ON GENERALIZATION OF HOPFIAN MODULES

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ABSTRACT

ON GENERALIZATION OF HOPFIAN MODULES

The notion of Hopfian modules are defined as a generalization of modules of finite length as the modules whose surjective endomorphisms are isomorphisms. These modules and several generalizations of them are extensively studied in the literature. The aim of this thesis is to review some known results and extends some results about generalized Hopfian and weakly Hopfian modules. It is shown that a module is Hopfian if and only if it is both generalized Hopfian and weakly Hopfian. Torsion-free abelian groups are weakly Hopfian. Any nonsingular uniform module is weakly Hopfian. Direct summands of weakly Hopfian modules is weakly Hopfian. It is shown that direct sum weak Hopfian modules is not necessarily weakly Hopfian.

ÖZET

HOPFIAN MODULLERİN GENELLEMELERİ ÜZERİNE

Hopfian modülleri kavramı; örten endomorfizmaları izomorfizma olan sonlu uzunluktaki modüllerin genelleştirilmesi olarak tanımlanmaktadır. Bu modüller ve bunların çeşitli genellemeleri literatürde kapsamlı olarak incelenmiştir. Bu tezin amacı, bilinen bazı sonuçların gözden geçirilmesi ve genel Hopfian ve zayıf Hopfian modülleri hakkında bazı sonuçların ortaya konulmasıdır. Bir modülün Hopfian modül olmasının gerek ve yeter koşulunun o modülün hem genel Hopfian hemde zayıf Hopfian olduğu gösterilmiştir. Serbest Torsion değişmeli gruplar zayıf Hopfiandır. Herhangi düzensiz tekil olmayan modüller zayıf Hopfian modüllerdir. Zayıf Hopfian modüllerin direk toplamı zayıf Hopfian modüldür. Direk toplamı zayıf Hopfian modül olan modüllerin zayıf Hopfian modül olmak zorunda olmadığı gösterilmiştir.

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LIST OF ABBREVIATIONS

R	an associative ring with unit unless otherwise stated
$\mathbb{Z}$, $\mathbb{Z}^+$	the ring of integers, the set of all positive integers
$\mathbb{Q}$	the field of rational numbers
$Hom_R(M, N)$	all R-module homomorphisms from M to N
$\oplus_{i \in I} M_i$	direct sum of R- modules M_i
$\prod_{i \in I} M_i$	direct product of R- modules M_i
$Kerf$	the kernel of the map f
imf	the image of the map f
$SocM$	the socle of the R-module M
$RadM$	the radical of the R-module M
$E(M)$	the injective envelope (hull) of a module M
$T(M)$	the torsion submodule of a module M
$Z(M)$	the singular submodule of a module M
$\ll$	small (or superfluous) submodule
$\trianglelefteq$	essential submodule
$Annl_R(X)$	$= \{r \in R \mid rX = 0\}$ = the *left* annihilator of a subset X of a *left* R-module M
$Annr_R(X)$	$= \{r \in R \mid Xr = 0\}$ = the *right* annihilator of a subset X of a *right* R-module M
$\cong$	isomorphic
$\leq$	submodule

CHAPTER 1

INTRODUCTION

Let R be a ring with identity. It is known that any surjective endomorphism of a module of finite length is an isomorphism. Hopfian right modules are the modules whose surjective endomorphisms are isomorphism. Hopfian modules are studied in (A. Ghorbani and A. Haghany, 2002). A right module is called generalized Hopfian if every surjective endomorphism is small i.e. have small kernel. Weakly Hopfian modules are defined as the right modules whose small surjective endomorphisms are isomorphism (Youngduo Wang, 2005). From the definitions it is clear that a right module is Hopfian if and only if it is generalized Hopfian and weakly Hopfian. A generalized Hopfian module need not be weakly Hopfian and vice versa. Pseudo-projective modules are weakly Hopfian. All torsion-free abelian groups are weakly Hopfian. Direct sum of weakly Hopfian (even quasi-projective) modules need not be weakly Hopfian.

We give examples in order to exhibit the relations between the classes of generalized Hopfian, weakly Hopfian and dual automorphism-invariant modules of (Surjeet Singh and Ashish K.Srivastava, 2012).

CHAPTER 2

PRELIMINARIES

In this chapter we give the basic definitions and results that are used in the sequel.

2.1. Definitions

Definition 2.1 *A submodule $N \leq M$ is called maximal if $N \neq M$ and it is not properly contained in any proper submodule of M.*

In a finitely generated R-module, every proper submodule is contained in a maximal submodule.

Definition 2.2 *A submodule K of an R-module M is called essential or large in M if for every nonzero submodule $L \leq M$, we have $K \cap L \neq 0$.*

Then M is called an essential extension of K and we write $K \trianglelefteq M$. A monomorphism $f : L \to M$ is said to be essential if Imf is an essential submodule of M.

Hence a submodule $K \leq M$ is essential if and only if the inclusion map $K \to M$ is an essential monomorphism. For example, in $\mathbb{Z}$ every non-zero submodule (=ideal) is essential.

Definition 2.3 *A submodule K of an R- module M is called superfluous or small in M, written $K \ll M$, if for every submodule $L \leq M$,the equality $K + L = M$ implies $L = M$.*

An epimorphism $f : M \to N$ is called superfluous if $Kerf \ll M$.

Obviously $K \ll M$ if and only if the canonical projection $M \to M/K$ is a superfluous epimorphism.

It is easy to see that e.g. in $\mathbb{Z}$ there are no non-zero superfluous submodules.

Definition 2.4 *A module M is called simple if it is non-zero and does not properly contain any non-zero submodule.*

Definition 2.5 *An R-module N is (finitely) generated by M or (finitely) M-generated if there exists an epimorphism $M^{(\mathcal{I})} \to N$ for some (finite) index set $\mathcal{I}$.*

Definition 2.6 *An R-module N is (finitely) co-generated by M or (finitely) M-co-generated if there exists a monomorphism $N \to M^{(\mathcal{I})}$ for some (finite) index set $\mathcal{I}$.*

Definition 2.7 *Let M be an R-module. As socle of M (= $Soc(M), SocM$) we denote the sum of all simple (minimal) submodules of M. If there is no minimal submodules in M we put $Soc(M) = 0$.*

$Soc(M)$ is a semi-simple submodule of M. Clearly, M is semi-simple if and only if $M = Soc(M)$. An important multiple characterization of the socle is given in the following proposition.

Proposition 2.1 *If M is a left R-module, then*

$$\begin{aligned} Soc\,(M) &= \Sigma\{K \leq M \mid K \text{ is minimal in } M\} \\ &= \bigcap\{L \leq M \mid L \text{ is essential in } M\}. \end{aligned}$$

Properties of the Socle ((Wisbauer, 1991), 21.2)
Let M be an R-module.

(1) For any morphism $f : M \to N$, we have $f(Soc(M)) \subset Soc(N)$.

(2) For any submodule $K \leq M$, we have $Soc(K) = K \cap Soc(M)$.

(3) $Soc(M) \trianglelefteq M$ if and only $Soc(K) \neq 0$ for every non-zero submodule $K \leq M$.

(4) $Soc(M)$ is an $\mathrm{End}_R(M)$-submodule, i.e. $Soc(M)$ is fully invariant in M.

(5) $Soc(\bigoplus_\Lambda M_\lambda) = \bigoplus_\Lambda Soc(M_\lambda)$.

Definition 2.8 *Dual to the socle we define as radical of an R-module M (= $Rad(M), RadM$) the intersection of all maximal submodules of M. If M has no maximal submodule we set $Rad(M) = M$.*

The characterization of the radical is given in the following proposition.

Proposition 2.2 *Let M be a left R-module. Then*

$$\begin{aligned} Rad\,(M) &= \bigcap\{K \leq M \mid K \text{ is maximal in } M\} \\ &= \Sigma\{L \leq M \mid L \text{ is superfluous in } M\}. \end{aligned}$$

Properties of the radical ((Wisbauer, 1991), 21.6)
Let M be an R-module.

(1) For a morphism $f : M \to N$ we have
(i) $f(RadM) \subset RadN$,
(ii) $Rad(M/RadM) = 0$, and
(iii) $f(RadM) = Rad(f(M))$, if $Ker(f) \subset RadM$.

(2) $RadM$ is an $\mathrm{End}_R(M)$-submodule of M (fully invariant).

(3) If every proper submodule of M is contained in a maximal submodule, then $RadM \ll M$ (e.g. if M is finitely generated).

(4) M is finitely generated if and only if $RadM \ll M$ and $M/RadM$ is finitely generated.

(5) If $M = \bigoplus_\Lambda M_\lambda$, then $RadM = \bigoplus_\Lambda RadM_\lambda$ and $M/RadM \simeq \bigoplus_\Lambda M_\lambda/RadM_\lambda$.

(6) If M is finitely cogenerated and $RadM = 0$, then M is semisimple and finitely generated.

(7) If $\overline{M} = M/RadM$ is semisimple and $RadM \ll M$, then every proper submodule of M is contained in a maximal submodule.

Definition 2.9 *The radical of ${}_RR$ is called the Jacabson radical of R, i.e.*

$$Jac(R) = Rad({}_RR)$$

As a fully invariant submodule of the ring, $Jac(R)$ is two-sided ideal in R.

