GENERALIZED GOLDEN-FIBONACCI CALCULUS AND APPLICATIONS

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by
Merve ÖZVATAN

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We approve the thesis of **Merve ÖZVATAN**

Examining Committee Members:

Prof. Dr. Oktay PASHAEV
Department of Mathematics, İzmir Institute of Technology

Prof. Dr. Engin BÜYÜKAŞIK
Department of Mathematics, İzmir Institute of Technology

Prof. Dr. Halil ORUÇ
Department of Mathematics, Dokuz Eylül University

03 July 2018

Prof. Dr. Oktay PASHAEV
Supervisor, Department of Mathematics
İzmir Institute of Technology

Prof. Dr. Engin BÜYÜKAŞIK
Head of the Department of
Mathematics

Prof. Dr. Aysun SOFUOĞLU
Dean of the Graduate School of
Engineering and Sciences

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ABSTRACT

GENERALIZED GOLDEN-FIBONACCI CALCULUS AND APPLICATIONS

In the present thesis the Golden-Fibonacci calculus is developed and several applications of this calculus are obtained. The calculus is based on the Golden derivative as a finite difference operator with Golden and Silver ratio bases, which allowed us to introduce Golden polynomials and Taylor expansion in terms of these polynomials. The Golden binomial and its expansion in terms of Fibonomial coefficients is derived. We proved that Golden binomials coincide with Carlitz' characteristic polynomials. By Golden Fibonacci exponential functions and related entire functions, the Golden-heat and the Golden-wave equations are introduced and solved. By introducing higher order Golden Fibonacci derivatives, related with powers of golden ratio, we develop the higher order Golden Fibonacci calculus. The higher order Fibonacci numbers, higher Golden periodic functions and higher Fibonomials appear as ingredients of this calculus. By using Golden Fibonacci exponential function, we introduce the generating function for new type of polynomials, the Bernoulli-Fibonacci polynomials and study their properties. As a geometrical application, the Apollonious type gaskets are described in terms of Fibonacci, Lucas and generalized Fibonacci numbers. Some mod 5 congruencies associated with Fibonacci and Lucas numbers are obtained.

ÖZET

GENELLEŞTİRİLMİŞ ALTIN-FİBONACCİ HESAPLAMASI VE UYGULAMALARI

Bu tezde, Altın-Fibonacci hesaplaması geliştirilmiş ve bu hesaplamanın çeşitli uygulamaları elde edilmiştir. Bu hesaplama, altın polinomları ve bu polinomlar cinsinden yazılan Taylor açılımını tanıtmamıza izin veren, altın ve gümüş oran tabanları ile sonlu bir fark operatörü olarak yazılan Altın türevine dayanır. Altın binomu ve altın binomun Fibonomial katsayıları cinsinden açılımı türetilmiştir. Altın binomlarının Carlitz'in karakteristik polinomları ile eşleştiğini ispatladık. Altın Fibonacci üstel fonksiyonları ve ilgili analitik fonksiyonları ile, Altın-ısı ve Altın-dalga denklemleri tanıtılmış ve çözülmüştür. Altın oranın kuvvetleri ile ilgili olan yüksek mertebeden Altın Fibonacci türevlerini tanımlayarak, yüksek mertebeden Altın Fibonacci hesaplamasını geliştiririz. Yüksek mertebeden Fibonacci sayıları, yüksek Altın periodik fonksiyonlar ve yüksek Fibonomialler bu hesaplamanın bileşenleri olarak görünür. Altın Fibonacci üstel fonksiyonunu kullanarak, yeni tip polinom olan Bernoulli-Fibonacci polinomları için üretim fonksiyonunu tanıttık ve bu polinomların özelliklerini inceledik. Geometriksel bir uygulama olarak, Apollonious'un teğet çemberler dizisi Fibonacci, Lucas ve genelleştirilmiş Fibonacci sayıları cinsinden tanımlanmıştır. Fibonacci ve Lucas sayıları ile ilişkili bazı mod 5 denklikleri elde edilmiştir.

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CHAPTER 1

INTRODUCTION

The Golden ratio φ is a special number determining the so called Golden proportion and approximately it is equal to 1.618 (Koshy, T., 2001). It appears in geometry, art, architecture and varies from sunflowers to proportions of human body. From mathematical point of view, Golden ratio is defined by Golden proportion or Golden section and it is related with the sequence of Fibonacci numbers F_n, where $n = 0, 1, 2, \ldots$ The Fibonacci numbers are given by the recursion relation $F_{n+1} = F_n + F_{n-1}$, with initial values $F_0 = 0$ and $F_1 = 1$. These numbers $(0, 1, 1, 2, 3, 5, 8, 13, \ldots)$ are named after Italian mathematician of Middle Ages, Leonardo of Pisa or Leonardo Fibonacci.

If we take the ratio of two consecutive Fibonacci numbers $\frac{F_{n+1}}{F_n}$, then in the limit $n \to \infty$, it becomes the Golden ratio φ. This is why these numbers are intrinsically related with Golden ratio and Golden proportion. Due to numereous applications in mathematics, science and art, Fibonacci numbers are also called as ”Nature's Perfect numbers” (Koshy, T., 2001).

The present thesis is devoted to description of Fibonacci numbers, their properties and applications by the so called Golden Fibonacci calculus. The main ingredient of this calculus is the Golden Fibonacci derivative as a finite difference derivative with Golden and Silver ratio as bases. This derivative was first introduced in paper (Pashaev O.K. and Nalci S., 2012). The derivative allows one to construct Golden binomials, and Taylor expansion in terms of Golden binomials. According to this expansion, the Golden exponential functions were introduced, and trigonometric and hyperbolic Golden functions, as solutions of Golden ordinary differential equation and partial differential equation have been discussed.

In the present work, we generalize this Golden Fibonacci calculus. Starting from definition of higher order Fibonacci numbers, as a q numbers with two bases φ^k and φ'^k, we show that they are integer numbers, appearing as the ratio of two Fibonacci numbers.

These higher Order Fibonacci numbers are related with higher order Fibonacci derivatives, allowing to derive corresponding higher Golden periodic functions, higher Fibonomials and higher Golden binomials. As we prove in present thesis, these higher order Fibonacci binomials coincide with Carlitz characteristic polynomials for the so called combinatorial matrices (Carlitz, L., 1965). Powers of these combinatorial matrices allow us to find mod 5 congruence relations for higher order Fibonacci numbers.

By Golden exponential function, we introduce generating function for new type of polynomials, called the Bernoulli-Fibonacci polynomials, and study their properties.

As a geometrical application of Fibonacci numbers, the problem of intersection (kissing) between circles in plane, called as Apollonious gasket, and Descartes theorem are studied. Specific radiuses of kissing circles, given by products of Fibonacci numbers, allow us to introduce Fibonacci, Lucas, and Generalized Apollonious gaskets.

The thesis is organized as follows.

In Chapter 2, we briefly review Fibonacci numbers and their generalizations, in Sections 2.1-2.4.

Problem of division of Fibonacci numbers, Section 2.5, lead us to Higher order Fibonacci numbers in Section 2.6. These numbers, as a special case of Fibonacci polynomials are considered in Section 2.7. The Cassini formula and generalizations are studied in Section 2.8.

In Chapter 3, we introduce the Golden derivatives, Section 3.1 and, formulate main properties, including Leibnitz rule and Golden periodic functions.

In Section 3.2, the generating function for Fibonacci numbers is derived by Golden derivative and in addition the entire generating function is obtained.

The Golden Taylor formula is studied in Section 3.3. The Golden exponential functions as entire functions are introduced in Section 3.4, and corresponding Golden trigonometric and hyperbolic functions are derived in Section 3.5. These functions represent solutions of Golden oscillator in Section 3.5.1. The Golden binomials are subject of Section 3.6. Remarkable limit of these binomials is derived in Section 3.7.

In Section 3.8, Fibonacci exponential function of two arguments is defined and in terms of this function, a solution of the Golden heat equation is obtained. Solution of the Golden wave equation is given in Section 3.9.

Chapter 4 is devoted to higher order Fibonacci derivatives and their applications. In Section 4.1, we start from definition of higher order Golden derivative, its main properties and corresponding periodic functions. Then, we introduce higher Fibonomial and higher order Golden binomials.

In Chapter 5, the Carlitz characteristic polynomials are introduced, as polynomials with roots given in terms of powers of golden ratio. These polynomials are related with combinatorial matrices. The main identity between higher order Golden binomials and Carlitz's characteristic polynomials is established in Section 5.1. Powers of combinatorial matrices and their properties are derived in Section 5.2.

In Chapter 6, the mod 5 congruence relations for higher order Fibonacci numbers with even index are derived.

In Chapter 7, by using Golden exponential function, we introduce and study Fibonacci analog of Bernoulli polynomials and numbers. We call them the Bernoulli-Fibonacci polynomials and numbers.

In Chapter 8, the Apollonious gaskets related with Fibonacci numbers are studied. Starting from definition of Apollonious gasket in Section 8.1, we derive Fibonacci-Apollonious gasket in Section 8.2. In Section 8.3, the Lucas Apollonious gasket and the general Apollonious gasket (Section 8.4) are introduced. In all these cases the recursion formula has the same form and the difference is only in initial conditions. By Descartes theorem, the radiuses of kissing circles in terms of Fibonacci numbers are obtained.

In Conclusion, Chapter 9, we summarize main results obtained in this thesis. Details of some calculations are presented in Appendices A, B and C.

CHAPTER 2

FIBONACCI NUMBERS

2.1. Fibonacci and Lucas Sequences

The Fibonacci sequence is defined by recursion formula;

$$F_{n+1} = F_n + F_{n-1}, \tag{2.1}$$

where $F_0 = 0$, $F_1 = 1$, $n = 1, 2, 3, \ldots$ First few Fibonacci numbers are;

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 \ldots$$

The sequence is named after Leonardo Fibonacci(1170-1250). Fibonacci numbers appear in Nature so frequently that they can be considered as Nature's Perfect Numbers. Also, another important Nature's number, the Golden ratio, which is seen in every area of life and art, and usually it is associated with aesthetics, is directly related to Fibonacci sequence.

