagator $K_M(q,t;q',t_0)$ given by (6.22) is the same with the propagator $K(q,t;q',t_0)$ found by WN-approach in (4.36), as expected.

Similarly, we can say that under conditions (6.24), the solutions $\psi_n^{(M)}(q,t)$ will coincide with solutions $\psi_n^{(L)}(q,t)$ obtained by Lewis-Riesenfeld method, when $\sigma(t)$ satisfies the initial conditions (5.48). This is exactly, the case $\sigma_0(t) = |\varepsilon_0(t)|$, and it is easy to check that the propagator $K^{(M)}(q,t;q',t_0)$ given by (6.22) will be the same with the propagator $K^{(L)}(q,t;q',t_0)$ given by (5.54).

CHAPTER 7

CONCLUSION

In the present thesis, we considered time-dependent Schrödinger equation for a quantum oscillator described by a quadratic, time-dependent hermitian Hamiltonian. We studied three methods for solving this Schrödinger equation: the Wei-Norman algebraic method, the Lewis-Riesenfeld method and the Malkin-Manko-Trifonov method.

The Wei-Norman algebraic method is also known as an Evolution operator approach for solving the initial value problem for the Schrödinger equation. Since the Hamiltonian is a linear combination of generators of $su(1,1)$ Lie algebra, then the evolution operator can be written as product of generators of the $SU(1,1)$ Lie group. Using this idea, we found the evolution operator and showed that it is completely determined by two-linearly independent real-valued solutions of the corresponding classical equation of motion. Then, application of the evolution operator to given initial function gives us the wave function solution of the Schrödinger equation.

The Lewis-Riesenfeld method is based on finding quadratic dynamical invariant for the time-dependent Schrödinger equation. The quadratic invariant is found as a linear combination of the $su(1,1)$ Lie algebra generators, where the time-dependent coefficients are completely determined by a solution of the corresponding Ermakov-Pinney nonlinear differential equation. The eigenvalues and eigenstates of the self-adjoint quadratic invariant are found by the same algebraic approach used for diagonalization of the standard harmonic oscillator. Then, the eigenstates of the invariant multiplied by a proper phase factor give us a complete set of orthonormal solutions to the Schrödinger equation and determine the propagator for the quantum evolution problem.

Malkin-Manko-Trifonov method is based on finding dynamical symmetries, which by definition are operators that map solutions of the Schrödinger equation to other solutions. In this approach a symmetry operator, linear in position and momentum, is completely determined by a complex-valued solution of the corresponding classical equation of motion. Then, a successive application of the dynamical symmetry to a Gaussian type solution of the Schrödinger equation gives us a complete set of orthonormal solutions, which are used also to find the propagator of the quantum system.

In the present thesis, we showed that the wave function solutions and propagators of the quantum parametric oscillator obtained by the above described different approaches are same, when in LR-approach the solution of the Ermakov-Pinney equation and in MMT-approach the complex solution of the classical equation of motion are satisfying proper initial conditions. In what follows, we write the initial-value problems for the ordinary differential equations, whose solutions determine the same time-evolved wave functions of the quantum problem under the same initial conditions.

Relations Between Solutions of Classical Equations of Motion

Evolution Operator Method

$$\begin{cases} \ddot{x} + \frac{\dot{\mu}}{\mu}\dot{x} + \omega^2(t)x = 0, \\ x_1(t_0) = x_0 \neq 0, \dot{x}_1(t_0) = 0, \\ x_2(t_0) = 0, \dot{x}_2(t_0) = 1/\mu(t_0)x_1(t_0) \end{cases}$$

x_1,x_2−linear independent real solutions

Solutions

$$\begin{cases} x_1(t) = \sqrt{m\omega_0}x_0\sigma_0(t)\cos\theta_0(t), \\ x_2(t) = \frac{1}{\sqrt{m\omega_0}x_0}\sigma_0(t)\sin\theta_0(t), \end{cases}$$

where $\theta_0(t) = \int_{t_0}^{t} \frac{1}{\mu(s)\sigma_0^2(s)}ds$.