Definition 2.10 *An element $x \in R$ is left quasi-regular in case $1 - x$ has a left inverse in R. Similarly $x \in R$ is right quasi-regular (quasi-regular) in case $1 - x$ has a right (two-sided) inverse in R.*

Proposition 2.3 ***Characterization of the Jacobson radical***
In a ring R with unit, $Jac(R)$ can be described as the

(a) *intersection of the maximal left ideals in R (= definition);*

(b) *sum of all superfluous left ideals in R;*

(c) *sum of all left quasi-regular left ideals;*

(d) *largest (left) quasi-regular ideal;*

(e) $\{r \in R \mid 1 - ar$ *is invertible for any* $a \in R\}$*;*

(f) *intersection of the annihilators of the simple left R-modules;*

(a*) *intersection of the maximal right ideals.*

*Replacing 'left' by 'right' further characterizations (**b***) - (**f***) are possible.*

2.2. Small Rings and Small Modules

Definition 2.11 *A right R-module M is called a small module if it is a small submodule in its injective hull* $E(M)$*, i.e* $M \ll E(M)$*.*

The following characterization of small module is well-known

Proposition 2.4 *((Wisbauer, 1991), Proposition 2.2)*
For a right R-module M, the followings are equivalent:

(i) *M is small.*

(ii) $M \ll E(M)$*.*

(iii) $M \ll E$ *for some injective right R-module E.*

(iv) $M \ll L$ *for some right R-module L containing M.*

Proposition 2.5 *If M is small then* M/N *is small for every* $N \leq M$*.*

Proof Suppose M is small i.e. $M \ll E(M)$. Let $N \leq M$, then $M/N \leq E(M)/N$. Let $L/N \leq E(M)/N$ such that $M/N + L/N = E(M)/N$, then $M + L = E(M)$. Since $M \ll E(M)$, $L = E(M)$. Hence $L/N = E(M)/N$ and M/L is small by Proposition 2.4. □

Definition 2.12 *A ring R is called left small if* ${}_RR$ *is a small module; e.g.* $\mathbb{Z}$ *is a small ring as it is small in* ${}_{\mathbb{Z}}\mathbb{Q}$*.*

Proposition 2.6 *((Ramamurthi, 1982), 3.3), ((Pareigis, 1966), 4.8)*
Let R be a ring and let $E(R)$ *be the injective hull of* ${}_RR$*. Then the following conditions are equivalent:*

(i) *R is a left small ring.*

(ii) *$Rad(M) = M$ for every injective left R-module M.*

(iii) *$Rad(E(R)) = E(R)$.*

2.3. Exact Sequences

Definition 2.13 *A finite or infinite sequence of R-maps and left R-modules*

$$\cdots \longrightarrow M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \longrightarrow \cdots$$

is called an exact sequence if $Im(f_{n+1}) = ker(f_n)$ for all n.

Proposition 2.7 *((Anderson and Fuller, 1992), Proposition 2.18)*

(i) *A sequence $0 \longrightarrow A \xrightarrow{f} B$ is exact if and only if f is injective.*

(ii) *A sequence $B \xrightarrow{g} C \longrightarrow 0$ is exact if and only if g is surjective.*

(iii) *A sequence $0 \longrightarrow A \xrightarrow{h} B \longrightarrow 0$ is exact if and only if h is an isomorphism.*

2.4. Noetherian and Artinian Rings

Definition 2.14 *A set $\mathcal{P}$ of submodules of M satisfies the ascending chain condition in case for every chain*

$$L_1 \leq L_2 \leq \ldots \leq L_n \leq \ldots$$

in $\mathcal{P}$, there is a positive number n with $L_{n+i} = L_n$ $i = 1, 2, \ldots$

Definition 2.15 *A set $\mathcal{P}$ of submodules of M satisfies the descending chain condition in case for every chain*

$$L_1 \geq L_2 \geq \ldots \geq L_n \geq \ldots$$

in $\mathcal{P}$, there is a positive number n with $L_{n+i} = L_n$ $i = 1, 2, \ldots$

Definition 2.16 *A module M is noetherian in case the lattice $\mathcal{P}(M)$ of all submodules of M satisfies the ascending chain condition. It is artinian in case $\mathcal{P}(M)$ satisfies the descending chain condition.*

Proposition 2.8 *((Anderson and Fuller, 1992) , Proposition 10.9)*
For a module M the following statements are equivalent:

(a) *M is noetherian;*

(b) *Every submodule of M is finitely generated;*

(c) *Every non-empty set of submodules of M has a maximal element.*

Proposition 2.9 *((Anderson and Fuller, 1992) , Proposition 10.10)*
For a module M the following statements are equivalent:

(a) *M is artinian;*

(b) *Every factor module of M is finitely co-generated;*

(c) *Every non-empty set of submodules of M has a minimal element.*

Corollary 2.1 *((Anderson and Fuller, 1992) , Corollary 10.11)*
Let M be a non-zero module:

(1) *If M is artinian , then M has simple submodule : in fact , $Soc(M)$ is an essential submodule;*

(2) *If M is noetherian , then M has maximal submodule : in fact , $Rad(M)$ is a superfluous submodule.*

Proposition 2.10 *((Anderson and Fuller, 1992), Proposition 10.12)*
Let

$$0 \to K \to M \to N \to 0$$

be an exact sequence of left R-modules. Then M is artinian (noetherian) if and only if both K and N are artinian (noetherian).

2.5. Composition Series

Definition 2.17 *Let M be a non-zero module. A finite chain of* $n+1$ *submodules of M*

$$M = M_0 > M_1 > M_2 > \ldots > M_n = 0$$

is called a composition series of length n for M provided that M_{i-1}/M_i *is simple* ($i = 1, 2, \ldots$)*; i.e. , provided each term in the chain is maximal in its predecessor.*

Proposition 2.11 *((Anderson and Fuller, 1992), Proposition 11.1)*
A non-zero module M has a composition series if and only if M is both artinian and noetherian.

Theorem 2.1 ***The Jordan-Hølder Theorem****((Anderson and Fuller, 1992),Theorem 11.3)*
If a module M has a composition series, then every pair of composition series for M are equivalent.

Proof If M has a composition series, then denote by $c(M)$ the minimum length of such a series for M. We shall induct on $c(M)$. Clearly, if $c(M) = 1$, there is no challenge. So assume that $c(M) = n > 1$ and that any module with a composition series of smaller length ha all of its composition series equivalent. Let

$$M = M_0 > M_1 > M_2 > \ldots > M_n = 0 \tag{2.1}$$

be composition series of minimal length for M and let

$$M = N_0 > N_1 > N_2 > \ldots > N_p = 0 \tag{2.2}$$

be second composition series for M. If $M_1 = N_1$, then by the induction hypothesis, since $c(M) \leq n-1$, the two series are equivalent. Suppose that $M_1 \neq N_1$. Since M_1 is maximal submodule of M , $M_1 + N_1 = M$. So

$$M/M_1 = (M_1 + N_1)/M_1 \cong N_1/(M_1 \cap N_1) \tag{2.3}$$

and

$$M/N_1 = (M_1 + N_1)/N_1 \cong M_1/(M_1 \cap N_1) \tag{2.4}$$

Thus $M_1 \cap N_1$ is maximal in both M_1 and N_1. Since M has a composition series, $M_1 \cap N_1$ also has composition series

$$M_1 \cap N_1 = L_0 > L_1 > L_2 > \ldots > L_k = 0$$

So

$$M_1 = L_0 > L_1 > L_2 > \ldots > L_k = 0$$

and

$$N_1 = L_0 > L_1 > L_2 > \ldots > L_k = 0$$

are composition series for M_1 and N_1, respectively. Since $c(M) < n$, every two composition series for M_1 are equivalent, so two series

$$M = M_0 > M_1 > M_2 > \ldots > M_n = 0$$

and

$$M = M_0 > M_1 > L_0 > \ldots > L_k = 0$$

are equivalent. In particular, $n = k+2$ and so $k = n-2 < n$. So $c(N_1) = k+1 < n$. Thus by our hypothesis, every two composition series for N_1 are equivalent. Thus the two series

$$M = N_0 > N_1 > N_2 > \ldots > N_p = 0$$

and

$$M = N_0 > N_1 > L_0 > \ldots > L_k = 0$$

are equivalent. But as we noted in 2.3 and 2.4

$$M/M_1 \cong N_1/L_0$$

and

$$M/N_1 \cong M_1/L_0$$

thus the series 2.1 and 2.2 are equivalent. $\square$

Definition 2.18 *Let M be non-zero module the composition length of M, $c(M)$, is defined unambiguously by*

$$c(M) = \begin{cases} 0, & \text{If } M = 0 \\ n, & \text{If } M \text{ has a composition series of length } n \end{cases}$$

If a module M is not of finite length, we say it is of infinite length and write

$$c(M) = \infty$$

Corollary 2.2 *((Anderson and Fuller, 1992), Corollary 11.4)*
Let K, M, and N be modules and suppose there is an exact sequence

$$0 \to K \to M \to N \to 0$$

of homomorphisms. Then

$$c(M) = c(N) + c(K)$$

Corollary 2.3 ***The Dimension Theorem*** *((Anderson and Fuller, 1992), Corollary 11.5)*
Let M be a module of finite length and let K and N be submodules of M. Then

$$c(K + N) + c(K \cap N) = c(K) + c(N)$$

Proof Clearly by the Second Isomorphism Theorem.

$$(K + N)/N \cong K/(K \cap N). \tag{2.5}$$

Then apply Corollary 2.2 to the two exact sequences

$$0 \to N \to K + N \to (K + N)/N \to 0$$

and

$$0 \to K \cap N \to K \to K/(K \cap N) \to 0$$

to get

$$c(K + N) - c(N) = c(K) - c(K \cap N)$$

□

Lemma 2.1 *((Anderson and Fuller, 1992), Lemma 11.6)*
Let M be a module and let f be an endomorphism of M.

(1) *If M is an artinian, then $Imf^n + ker(f)^n = M$ for some n, whence f is an automorphism if and only if it is monic;*

(2) *If M is a noetherian, then $Imf^n \cap ker(f)^n = 0$ for some n, whence f is an automorphism if and only if it is epic.*

Proposition 2.12 (Fitting's Lemma) *((Anderson and Fuller, 1992), Proposition 11.7)*
If M is a module of finite length n and if f is an endomorphism of M, then

$$M = Imf^n \oplus ker(f)^n.$$

Proof From Proposition 2.11, M is both artinian and noetherian, so by Lemma 2.1, there is an m with $M = Imf^n \oplus ker(f)^n$. But since M has length n, both $Imf^n = Imf^m$ and $ker(f)^n = ker(f)^m$. □

2.6. Singular Submodule

Given any right module M, the *singular submodule of M* is the set

$$Z(M) = \{m \in M : mI = 0 \ for\ some\ essential\ right\ ideal\ I\ of\ R\}.$$

Equivalently, $Z(M)$ is the set of those $m \in M$ for which the right ideal $ann_R(m) = \{r \in R : mr = 0\}$ is essential in R. An R-module M is called singular if $Z(M) = M$, and it is called a nonsingular module if $Z(M) = 0$. A ring R is called a right nonsingular ring if R is nonsingular as a right R-module. $Z_r(R)$ will be used for $Z(R_R)$. Similarly, we say that R is *left nonsingular ring* if $Z_l(R) = 0$.