There is another famous sequence, which is called the Lucas sequence. The Lucas numbers give the sequence of integers, defined by same recurrence formula;

$$L_{n+1} = L_n + L_{n-1},$$

but with different initial values $L_0 = 2$, $L_1 = 1$. First few of Lucas numbers are,

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots$$

There is a relation between Fibonacci and Lucas numbers, given by formula,

$$\boxed{L_n = F_{n-1} + F_{n+1}} \tag{2.2}$$

and meaning that Lucas sequence is addition of two Fibonacci sequences.

2.2. Binet Formula

Formula giving Fibonacci number F_n for given n is called the Binet formula. To derive Binet Formula for Fibonacci numbers, let $F_n = \lambda^n$, which by substituting in the recursion formula (2.1) gives us;

$$\lambda^{n+1} = \lambda^n + \lambda^{n-1} \quad \Rightarrow \quad \lambda = 1 + \frac{1}{\lambda} \quad \Rightarrow \quad \lambda^2 = \lambda + 1$$

This quadratic equation has characteristic roots denoted by φ and φ', having the values;

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1,6180339\ldots \quad \text{and} \quad \varphi' = \frac{1-\sqrt{5}}{2} \approx -0,6180339\ldots$$

Number φ and φ' are called the Golden and Silver ratio, respectively.
Also, from the quadratic equation, it can be seen that; $\varphi\varphi' = -1$ & $\varphi + \varphi' = 1$.
Then, the solution F_n is a linear combination;

$$F_n = c_1 \ \varphi^n + c_2 \ \varphi'^n,$$

with arbitrary $c_1, \ c_2$ constants. By using initial values $F_0 = 0$, $F_1 = 1$ constants c_1 and c_2 can be fixed as;

$$c_1 + c_2 = F_0 = 0, \qquad c_1\varphi + c_2\varphi' = F_1 = 1 \quad \Rightarrow \quad c_1 = \frac{1}{\varphi - \varphi'}, \quad c_2 = -\frac{1}{\varphi - \varphi'}.$$

Due to this, Fibonacci numbers-F_n can be expressed explicitly by Binet Formula;

$$F_n = \frac{\varphi^n - \varphi'^n}{\varphi - \varphi'}. \tag{2.3}$$

The Binet type formula for Lucas sequence can be derived by the same logic in the form;

$$L_n = \varphi^n + \varphi'^n.$$

From formula (2.3), due to irrational character of φ and φ', it is not evident at all that F_n are integer numbers. Though it is clear from the recursion formula (2.1).

Binet formula allows one to find Fibonacci numbers directly, without using recursions. For example, to find F_{20} by using Binet formula we have;

$$F_{20} = \frac{\varphi^{20} - \varphi'^{20}}{\varphi - \varphi'} = 6765.$$

Binet formula allows one to define also Fibonacci numbers for negative n;

$$F_{-n} = \frac{\varphi^{-n} - \varphi'^{-n}}{\varphi - \varphi'} = \frac{\frac{1}{\varphi^n} - \frac{1}{\varphi'^n}}{\varphi - \varphi'} = \frac{\varphi'^n - \varphi^n}{\varphi - \varphi'} \cdot \frac{1}{(\varphi\varphi')^n}$$

and since $\varphi\varphi' = -1$,

$$\Rightarrow \qquad = -\frac{\varphi^n - \varphi'^n}{\varphi - \varphi'}\frac{1}{(-1)^n} = -F_n\frac{1}{(-1)^n} = -F_n(-1)^n = (-1)^{n+1}F_n$$

So, we have;

$$\boxed{F_{-n} = (-1)^{n+1}F_n} \tag{2.4}$$

For Lucas sequence, similar calculations gives;

$$L_{-n} = \varphi^{-n} + \varphi'^{-n} = \frac{\varphi^n + \varphi'^n}{(\varphi\varphi')^n} = (-1)^n L_n \Rightarrow \boxed{L_{-n} = (-1)^n L_n} \tag{2.5}$$

Definition 2.1 *Fibonacci and Lucas numbers for negative integers n are respectively defined by relations,*

$$\begin{aligned} F_{-n} &= (-1)^{n+1} F_n, \\ L_{-n} &= (-1)^n L_n \end{aligned}$$

If we take any two successive Fibonacci numbers, their ratio while going to infinity become more and more close to the Golden Ratio "φ".

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{\varphi^{n+1} - \varphi'^{n+1}}{\varphi^n - \varphi'^n} = \lim_{n\to\infty} \frac{\varphi^{n+1}}{\varphi^n} = \varphi. \tag{2.6}$$

The same result is valid for Lucas numbers;

$$\lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \lim_{n\to\infty} \frac{\varphi^{n+1} + \varphi'^{n+1}}{\varphi^n + \varphi'^n} = \lim_{n\to\infty} \frac{\varphi^{n+1}}{\varphi^n} = \varphi. \tag{2.7}$$

Proposition 2.1 *Any integer power of Golden and Silver ratios can be expressed in terms of Fibonacci numbers as;*

$$\varphi^n = \varphi F_n + F_{n-1} \ \ \textit{and} \ \ \varphi'^n = \varphi' F_n + F_{n-1}. \tag{2.8}$$

Proof Proof will be done by using Principle of Mathematical induction. For $n = 1$, $\varphi = \varphi$. Assume that for $n \in \mathbb{Z}$, $\varphi^n = \varphi F_n + F_{n-1}$ is true. Then the case $n + 1$ gives;

$$\begin{aligned} \varphi^{n+1} &= (\varphi^n)\varphi = (\varphi F_n + F_{n-1})\varphi = \varphi^2 F_n + \varphi F_{n-1} = (\varphi + 1)F_n + \varphi F_{n-1} \\ &= \varphi(F_n + F_{n-1}) + F_n = \varphi F_{n+1} + F_n. \end{aligned}$$

Since, $\varphi^{n+1} = \varphi F_{n+1} + F_n$ is obtained, the proof is done. Similarly, $\varphi'^{n+1} = \varphi' F_{n+1} + F_n$ can be proved. □

2.3. Trigonometric Representation of Fibonacci Numbers

Starting from Binet Formula (2.3), we can derive following formula for Fibonacci numbers in summation form;

$$F_n = 2^{n-1} \sum_{k=0}^{n-1} (-1)^k \cos^{n-k-1}\left(\frac{\pi}{5}\right) \sin^k\left(\frac{\pi}{10}\right). \tag{2.9}$$

For derivation of this formula, see Appendix A.1.

2.4. Generalized Fibonacci Numbers

Fibonacci and Lucas numbers are numbers determined by the same recursion formula (2.1) but with different initial values. Here, we are going to generalize these numbers, by choosing different initial numbers G_0 and G_1, but preserving the recursion formula (2.1). We can call them as Generalized Fibonacci numbers. For example, if $G_0 = 0$, $G_1 = 4$, we have the sequence;

$$4, 4, 8, 12, 20, 32, 52, ...$$

Definition 2.2 *Generalized Fibonacci number sequence is defined by the recursion formula,*

$$G_{n+1} = G_n + G_{n-1}$$

and an arbitrary initial numbers G_0, G_1.

To get Binet type formula for these numbers, we substitute $G_n = \beta^n$ to the recursion formula,

$$\beta^{n+1} = \beta^n + \beta^{n-1}.$$

After cancelling the powers β^n's gives us;

$$\beta^2 = \beta + 1 \Rightarrow \beta_1 = \varphi \text{ and } \beta_2 = \varphi'.$$

Then, generic G_n can be written as a linear combination of these solutions with arbitrary constants c_1, c_2:

$$G_n = c_1 \ \varphi^n + c_2 \ \varphi'^n.$$

Constants $c_1,\ c_2$ can be fixed by the initial values G_0 , G_1;

$$c_1 + c_2 = G_0, \quad c_1\varphi + c_2\varphi' = G_1 \quad \Rightarrow \quad c_1 = \frac{G_1 - \varphi' G_0}{\varphi - \varphi'}, \ \ c_2 = -\frac{G_1 - \varphi G_0}{\varphi - \varphi'}.$$

After substituting c_1 and c_2,

$$G_n = \frac{(G_1 - \varphi' G_0)\varphi^n - (G_1 - \varphi G_0)\varphi'^n}{\varphi - \varphi'}.$$

Proposition 2.2 *The Binet formula for Generalized Fibonacci numbers G_n satisfying $G_{n+1} = G_n + G_{n-1}$ and initial values G_0 , G_1 is;*

$$G_n = \frac{(G_1 - \varphi' G_0)\varphi^n - (G_1 - \varphi G_0)\varphi'^n}{\varphi - \varphi'}. \tag{2.10}$$

In particular cases;

- If $G_0 = 0$ and $G_1 = 1$, our equation becomes Binet formula (2.3) for Fibonacci numbers.

- If $G_0 = 2$ and $G_1 = 1$, then Lucas sequence appears.

Formula (2.10) allows us to represent generalized Fibonacci number sequence as a linear combination of two Fibonacci sequences;

$$\boxed{G_n = G_1 F_n + G_0 F_{n-1}} \tag{2.11}$$

2.5. Addition and Division of Fibonacci Numbers

The Addition Formula for Fibonacci numbers is given by the following proposition.