Lewis-Riesenfeld Method

$$\begin{cases} \ddot{\sigma}(t) + \frac{\dot{\mu}}{\mu}\dot{\sigma}(t) + \omega^2(t)\sigma(t) = \frac{1}{\mu^2\sigma^3(t)}, \\ \sigma(t_0) = \frac{1}{\sqrt{m\omega_0}}, \dot{\sigma}(t_0) = 0. \end{cases}$$

$$\sigma(t) = \frac{1}{\sqrt{m\omega_0}}\sqrt{\frac{x_1^2(t) + m^2\omega_0^2x_0^4x_2^2(t)}{x_0^2}} = \sigma_0(t)$$

$\sigma(t)$−real solution of Ermakov-Pinney equation

Manko-Dodonov Approach

$$\begin{cases} \ddot{\varepsilon}(t) + \frac{\dot{\mu}}{\mu}\dot{\varepsilon}(t) + \omega^2(t)\varepsilon(t) = 0, \\ \varepsilon(t_0) = \frac{1}{\sqrt{m\omega_0}}, \dot{\varepsilon}(t_0) = i\frac{\sqrt{m\omega_0}}{\mu(t_0)}. \end{cases}$$

$$\begin{aligned} \varepsilon(t) &= \frac{x_1(t)}{\sqrt{m\omega_0}x_0} + i\sqrt{m\omega_0}x_0x_2(t) = \varepsilon_0(t) \\ &= |\varepsilon(t)|\, e^{i\theta(t)} = \sigma(t)e^{i\theta(t)} \end{aligned}$$

$\varepsilon(t)$−complex solution

where $\theta(t) = \int_{t_0}^{t} \frac{1}{\mu(s)|\varepsilon(s)|^2}ds$.

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APPENDIX A

UNBOUNDED OPERATORS ON HILBERT SPACES

This appendix includes basic properties of unbounded operators which are used in the thesis.

Definition A.1 *Let X and Y be normed spaces and* $\hat{A} : D(\hat{A}) \to Y$ *is a linear operator where* $D(\hat{A}) \subset X$*. The operator* $\hat{A}$ *is called an unbounded operator if there exists a sequence* $\{x_n\} \in D(\hat{A})$ *with* $\|x_n\| = 1$ *for all* $n \in \mathbb{N}$ *which implies* $\|\hat{A}x_n\| \to \infty$ *(Debnath and Mikusiński, 2005).*

Definition A.2 *An operator defined in a normed space X is called densely defined if its domain is a dense subset of X, that is,* $\overline{D(\hat{A})} = X$*.*

Definition A.3 *The adjoint* $\hat{A}^\dagger$ *of densely defined operator* $\hat{A}$ *in a Hilbert space H is the operator defined on the set of all* $y \in H$ *for which* $\langle \hat{A}x|y\rangle$ *is a continuous functional on* $D(\hat{A})$ *such that*

$$\langle \hat{A}x|y\rangle = \langle x|\hat{A}^\dagger y\rangle, \quad for\ all\ x \in D(\hat{A})\ and\ y \in D(\hat{A}^\dagger).$$

Definition A.4 *A densely defined unbounded operator* $\hat{A}$ *in Hilbert space H is symmetric (Hermitian) if*

$$\langle \varphi|\hat{A}\Psi\rangle \quad = \quad \langle \hat{A}\varphi|\Psi\rangle,$$

for all $\varphi, \Psi \in D(\hat{A})$*.*

Definition A.5 *A densely defined operator* $\hat{A}$ *in Hilbert space H is self-adjoint if*

$$D(\hat{A}^\dagger) = D(\hat{A}),$$

and $\hat{A}^\dagger\varphi = \hat{A}\varphi$ *for all* $\varphi \in D(\hat{A})$*.*

APPENDIX B

COMMUTATION RELATIONS OF THE LIE GROUP GENERATORS

In this appendix, we calculate commutators (3.31). Acting on arbitrary function $f(q)$ we have

$$\begin{aligned}
[\hat{K}_-, \hat{K}_+]f(q) &= \frac{-i}{2}\frac{\partial^2}{\partial q^2}\frac{i}{2}q^2 f - \frac{i}{2}q^2(\frac{-i}{2})\frac{\partial^2}{\partial q^2}f \\
&= \frac{1}{4}\frac{\partial}{\partial q}(2qf + q^2 f') - \frac{1}{4}q^2 f'' \\
&= \frac{1}{4}(2f + 2qf' + 2qf' + q^2 f'') - \frac{1}{4}q^2 f'' \\
&= \frac{f}{2} + q\frac{\partial}{\partial q}f
\end{aligned} \tag{B.1}$$

Then this expression becomes,

$$[\hat{K}_-, \hat{K}_+]f = (\frac{1}{2} + q\frac{\partial}{\partial q})f$$

which implies

$$\begin{aligned}
[\hat{K}_-, \hat{K}_+] &= 2\frac{1}{2}(\frac{1}{2} + q\frac{\partial}{\partial q}) \\
&= 2\hat{K}_0
\end{aligned} \tag{B.2}$$

By the same way, other commutation relations can be calculated.