Proposition 2.13 *(Goodearl, 1976)*
The following hold for any ring R.

(1) A module N is nonsingular if and only if $Hom(M, N) = 0$ for all singular modules M.

(2) If R is a right semi-hereditary ring, then $Z_r(R) = 0$.

(3) If $Z_r(R) = 0$, then $Z(M/Z(M)) = 0$ for all right R-modules M.

(4) If $N \leq M$, then $Z(N) = N \cap Z(M)$.

(5) Suppose that $Z_r(R) = 0$. A right module M is singular if and only if $Hom(M, N) = 0$ for all nonsingular right modules N.

Let M be an R-module and $N \leq M$. If N is an essential submodule of M, then M/N is singular. Converse is not true in general. For example, let $M = \mathbb{Z}/2\mathbb{Z}$ and $N = 0$. M/N is singular but N is not an essential submodule of M. The following Proposition shows that when the converse true.

Proposition 2.14 *((Goodearl, 1976), Proposition 1.21)*
Let M be a nonsingular module and $N \leq M$. Then M/N is singular if and only if N is an essential submodule of M.

The class of all singular right modules is closed under submodules, factor modules and direct sums. On the other hand, the class of all nonsingular right modules is closed under submodules, direct products, essential extensions, and module extensions.

Proposition 2.15 *((Goodearl, 1976), Proposition 1.24)*
If M is any simple right R-module, then M is either singular or projective, but not both.

2.7. Uniform Modules and Local Modules

Definition 2.19 *A submodule K of a non-zero module M is said to be large or essential if $K \cap L \neq 0$ for every non-zero submodule $L \leq M$. If all non-zero submodules of M are large in M, then M is called uniform.*

Proposition 2.16 *((John Clark, Narayanaswani Vanaja, Cristian Lomp, Robert Wisbauer, 2006), 1.6)*
For M the following are equivalent.

(a) *M is uniform;*

(b) *Every non-zero submodule of M is indecomposable;*

(c) *For any module morphisms $f : K \to M$, $g : M \to N$, where $f \neq 0$, gf injective implies that f and g are injective.*

Proof

$(a) \Leftrightarrow (b)$ For any non-zero submodules $N, K \leqslant M$ with $N \cap K = 0$, the submodule $N + K = N \oplus K$ is decomposable.

$(a) \Rightarrow (c)$ Let gf be injective. Then clearly f is injective. Suppose g is not injective. Then $Imf \cap ker(g) \neq 0$ and this implies $ker(gf) \neq 0$, a contradiction.

$(c) \Rightarrow (a)$ Let $N, K \leqslant M$ with $N \neq 0$ and $N \cap K = 0$. Then the composition of the canonical maps $N \to M \to M/K$ is injective and now (c) implies that $M \to M/K$ is injective, that is $K = 0$.

□

Definition 2.20 *If a module M has a largest submodule, i.e. a proper submodule which contains all other proper submodules, then M is called a local module.*

Such a submodule has to be equal to the radical of M and in this case $Rad(M) \ll M$.

M is local if and only if it is cyclic and every non-zero factor submodule of M is indecomposable.

A cyclic and self-projective module M is local if and only if $End(M)$ is a local ring((Wisbauer, 1991), 41.4).

2.8. Closed Co-closed Submodules

Definition 2.21 *Given a submodule K of M, a submodule $L \leqslant M$ is called a complement of K in M if it is maximal in the set of all submodules $L^{'} \leqslant M$ with $K \cap L^{'} = 0$.*

By the Zorn's lemma, every submodule has a complement in M. A submodule $L \leqslant M$ is called a complement submodule provided it is the complement of some submodule of M. If L is a complement of K in M, then there is a complement H of L in M that contains K. By construction, $K \trianglelefteq H$ and H has no proper essential extension in M; thus H is called an essential closure of K in M.

Definition 2.22 *Let M be any module. A module K of M is closed in M if K has no proper essential extension in M, i.e. whenever L is a submodule of M such that K is essential in L then $K = L$.*

Proposition 2.17 *((John Clark, Narayanaswani Vanaja, Cristian Lomp, Robert Wisbauer, 2006), 1.10)*
For a submodule K of M the following are equivalent.

(a) *K is a closed submodule;*

(b) *K is a complement submodule ;*

(c) *$K = H \cap M$, where H is a maximal essential extension of K in M.*

Proposition 2.18 *((John Clark, Narayanaswani Vanaja, Cristian Lomp, Robert Wisbauer, 2006), 1.11)*
Let the submodule K of M be a complement of $L \leqslant M$. Then,

(1) $(K + L)/K \trianglelefteq M/K$ *and* $K + L \trianglelefteq M$.

(2) *If* $U \trianglelefteq M$ *and* $K \leq U$, *then* $U/K \trianglelefteq M/K$.

Definition 2.23 *A submodule L of M is called co-closed in M if L has no proper submodule K for which $K \leq L$ is co-small in M.*

Thus L is co-closed in M if and only if for any proper submodule K of L, there is a submodule N of M such that $L + N = M$ but $K + N \neq M$.

Proposition 2.19 *((John Clark, Narayanaswani Vanaja, Cristian Lomp, Robert Wisbauer, 2006), 3.7)*
Let $K \leqslant L \leqslant M$ be submodules.

(1) *If L is co-closed in M, then L/K is co-closed in M/K.*

(2) *If $K \ll L$ and L/K is co-closed in M/K, then L is co-closed in M.*

(3) *If $L \leqslant M$ is co-closed, then $K \ll M$ implies $K \ll L$; hence $Rad(L) = L \cap Rad(M)$.*

(4) *If L is hallow, then either $L \leqslant M$ is co-closed in M or $L \ll M$.*

(5) *If $f : M \to N$ is a small epimorphism and L is co-closed in M, then $(L)f$ is co-closed in N.*

(6) *If K is co-closed in M, then K is co-closed in L and the converse is true if L is co-closed in M.*

Definition 2.24 *A commutative ring R is called distributive if*

$$A \cap (B + C) = A \cap B + A \cap C$$

for all ideals A, B, C of R.

In general closed submodules need not be co-closed and a co-closed submodules need not be closed.

Lemma 2.2 *((Helmut Zöschinger), Lemma)*
For a commutative noetherian ring R, the followings are equivalent:

(i) *In every R-module M every closed submodule is co-closed.*

(ii) *In every R-module M every co-closed submodule is closed.*

(iii) *R is distributive.*

2.9. V-Rings

In this section we introduce a class of rings called right V-rings. A ring R is a right V-ring if every simple right R-module is injective.

Theorem 2.2 *((Lam, 1999), Theorem 3.75)*
For any ring R, the following are equivalent:

(1) *R is a right V-ring;*

(2) *any right ideal $A \subsetneq R$ is an intersection of maximal right ideals;*

(3) *for any right R-module M, $Rad(M) = 0$.*

Proof

(3) $\Rightarrow$ (2) Follows by applying (3) to the module $M = R/A$.

(2) $\Rightarrow$ (1) We shall show that any simple right R-module S is injective by applying Baer's Test to S. Thus, consider any homomorphism $f : B \to S$, where $B \subseteq R$ is any right ideal. In order to extend f to R, we may assume that $f \neq 0$. Fix an element $x \notin A := ker(f)$. By (2), there exists a maximal right ideal $m \supseteq A$ not containing x. Since $B/A \cong S$ is simple, we have $m \cap B = A$, and clearly, $B + m = R$. We can then extend f to $g : R \to S$ by defining $g(b + m) = f(b)$ for any $b \in B$ and any $m \in M$.

(1) $\Rightarrow$ (3) We are supposed to show here that any $x \in M$ is excluded by some maximal submodule. The cyclic module xR certainly has a maximal submodule, so there exists a surjection $h : xR \to S$ for some simple module S. Since S is assumed to be injective, h extends to some homomorphism $h' : M \to S$. Now $ker(h')$ is a maximal submodule of M excluding x.

□

CHAPTER 3

INJECTIVE, PROJECTIVE AND QUASI-PROJECTIVE MODULES

In this chapter we give the definitions and main properties and characterization of projective, injective and quasi-projective modules.

3.1. Injective Modules

Definition 3.1 *A left R-module E is injective if, whenever i is an injection, a dashed arrow exists making the following diagram commute.*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[dl, dashed, "g"] \\ & E & \end{tikzcd}$$

Proposition 3.1 *((Rotman, 2009), Proposition 3.28)*

(i) *If $(E_k)_{k\in K}$ is a family of injective left R-modules, then $\prod_{k\in K} E_k$ is also an injective left R-module.*

(ii) *Every direct summand of an injective left R-module E is injective.*

Proof

(i) Consider the diagram in which $E = \prod E_k$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[dl, dashed, "g"] \\ & E & \end{tikzcd}$$

Let $p_k : E \to E_k$ be the kth projection, so that $p_k f : A \to E_k$. Since E_k is an injective module, there is $g_k : B \to E_k$ with $g_k i = p_k f$. Now define $g : B \to E$ by

$g : b \mapsto (g_k(b))$. The map g does extend f, for if $b = i(a)$, then

$$g(i(a)) = (g_k(i(a))) = (p_k f(a)) = f(a),$$

because $x = (p_k(x))$ for every x in the product.

(ii) Assume that $E = E_1 \oplus E_2$, let $i : E_1 \to E$ be the inclusion, and let $p : E \to E_1$ be the projection (so that $pi = 1_{E_1}$).

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "j"] \arrow[d, "f"] & C \arrow[d, dashed, "g_0"] \\
& E_1 \arrow[r, shift left, "i"] & E_1 \oplus E_2 \arrow[l, shift left, "p"]
\end{tikzcd}$$

Then, the proof can be completed easily using the diagram .

□

Corollary 3.1 *Any finite direct sum of injective left R-modules is injective.*

Proof The direct sum of finitely many modules coincides with the direct product. □

Theorem 3.1 ***(Baer Criterion)*** *((Rotman, 2009), Theorem 3.30)*
A left R-module E is injective if and only if every map $f : I \to E$, where I is an ideal in R, can be extended to R.

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "f"] & R \arrow[dl, dashed, "g"] \\
& E &
\end{tikzcd}$$

Proof Since any left ideal I is a submodule of R, the existence of an extension g of f is just special case of the definition of injectivity of E.