Proposition 2.3 (Addition formula)

$$F_{n+m} = F_m F_{n+1} + F_n F_{m-1} \quad where \quad m, n \in \mathbb{Z} \tag{2.12}$$

Proof First, begin to write corresponding Binet Formula for F_{n+m},

$$\begin{aligned} F_{n+m} &= \frac{\varphi^{n+m} - \varphi'^{n+m}}{\varphi - \varphi'} = \frac{\varphi^n \varphi^m}{\varphi - \varphi'} - \frac{\varphi'^n \varphi'^m}{\varphi - \varphi'} \\ &= \varphi^n \frac{(\varphi^m - \varphi'^m + \varphi'^m)}{\varphi - \varphi'} - \varphi'^m \frac{(\varphi'^n - \varphi^n + \varphi^n)}{\varphi - \varphi'} \\ &= \varphi^n \left(F_m + \frac{\varphi'^m}{\varphi - \varphi'} \right) + \varphi'^m \left(F_n - \frac{\varphi^n}{\varphi - \varphi'} \right) \\ &= \varphi^n F_m + \frac{\varphi^n \varphi'^m}{\varphi - \varphi'} + \varphi'^m F_n - \frac{\varphi'^m \varphi^n}{\varphi - \varphi'} \\ &= \varphi^n F_m + \varphi'^m F_n. \end{aligned}$$

Therefore, we obtain;

$$F_{n+m} = \varphi^n F_m + \varphi'^m F_n. \tag{2.13}$$

Substituting $\varphi^n = \varphi F_n + F_{n-1}$ and $\varphi'^m = \varphi' F_m + F_{m-1}$ gives,

$$\begin{aligned} F_{n+m} &= \varphi^n F_m + \varphi'^m F_n = (\varphi F_n + F_{n-1}) F_m + (\varphi' F_m + F_{m-1}) F_n \\ &= \varphi F_n F_m + F_{n-1} F_m + \varphi' F_m F_n + F_{m-1} F_n \\ &= (\varphi + \varphi') F_n F_m + F_{n-1} F_m + F_{m-1} F_n \\ &= F_n F_m + F_{n-1} F_m + F_{m-1} F_n \\ &= (F_n + F_{n-1}) F_m + F_{m-1} F_n \\ &= F_{n+1} F_m + F_{m-1} F_n. \end{aligned}$$

□

In the addition formula (2.12), if we denote $n + m = N$, then $n = N - m$, and we get;

$$F_N = F_{N-m}F_{m-1} + F_mF_{N-m+1}$$

After writing $N = n$, it gives;

$$\boxed{F_n = F_{n-m}F_{m-1} + F_mF_{n-m+1}} \tag{2.14}$$

Again denoting $n + m = N$, but now $m = N - n$, we get another partition of F_n;

$$F_N = F_{N-n}F_{n+1} + F_nF_{N-n-1}.$$

After writing $n = m$, it gives;

$$F_N = F_{N-m}F_{m+1} + F_mF_{N-m-1}.$$

Finally, denoting $N = n$;

$$\boxed{F_n = F_{n-m}F_{m+1} + F_mF_{n-m-1}}. \tag{2.15}$$

Equations (2.14) and (2.15) give two different partitions of Fibonacci number F_n.

From addition formula (2.12), Fibonacci numbers for even n are;

$$\boxed{F_{2k} = F_k\, L_k} \tag{2.16}$$

where L_k- Lucas numbers.

Indeed, $F_{2k} = F_{k+k} = F_k\, F_{k-1} + F_{k+1}\, F_k = F_k\, (F_{k-1} + F_{k+1}) = F_k\, L_k$

Also,

$$\boxed{F_{3k} = F_k \ (F_{2k-1} + F_{k+1} \ L_k)} \tag{2.17}$$

It is easy to see from;

$$F_{3k} \quad = \quad F_{k+2k} = F_k \ F_{2k-1} + F_{k+1} \ (F_{2k}) = F_k \ F_{2k-1} + F_{k+1} \ (F_k \ L_k) = F_k(F_{2k-1} + F_{k+1} \ L_k)$$

To continue,

$$\boxed{F_{4k} = F_k \ \ L_k \ \ L_{2k}} \tag{2.18}$$

is obtained from,

$$F_{4k} = F_{2k+2k} = F_{2k} \ F_{2k-1} + F_{2k+1} \ F_{2k} = F_{2k}(F_{2k-1} + F_{2k+1}) = F_{2k} \ \ L_{2k} = F_k \ \ L_k \ \ L_{2k}.$$

From above results, we have next divisibility property of F_{nk}.

Proposition 2.4 *F_{nk} is divisible by F_k.*

Proof Proof by induction on n. For $n = 1$, F_k is divisible by F_k, clearly. For n, suppose $F_{nk} = F_k \ X(k, n)$, where $X(k, n) \in \mathbb{Z}$. For the case $n + 1$, by using the equality (2.12), we have;

$$\begin{aligned} F_{(n+1)k} = F_{k+nk} \ &\overset{(2.12)}{=} \ F_k \ F_{nk-1} + [F_{nk}] \ \ F_{k+1} \\ &= \ F_k \ F_{nk-1} + \ \ [F_k \ X(k, n)] \ \ F_{k+1} \\ &= \ F_k(\ F_{nk-1} + \ \ X(k, n) \ F_{k+1}). \end{aligned} \tag{2.19}$$

This is why $F_{(n+1)k}$ is divisible by F_k. So, by principle of mathematical induction, F_{nk} is divisible by F_k, for any n. □

2.6. Higher Order Fibonacci Numbers

Since F_{nk} is divisible by F_k, the ratio $\frac{F_{nk}}{F_k}$ is an integer. These numbers we call the "Higher Order Fibonacci numbers".

Definition 2.3 *Higher order Fibonacci numbers are defined as,*

$$F_n^{(k)} = \frac{F_{nk}}{F_k}. \tag{2.20}$$

Then, all Higher order Fibonacci numbers are integer.

Proposition 2.5 *Binet type formula for Higher order Fibonacci numbers is,*

$$F_n^{(k)} = \frac{(\varphi^k)^n - (\varphi'^k)^n}{\varphi^k - \varphi'^k} \tag{2.21}$$

Proof It is derived simply by using the Binet formula,

$$F_{nk} = \frac{(\varphi^k)^n - (\varphi'^k)^n}{\varphi - \varphi'} = \frac{(\varphi^k)^n - (\varphi'^k)^n}{\varphi^k - \varphi'^k}\frac{\varphi^k - \varphi'^k}{\varphi - \varphi'} = \frac{(\varphi^k)^n - (\varphi'^k)^n}{\varphi^k - \varphi'^k} \; F_k.$$

Thus, Higher Order Fibonacci numbers are written as a ratio;

$$\boxed{F_n^{(k)} = \frac{(\varphi^k)^n - (\varphi'^k)^n}{\varphi^k - \varphi'^k} = \frac{F_{nk}}{F_k}.} \tag{2.22}$$

□

Due to above definition, we have formula for factorization of Fibonacci numbers;

$$F_{nk} = F_k \; F_n^{(k)}.$$

For example, Higher order Fibonacci numbers $F_n^{(3)}$ for $k = 3$ are given by;

$$F_0^{(3)} = \frac{F_0}{F_3} = 0, F_1^{(3)} = \frac{F_3}{F_3} = 1, F_2^{(3)} = \frac{F_6}{F_3} = 4, F_3^{(3)} = \frac{F_9}{F_3} = 17, F_4^{(3)} = \frac{F_{12}}{F_3} = 72, \ldots \tag{2.23}$$

Now, we are going to derive the recursion relation formula for Higher Order Fibonacci numbers. It is given by the next theorem.

Theorem 2.1 *(Recurrence relation for Higher Order Fibonacci numbers)*

$$F^{(k)}_{n+1} = L_k F^{(k)}_n + (-1)^{k-1} F^{(k)}_{n-1}. \tag{2.24}$$

This formula is particular case of more general relation, given by Theorem 2.2.

Theorem 2.2

$$F_{k(n+1)+\alpha} = L_k F_{kn+\alpha} + (-1)^{k-1} F_{k(n-1)+\alpha} \quad \textit{where} \quad \alpha = 0, 1, \ldots, k-1. \tag{2.25}$$

Proof First we prove Theorem 2.2.

$$\begin{aligned}
F_{kn+k+\alpha} &= \frac{1}{\varphi - \varphi'}\left[\varphi^{kn+k+\alpha} - \varphi'^{kn+k+\alpha}\right] \\
&= \frac{1}{\varphi - \varphi'}\left[\varphi^{kn+\alpha}\varphi^{k} - \varphi'^{kn+\alpha}\varphi'^{k}\right] \\
&= \frac{1}{\varphi - \varphi'}\left[\varphi^{kn+\alpha}\varphi^{k} + (-\varphi'^{kn+\alpha}\varphi^{k} + \varphi'^{kn+\alpha}\varphi^{k}) - \varphi'^{kn+\alpha}\varphi'^{k}\right] \\
&= \frac{1}{\varphi - \varphi'}\left[(\varphi^{kn+\alpha} - \varphi'^{kn+\alpha})\varphi^{k} + \varphi'^{kn+\alpha}\varphi^{k} - \varphi'^{kn+\alpha}\varphi'^{k}\right] \\
&= F_{kn+\alpha}\varphi^{k} + \frac{1}{\varphi - \varphi'}\left[\varphi'^{kn+\alpha}\varphi^{k} - \varphi'^{kn+\alpha}\varphi'^{k}\right] \\
&= F_{kn+\alpha}\left[\varphi^{k} + (-\varphi'^{k} + \varphi'^{k})\right] + \frac{1}{\varphi - \varphi'}\left[\varphi'^{kn+\alpha}\varphi^{k} - \varphi'^{kn+\alpha}\varphi'^{k}\right] \\
&= F_{kn+\alpha}(\varphi^{k} + \varphi'^{k}) - F_{kn+\alpha}\varphi'^{k} + \frac{1}{\varphi - \varphi'}\left[\varphi'^{kn+\alpha}\varphi^{k} - \varphi'^{kn+\alpha}\varphi'^{k}\right] \\
&= F_{kn+\alpha}L_k + \frac{1}{\varphi - \varphi'}\left[\varphi'^{kn+\alpha}\varphi'^{k} - \varphi^{kn+\alpha}\varphi'^{k} + \varphi'^{kn+\alpha}\varphi^{k} - \varphi'^{kn+\alpha}\varphi'^{k}\right] \\
&= L_k F_{kn+\alpha} + \frac{1}{\varphi - \varphi'}\left[\varphi'^{kn+\alpha}\varphi^{k} - \varphi^{kn+\alpha}\varphi'^{k}\right] \\
&= L_k F_{kn+\alpha} + \frac{\varphi^{k}\varphi'^{k}}{\varphi - \varphi'}\left[\varphi'^{kn-k+\alpha} - \varphi^{kn-k+\alpha}\right] \\
&= L_k F_{kn+\alpha} - \frac{\varphi^{k}\varphi'^{k}}{\varphi - \varphi'}\left[\varphi^{kn-k+\alpha} - \varphi'^{kn-k+\alpha}\right] \\
&= L_k F_{kn+\alpha} - \frac{(\varphi\varphi')^{k}}{\varphi - \varphi'}\left[\varphi^{kn-k+\alpha} - \varphi'^{kn-k+\alpha}\right] \quad \text{since} \quad (\varphi\varphi')^{k} = (-1)^{k}, \\
&= L_k F_{kn+\alpha} - (-1)^{k}\left[\frac{\varphi^{kn-k+\alpha} - \varphi'^{kn-k+\alpha}}{\varphi - \varphi'}\right]
\end{aligned}$$

$$
\begin{aligned}
&= L_k F_{kn+\alpha} + (-1)^{k+1} F_{kn-k+\alpha} \text{ and since } (-1)^{k+1}(-1)^{-2} = (-1)^{k-1}, \\
&= L_k F_{kn+\alpha} + (-1)^{k-1} F_{kn-k+\alpha}
\end{aligned}
$$