APPENDIX C

THE FOURIER TRANSFORM

Definition C.1 *The Fourier transform of an integrable function f, denoted by $\hat{f}$ or $\mathcal{F}\{f(x)\}$ is given by following integral*

$$\hat{f}(\xi) = \mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx \tag{C.1}$$

Theorem C.1 ***(Fourier Inversion Theorem)*** *For any integrable function f, the inverse fourier transform can be expressed as an integral*

$$f(x) = \mathcal{F}^{-1}\{f(\xi)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) e^{ix\xi} d\xi \tag{C.2}$$

Theorem C.2 *Let f, g be an integrable function and constants $\alpha, \beta \in \mathbb{C}$. Then*

(Linearity) $\mathcal{F}\{\alpha f(x) + \beta g(x)\} = \alpha\mathcal{F}\{f(x)\} + \beta\mathcal{F}\{g(x)\}$,

(Translation) $\mathcal{F}\{f(x-a)\} = \hat{f}(\xi)e^{-ia\xi}, a \in \mathbb{R}$,

(Modulation) $\mathcal{F}\{e^{i\alpha x} f(x)\} = \hat{f}(\xi - \alpha)$,

(Scaling) $\mathcal{F}\{f(\alpha x)\} = \frac{1}{|\alpha|}\hat{f}\left(\frac{\xi}{\alpha}\right), \alpha \neq 0$,

(Conjugation) $\mathcal{F}\{\overline{f(x)}\} = \overline{f(-\xi)}$.

Proposition C.1 *If $g(x) = e^{-x^2/2}H_n(x)$, then Fourier transform of $g(x)$ will be*

$$\mathcal{F}\{g(x)\} = (-i)^n \exp\left[-\frac{\xi^2}{2}\right] H_n(\xi) \tag{C.3}$$

where $H_n(\xi)$ is the n-th Hermite polynomial for all n=0,1,2,3,...

Proof Consider the exponential generating function of Hermite Polynomials,i.e.,

$$\exp[-t^2 + 2xt] = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!}. \tag{C.4}$$

Multiplying this equation with $\exp(-x^2/2)$, gives

$$\exp[-\frac{1}{2}x^2 + 2xt - t^2] = \sum_{n=0}^{\infty} \exp[-\frac{1}{2}x^2] \frac{H_n(x)t^n}{n!}. \tag{C.5}$$

Taking the Fourier transform of the left side of the above equation, we have

$$\begin{aligned}
\mathcal{F}\left\{ e^{-\frac{1}{2}x^2+2xt-t^2} \right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} e^{-\frac{1}{2}x^2+2xt-t^2} dx, \\
&= e^{t^2-2it\xi-\frac{\xi^2}{2}}, \\
&= \sum_{n=0}^{\infty} e^{-\frac{\xi^2}{2}} H_n(\xi) \frac{(-it)^n}{n!}. \qquad \text{(C.6)}
\end{aligned}$$

and the Fourier transform of the right side will be

$$\mathcal{F}\left\{ \sum_{n=0}^{\infty} \exp[-\frac{1}{2}x^2] \frac{H_n(x)t^n}{n!} \right\} = \sum_{n=0}^{\infty} \mathcal{F}\left\{ \exp[-\frac{1}{2}x^2] H_n(x) \right\} \frac{t^n}{n!}. \tag{C.7}$$

Equating both sides, equation(C.6) and equation(C.7) gives

$$\begin{aligned}
\sum_{n=0}^{\infty} e^{-\frac{\xi^2}{2}} H_n(\xi) \frac{(-it)^n}{n!} &= \sum_{n=0}^{\infty} \mathcal{F}\left\{ \exp[-\frac{1}{2}x^2] H_n(x) \right\} \frac{t^n}{n!}, \\
\Longrightarrow \qquad \mathcal{F}\left\{ \exp[-\frac{1}{2}x^2] H_n(x) \right\} &= (-i)^n \exp\Big[-\frac{\xi^2}{2} \Big] H_n(\xi).
\end{aligned}$$

This result completes the proof. □

APPENDIX D

EXPONENTIAL OPERATORS

This section provides the necessary calculations mostly needed to treat time dependent problems. Following equalities show how operators act on a given function.