Suppose we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] & B \\
& E &
\end{tikzcd}$$

where A is a submodule of a left R-module B. For notational convenience, let us assume i is the inclusion [this assumption amounts to permitting us to write a instead of $i(a)$ whenever $a \in A$]. We are going to use Zorn's lemma. Let $\mathcal{X}$ be the set of all ordered pairs (A', g'), where $A \subseteq A' \subseteq B$ and $g' : A' \to E$ extends f; that is $g'|_A = f$. Note that $\mathcal{X} \neq \emptyset$, because $(A, f) \in \mathcal{X}$. Partially order $\mathcal{X}$ by defining

$$(A^{'}, g^{'}) \leq (A^{''}, g^{''})$$

to mean $A^{'} \subseteq A^{''}$ and $g^{''}$ extends $g^{'}$. We may think that chains in X have upper bounds in $\mathcal{X}$; hence, Zorn's lemma applies, and there exists a maximal element (A_0, g_0) in $\mathcal{X}$. If $A_0 = B$, we are done, and so we may assume that there is some $b \in B$ with $b \notin (A_0)$.
Define

$$I = \{r \in R : rb \in A_0\}.$$

It easy to see that I is a left ideal in R. Define $h : I \to E$ by

$$h(r) = g_0(rb).$$

By hypothesis, there is a map $h^* : R \longrightarrow E$ extending h. Finally, define $A_1 = A_0 + < b >$ and $g_1 : A_1 \longrightarrow E$ by

$$g_1(a_0 + rb) = g_0(a_0) + rh^*(1),$$

where $a_0 \in A_0$ and $r \in R$.
Let us show that g_1 is well defined. If $a_0 + rb = a_0' + r'b$, then $(r - r')b = a_0' - a_0 \in A_0$; it follows that $r - r' \in I$. Therefore, $g_0((r - r')b)$ and $h(r - r')$ are defined, and we have

$$g_0(a_0' - a_0) = g_0((r - r')b) = h(r - r') = h^*(r - r') = (r - r')h^*(1).$$

Thus $g_0(a_0') - g_0(a_0) = rh^*(1) - r'h^*(1)$ and $g_0(a_0') + r'h^*(1) = g_0(a_0) + rh^*(1)$, as desired. Clearly $g_1 a_0 = g_0 a_0$ for all $a_0 \in A_0$, so that the map g_1 extends g_0. We conclude that $(A_0, g_0) \preceq (A_1, g_1)$, contradicting the maximality of (A_0, g_0). Therefore $A_0 = B$, the map g_0 is lifting of f, and E is injective. $\square$

Definition 3.2 *Let M be an R-module over a domain R. If $r \in R$ and $m \in M$, then we say that m is divisible by r if there is some $m' \in M$ with $m = rm'$. We say that M is a divisible module if each case $m \in M$ is divisible by every nonzero $r \in R$.*

If R is a domain, $r \in R$ and M is an R-module, then the function $\varphi_r : M \to M$, defined by $\varphi_r : m \mapsto rm$, is an R-map. It is clear that M is divisible module if and only if φ_r is surjective for every $r \neq 0$.

Lemma 3.1 *Let G be an abelian group. Then the followings are equivalent:*

(i) *G is an injective.*

(ii) *G is a divisible.*

(iii) *$Rad(G) = G$.*

Lemma 3.2 *((Rotman, 2009), Lemma 3.33)*
If R is a domain, then every injective R-module E is a divisible module.

Corollary 3.2 *Let R be a principal ideal domain.*

(i) *An R-module E is injective if and only if it is divisible.*

(ii) *Every quotient of an injective R-module E is itself injective.*

Corollary 3.3 *((Rotman, 2009), Corollary 3.36)*
Every abelian group M can be imbedded as a subgroup of some injective abelian group.

Theorem 3.2 *((Rotman, 2009), Theorem 3.38)*
For every ring R, every left R-module M can be imbedded as a submodule of an injective left R-module.

Theorem 3.3 *((Rotman, 2009), Theorem 3.39)*
If R is a ring for which every direct sum of injective left R-modules is an injective module, then R is left Noetherian.

Definition 3.3 *Let M and E be left R-modules. Then E is an essential extension of M if there is an injective R-map $\alpha : M \to E$ with $S \cap \alpha(M) \neq \{0\}$ for every nonzero submodule $S \subseteq E$. If also $\alpha(M) \subsetneq E$ is called a proper essential extension of M.*

Lemma 3.3 *((Rotman, 2009), 3.44)*
Given a left R-module M, the following conditions are equivalent for a module $E \supseteq M$.

(i) *E is a maximal essential extension of M; that is, no proper extension of E is an essential extension of M.*

(ii) *E is an injective module and E is an essential extension of M.*

(iii) *E is an injective module and there is no proper injective intermediate submodule E'; that is, there is no injective E' with $M \subseteq E' \subsetneq E$.*

Definition 3.4 *If M is a left R-module, then a left R-module E containing M is an injective envelope of M, denoted by $E(M)$ if any of equivalent conditions in Lemma 3.3 hold.*

3.2. Projective Modules

Definition 3.5 *A left R-module P is projective if, whenever p is surjective and h is any map, there exists a lifting g; that is, there exists a map g making the following diagram commute:*

$$\begin{tikzcd} & B \arrow[dl, dashed, "g"'] \arrow[d, "h"] & \\ C \arrow[r, "p"'] & A'' \arrow[r] & 0 \end{tikzcd}$$

Proposition 3.2 *((Anderson and Fuller, 1992) , Proposition 17.2)*
The following statements about a left R-module P are equivalent;

(a) *P is projective;*

(b) *Every epimorphism $M \to P \to 0$ splits;*

(c) *P is isomorphic to a direct-summand of a free left R-module.*

3.3. Quasi-Projective Modules

In this section we outline some properties of quasi-projective modules. Most of the results of this section can be found in ((L.Fuchs and K.M.Rangaswamy , 1970)).

Definition 3.6 *A module M over a ring R is called quasi-projective if for every submodule N of M and for every R-module homomorphism $\varphi : M \to M/N$, then there is an R-endomorphism ψ of M making the diagram*

$$\begin{tikzcd} & M \arrow[dl, dotted, "\psi"'] \arrow[d, "\varphi"] \\ M \arrow[r, "\eta"] & \frac{M}{N} \end{tikzcd}$$

commute where η denotes the natural map.

Lemma 3.4 *((L.Fuchs and K.M.Rangaswamy , 1970), Lemma 1)*
Every direct summand of a quasi-projective module is quasi-projective.

Lemma 3.5 *((L.Fuchs and K.M.Rangaswamy , 1970), Lemma 2)*
If M is quasi-projective and N is fully invariant submodule of M, then M/N is likewise quasi-projective.

Lemma 3.6 *((L.Fuchs and K.M.Rangaswamy , 1970), Lemma 3)*
If $M_i (i \in I)$ are quasi-projective R-module such that, for very submodule N of the direct sum $M = \bigoplus_{i \in I} M_i$, $N_i = \bigoplus (N \cap M_i)$ hold, then M is again quasi-projective.

Proof Hypothesis implies that every quotient module M/N of M is of the form $\bigoplus(M_i/N_i)$ with $N_i \subseteq M_i$. Every homomorphism $M_i \to M_j/N_j$ with $i \neq j$ must be trivial, because otherwise there exist submodules N'_i and N'_j such that $M_i/N'_i \cong N'_j/N_j$ are non-zero submodules and so there is a sub-direct sum of M_i and $N!_j$ which is not their direct sum. Thus every $\psi : \bigoplus(M_i) \to \bigoplus(M_i/N_i)$ acts coordinate-wise whence the quasi-projectivity of M is obvious. □

Lemma 3.7 *((L.Fuchs and K.M.Rangaswamy , 1970), Lemma 4)*
If N is submodule of quasi-projective module M such that M/N is isomorphic to a direct summand of M, then N itself is a summand of M.

Proof Let A be a summand of M with $\pi : M \to A$, $\rho : A \to M$ as projection and injection maps and let $\alpha : A \to M/N$ be an isomorphism. For the natural map $\eta : M \to M/N$, there exists a $\phi : M \to M$ rendering

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[d, "\phi"] & A \arrow[d, "\alpha"] \\ M \arrow[r, "\eta"] & M/N \end{tikzcd}$$

commutative, i.e. $\eta\phi = \alpha\pi$. Define $M/N \to M$ as $\phi\rho\alpha^{-1}$; then $\eta\phi\rho\alpha^{-1} = \alpha\pi\rho\alpha^{-1}$ is the identity map of M/N. Hence the sequence $0 \to N \to M \to M/N \to 0$ splits. □

Lemma 3.8 *((L.Fuchs and K.M.Rangaswamy , 1970), Lemma 5)*
Let N be a submodule of the quasi-projective module M such that there exists an epimorphism $\varepsilon : N \to M$, then M is isomorphic to a direct summand of N.

Proof Let $K = Ker\varepsilon$ and let $\psi : N/K \to M$ be the isomorphism induced by ε , clearly $\varphi : M \to M/K$ is a monomorphism, then $\psi\varphi : N/K \to M/K$ is an identity on N/K. Since M is quasi-projective module, there exists $\beta : M \to M$ such that $\eta\beta = \varphi$ where $\beta(M) \subset \eta^{-1}(N/K) = N$ and $\eta : M \to M/K$ is natural map. For $\beta\psi : N/K \to N$, $\eta\beta\psi = \varphi\psi$ acts identically on N/K. Hence $0 \to K \to N \to N/K \to 0$ is splitting, implies $N = K \oplus N/K$. Since $N/K \cong M$, M is isomorphic to a direct summand of N. □

Lemma 3.9 *((L.Fuchs and K.M.Rangaswamy , 1970) , Lemma 6)*
If N is a submodule in a quasi-projective module M, then cardinality of $E(M/N)$ does not exceed that of $E(M)$.

The structure of quasi-projective abelian groups is given in the following theorem.