Therefore,

$$F_{k(n+1)+\alpha} = L_k F_{kn+\alpha} + (-1)^{k-1} F_{k(n-1)+\alpha} \tag{2.26}$$

is obtained. By choosing $\alpha = 0$ and dividing both sides of the equation with F_k gives us the desired recursion formula (2.24),

$$F_{n+1}^{(k)} = L_k F_n^{(k)} + (-1)^{k-1} F_{n-1}^{(k)}.$$

□

The total set of Fibonacci numbers F_n is the sum of subsets for each k;
$\underline{k = 2:}$ $F_{2n},\ F_{2n+1}$
$\underline{k = 3:}$ $F_{3n},\ F_{3n+1},\ F_{3n+2}$
$\vdots$
$\underline{k = k:}$ $F_{kn},\ F_{kn+1}, \ldots,\ F_{kn+(k-1)}$
Equation (2.26) says that for given k, the subsequences $F_{kn},\ F_{kn+1},\ F_{kn+2}, \ldots,\ F_{kn+(k-1)}$, satisfy the same recursion formula;

$$F_{k(n+1)+\alpha} = L_k F_{kn+\alpha} + (-1)^{k-1} F_{k(n-1)+\alpha}$$

This type of recursion formula is special case Fibonacci polynomials, which we are studying in Section 2.7.

Example 2.1 *Let us think sequences with $k = 3$;*
$\underline{\alpha = 0}$ $\Rightarrow$ $F_{3n} = 0, 2, 8, 34, \ldots$
$\underline{\alpha = 1}$ $\Rightarrow$ $F_{3n+1} = 1, 3, 13, 55, \ldots$
$\underline{\alpha = 2}$ $\Rightarrow$ $F_{3n+2} = 1, 5, 21, 89, \ldots$

To generate each of three sequences, we have the same recursion relation,

$$F_{3(n+1)+\alpha} = L_3 F_{3n+\alpha} + (-1)^{3-1} F_{3(n-1)+\alpha}, \quad \text{where } \alpha = 0, 1, 2.$$

Also, we can notice that all of the sequences F_{3n},F_{3n+1},F_{3n+2} starts with different initial values. In their union set, they cover the whole Fibonacci sequence.

Example 2.2 *For $k = 3$, with the initial numbers $F_0^{(3)} = 0$ and $F_1^{(3)} = 1$, we can derive the Higher Order Fibonacci number sequence given in (2.23) by using 3^{rd} Lucas number and alternating sign function, i.e,* $F_{n+1}^{(3)} = L_3 F_n^{(3)} + (-1)^{3-1} F_{n-1}^{(3)}$.

n=1: $F_2^{(3)} = 4 \cdot 1 + (-1)^2 \cdot 0 = 4$
n=2: $F_3^{(3)} = 4 \cdot 4 + (-1)^2 \cdot 1 = 17$
n=3: $F_4^{(3)} = 4 \cdot 17 + (-1)^2 \cdot 4 = 72$
n=4: $F_5^{(3)} = 4 \cdot 72 + (-1)^2 \cdot 17 = 305$
$\vdots$

Proposition 2.6 *By extending n and k to negative integer numbers for Higher order Fibonacci numbers $F_n^{(k)}$, the formulas can be derived as;*

$$F_{-n}^{(k)} = (-1)^{kn+1} F_n^{(k)} \tag{2.27}$$

$$F_n^{(-k)} = (-1)^{(n+1)k} F_n^{(k)} \tag{2.28}$$

$$F_{-n}^{(-k)} = (-1)^{k+1} F_n^{(k)} \tag{2.29}$$

Proof

$$\begin{aligned} F_{-n}^{(k)} &= \frac{(\varphi^k)^{-n} - (\varphi'^k)^{-n}}{\varphi^k - \varphi'^k} = \frac{1}{\varphi^k - \varphi'^k}\left(\frac{1}{\varphi^{kn}} - \frac{1}{\varphi'^{kn}}\right) = \frac{1}{\varphi^k - \varphi'^k}\left(-\frac{\varphi^{kn} - \varphi'^{kn}}{(\varphi\varphi')^{kn}}\right) \\ &= \frac{(\varphi^k)^n - (\varphi'^k)^n}{\varphi^k - \varphi'^k}(-1)^{kn+1} = (-1)^{kn+1} F_n^{(k)} \end{aligned}$$

As a special case, by choosing $k = 1$, we obtain our previous result (2.4).

$$\begin{aligned} F_n^{(-k)} &= \frac{(\varphi^{-k})^n - (\varphi'^{-k})^n}{\varphi^{-k} - \varphi'^{-k}} = \frac{\varphi'^{kn} - \varphi^{kn}}{(\varphi\varphi')^{kn}} \cdot \frac{(\varphi\varphi')^k}{\varphi'^k - \varphi^k} = \frac{\varphi'^{kn} - \varphi^{kn}}{(\varphi\varphi')^{kn}} \cdot \frac{(\varphi\varphi')^k}{\varphi'^k - \varphi^k}\frac{(-1)^{kn}}{(-1)^{kn}} \\ &= \frac{\varphi^{kn} - \varphi'^{kn}}{\varphi^k - \varphi'^k}(-1)^{(n+1)k} = (-1)^{(n+1)k} F_n^{(k)} \end{aligned}$$

$$
\begin{aligned}
F_{-n}^{(-k)} &= \frac{(\varphi^{nk}) - (\varphi'^{nk})}{\varphi^{-k} - \varphi'^{-k}} = \frac{\varphi^{nk} - \varphi'^{nk}}{\varphi^{k} - \varphi'^{k}} \frac{\varphi^{k} - \varphi'^{k}}{\varphi^{-k} - \varphi'^{-k}} = F_n^{(k)} \frac{\varphi^{k} - \varphi'^{k}}{\frac{1}{\varphi^k} - \frac{1}{\varphi'^k}} = F_n^{(k)} \frac{\varphi^{k} - \varphi'^{k}}{\varphi'^{k} - \varphi^{k}} (\varphi\varphi')^k \\
&= (-1)^{k+1} F_n^{(k)}
\end{aligned}
$$

□

These formulas determine $F_n^{(k)}$ for each $k \in \mathbb{Z}$, and each $n \in \mathbb{Z}$.

There is an important proposition, which gives relation between powers of Golden-Silver ratio and Higher order Fibonacci numbers.

Proposition 2.7 *For $k \in Z$ and $n \in Z$,*

$$
(\varphi^k)^n = \varphi^k F_n^{(k)} + (-1)^{k+1} F_{n-1}^{(k)} \tag{2.30}
$$

$$
(\varphi'^k)^n = \varphi'^k F_n^{(k)} + (-1)^{k+1} F_{n-1}^{(k)} \tag{2.31}
$$

Proof By definition of Higher Order Fibonacci numbers, we need to prove,

$$
(\varphi^k)^n = \varphi^k \frac{F_{nk}}{F_k} + (-1)^{k+1} \frac{F_{(n-1)k}}{F_k}.
$$

To prove it, we will show the equality;

$$
F_k (\varphi^k)^n = \varphi^k F_{nk} + (-1)^{k+1} F_{(n-1)k}.
$$

Starting from right hand side, it gets;

$$
\begin{aligned}
\varphi^k F_{nk} + (-1)^{k+1} F_{(n-1)k} &= \varphi^k \frac{\varphi^{nk} - \varphi'^{nk}}{\varphi - \varphi'} + (-1)^{k+1} \frac{\varphi^{(n-1)k} - \varphi'^{(n-1)k}}{\varphi - \varphi'} \\
&= \frac{1}{\varphi - \varphi'} \left[\varphi^{nk+k} - \varphi'^{nk}\varphi^k + (-1)^{k+1}\varphi^{kn-k} + (-1)^k \varphi'^{kn-k}\right] \\
&= \varphi^{kn} \frac{\varphi^k + (-1)^{k+1}\varphi^{-k}}{\varphi - \varphi'} + \frac{(-1)^k \varphi'^{kn-k} - \varphi'^{nk}\varphi^k}{\varphi - \varphi'} \\
&= \varphi^{kn} \frac{\varphi^k - (-1)^k \frac{1}{\varphi^k}}{\varphi - \varphi'} + \frac{\varphi'^{kn}\left((-1)^k (\varphi')^{-k} - \varphi^k\right)}{\varphi - \varphi'} \\
&= \varphi'^{kn} \frac{\varphi^k - (\frac{1}{\varphi})^k}{\varphi - \varphi'} + \frac{\varphi'^{kn}}{\varphi - \varphi'} \left((-1)^k \frac{(-1)^{-k}}{(\varphi)^{-k}} - \varphi^k\right) \\
&= \varphi^{kn} \frac{\varphi^k - \varphi'^k}{\varphi - \varphi'} + \frac{\varphi'^{kn}}{\varphi - \varphi'} \left(\varphi^k - \varphi^k\right) = \varphi^{kn} F_k.
\end{aligned}
$$

With the similar logic, $(\varphi'^k)^n = \varphi'^k\, F_n^{(k)} + (-1)^{k+1}\, F_{n-1}^{(k)}$ can be proved, also. □

2.7. Fibonacci Polynomials

Here, we modify recursion formula (2.1) by introducing two arbitrary coefficients p and q;

$$\boxed{F_{n+1} = pF_n + qF_{n-1}} \tag{2.32}$$

By choosing initial values $F_0 = 0$, $F_1 = 1$, the corresponding sequence will depend on two numbers p and q. This sequence of two variable polynomials-$F_n(p,q)$ is called the Fibonacci polynomials.