Shift(Translation) Operator:

$$\exp\left[\lambda\frac{d}{dq}\right]f(q) = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\frac{d^n}{dq^n}f(q) = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}f^{(n)}(q) = f(q+\lambda). \tag{D.1}$$

where λ is a parameter (constant).

Dilatation Operator:

$$\exp\left[\lambda q\frac{d}{dq}\right]f(q) = f(e^{\lambda}q). \tag{D.2}$$

Proposition D.1 *For a given function $f_0(q)$ of a real variable q, we have*

$$\exp\left[-\frac{i\lambda}{2}\frac{\partial^2}{\partial q^2}\right]f_0(q) = f(q,\lambda),$$

where $f(q,z)$ satisfies the IVP for Schrödinger equation

$$\frac{1}{2}\frac{\partial^2}{\partial q^2}f(q;z) = i\frac{\partial}{\partial z}f(q;z), \tag{D.3}$$

$$f(q,z)|_{z=0} = f(q;0) \equiv f_0(q). \tag{D.4}$$

Proof : If $f(q,z)$ satisfies (D.3), then we have also

$$\exp\left[-\frac{i\lambda}{2}\frac{\partial^2}{\partial q^2}\right]f(q;z) = \exp\left[\lambda\frac{\partial}{\partial z}\right]f(q;z),$$

and it follows that

$$\begin{aligned}
\exp\left[-\frac{i\lambda}{2}\frac{\partial^2}{\partial q^2}\right]f_0(q) &= \exp\left[-\frac{i\lambda}{2}\frac{\partial^2}{\partial q^2}\right]f(q;0) = \exp\left[\lambda\frac{\partial}{\partial z}\right]f(q;z)|_{z=0},\\
&= f(q;z+\lambda)|_{z=0} = f(q,\lambda).
\end{aligned}$$

□

Now, we apply Proposition $C.1$ for three special choices of the function $f_0(q)$.

- Let $f_0(q) = \varphi_n(q) \equiv N_n e^{\frac{-m\omega_0}{2\hbar}q^2} H_n\Big(\sqrt{\frac{m\omega_0}{\hbar}}q\Big)$, where $N_n = (2^n n!)^{-1/2}(\frac{m\omega_0}{\hbar\pi})^{1/4}$ for $n = 0, 1, 2, 3, ...,$. Then we need to solve the following Schrödinger equation:

$$\begin{cases} \frac{1}{2}\frac{\partial^2}{\partial q^2} f_n(q;z) = i\frac{\partial}{\partial z} f_n(q;z),\\ f_n(q;0) = \varphi_n(q), \end{cases}$$

Taking fourier transform with respect to variable q of the above IVP we get

$$\begin{cases} \frac{\partial}{\partial z}\tilde{f}_n(\xi;z) = \frac{i}{2}\xi^2\tilde{f}_n(\xi;z),\\ \tilde{f}_n(\xi;0) = F\Big\{N_n e^{\frac{-m\omega_0}{2\hbar}q^2} H_n\Big(\sqrt{\frac{m\omega_0}{\hbar}}q\Big)\Big\} = \frac{(-i)^n}{\sqrt{\frac{m\omega_0}{\hbar}}}\exp\Big[\frac{-\hbar\xi^2}{2m\omega_0}\Big]H_n\Big(\frac{\sqrt{\hbar}\xi}{\sqrt{m\omega_0}}\Big)\Big). \end{cases}$$