Theorem 3.4 *((L.Fuchs and K.M.Rangaswamy , 1970), Theorem)*
An abelian group A is quasi-projective if, and only if it is;

(1) *free or*

(2) *a torsion group such that every p-component A_p is a direct sum of cyclic groups of the same order p^n where p is prime.*

Proof Clearly free groups F are quasi-projective, then p^nF is fully invariant submodule of F so from Lemma 3.5, the groups F/p^nF are quasi-projective. From Lemma 3.6, a direct sum of groups F/p^nF with different primes p is quasi-projective. Hence the sufficiency holds because F/p^nF is a direct sum of cyclic groups of order p^n.

Conversely, suppose A is quasi-projective. If A is torsion, then from Lemma 3.4, every A_p is quasi-projective. If A_p is not reduced, then it includes a summand of $\mathbb{Z}_{p^\infty}$. From Lemma 3.4 and Lemma 3.7, every proper subgroup of $\mathbb{Z}_{p^\infty}$ have to be summand of $\mathbb{Z}_{p^\infty}$ which is impossible, thus A_p is reduced. It cannot have a summand of the form $\mathbb{Z}_{p^a} \oplus \mathbb{Z}_{p^b}$ with $a < b$, because this cannot be quasi-projective as a result of the existence of an epimorphism $\mathbb{Z}_{p^a} \to \mathbb{Z}_{p^b}$ whose kernel is not a summand. Hence the basic subgroups C_p of A_p are direct sums of cyclic groups of the same orders p^n, and C_p is now a direct sum of A_p but A_p is reduced ,so $C_p = A_p$.

If A is torsion-free, we divide two parts whether A has finite or infinite rank. If A has finite rank, then let F be a free subgroup of rank k in A. Now $E(A)$ is countable, thus from Lemma 3.9 $E(A/F)$ is nearly countable. Since A/F is torsion, this holds only if A/F is finite in which case A is free. If A have infinite rank, then let F be a free subgroup of A of the same rank as A. So there exists an epimorphism $F \to A$. From Lemma 3.8, we conclude that A is isomorphic to summand of F and therefore A is free.

Finally, we indicate that A can not be mixed. If T is torsion part of A, then from Lemma 3.5 A/T is quasi-projective and so free by what has been proved,i.e. $A = T \oplus F$ where quasi-projective T and free F. If neither $T = 0$ nor $F = 0$, then there is a cyclic direct summand $\mathbb{Z}_{p^n}$ of T and an epimorphism $\psi : F \to \mathbb{Z}_{p^n}$ whose kernel is not a direct sum of F, contradict with Lemma 3.7 $\square$

CHAPTER 4

DUAL AUTOMORPHISM-INVARIANT MODULES AND DEDEKIND FINITE MODULES

In this chapter we outline some properties of dual automorphism-invariant modules and Dedekind finite modules. We give the main properties and characterization of dual automorphism-invariant modules and dedekind finite modules.

4.1. Dual Automorphism-Invariant Modules

Definition 4.1 *A right R-module M is called a dual-automorphism-invariant module if whenever K_1 and K_2 are small submodules of M , then any epimorphism $\eta : M/K_1 \to M/K_2$ with small kernel lifts to an endomorphism φ of M.*

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d] & M \arrow[d] \\ M/K_1 \arrow[r, "\eta"] & M/K_2 \end{tikzcd}$$

Lemma 4.1 *((Surjeet Singh and Ashish K.Srivastava, 2012), Lemma 1)*
Let M be a dual-automorphism-invariant module. If $\varphi : M \to M$ is an epimorphism with small kernel, then φ is an automorphism.

Proof Suppose $\varphi : M \to M$ is an epimorphism with small kernel. Write $K = Ker(\varphi)$. Then $\psi : M/K \to M$ is an isomorphism. Consider $\psi^{-1} : M \to M/K$. Since M is a dual automorphism-invariant module, by definition, ψ^{-1} lifts to an endomorphism $\beta : M \to M$. We have $\beta(M) + K = M$. Since $K \ll M$, we get $\beta(M) = M$. This implies β is an epimorphism. Then for every $m \in M$, $\psi^{-1}(m) = \beta(m) + K$. Now $m = \psi\psi^{-1}(m) = \psi(\beta(m) + K) = \varphi\beta(m)$. This shows that $\varphi\beta = 1_M$. Hence $\varphi^{-1} = \beta$. □

Proposition 4.1 *((Surjeet Singh and Ashish K.Srivastava, 2012), Proposition 3)*
Let R be a right V-ring. Then every right R-module is a dual automorphism-invariant.

Proof Suppose R is V-ring. Let M be a right R-module. We wish to show that M has no small submodule. Let $0 \neq x \in M$, set $\Omega = \{N \leqslant M : x \notin N\}$. Then take a chain $N_i \in \Omega$ such that $n \notin N_i$. Let $\bigcup N_i = K \leqslant M$. Then $K \in \Omega$. Clearly K is upper-bound for N_i. By the **Zorn's Lemma** Ω has a maximal submodule say N. Let $N \lneqq L < M$, then $x \in L \Rightarrow xR \in L$.

$$(xR + N)/N \leqslant L/N \leqslant M/N.$$

Then $(xR + N)/N \leqslant M/N$. Therefore,

$$\Omega = \{N \leqslant M : x \notin N\} = (xR + N)/N.$$

Clearly $(xR + N)/N$ is a simple. Since R is V-ring, $(xR + N)/N$ is injective. Then

$$0 \to (xR + N)/N \to M/N \to (M/N)/((xR + N)/N) \to 0$$

splits. So $(xR + N)/N$ is direct summand of M/N,

$$(xR + N)/N \oplus (M/N)/((xR + N)/N) = M/N.$$

Thus $M = xR + N$. This implies that M has no small submodule and consequently M is dual automorphism-invariant module. □

Lemma 4.2 *((Surjeet Singh and Ashish K.Srivastava, 2012), Lemma 4)*
Let M_1, M_2 be right R-modules. If $M = M_1 \oplus M_2$ is dual automorphism-invariant, then any homomorphism $f : M_1 \to M_2/K_2$ with K_2 small in M_2 and $ker f$ small in M_1 lifts to a homomorphism $g : M_1 \to M_2$.

Theorem 4.1 *((Surjeet Singh and Ashish K.Srivastava, 2012), Theorem 5)*
A ring R is a right V-ring if and only if every finitely generated right R-module is dual automorphism-invariant.

Proof Suppose every finitely generated right R-module is dual automorphism-invariant. We wish to show that R is a right V-ring. Assume to the contrary that R is not a V-ring. Then there exists a simple right R-module S such that S is not injective. Let $E(S)$ be the injective hull of S. Then $E(S) \neq S$. Choose any $X \in E(S)\ S$. Then S is small in xR and xR is uniform. Let $A = ann_r(x)$. As S is a submodule of $xR \cong R/A$, we may take $S = B/A$ for some $A \subset B \subset R_R$. Consider $M = \frac{R}{A} \times \frac{R}{B}$. As M is finitely generated, by hypothesis M is dual automorphism-invariant. We have the identity homomorphism $1_{R/B} : R/B \to R/A \cong \frac{R/A}{R/B}$ where $ker(1_{R/B}) = 0$ is small in R/B and B/A small in R/A. By Lemma 4.2, the identity map on R/B can be lifted to a homomorphism $\eta : \frac{R}{B} \to \frac{R}{A}$. Thus $Im(\eta)$ is summand of R/A which is a contradiction to the fact that $R/A(\cong xR)$ is uniform. Hence R is a right V-ring.

The converse is obvious from Proposition 4.1. □

Definition 4.2 *A module M is called a pseudo-projective module if for every submodule N of M, any homomorphism $\varphi : M \to M/N$ can be lifted to a homomorphism $\psi : M \to M$, that is the diagram below commutes.*

$$\begin{tikzcd} & M \arrow[dl, dotted, "\psi"'] \arrow[d, "\varphi"] \\ M \arrow[r, "\eta"] & \frac{M}{N} \end{tikzcd}$$

Proposition 4.2 *((Surjeet Singh and Ashish K.Srivastava, 2012), Proposition 7)*
Any pseudo-projective module is a dual automorphism-invariant.

Proof Suppose M is a pseudo-projective module. Let L_1, L_2 be two small submodules of M and $\phi : M/L_1 \to M/L_2$ be an epimorphism. Let $\pi_1 : M \to M/L_1$ be a natural mapping. As M is pseudo-projective, $\phi\pi_1$ lifts to an endomorphism η of M. Let $\pi_2 : M \to M/L_2$ be a natural mapping. Then $\pi_2\eta = \phi\pi_1$. Therefore $\pi_2\eta(L_1) = \phi\pi_1(L_1) = 0$ gives $\eta(L_1) \subseteq L_2$. Hence η is lifting of ϕ. So M is a dual automorphism-invariant module. □

Proposition 4.3 *((Surjeet Singh and Ashish K.Srivastava, 2012), Proposition 11)*
Any direct summand of a dual automorphism-invariant module is dual automorphism-invariant.

Proof Let M be a dual automorphism-invariant right R-module and let $M = A \oplus B$. Let K_1, K_2 be two small submodules of A and $\sigma : A/K_1 \to A/K_2$ be an epimorphism with $ker(\sigma) \ll A/K_1$. Clearly, K_1, K_2 are small in M and $\sigma' = \sigma \oplus 1_B : M/K_1 \to M/K_2$ is an epimorphism with $ker(\sigma') \ll M/K_1$. Since M is a dual automorphism-invariant, σ' lifts to an endomorphism η of M. For the inclusion map $i_1 : A \to M$ and the projection $\pi_1 : M \to A$, the map $\pi_1 \eta i_1 : A \to A$ lifts σ. Hence A is dual automorphism-invariant. This shows that any direct summand of a dual automorphism-invariant module is a dual automorphism-invariant module. □

Lemma 4.3 *((Surjeet Singh and Ashish K.Srivastava, 2012), Lemma 18)*
Let G be a torsion abelian group such that G is dual automorphism-invariant. Then G is reduced.