For solution of this equation, let $F_n(p,q) = \gamma^n$. Then, with the recursion formula (2.32) characteristic equation becomes $\gamma^2 = p\gamma + q$. By denoting our roots as a and b, we have $a + b = p$ and $ab = -q$. Then, Fibonacci polynomial $F_n(p,q)$ is written as;

$$F_n(p,q) = c_1 a^n + c_2 b^n. \tag{2.33}$$

With the help of our initial values, coefficients can be found as;

$$c_1 + c_2 = F_0(p,q) = 0, \quad c_1 a + c_2 b = F_1(p,q) = 1 \;\Rightarrow\; c_1 = \frac{1}{a-b}, \;\; c_2 = -\frac{1}{a-b}$$

By this way, we obtain Binet type formula for Fibonacci polynomials as;

$$F_n(p,q) = \frac{a^n - b^n}{a-b}, \tag{2.34}$$

where $a, b = \frac{p}{2} \pm \frac{\sqrt{p^2-4q}}{2}$.

In the recursion formula (2.32), if p and q are arbitrary integer numbers, then we get the sequence of integer numbers;

$F_0(p,q) = 0$

$F_1(p,q) = 1$

$F_2(p,q) = p$
$F_3(p,q) = p^2 + q$
$F_4(p,q) = p(p^2 + q) + qp$
$F_5(p,q) = p^2(p^2 + 2q) + q(p^2 + q)$
$F_6(p,q) = p^3(p^2 + 2q) + 2qp(p^2 + q) + q^2 p$
$F_7(p,q) = p^4(p^2 + 2q) + 3p^4 q + 6p^2 q^2 + q^3$
$\vdots$

which we call Fibonacci polynomial numbers.

Also, when we choose $p = q = 1$, the recursion relation will be standard Fibonacci recursion and Fibonacci numbers. Therefore, $F_n(1,1) = F_n$.

2.8. Cassini Formula and Generalizations

In Chapter 5, following matrix is introduced

$$A_2^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$

Determinant of this matrix $det(A_2^n) = F_{n+1}F_{n-1} - F_n^2$ can be calculated by using Cassini's Formula.

Proposition 2.8 *(Cassini's Formula) For every positive integer n,*

$$\boxed{F_{n-1}F_{n+1} - F_n^2 = (-1)^n} \tag{2.35}$$

Proof (Koshy, T., 2001) Proof will be done by using Principal of Mathematical induction. For $n = 1$, we have;

$$F_0 F_2 - F_1^2 = 0 \cdot 1 - 1^2 = (-1)^1 = -1.$$

Suppose that it is true for all $k \geq 1$, that is for $n = k$,

$$F_{k-1}F_{k+1} - F_k^2 = (-1)^k. \tag{2.36}$$

Then for the case $n = k + 1$, we have;

$$\begin{aligned}
F_{(k+1)-1}F_{(k+1)+1} - F_{k+1}^2 &= F_kF_{k+2} - F_{k+1}^2 = (F_{k+1} - F_{k-1})(F_k + F_{k+1}) - F_{k+1}^2 \\
&= F_kF_{k+1} + F_{k+1}^2 - F_kF_{k-1} - F_{k-1}F_{k+1} - F_{k+1}^2 \\
&= F_kF_{k+1} - F_kF_{k-1} - F_{k-1}F_{k+1} - F_k^2 + F_k^2 \\
&= F_kF_{k+1} - F_kF_{k-1} - F_k^2 - \left(F_{k-1}F_{k+1} - F_k^2\right) \\
&\overset{(2.36)}{=} F_kF_{k+1} - F_kF_{k-1} - F_k^2 - (-1)^k \\
&= F_kF_{k+1} - F_k(F_{k-1} + F_k) - (-1)^k \\
&= F_kF_{k+1} - F_kF_{k+1} - (-1)^k \\
&= (-1)^{k+1}
\end{aligned}$$

□

Now, we can derive similar formula for Fibonacci Polynomials, $F_n(p,q)$.
By using Fibonacci Polynomials from previous section, we start with;

n=1: $F_0(p,q)\ F_2(p,q) - F_1^2(p,q) = 0 \cdot p - 1^2 = -1$

n=2: $F_1(p,q)\ F_3(p,q) - F_2^2(p,q) = 1 \cdot (p^2 + q) - p^2 = q$

n=3: $F_2(p,q)\ F_4(p,q) - F_3^2(p,q) = p \cdot (p(p^2 + q) + qp) - (p^2 + q)^2 = -q^2$

n=4: $F_3(p,q)\ F_5(p,q) - F_4^2(p,q) = (p^2+q)\cdot(p^2(p^2+2q)+q(p^2+q)) - (p(p^2+q)+qp)^2 = q^3$

$\vdots$

Thus, generalized Cassini formula can be claimed and proved as a next proposition.

Proposition 2.9 *Cassini's formula for Fibonacci polynomials is given by,*

$$F_{n-1}(p,q)\ \ F_{n+1}(p,q) - F_n^2(p,q) = (-1)^n q^{n-1} \tag{2.37}$$

Proof Substituting the Binet type formula for Fibonacci polynomials,

$$\begin{aligned}
F_{n-1}(p,q)F_{n+1}(p,q) - F_n^2(p,q) &= \frac{a^{n-1} - b^{n-1}}{a-b} \cdot \frac{a^{n+1} - b^{n+1}}{a-b} - \left(\frac{a^n - b^n}{a-b}\right)^2 \\
&= \left(\frac{1}{a-b}\right)^2 \left[a^{2n} - a^{n-1}b^{n+1} - a^{n+1}b^{n-1} + b^{2n} - (a^{2n} - 2(ab)^n + b^{2n})\right] \\
&= \left(\frac{1}{a-b}\right)^2 \left[(ab)^n \left(-\frac{b}{a} - \frac{a}{b} + 2\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{a-b}\right)^2 \left[(ab)^n \left(-\frac{(a^2-2ab+b^2)}{ab}\right)\right] \\
&= \left(\frac{1}{a-b}\right)^2 (ab)^n \left(-\frac{(a-b)^2}{ab}\right) \\
&= -(ab)^{n-1} \\
&\overset{(ab=-q)}{=} -(-q)^{n-1} \\
&= (-1)^n (q)^{n-1}
\end{aligned}
$$

□

If $p = q = 1$, then equation (2.37) reduces to the equation (2.35).

Now, let's define $A_2(p,q)$ matrix by using Fibonacci polynomials introduced in Section 2.7.

Definition 2.4 *Let*

$$
A_2(p,q) = \begin{pmatrix} F_0(p,q) & F_1(p,q) \\ qF_1(p,q) & F_2(p,q) \end{pmatrix} \tag{2.38}
$$

Arbitrary n^{th} power of $A_2(p,q)$ matrix is found by next proposition.

Proposition 2.10 *For every positive integer n, we have;*

$$
[A_2(p,q)]^n = \begin{pmatrix} qF_{n-1}(p,q) & F_n(p,q) \\ qF_n(p,q) & F_{n+1}(p,q) \end{pmatrix} \tag{2.39}
$$

Proof Proof will be done by Principal of Mathematical Induction on n. For $n = 1$, we have;

$$
A_2(p,q) = \begin{pmatrix} F_0(p,q) & F_1(p,q) \\ qF_1(p,q) & F_2(p,q) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}.
$$

Suppose that for $n = k$,

$$
[A_2(p,q)]^k = \begin{pmatrix} qF_{k-1}(p,q) & F_k(p,q) \\ qF_k(p,q) & F_{k+1}(p,q) \end{pmatrix}
$$

is true. For the case $n = k + 1$, we obtain;

$$\begin{aligned}[A_2(p,q)]^{k+1} &= [A_2(p,q)]^k A_2(p,q) = \begin{pmatrix} qF_{k-1}(p,q) & F_k(p,q) \\ qF_k(p,q) & F_{k+1}(p,q) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix} \\ &= \begin{pmatrix} qF_k(p,q) & qF_{k-1}(p,q) + pF_k(p,q) \\ qF_{k+1}(p,q) & qF_k(p,q) + pF_{k+1}(p,q) \end{pmatrix} \\ &\overset{(2.32)}{=} \begin{pmatrix} qF_k(p,q) & F_{k+1}(p,q) \\ qF_{k+1}(p,q) & F_{k+2}(p,q) \end{pmatrix}.\end{aligned}$$

□

The equation (2.37) will be helpful if we calculate the determinant of the matrix $[A_2(p,q)]^n$;

$$\begin{aligned} det\left([A_2(p,q)]^n\right) &= \begin{vmatrix} qF_{n-1}(p,q) & F_n(p,q) \\ qF_n(p,q) & F_{n+1}(p,q) \end{vmatrix} \\ &= q\left[F_{n-1}(p,q) \cdot F_{n+1}(p,q) - F_n^2(p,q)\right] \\ &\overset{(2.37)}{=} q\left[(-1)^n q^{n-1}\right] \\ &= (-1)^n q^n = (-q)^n \end{aligned}$$

Thus;

$$q\left[F_{n-1}(p,q) \cdot F_{n+1}(p,q) - F_n^2(p,q)\right] = (-q)^n \tag{2.40}$$

or,

$$\boxed{det\left([A_2(p,q)]^n\right) = (-1)^n q^n} \tag{2.41}$$

CHAPTER 3

FIBONACCI CALCULUS

In this Chapter, we follow notations and some results from (Pashaev O.K. and Nalci S., 2012).