First solving this system, then taking inverse Fourier transform we can find $f_n(q;z)$ as following:

$$\begin{aligned}
f_n(q;z) &= N_n \times \frac{1}{(1+(\frac{m\omega_0}{\hbar}z)^2)^{1/4}} \times \exp\left[-i\left(\frac{\frac{m\omega_0}{\hbar}z}{1+(\frac{m\omega_0}{\hbar}z)^2}\right)\frac{m\omega_0}{2\hbar}q^2\right]\\
&\quad \times \exp\left[i\left(n+\frac{1}{2}\right)\arctan(\frac{m\omega_0}{\hbar}z)\right] \times \exp\left[-\left(\frac{1}{1+(\frac{m\omega_0}{\hbar}z)^2}\right)\frac{m\omega_0}{2\hbar}q^2\right]\\
&\quad \times H_n\left(\left(\frac{1}{(1+(\frac{m\omega_0}{\hbar}z)^2)^{1/2}}\right)\sqrt{\frac{m\omega_0}{\hbar}}q\right), \qquad \text{(D.5)}
\end{aligned}$$

with z real and so that $f_n(q;0) = \varphi_n(q)$. As a result, we have

$$\exp\left[\frac{-i}{2}\lambda\frac{\partial^2}{\partial q^2}\right]\varphi_n(q) = f_n(q,\lambda).$$

- If $f(q;0) = \delta(q-q')$ then we need to solve IVP,

$$\begin{cases} \left[\frac{1}{2}\frac{\partial^2}{\partial q^2}\right] f(q;z) = i\left[\frac{\partial}{\partial z}\right] f(q;z), \\ f(q;0) = \delta(q-q'). \end{cases}$$

By Fourier transform we get

$$\begin{cases} \frac{\partial}{\partial z}\tilde{f}(\xi;z) = \frac{i}{2}\xi^2 \tilde{f}(\xi;z), \\ \tilde{f}(\xi;0) = F\{\delta(q-q')\} = \frac{1}{\sqrt{2\pi}}e^{-i\xi q'}. \end{cases}$$

Solving this system and taking inverse Fourier transform we have

$$f(q;z) \quad = \quad \frac{1}{\sqrt{2\pi}}\sqrt{\frac{i}{z}}\exp\left[\frac{-i}{2z}(q-\acute{q})^2\right]. \tag{D.6}$$

As a result, following equality is valid,

$$\exp\left[\frac{-i}{2}\lambda\frac{\partial^2}{\partial q^2}\right]\delta(q-q') = f(q,\lambda) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{i}{\lambda}}\exp\left[\frac{-i}{2\lambda}(q-q')^2\right].$$

- Let $f(q;0) = \exp\left[\frac{-i}{2}\mathrm{f}(t)e^{-2h(t)}q^2\right]\delta(e^{-h(t)}q-q')$ where $\mathrm{f}(t)$ and $h(t)$ are real-valued functions. Then we need to solve IVP,

$$\begin{cases} \left[\frac{1}{2}\frac{\partial^2}{\partial q^2}\right] f(q;z) = i\left[\frac{\partial}{\partial z}\right] f(q;z), \\ f(q;0) = \exp\left[\frac{-i}{2}\mathrm{f}(t)e^{-2h(t)}q^2\right]\delta(e^{-h(t)}q-q'). \end{cases}$$

By Fourier transform, above IVP will be

$$\begin{cases} \frac{\partial}{\partial z}\tilde{f}(\xi;z) = \frac{i}{2}\xi^2 \tilde{f}(\xi;z), \\ \tilde{f}(\xi;0) = F\left\{e^{\frac{-i}{2}\mathrm{f}(t)e^{-2h(t)}q^2}\delta(e^{-h(t)}q-q')\right\} = \frac{1}{\sqrt{2\pi}}\exp\left[-i\xi e^{h(t)q'}\right]\exp\left[\frac{-i}{2}\mathrm{f}(t)q'^2\right]. \end{cases}$$

Solving this system and taking inverse Fourier transform we get

$$f(q;z) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{i}{z}}\exp\left[\frac{-i}{2}\mathrm{f}(t)q'^2\right]\exp\left[\frac{-i}{2z}(q-e^{h(t)}\acute{q})^2\right]. \tag{D.7}$$

As a result, we have

$$\exp\left[\frac{-i}{2}\lambda\frac{\partial^2}{\partial q^2}\right]\exp\left[\frac{-i}{2}\mathrm{f}(t)e^{-2h(t)}q^2\right]\delta(e^{-h(t)}q-q') = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{i}{\lambda}}\exp\left[\frac{-i}{2}\mathrm{f}(t)q'^2\right] \times\exp\left[\frac{-i}{2\lambda}(q-e^{h(t)}\acute{q})^2\right].$$