Proof Assume the contrary that G is not reduced. Then we have $G \cong \bigoplus_{m_p} \mathbb{Z}_{p^\infty}$ for some prime number p. Consider $H = \mathbb{Z}_{p^\infty}$. Its every proper subgroup is small. Let $A \subsetneq B$ be two proper subgroups of H. There exists an isomorphism $\sigma : H/A \to H/B$. Since every summand of dual automorphism-invariant module is dual automorphism-invariant, H is dual automorphism-invariant. Therefore σ lifts to an endomorphism η of H. Then $\sigma(A) = B$. This gives a contradiction as order of A is less than order of B. Hence G is reduced. □

Theorem 4.2 *((Surjeet Singh and Ashish K.Srivastava, 2012), Theorem 20)*
Let G be a torsion abelian group. Then the followings are equivalent:

(i) *G is dual-automorphism-invariant;*

(ii) *G is quasi-projective.*

Proof

$(i) \Rightarrow (ii)$ Since any abelian group is a direct sum of a divisible group and a reduced group, in view of Lemma 4.3, it follows that G is reduced. Let p be prime number. Consider the p-component G_p of G. Suppose $G_p \neq 0$. As G_p is reduced, $G_p = A_1 \oplus L$ where A_1 is a non-zero cyclic p-group. Now $o(A_1) = p^n$ for some $n > 0$. If $L = 0$, we get that G_p is quasi-projective. Suppose $L \neq 0$. Then $L = A_2 \oplus L_1$ where A_2 is a non-zero cyclic p-group. By Proposition 4.3, $A_1 \oplus A_2$ is dual automorphism-invariant. As every subgroup of A_1 or A_2 is small, it follows that A_1 is A_2-projective and A_2 is A_1-projective. Hence $A_1 \oplus A_2$ is quasi-projective. This gives $A_1 \cong A_2$. Hence we get G_p is a direct sum of copies of A_1. So G_p is quasi-projective. This proves that G itself is quasi-projective.

$(ii) \Rightarrow (i)$ This follows from Theorem 4.2.

□

Theorem 4.3 *((Surjeet Singh and Ashish K.Srivastava, 2012), Theorem 22)*
Let G be a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$. Then the following conditions are equivalent:

(i) *G is dual-automorphism-invariant;*

(ii) *The number of primes p for which $G_p = \{x \in G : p^n x \in \mathbb{Z}\} = \mathbb{Z}$ is not finite;*

(iii) $J(G) = 0$.

4.2. Dedekind Finite Modules

Definition 4.3 *A ring R is called dedekind finite ring if $ba = 1$ whenever $ab = 1$. Equivalently, R is dedekind finite ring if whenever a is left or right invertible, then a is invertible.*

Clearly $ab = 1$ implies that ba is non-zero idempotent, so R is dedekind finite if and only if R is not isomorphic to any proper left or right ideal direct summand.

We shall denote the class of dedekind finite rings by DF.

DF is closed under direct product, finite direct sums and sub-rings, but not under homomorphic images.

Proposition 4.4 *((Simion Breaz, Grigore Călugărenu and Philip Schulyz, 2011), Proposition 2.1)*
If R has no right or left zero divisor, then $R \in DF$.

Proof Suppose R has no right or left zero divisor. Let a be not right zero divisor and $ab \in R$. Then $a(ab - ba) = 0$, so $ab = ba = 1$. For left zero divisor proof is similar. □

Definition 4.4 *Let R be any unital ring and M be a unital R-module. M is called DF-module if its ring of endomorphism, then $End(M)$ is a dedekind finite.*

Consequently, M is DF-module if and only if M is not isomorphic to any proper direct summand of itself.
Some properties of DF-modules are given in the following proposition.

Proposition 4.5 *((Simion Breaz, Grigore Călugărenu and Philip Schulyz, 2011), Proposition 3.2)*
Let M be an R-module.

(1) *There is a monomorphism $f \in End(M)$ with Imf a proper direct summand if and only if there is an epimorphism $g \in End(M)$ with $ker(g)$ a proper direct summand.*

(2) *If M is a DF-module, then so is any direct summand of M.*

(3) *If M is the direct sum of infinitely many copies of the same non-zero module, then M is not DF-module.*

(4) *Let $f \in End(M)$ be a monomorphism of M onto a direct summand, and let N be fully invariant submodule which is a DF-module. Then $f(N) = N$.*

(5) *Let N be fully invariant submodule of M. If M and M/N are DF-modules, then M is DF-module.*

(6) *M is the direct sum of fully invariant submodules N_i, then M is DF-module if and only if every N_i is.*

(7) *If M is the direct product of (infinitely many) fully invariant submodules N_i, then M is DF-module if and only if every N_i is.*

Theorem 4.4 *((Simion Breaz, Grigore Câlugârenu and Philip Schulyz, 2011), Theorem 3.4)*
Let M be R-module and N be a fully invariant submodule of M.

(1) *If N is essential DF-module, then M is DF.*

(2) *If N is superfluous and M/N is DF, then M is DF.*

Proof

(1) Suppose N is essential DF-module. Let f, g be endomorphisms of M such that $fg = 1$. Then $f \ \ : f|_N$ and $g \ \ : g|_N$ are restriction of endomorphisms of N such that $f \ g \ = 1_N$. Since N is large and $0 = kerg \ = ker(g) \cap N$, $kerg = 0$, so M is DF.

(2) Suppose N is small and M/N is DF. Let f^- and g^- be the induced endomorphism of M/N. Since M/N is DF, $f^-g^- = 1_{M/N}$. Then we conclude that f^- is an epimorphism, so $f(M) + N = M$. As N is small, hence $f(M) = M$. This implies that M is DF.

□

Corollary 4.1 *((Simion Breaz, Grigore Câlugârenu and Philip Schulyz, 2011), Corollary 3,5)*
Let M be an R-module.

(1) *If $Soc(M)$ is essential and DF, then M is DF.*

(2) *If $Rad(M)$ is superfluous and $M/Rad(M)$ is DF, then M is DF.*

CHAPTER 5

GENERALIZED HOPFIAN MODULES

Hopfian modules, Generalized Hopfian modules introduced in (A. Ghorbani and A. Haghany, 2002). In this section, we outline some properties of Hopfian and generalized Hopfian modules. We give the relation between Hopfian and generalized Hopfian modules.

Definition 5.1 *Let R be a ring and M be a right R-module. If every surjective R-endomorphism of M is an isomorphism, then M is called Hopfian.*

Definition 5.2 *Let R be a ring and M be a right R-module. If every surjective R-endomorphism of M has a small kernel in M, then M is called generalized Hopfian (gH for short).*

Every Hopfian modules is gH. But the converse is not true in general.

Example 5.1 *Consider the prüfer group $\mathbb{Z}_{p^\infty}$ for some prime p. Then each epimorphism of $\mathbb{Z}_{p^\infty}$ is small, so $\mathbb{Z}_{p^\infty}$ is gH. On the other hand, the map $f : \mathbb{Z}_{p^\infty} \to \mathbb{Z}_{p^\infty}$ given by $f(a) = pa$ is an epimorphism with $ker(f) = \{a \in \mathbb{Z}_{p^\infty} : pa = 0\}$. Since $ker(f) \ll \mathbb{Z}_{p^\infty}$, f is an epimorphism which is not a isomorphism. So $\mathbb{Z}_{p^\infty}$ is not Hopfian.*

Theorem 5.1 *((A. Ghorbani and A. Haghany, 2002), Theorem 1.1)*
The following are equivalent conditions on right R-module M.

(1) *M is gH.*

(2) *For any surjective endomorphism f of M, if $N \ll M$, then $f^{-1}(N) \ll M$.*

(3) *If $N \ll M$ and there is an R-epimorphism $M/N \to M$, then $N \ll M$.*

(4) *If N is a proper submodule of M and if f is a surjective endomorphism of M, then $f(N) \neq M$.*

(5) *M is Dedekind finite and the kernel of any surjective endomorphism of M is either small or direct summand.*

(6) *There exist a fully invariant submodule N of M such that M/N is gH.*

(7) *For any right R-module X, if there is an epimorphism $M \to M \oplus X$, then $X = 0$.*

Proof We wish prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1) $\Leftrightarrow$ (6) and (1) $\Leftrightarrow$ (7).

(1) $\Rightarrow$ (2) Suppose M is gH. Let $f : M \to M$ be an surjective endomorphism and $N \ll M$. Assume L is submodule of M containing $ker(f)$ with $f^{-1}(N)/ker(f) + L/ker(f) = M/ker(f)$. Then $f^{-1}(N) + L = M$, hence $N + f(L) = M$ so $f(L) = M$. This implies that $f(L) = f(M)$ which also implies $L = M$ because $ker(f) \subseteq L$. Thus $f^{-1}(N)/ker(f)$ is small submodule of $M/ker(f)$. Since $ker(f) \ll M$, $f^{-1}(N) \ll M$.

(2) $\Rightarrow$ (3) Suppose (2) is hold. Assume that $f : M/N \to M$ is an epimorphism. Let $\pi : M \to M/N$ be the canonical epimorphism. Then $f\pi : M \to M$ is a surjective endomorphism of M. From (2), $ker(f\pi) = (f\pi)^{-1}(0)$ is superfluous submodule of M. Since $N \ll ker(f\pi)$, $N \ll M$.

(3) $\Rightarrow$ (4) Suppose (3) is hold. Assume that N is a proper submodule of M and $f : M \to M$ is an epimorphism with $f(N) = M$. Then $M = ker(f) + N$; indeed $f^{-} : M/ker(f) \to M$ is an epimorphism. From (3), $ker(f) \ll M$. Hence $M = N$ which contradicting the assumption.

(4) $\Rightarrow$ (5) Suppose (4) is hold. Firstly we want to show that $End(M)$ is a directly finite ring, thus M will be a Dedekind finite module. Assume $\eta\beta = 1$ where η and β are in $End(M)$. Clearly β is an injective η is a surjective. Since $\eta(Im\beta) = M$, from (4) $Im\beta = M$, and thus β is invertible. It implies that $\beta\eta = 1$. Then assume $f : M \to M$ is an epimorphism. Let $ker(f) + N = M$ where N is any submodule of M. Then $f(N) = M$ and hence from (4), $N = M$. We proved that $ker(f) \ll M$.

(5) $\Rightarrow$ (1) Suppose (5) is hold. Assume that M is not gH and $f : M \to M$ is an epimorphism whose kernel is not superfluous submodule of M. From (5) there is $L \leq M$ with $L \oplus ker(f) = M$. Hence we get $M \cong M/ker(f) \cong L$, and therefore $M \oplus kerf \cong M$. Since M is dedekind finite the last isomorphism shows that $ker(f) = 0$ which contradicting the hypothesis. Hence $ker(f) \ll M$.