3.1. Golden Derivative

Definition 3.1 *The Golden derivative operator* D_F^x *acts on arbitrary function* $f(x)$ *according to formula;*

$$D_F^x[f(x)] = \frac{f(\varphi x) - f(-\frac{x}{\varphi})}{\left(\varphi - \left(-\frac{1}{\varphi}\right)\right)x} = \frac{f(\varphi x) - f(\varphi' x)}{(\varphi - \varphi')\, x} \tag{3.1}$$

The Golden derivative operator is a linear operator since for every pair of functions f and g and scalar λ, the following properties hold;

- $D_F^x(f(x) + g(x)) = D_F^x(f(x)) + D_F^x(g(x))$
- $D_F^x(\lambda f(x)) = \lambda\, D_F^x(f(x))$

3.1.1. Golden Leibnitz Rule

By using definition of Golden derivative, the Golden Leibnitz Rule can be derived in the following way;

$$\begin{aligned}
D_F^x(f(x)g(x)) &= \frac{f(\varphi x)g(\varphi x) - f\left(-\frac{x}{\varphi}\right)g\left(-\frac{x}{\varphi}\right)}{(\varphi - \varphi')x} \\
&= \frac{\left(f(\varphi x) - f\left(-\frac{x}{\varphi}\right) + f\left(-\frac{x}{\varphi}\right)\right)g(\varphi x) - f\left(-\frac{x}{\varphi}\right)g\left(-\frac{x}{\varphi}\right)}{(\varphi - \varphi')x} \\
&= \frac{f(\varphi x) - f\left(-\frac{x}{\varphi}\right)}{(\varphi - \varphi')x}g(\varphi x) + \frac{f\left(-\frac{x}{\varphi}\right)g(\varphi x)}{(\varphi - \varphi')x} - \frac{f\left(-\frac{x}{\varphi}\right)g\left(-\frac{x}{\varphi}\right)}{(\varphi - \varphi')x}
\end{aligned}$$

$$
\begin{aligned}
&= D_F^x(f(x))\, g(\varphi x) + f\left(-\frac{x}{\varphi}\right) \left(\frac{g(\varphi x) - g\left(-\frac{x}{\varphi}\right)}{(\varphi - \varphi')x}\right) \\
&= D_F^x(f(x))\, g(\varphi x) + f\left(-\frac{x}{\varphi}\right) D_F^x(g(x))
\end{aligned}
$$

This is why we have the following proposition.

Proposition 3.1 *(Golden Leibnitz Rule)*

$$
\begin{aligned}
1)\ D_F^x(f(x)g(x)) &= D_F^x(f(x))\, g(\varphi x) + f\left(-\frac{x}{\varphi}\right) D_F^x(g(x)) && (3.2) \\
2)\ D_F^x(f(x)g(x)) &= D_F^x(f(x))\, g\left(-\frac{x}{\varphi}\right) + f(\varphi x)\, D_F^x(g(x)) && (3.3) \\
3)\ D_F^x(f(x)g(x)) &= D_F^x(f(x)) \left(\frac{g(\varphi x) + g\left(-\frac{x}{\varphi}\right)}{2}\right) \\
&+ D_F^x(g(x)) \left(\frac{f(\varphi x) + f\left(-\frac{x}{\varphi}\right)}{2}\right) && (3.4) \\
4)\ D_F^x(f(x)g(x)) &= \left(\alpha f\left(-\frac{x}{\varphi}\right) + (1-\alpha) f(\varphi x)\right) D_F^x(g(x)) \\
&+ \left(\alpha g(\varphi x) + (1-\alpha) g\left(-\frac{x}{\varphi}\right)\right) D_F^x(f(x)) && (3.5)
\end{aligned}
$$

From the definition of Golden derivative (3.1), by symmetry, we can interchange $\varphi \leftrightarrow \varphi'$ to get 2). Formulas 1) and 2) can be rewritten in explicitly symmetrical form 3). By multiplying (3.2) with α, (3.3) with $(1-\alpha)$ and adding them, more general form of Golden Leibnitz formula is obtained, which is given with an arbitrary α in 4). By choosing $\alpha = 1$, we have (3.2), and for $\alpha = \frac{1}{2}$, (3.4) is obtained.

Example 3.1 *For function $F(x) = x^5$, golden derivative is obtained as;*

$$
D_F^x(x^5) = F_5\, x^4 = 5x^4.
$$

As an another way, by using (3.2) and choosing $f(x) = x^2$, $g(x) = x^3$;

$$
\begin{aligned}
D_F^x(x^5) &= D_F^x(x^2 \cdot x^3) = D_F^x(x^2)\, (\varphi x)^3 + (\varphi' x)^2\, D_F^x(x^3) = F_2\, x(\varphi x)^3 + (\varphi' x)^2 F_3 x^2 \\
&= \varphi^3 x^4 + 2\varphi'^2 x^4 = (\varphi^3 + 2\varphi'^2)x^4 = (2\varphi + 1 + 2(\varphi' + 1))x^4 = 5x^4.
\end{aligned}
$$

So,the same result is obtained by using Golden Leibnitz rule.

Now we may compute the Golden derivative of the quotient of $f(x)$ and $g(x)$ as;

$$D_F^x(f(x)) \quad = \quad D_F^x\left(g(x)\frac{f(x)}{g(x)}\right)$$

and by using (3.2),

$$D_F^x(f(x)) \quad = \quad D_F^x(g(x))\,\frac{f(\varphi x)}{g(\varphi x)} + g\left(-\frac{x}{\varphi}\right)\,D_F^x\left(\frac{f(x)}{g(x)}\right).$$

If we leave alone $D_F^x\left(\frac{f(x)}{g(x)}\right)$ in the right hand side, we can get the Golden derivative of the quotient.

Proposition 3.2

$$D_F^x\left(\frac{f(x)}{g(x)}\right) = \frac{D_F^x(f(x))\,g(\varphi x) - f(\varphi x)\,D_F^x(g(x))}{g(\varphi x)\,g\left(-\frac{x}{\varphi}\right)} \tag{3.6}$$

Similar to the product rule, the quotient rule can be written in several different forms.

3.1.2. Golden Periodic Function

Definition 3.2 *Function $A(x)$ is called the Golden periodic function if;*

$$D_F^x\ (A(x)) = 0 \qquad \Longleftrightarrow \qquad A(\varphi x) = A\left(-\frac{x}{\varphi}\right) \tag{3.7}$$

or,

$$A(\varphi x) = A(\varphi' x) \qquad \Longleftrightarrow \qquad A(\varphi^2 x) = A(-x) \tag{3.8}$$

The Golden periodic function has interesting symmetry; rescaling argument x by φ^2 in positive direction is equivalent to value of function at reflected point $-x$.

As an example, we consider;

$$A(x) = \sin\left(\frac{\pi}{\ln\varphi}\ln|x|\right). \tag{3.9}$$

This function is Golden periodic. (See the Appendix B.1.) And, it is an even function $A(x) = A(-x)$. As a result, for this function we have golden periodicity condition in the form;

$$A(\varphi^2 x) = A(x). \tag{3.10}$$

So, this function satisfies self similarity property with φ^2 scaling factor. It is seen from Figure 3.1 and Figure 3.2 that rescaling interval of x by φ^2 does not change shape of the function. This is the property of Golden self-similar even function.

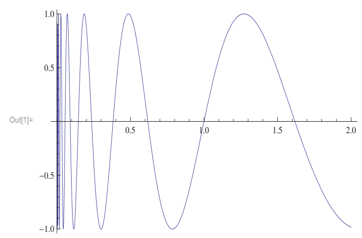

Figure 3.1. Graph of the function $A(x)$ on interval $0 \leq x \leq 2$

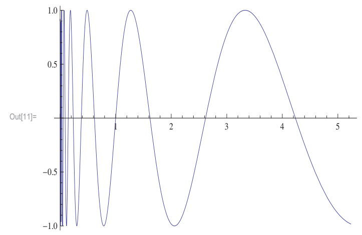

Figure 3.2. Graph of the function $A(x)$ on interval $0 \leq x \leq 2\varphi^2$

3.2. Generating Functions for Fibonacci Numbers

Example 3.2 *Application of the Golden derivative operator D_F^x on x^n generates Fibonacci numbers;*

$$D_F^x(x^n) = \frac{(\varphi x)^n - (\varphi' x)^n}{(\varphi - \varphi')x} = \frac{\varphi^n - \varphi'^n}{\varphi - \varphi'} x^{n-1} = F_n \; x^{n-1}$$

So, Fibonacci numbers can be represented also as,

$$\boxed{F_n = \frac{D_F^x(x^n)}{x^{n-1}}} \tag{3.11}$$

Definition 3.3 *The function $F(x)$,*

$$F(x) = \sum_{n=0}^{\infty} F_n x^n \tag{3.12}$$

is called the generating function of Fibonacci numbers F_n. According to Taylor formula;

$$F_n = \frac{1}{n!} \frac{d^n}{dx^n} F(x) \bigg|_{x=0} \tag{3.13}$$

in a disk of analyticity around $x = 0$. Explicit form of the series is;

$$F(x) = x + x^2 + 2x^3 + 3x^4 + 5x^5 + \ldots \tag{3.14}$$

Proposition 3.3 *Generating function $F(x)$ which is convergent in domain $|x| < |\varphi'|$ has explicit representation;*

$$F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}. \tag{3.15}$$

Proof To find the domain of convergency, we apply the ratio test:

$$\rho = \lim_{n\to\infty}\left|\frac{F_{n+1}x^{n+1}}{F_n x^n}\right| = \lim_{n\to\infty}\left|\frac{F_{n+1}}{F_n}\right| \lim_{n\to\infty}|x| = \varphi|x|$$

For convergency $\rho < 1$ implies,

$$|x| < \frac{1}{\varphi} = |\varphi'|$$

and as follows $|x| < |\varphi'| < 1$. By using Golden derivative, we have;

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} F_n x^n = x \sum_{n=0}^{\infty} F_n x^{n-1} \overset{(3.11)}{=} x \sum_{n=0}^{\infty} D_F^x(x^n) = x\, D_F^x \sum_{n=0}^{\infty} x^n \\ &\overset{|x|<1}{=} x\, D_F^x\left(\frac{1}{1-x}\right) \\ &\overset{(3.1)}{=} x\,\frac{\left(\frac{1}{1-\varphi x} - \frac{1}{1-\varphi' x}\right)}{(\varphi-\varphi')x} \\ &= \frac{x}{(1-\varphi x)(1-\varphi' x)} = \frac{x}{1-x-x^2}. \end{aligned}$$

□

Corollary 3.1 *$F(x)$ is meromorphic function with one zero at $x = 0$ and two single poles at $x = -\varphi$ and $x = -\varphi'$. Indeed,*

$$1-\varphi x = 0 \Rightarrow x = \frac{1}{\varphi} = -\varphi'$$
$$1-\varphi' x = 0 \Rightarrow x = \frac{1}{\varphi'} = -\varphi$$

3.2.1. Entire Generating Function

In previous section, we considered generating function $F(x)$ for Fibonacci numbers in disk $|x| < |\varphi'|$. Here, we introduce generating function for Fibonacci numbers which is entire function.