APPENDIX E

HERMITE POLYNOMIALS

Proposition E.1 *For real or complex valued function $\zeta(t)$, one has operator identities*

$$(a) \qquad \zeta(t)q - \frac{\partial}{\partial q} = -\exp\left[\frac{\zeta(t)}{2}q^2\right]\frac{\partial}{\partial q}\exp\left[\frac{-\zeta(t)}{2}q^2\right], \tag{E.1}$$

$$(b) \qquad \left(\zeta(t)q - \frac{\partial}{\partial q}\right)^n = (-1)^n \exp\left[\frac{\zeta(t)}{2}q^2\right]\frac{\partial^n}{\partial q^n}\exp\left[\frac{-\zeta(t)}{2}q^2\right]. \tag{E.2}$$

Proof :

(a) We will directly apply RHS of the equation (E.1) to arbitrary function $f(q)$,

$$\begin{aligned} -\exp\left[\frac{\zeta(t)}{2}q^2\right]\frac{\partial}{\partial q}\left(\exp\left[\frac{-\zeta(t)}{2}q^2\right]f(q)\right) &= -\exp\left[\frac{\zeta(t)}{2}q^2\right]\Bigg(-\zeta(t)qf(q)\exp\left[\frac{-\zeta(t)}{2}q^2\right] \\ &+ \exp\left[\frac{\zeta(t)}{2}q^2\right]\exp\left[\frac{-\zeta(t)}{2}q^2\right]\frac{\partial f(q)}{\partial q}\Bigg), \\ &= \zeta(t)qf(q) - \frac{\partial f(q)}{\partial q} = \left(\zeta(t)q - \frac{\partial}{\partial q}\right)f(q) \end{aligned} \tag{E.3}$$

It implies the desired equality (E.1). In addition, it can be proved by using the Hausdorff identity (3.35).

(b) Applying $\zeta(t)q - \partial/\partial q$ n times and using (E.1), we get the equality (E.2). □

Proposition E.2 *For $\alpha > 0$, one has*

$$\frac{d^n}{dq^n}e^{-\alpha q^2} = (-1)^n(\sqrt{\alpha})^n H_n(\sqrt{\alpha}q)e^{-\alpha q^2}, \tag{E.4}$$

where $H_n(\sqrt{\alpha}q)$ are the Hermite Polynomials.

Proof :Mathematical induction will be used for the proof. Firstly for $n = 1$, we have

$$\frac{d}{dq}e^{-\alpha q^2} = -2\alpha q e^{-\alpha q^2} = (-1)\sqrt{\alpha}H_1(\sqrt{\alpha}q)e^{-\alpha q^2}(-1)^n(\sqrt{\alpha})^n H_n(\sqrt{\alpha}q)e^{-\alpha q^2} \tag{E.5}$$

which shows that equation (E.4) is valid. Now we will prove that equation (E.4) holds for $n = k + 1$, by assuming that it is true for $n = k$. Consider the following equations,

$$\begin{aligned}
\frac{d^{k+1}}{dq^{k+1}} e^{-\alpha q^2} &= \frac{d}{dq}\left((-1)^k(\sqrt{\alpha})^k H_k(\sqrt{\alpha} q) e^{-\alpha q^2}\right), \\
&= (-1)^k(\sqrt{\alpha})^k \frac{d}{dq}\left(H_k(\sqrt{\alpha} q) e^{-\alpha q^2}\right), \\
&= (-1)^k(\sqrt{\alpha})^k \left(\frac{d}{dq} H_k(\sqrt{\alpha} q) - 2\alpha q H_k(\sqrt{\alpha} q)\right) e^{-\alpha q^2}.
\end{aligned} \tag{E.6}$$

Now, using the recursion relation for Hermite polynomials, we have,

$$\frac{d^{k+1}}{dq^{k+1}} e^{-\alpha q^2} = (-1)^{n+1}(\sqrt{\alpha})^{k+1} H_{k+1}(\sqrt{\alpha q}) e^{-\alpha q^2}, \tag{E.7}$$

which shows that for $n = k + 1$, equation (E.4) is valid. Thus, we completed the proof. □