(1) $\Rightarrow$ (6) Trivial, take $N = 0$.

(6) $\Rightarrow$ (1) Assume that N is fully invariant submodule such that $N \ll M$ and M/N is gH. Let f be an epimorphism of endomorphism of M. Then $g : M/N \to M/N$ defined by $g(m)+N = f(m)+N$ is a well defined epimorphism with $ker(g) \ll M/N$. Assume that $kerg = L/N$ for some submodule L of M. Then $L/N \ll M/N$ and so $N \ll M$, thus $L \ll M$ Since $ker(f) \leqslant L$, we get $ker(f) \ll M$.

(1) $\Rightarrow$ (7) Assume that M is gH and $f : M \to M \oplus X$ is an epimorphism. Let $\pi_1 : M \to M$ denotes projection onto the first component and $\tau : X \to X \oplus M$ be the natural injection. From (1), we get $ker(\pi_1 f) = f^{-1}(\tau(X)) \ll M$. Then

$$\tau(X) = f[f^{-1}(\tau(X))] = f(ker(\pi_1 f)) \ll f(M) = M \oplus X$$

so $X = 0$.

(7) $\Rightarrow$ (1) Suppose (7) is hold. Assume that $f : M \to M$ is an epimorphism and $ker(f) + L = M$ for some submodule M. Then

$$\begin{aligned} M/L \cap ker(f) &= ker(f)/L \cap ker(f) \oplus L/L \cap ker(f) \cong M/L \oplus M/ker(f) \\ &\cong M/L \oplus M \end{aligned}$$

Hence an epimorphism $M \to M/L \oplus M$ is defined by $m \to (m + L, f(m))$, so by hypothesis $M/L = 0$. This means $L = M$.

□

Corollary 5.1 *((A. Ghorbani and A. Haghany, 2002), Corollary 1.2)*
Let M be a gH module and f be a surjective endomorphism of M.

(i) *If $N \leqslant M$, we have $N \ll M$ iff $f(N) \ll M$.*

(ii) *$Jac(M) = \sum f(N) = \sum f^{-1}$ where N runs through the set of all small submodules of M.*

Corollary 5.2 *((A. Ghorbani and A. Haghany, 2002), Corollary 1.3)*
A direct summand of a gH module is a gH.

Corollary 5.3 *((A. Ghorbani and A. Haghany, 2002), Corollary 1.4)*
For a right R-module M consider the following statements.

(i) *M is Hopfian.*

(ii) *M is gH.*

(iii) *M is Dedekind finite.*

Proof The proof of $(i) \Rightarrow (ii) \Rightarrow (iii)$ comes from Theorem 5.1. Now to complete the proof we need to show $(iii) \to (i)$. Suppose that M is Dedekind finite and $f : M \to M$ is an epimorphism. Then there is an endomorphism g of M such that $fg = 1$. Since $End(M)$ is a directly finite ring, we get $gf = 1$. Hence f is an isomorphism, and M is a Hopfian. □

Proposition 5.1 *((A. Ghorbani and A. Haghany, 2002), Proposition 1.8)*
If M is a quasi-projective module, then M is gH iff so is M/N for any small submodule N of M.

Proof The "If" part is trivial, just take $N = 0$. Now we will show that the other part. Suppose that M is gH and $N \ll M$. Let $f : M/N \to M/N$ be an epimorphism and $\pi : M \to M/N$ be the canonical epimorphism. Then we get $f\pi : M \to M/N$. Since M is quasi-projective, there exists an endomorphism g of M such that $\pi g = f\pi$. From this inequality we get g is epic, as π is small. From Corollary 5.3, M is Hopfian, thus g is an isomorphism. We get

$$f(x + N) = f\pi(x) = \pi g(x) = g(x) + N$$

so $g(N) \leq N$, and $kerf = L/N$ where $L = \{x \in M : g(z) \in N\} = g^{-1}(N)$. Since g^{-1} is a homomorphism and $N \ll M$, $g^{-1}(N) \ll M$, in the end $kerf = g^{-1}(N)/N \ll M/N$. Hence M/N is gH. □

Proposition 5.2 *((A. Ghorbani and A. Haghany, 2002), Proposition 1.11)*
Suppose that M/N is gH whenever N is non-zero submodule of M. Then M is gH.

Proposition 5.3 *((A. Ghorbani and A. Haghany, 2002), Proposition 1.15)*
A non-zero module that satisfies d.c.c on small submodules is gH.

Proposition 5.4 *((A. Ghorbani and A. Haghany, 2002), Proposition 1.16)*
Let N be a fully invariant submodule of M such that M/N is Hopfian. If N is Hopfian (respectively gH) then so is M.

Proof Let N be a fully invariant submodule of M with M/N is Hopfian. Suppose N is Hopfian. Then let $f : M \to M$ be an epimorphism and $g : M/N \to M/N$ be the induced map which is surjective. Since N is Hopfian, g is an isomorphism, thus $N = f^{-1}(N)$. Therefore $f\,|_N : N \to N$ is an epimorphism. Thus $kerf \cap N = 0$. Since $kerf \leq N$, M is gH. □

Proposition 5.5 *((A. Ghorbani and A. Haghany, 2002), Proposition 1.17)*
Let P be a property of modules preserves under isomorphism. If a module M has the property P satisfies a.c.c on non-zero (respectively non-small) submodules N such that M/N has the property P, then M is Hopfian (respectively gH).

Proof Assume that M is not Hopfian. Then there is a submodule K_1 with $K_1 \neq 0$ and $M/K_1 \cong M$. Thus M/K_1 is not Hopfian but satisfies P. Thus there is a submodule $K_2 \subseteq K_1$ with $K_2 \neq K_1$ and $M/K_2 \cong M/K_1$. Hence we have $0 \subset K_1 \subset K_2$ with $M/K_i \cong M$ for $i = 1, 2$. Repeating the process until a chain of submodules of structure which contradicts our assumption. Thats why M is Hopfian. □

Corollary 5.4 *((A. Ghorbani and A. Haghany, 2002), Corollary 1.18)*
If a module M satisfies a.c.c on small submodules, then it is generalized Hopfian.

Corollary 5.5 *((A. Ghorbani and A. Haghany, 2002), Corollary 1.21)*
If M has a a.c.c on its non-zero submodules N such that M/N is not Hopfian, then M is Hopfian.

CHAPTER 6

WEAKLY HOPFIAN MODULES

In this section we study weakly Hopfian modules.

Definition 6.1 *Let M be a module. M is called weakly Hopfian module (for short WH-module) if any small surjection of M is an isomorphism.*

Proposition 6.1 *For a non-zero module M, The following statements are equivalent:*

(i) *M is a WH-module.*

(ii) *$\frac{M}{K} \cong M$ for any small submodule K of M if and only if $K = 0$.*

Proof

$(i) \Rightarrow (ii)$ Suppose $M \cong \frac{M}{K}$ for some $K \ll M$. Let $\psi : \frac{M}{K} \to M$ and $\pi : M \to \frac{M}{K}$ be two isomorphisms. Then the map

$$M \xrightarrow{\pi} \frac{M}{K} \xrightarrow{\psi} M .$$

is an epimorphism with $ker(\psi\pi) = K$. Then $\psi\pi$ is a small epimorphism. So $\psi\pi$ is an isomorphism by (i), and so $K = 0$.

$(ii) \Rightarrow (i)$ Suppose (ii) hold. Let $f : M \to M$ be small epimorphism. Then $M \cong \frac{M}{ker(f)}$ by first isomorphism theorem. From (ii), we get $ker(f) = 0$. This shows f is an isomorphism. Hence M is a WH.

□

Theorem 6.1 *((Youngduo Wang, 2005), Theorem 3,1)*
Let M be a module. If M satisfies condition that M/N is WH for every small submodule, $0 \neq N \leq M$, then M is WH.

Proof Suppose that M satisfies the condition given in theorem and M is not weekly Hopfian. Then there exists a small epimorphism of M which is not isomorphism. Let $K = kerf$. Then $kerf \neq 0$ and $g : M/K \to M$ is an isomorphism. Thus $\eta g : M/N \to M/N$ is a small epimorphism which is not an isomorphism where $\eta : M \to M/N$ is a canonical epimorphism. This contradicts with our assumption. Hence M is WH. □

Lemma 6.1 *((Wisbauer, 1991), Lemma 19.3)*
Let K, L, N and M be R-modules. If $f : M \to N$ and $g : N \to L$ are two epimorphisms, then fg is superfluous if and only if f and g are superfluous.

Theorem 6.2 *((Youngduo Wang, 2005), Theorem 3.5)*
Let M be a module with a.c.c on small submodules. Then M is a WH.

Proof Let f be small epimorphism of endomorphism of M. Let M satisfies $a.c.c$ on small submodules. Then $ker(f) \leq ker(f)^2 \leq ker(f)^3 \leq \dots$ is an ascending chain of small submodules of M by Lemma 6.1. Since M satisfies $a.c.c$, there exists a positive number n such that $ker(f)^n = ker(f)^{n+1}$. Now we wish to show that $ker(f) = 0$. Let take $x \in ker(f)$, then $ker(f(x)) = 0$. Since f is an epimorphism, there exists $m_1 \in M$ such that $f(m_1) = x$. Then again f is an epimorphism, there exists $m_2 \in M$ such that $f(m_2) = m_1$. Repeating the process, we get $m_{n-1} \in M$ such that $f(m_n) = m_{n-1}$. Hence $f(m_1) = f^2(m_2) = f^3(m_3) = \dots = f^n(m_n) = x$. Since $x \in ker(f)$, $f(x) = f(f^n(m_n)) = 0$. This implies that $f^{n+1}(m_n) = 0$. Hence $m_n \in ker(f)^{n+1} = ker(f)^n$. Therefore $f^n(m_n) = 0 = x$. □

Proposition 6.2 *Dual automorphism-invariant modules are WH.*

Proof Let M be a dual automorphism-invariant module and $f : M \to M$ small epimorphism. We wish to show that there is g such that $fg = 1_M$. Let $K = ker(f)$. Then f induces an isomorphism $\varphi : M/K \to M$. Consider $\varphi^{-1} : M \to M/K$. Since M is dual automorphism, φ^{-1} lifts to an endomorphism g of M. We have $g(M)+K = M$. As $K \ll M$, $g(M) = M$. Thus g is an epimorphism.Then for any $x \in M$, $\varphi^{-1}(x) = g(x) + K$. Now $x = \varphi\varphi^{-1} = f(g(x))$. This proves $fg = 1_M$. □

Converse of this proposition is not true in general.