By calculating the Golden derivative of exponential function e^x, in power series form;

$$\begin{aligned} D_F^x(e^x) &= D_F^x\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right) = D_F^x\left(\frac{1}{0!}+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...\right) \\ &= \sum_{n=1}^{\infty} D_F^x\left(\frac{x^n}{n!}\right) = \sum_{n=1}^{\infty}\frac{D_F^x(x^n)}{n!} = \sum_{n=1}^{\infty} F_n\frac{x^{n-1}}{n!} = \sum_{n=0}^{\infty} F_{n+1}\frac{x^n}{(n+1)!} \end{aligned}$$

so that,

$$D_F^x(e^x) = \sum_{n=0}^{\infty} F_{n+1}\frac{x^n}{(n+1)!}. \tag{3.16}$$

As easy to see by ratio test, the right hand side is an entire function. From another side, calculating $D_F^x(e^x)$ by using Golden derivative formula (3.1) gives;

$$D_F^x(e^x) = \frac{e^{\varphi x}-e^{\varphi' x}}{(\varphi-\varphi')x} = \frac{e^{\left(\frac{1+\sqrt{5}}{2}\right)x}-e^{\left(\frac{1-\sqrt{5}}{2}\right)x}}{\left(\frac{1+\sqrt{5}}{2}-\frac{1-\sqrt{5}}{2}\right)x} = \frac{e^{\frac{1}{2}x}\left(e^{\frac{\sqrt{5}}{2}x}-e^{-\frac{\sqrt{5}}{2}x}\right)}{\sqrt{5}x} = \frac{e^{\frac{x}{2}}2\sinh\left(\frac{\sqrt{5}}{2}x\right)}{\sqrt{5}x}.$$

Thus, by this alternative way, we obtain;

$$D_F^x(e^x) = \frac{2e^{\frac{x}{2}}\sinh\left(\frac{\sqrt{5}}{2}x\right)}{\sqrt{5}x}. \tag{3.17}$$

By equating both results (3.16) & (3.17), we obtain identity;

$$\boxed{\sum_{n=0}^{\infty}\frac{F_{n+1}}{(n+1)!}x^n = e^{\frac{x}{2}}\;\frac{\sinh\left(\frac{\sqrt{5}}{2}x\right)}{\frac{\sqrt{5}}{2}x}} \tag{3.18}$$

This relation can be considered as entire generating function of Fibonacci numbers(up to factorial).

By setting $x = 1$, the sum of the series;

$$\sum_{n=1}^{\infty} \frac{F_n}{n!} = e^{\frac{1}{2}} \frac{\sinh\left(\frac{\sqrt{5}}{2}\right)}{\frac{\sqrt{5}}{2}} \tag{3.19}$$

is obtained.

After replacing $x \to ix$ in equation (3.18), we get,

$$\sum_{n=0}^{\infty} \frac{F_{n+1}}{(n+1)!}(i)^n x^n = e^{\frac{ix}{2}} \frac{\sinh\left(i\frac{\sqrt{5}}{2}x\right)}{i\frac{\sqrt{5}}{2}x}. \tag{3.20}$$

By using $\sinh(ix) = i\sin(x)$ at the right hand side of equality, and splitting the sum at the left hand side to even and odd parts with $n = 2k$ and $n = 2k+1$ gives;

$$\sum_{k=0}^{\infty} \frac{F_{2k+1}}{(2k+1)!}(i)^{2k} x^{2k} + \sum_{k=0}^{\infty} \frac{F_{2k+2}}{(2k+2)!}(i)^{2k+1} x^{2k+1} = e^{\frac{ix}{2}} \frac{\sin\left(\frac{\sqrt{5}}{2}\right)x}{\frac{\sqrt{5}}{2}x} \tag{3.21}$$

Since $i^2 = -1$,

$$\sum_{k=0}^{\infty} \frac{F_{2k+1}}{(2k+1)!}(-1)^k x^{2k} + i\sum_{k=0}^{\infty} \frac{F_{2k+2}}{(2k+2)!}(-1)^k x^{2k+1} = e^{\frac{ix}{2}} \frac{\sin\left(\frac{\sqrt{5}}{2}x\right)}{\frac{\sqrt{5}}{2}x} \tag{3.22}$$

Writing $e^{\frac{ix}{2}} = \cos\left(\frac{x}{2}\right) + i\sin\left(\frac{x}{2}\right)$,

$$\sum_{k=0}^{\infty} \frac{F_{2k+1}}{(2k+1)!}(-1)^k x^{2k} + i\sum_{k=0}^{\infty} \frac{F_{2k+2}}{(2k+2)!}(-1)^k x^{2k+1} = \cos\left(\frac{x}{2}\right)\frac{\sin\left(\frac{\sqrt{5}}{2}x\right)}{\frac{\sqrt{5}}{2}x} + i\sin\left(\frac{x}{2}\right)\frac{\sin\left(\frac{\sqrt{5}}{2}x\right)}{\frac{\sqrt{5}}{2}x}$$

Now, equating real and imaginary parts, we have new identities, as generating functions for even and odd order Fibonacci numbers,

$$\boxed{\sum_{k=0}^{\infty} \frac{F_{2k+1}}{(2k+1)!}(-1)^k x^{2k} = \cos\left(\frac{x}{2}\right) \frac{\sin\left(\frac{\sqrt{5}}{2}x\right)}{\frac{\sqrt{5}}{2}x}} \tag{3.23}$$

and,

$$\boxed{\sum_{k=0}^{\infty} \frac{F_{2k+2}}{(2k+2)!}(-1)^k x^{2k+1} = \sin\left(\frac{x}{2}\right) \frac{\sin\left(\frac{\sqrt{5}}{2}x\right)}{\frac{\sqrt{5}}{2}x}} \tag{3.24}$$

Functions in (3.23) and (3.24) are entire functions, giving several interesting identities for different values of x.

From (3.23) follow identities for:

1) $x = \pi$,

$$\sum_{k=0}^{\infty} \frac{F_{2k+1}}{(2k+1)!}(-1)^k \pi^{2k} = 0 \tag{3.25}$$

2) $x = \frac{2\pi}{\sqrt{5}}$,

$$\sum_{k=0}^{\infty} \frac{F_{2k+1}}{(2k+1)!}(-1)^k \frac{(2\pi)^{2k}}{5^k} = 0 \tag{3.26}$$

3) $x = \frac{\pi}{\sqrt{5}}$,

$$\sum_{k=0}^{\infty} \frac{F_{2k+1}}{(2k+1)!}(-1)^k \frac{\pi^{2k}}{5^k} = \frac{2\cos\left(\frac{\pi}{2\sqrt{5}}\right)}{\pi} \tag{3.27}$$

4) $x = 2\pi$,

$$\sum_{k=0}^{\infty} \frac{F_{2k+1}}{(2k+1)!}(-1)^k (2\pi)^{2k} = -\frac{\sin\left(\sqrt{5}\pi\right)}{\sqrt{5}\pi} \tag{3.28}$$

5) $x = 1$,

$$\sum_{k=0}^{\infty} \frac{F_{2k+1}}{(2k+1)!}(-1)^k = \cos\left(\frac{1}{2}\right) \frac{\sin\left(\frac{\sqrt{5}}{2}\right)}{\frac{\sqrt{5}}{2}} \tag{3.29}$$

Also, from equation (3.24) follow identities for:

1) $x = \pi$,

$$\sum_{k=0}^{\infty} \frac{F_{2k+2}}{(2k+2)!}(-1)^k \pi^{2k+1} = \frac{\sin\left(\frac{\sqrt{5}}{2}\pi\right)}{\frac{\sqrt{5}}{2}\pi} \tag{3.30}$$

2) $x = \frac{2\pi}{\sqrt{5}}$,

$$\sum_{k=0}^{\infty} \frac{F_{2k+2}}{(2k+2)!}(-1)^k \left(\frac{2\pi}{\sqrt{5}}\right)^{2k+1} = 0 \tag{3.31}$$

3) $x = \frac{\pi}{\sqrt{5}}$,

$$\sum_{k=0}^{\infty} \frac{F_{2k+2}}{(2k+2)!}(-1)^k \left(\frac{\pi}{\sqrt{5}}\right)^{2k+1} = \frac{2\sin\left(\frac{\pi}{2\sqrt{5}}\right)}{\pi} \tag{3.32}$$

4) $x = 2\pi$,

$$\sum_{k=0}^{\infty} \frac{F_{2k+2}}{(2k+2)!}(-1)^k (2\pi)^{2k+1} = 0 \tag{3.33}$$

5) $x = 1$,

$$\sum_{k=0}^{\infty} \frac{F_{2k+2}}{(2k+2)!}(-1)^k = \sin\left(\frac{1}{2}\right) \frac{\sin\left(\frac{\sqrt{5}}{2}\right)}{\frac{\sqrt{5}}{2}} \tag{3.34}$$

3.3. Golden Taylor Formula

Taylor expansion of arbitrary polynomial to the set of polynomials is determined by Theorem (Kac, V. and Cheung, P., 2002). Here, we apply this theorem to Golden polynomials.