Example 6.1 *Every non-zero endomorphism of $\mathbb{Q}$ is an isomorphism. Let $f : \mathbb{Q} \to \mathbb{Q}$ be a non-zero endomorphism. For any $\frac{a}{b} \in \mathbb{Q}$, $b \neq 0$. $f(1) = f(\frac{b}{b}) = bf(\frac{1}{b})$. That is $f(\frac{1}{b}) = \frac{1}{b}f(1)$. Thus $f(\frac{a}{b}) = af(\frac{1}{b}) = \frac{a}{b}f(1)$. This shows that f is a monomorphism and an epimorphism. Hence f is an isomorphism.*

The following example shows that $\mathbb{Q}$ is WH but not dual automorphism-invariant by Theorem 4.3.
In order to show that quasi-projective modules are WH, we use the following lemma.

Lemma 6.2 *((Anderson and Fuller, 1992), 5.1)*

Let $f : M \to N$ and $f' : N \to M$ be homomorphism such that

$$ff' = 1_N$$

Then

$$M = ker(f) \oplus Imf'.$$

Moreover f is an epimorphism and f' is a monomorphism.

Proof Let $n \in N$. Then $f(f'(n)) = n$. So f is epic.
$f'(n) = 0 \Rightarrow n = f(f'(n)) \Rightarrow f(0) = 0$. So f' is a monomorphism.
Let $x \in M$, then

$$\begin{aligned} f(x - f'f(x)) &= f(x) - ff'f(x) = f(x) - f(x) = 0 \\ &\Rightarrow x - f'f(x) \in ker(f) \\ &\Rightarrow x = x - f'f(x) + f'f(x) \in ker(f) + Imf \\ &\Rightarrow M = ker(f) + Imf. \end{aligned}$$

Let $a \in ker(f) \cap Imf$. Then $f(a) = 0$ and $a = f'(b)$ for some $b \in N$. Then

$$b = f(f'(b)) = f(a) = 0$$

and so

$$a = f'(b) = f'(0) = 0$$

Thus $ker(f) \cap Imf = 0$ and $M = ker(f) \oplus Imf$. □

Corollary 6.1 *Quasi-projective modules are WH-modules.*

Proof Let M be a quasi-projective module. Suppose $M \cong \frac{M}{K}$ for some $K \ll M$. We shall prove that $K = 0$, and so M is WH by Proposition 6.1. Let $\phi : \frac{M}{K} \to M$ be an isomorphism. The map $\phi\pi : M \to M$, where $\pi : M \to \frac{M}{K}$ is canonical epimorphism has kernel K i.e. $ker(\phi\pi) = K$. Since M is quasi-projective, there is a $g : M \to M$ which makes the following diagram commutative.

$$\begin{tikzcd} & M \arrow[dl, dotted, "g"'] \arrow[d, "1_M"] \\ M \arrow[r, "\phi\pi"] & M \end{tikzcd}$$

By Lemma 6.2, $M = ker(\phi\pi) \oplus Im(g)$. Since $ker(\phi\pi) = K \ll M$, we must have $K = 0$. Hence M is a WH. □

Proposition 6.3 *((Youngduo Wang, 2005), Proposition 3.3)*
Let M be a WH and K is a direct summand of M. Then K is WH.

Proof Let M be a WH and $M = K \oplus K^{'}$. Suppose that any small epimorphism $f : M \to M$ is an isomorphism. Let $\alpha : K \to K$ be small epimorphism and take $1_{K'} : K^{'} \to K^{'}$. Firstly we will show that α has a small kernel.

Define

$$\begin{aligned} \alpha(k) &= k \\ 1_{K'}(k^{'}) &= k^{'} \end{aligned}$$

Consider $\psi : K \oplus K^{'} \to K \oplus K^{'}$ is defined by $\psi(k, k^{'}) = (\alpha(k), k^{'})$. Thus

$$ker(\psi) = ker(\alpha) \oplus 0 \ll K \oplus M.$$

Next we will show that α is a monomorphism.

$$\begin{aligned} \psi(k, k^{'}) = 0 &\Rightarrow (\alpha(k), k^{'}) = 0 \\ &\Rightarrow (k, k^{'}) = 0 \end{aligned}$$

thus α is monomorphism. Therefore α is an isomorphism. □

Direct sum of WH-modules need not be WH, as it is shown in the following example.

Example 6.2 *Let p be prime number and*

$$A = (\bigoplus_{n=1}^{\infty} < a_n >) \oplus (\bigoplus_{k=1}^{\infty} < b_k >)$$

where $< a_i > \cong \mathbb{Z}_p$ *and* $< b_i > \cong \mathbb{Z}_{p^2}$ *for each i. Set* $A_1 = (\bigoplus_{n=1}^{\infty} < a_n >)$ *and* $A_2 = (\bigoplus_{k=1}^{\infty} < b_k >)$*. Since* $Rad(A_1) = 0$*,* $A_1 = (\bigoplus_{n=1}^{\infty} < a_n >)$ *is WH. By Theorem 3.4,* $A_2 = (\bigoplus_{k=1}^{\infty} < b_k >)$ *is quasi-projective. Thus* A_2 *is WH by Corollary 6.1. Consider the map*

$$\begin{aligned} f : A &\rightarrow A \\ f(a_i) &= a_{i+1}, \textit{for each} \quad i \geq 1 \\ f(b_1) &= a_1 \\ f(b_i) &= b_{i-1}, \textit{for each} \quad i \geq 2 \end{aligned}$$

Then f is a well defined epimorphism. From the definition of f $0 \neq ker(f) = p < b_1 >$*. Since* $\mathbb{Z}_{p^2}$ *is local,* $< b_i >$ *is local for each* $i \geq 1$*. So*

$$0 \neq ker(f) \;=\; p < b_1 > \;\ll\; < b_i > \;\leqslant A.$$

Thus f is a small epimorphism of A, which is not an isomorphism. Hence A is not WH.

Proposition 6.4 *Let G be a module of finite length and H be a module with* $Rad(H) = 0$*, Then* $M = G \oplus H$ *is WH.*

Proof Let $f : M \rightarrow M$ be small epimorphism. Then $ker(f) \ll M \Rightarrow ker(f) \lneqq G \oplus 0$.

$$\frac{G \oplus H}{ker(f)} \cong \frac{G}{ker(f)} \oplus H \cong G \oplus H \quad \Rightarrow \quad G \cong \frac{G}{ker(f)}.$$

Let consider $0 \to ker f \to G \to \frac{G}{ker f} \to 0$ exact sequence. Then we have $c(G) = c(ker(f)) + c(\frac{G}{ker(f)})$. Since $G \cong \frac{G}{ker(f)}$, $c(G) = c(\frac{G}{ker(f)})$. Therefore $c(ker(f)) = 0$. This gives f is an isomorphism. This shows M is a WH-module. □

Corollary 6.2 *All finitely generated abelian groups are WH.*

Proof

$$A = \mathbb{Z}_{p^{n_1}} \oplus \mathbb{Z}_{p^{n_2}} \oplus \mathbb{Z}_{p^{n_3}} \dots \mathbb{Z}_{p^{n_n}} \oplus \mathbb{Z}^I$$

For simplicity let $A_1 = \mathbb{Z}_{p^{n_1}} \oplus \mathbb{Z}_{p^{n_2}} \oplus \mathbb{Z}_{p^{n_3}} \dots \mathbb{Z}_{p^{n_n}}$ and $A_2 = \mathbb{Z}^I$. Then A_1 is finite part and $Rad(A_2) = 0$ by Proposition 6.4 A is WH. □

Proposition 6.5 *Every non-singular uniform right R-module is WH.*

Proof Suppose M is non-singular uniform right R-module. Let $f : M \to M$ be an small epimorphism. i.e. $ker(f) \ll M$. Suppose $ker(f) \neq 0$. Then $ker(f) \trianglelefteq M$ because M is a uniform module. So $\frac{M}{ker(f)}$ is singular by Proposition 2.14. Since f is an epimorphism, by first isomorphism theorem $\frac{M}{ker(f)} \cong M$. This is impossible because $\frac{M}{ker(f)}$ is singular and M is non-singular. Therefore $ker(f)$ must be zero. So f is an isomorphism. Hence M is a WH. □

Proposition 6.6 *Every torsion free abelian group is WH.*

Proof Let G be torsion free abelian group and $f : G \to G$ be a small epimorphism. Firstly we will show that $ker(f)$ is closed in G.
Since f is an epimorphism, $G \cong \frac{G}{ker(f)}$. Then $\frac{G}{ker(f)}$ is torsion free. Then $ker(f)$ is closed submodule of G.
By Lemma 2.2, $ker(f)$ is a co-closed in G. Now we have $ker(f) \ll G$ and $ker(f)$ is co-closed submodule of G. Therefore $Rad(ker(f)) = ker(f)$ by Proposition 2.19 (3). Then $ker(f)$ is injective by Lemma 3.1. So $ker(f)$ is direct-summand of G. But $ker(f) \ll G$. Thus we must have $ker(f) = 0$. This proves that G is a WH. □

Proposition 6.7 *M is a Hopfian module if and only if M is both a generalized Hopfian module and a WH-module.*

Proof

$\Rightarrow$ Clear.

$\Leftarrow$ Let $f : M \to M$ be an epimorphism. Since M is generalized Hopfian $ker(f) \ll M$. Then $ker(f) = 0$ by the WH assumption. Hence f is an isomorphism. Therefore M is Hopfian.

$\square$

CHAPTER 7

CONCLUSION

In these thesis the relation between the classes of generalized Hopfian, weak Hopfian and dual automorphism-invariant modules are given. Some new results about weak Hopfian modules are proved.
For further studies we shall be interested in the following problems: (1) Characterize the rings whose nonsingular right modules are weak Hopfian. Note that the ring of integers is an example of such ring. (2) Characterize the rings whose right modules are weak Hopfian. (3) Describe the weak Hopfian torsion abelian groups.

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