Theorem 3.1 *(Golden Taylor Expansion)*

The Golden derivative operator D_F^x as a linear operator acts on the space of polynomials, and

$$P_n(x) \equiv \frac{x^n}{F_n!} \equiv \frac{x^n}{F_1 \cdot F_2 \ldots F_n}$$

satisfy the following conditions:

(i) $P_0(0) = 1$ *and* $P_n(0) = 0$ *for any* $n \geq 1$;

(ii) $deg(P_n) = n$;

(iii) $D_F^x(P_n(x)) = P_{n-1}(x)$ *for any* $n \geq 1$, *and* $D_F^x(1) = 0$.

Then, for any polynomial f(x) of degree N, one has the following Taylor formula;

$$f(x) = \sum_{n=0}^{N} (D_F^x)^n f(0) P_n(x) = \sum_{n=0}^{N} (D_F^x)^n f(0) \frac{x^n}{F_n!}.$$

In the limit $N \to \infty$ (if it exists) this formula determines expansion of the function;

$$f(x) = \sum_{n=0}^{\infty} (D_F^x f)^n(0) \frac{x^n}{F_n!}.$$

If an infinite series,

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{F_n!} \tag{3.35}$$

is convergent in some domain, then it determines function,

$$f_F(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{F_n!} \tag{3.36}$$

in this domain, which we call the Golden-Fibonacci function.

Let's check the convergency of functions;

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \quad \text{and} \quad f_F(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{F_n!} \tag{3.37}$$

By the ratio test, we have;

$$\rho = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{(n+1)!}\frac{n!}{a_nx^n}\right| = |x|\lim_{n\to\infty}\left|\frac{1}{n+1}\right|\left|\frac{a_{n+1}}{a_n}\right| \tag{3.38}$$

$$\rho_F = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{F_{n+1}!}\frac{F_n!}{a_nx^n}\right| = |x|\lim_{n\to\infty}\left|\frac{1}{F_{n+1}}\right|\left|\frac{a_{n+1}}{a_n}\right|. \tag{3.39}$$

The second limit implies,

$$\lim_{n\to\infty}\left|\frac{1}{F_{n+1}}\right|\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{F_{n+1}}\right|\left|\frac{1}{n+1}\right|\left|\frac{a_{n+1}}{a_n}\right|.$$

If $f(x)$ is entire function, then $\lim_{n\to\infty}\left|\frac{1}{n+1}\right|\left|\frac{a_{n+1}}{a_n}\right| \to 0$. Since $\lim_{n\to\infty}\left|\frac{n+1}{F_{n+1}}\right| = 0$, then as follows function $f_F(x)$ is also entire. This means that to every entire function $f(x)$ we can relate another entire function $f_F(x)$.

As an example, here we consider e^x which is entire function. Then, corresponding Golden exponential as,

$$e_F^x = \sum_{n=0}^{\infty}\frac{x^n}{F_n!} \tag{3.40}$$

is also entire function.

3.4. Golden Exponential Functions

The 1^{st} and 2^{nd} type of Golden Exponential functions are defined as;

$$\boxed{e_F^x \equiv \sum_{n=0}^{\infty}\frac{x^n}{F_n!}} \tag{3.41}$$

and

$$\boxed{E_F^x \equiv \sum_{n=0}^{\infty}(-1)^{\frac{n(n-1)}{2}}\frac{x^n}{F_n!}} \tag{3.42}$$

Both of these are entire functions.

Definition 3.4

$$F_n! = \begin{cases} 1, & \textit{if } n = 0; \\ F_1 F_2 F_3 \ldots F_n, & n \geq 1. \end{cases} \tag{3.43}$$

Proposition 3.4 *There is a relation between these functions:*

$$E_F^x \equiv e_{-F}^x$$

Proof To show this, we write;

$$\begin{aligned} E_F^x &= \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{x^n}{F_n!} = \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{x^n}{F_n!} \frac{(-1)^{\frac{n(n-1)}{2}}}{(-1)^{\frac{n(n-1)}{2}}} = \sum_{n=0}^{\infty} (-1)^{n(n-1)} \frac{x^n}{(-1)^{\frac{n(n-1)}{2}} F_n!} \\ &= \sum_{n=0}^{\infty} \frac{x^n}{(-1)^{\frac{n(n-1)}{2}} F_n!}. \end{aligned}$$

By using (2.4), we calculate,

$$\begin{aligned} F_{-n}! &= F_{-n} F_{-n+1} \ldots F_{-1} \overset{(2.4)}{=} (-1)^{n-1} F_n \, (-1)^{n-2} F_{n-1} \ldots F_1 \\ &= (-1)^{(n-1)+(n-2)+\ldots+1} F_n! = (-1)^{\frac{n(n-1)}{2}} F_n! \end{aligned}$$

and obtain that,

$$F_{-n}! = (-1)^{\frac{n(n-1)}{2}} F_n! \tag{3.44}$$

Then, it gives;

$$E_F^x = \sum_{n=0}^{\infty} \frac{x^n}{(-1)^{\frac{n(n-1)}{2}} F_n!} \overset{(3.44)}{=} \sum_{n=0}^{\infty} \frac{x^n}{F_{-n}!} \equiv e_{-F}^x \tag{3.45}$$

□

Graphs of two exponential functions can be seen in the Figures 3.3 and 3.4.

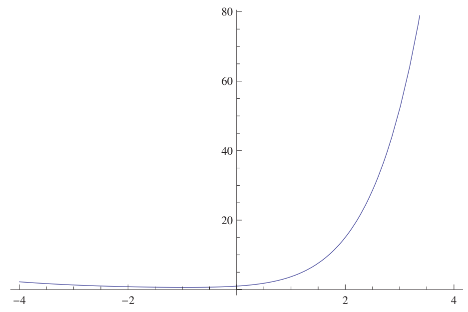

Figure 3.3. Graph of the function e_F^x

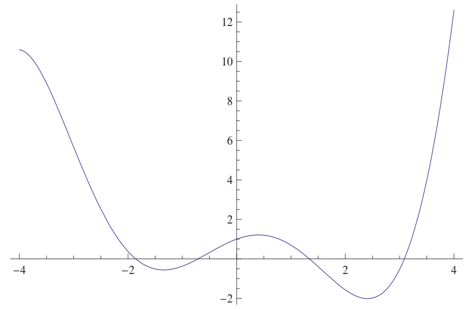

Figure 3.4. Graph of the function E_F^x

Conjecture: Function e_F^x has no zeros, but function E_F^x has infinitely many zeros and these zeros are located at Fibonacci numbers.

Theorem 3.2 *The Golden derivative of Golden exponential functions is found as;*

$$D_F^x(e_F^{kx}) = ke_F^{kx} \tag{3.46}$$

$$D_F^x(E_F^{kx}) = kE_F^{-kx} \tag{3.47}$$

for an arbitrary k.

Proof

$$\begin{aligned}
D_F^x(e_F^{kx}) = D_F^x\left(\sum_{n=0}^{\infty}\frac{(kx)^n}{F_n!}\right) &= D_F^x\left(\frac{1}{F_0!}+\frac{kx}{F_1!}+\frac{k^2x^2}{F_2!}+\frac{k^3x^3}{F_3!}+\ldots\right)\\
&= D_F^x\left(\sum_{n=1}^{\infty}\frac{k^nx^n}{F_n!}\right)=\sum_{n=1}^{\infty}k^n\frac{D_F^x(x^n)}{F_n!}\\
&= \sum_{n=1}^{\infty}\frac{k^nF_nx^{n-1}}{F_n!}=\sum_{n=1}^{\infty}\frac{k^nx^{n-1}}{F_{n-1}!}\\
&= \sum_{n=0}^{\infty}\frac{k^{n+1}x^n}{F_n!}=k\sum_{n=0}^{\infty}\frac{(kx)^n}{F_n!}\\
&= ke^{kx}.
\end{aligned}$$

$$\begin{aligned}
D_F^x(E_F^{kx}) = D_F\left(\sum_{n=0}^{\infty}(-1)^{\frac{n(n-1)}{2}}\frac{x^n}{F_n!}\right) &= D_F^x\left(\frac{1}{F_0!}+\frac{kx}{F_1!}-\frac{k^2x^2}{F_2!}-\frac{k^3x^3}{F_3!}+\ldots\right)\\
&= D_F^x\left(\sum_{n=1}^{\infty}(-1)^{\frac{n(n-1)}{2}}\frac{k^nx^n}{F_n!}\right)\\
&= \sum_{n=1}^{\infty}(-1)^{\frac{n(n-1)}{2}}k^n\frac{D_F^x(x^n)}{F_n!}\\
&= \sum_{n=1}^{\infty}(-1)^{\frac{n(n-1)}{2}}k^n\frac{F_nx^{n-1}}{F_n!}\\
&= \sum_{n=1}^{\infty}(-1)^{\frac{n(n-1)}{2}}k^n\frac{x^{n-1}}{F_{n-1}!}\\
&= \sum_{n=0}^{\infty}(-1)^{\frac{(n+1)n}{2}}k^{n+1}\frac{x^n}{F_n!}\\
&= k\sum_{n=0}^{\infty}(-1)^{\frac{(n^2+n)}{2}}k^n\frac{x^n}{F_n!}\\
&= k\sum_{n=0}^{\infty}(-1)^{\frac{(n^2+n)-n+n}{2}}\frac{(kx)^n}{F_n!}\\
&= k\sum_{n=0}^{\infty}(-1)^{\frac{n(n-1)}{2}}(-1)^n\frac{(kx)^n}{F_n!}\\
&= k\sum_{n=0}^{\infty}(-1)^{\frac{n(n-1)}{2}}\frac{(-kx)^n}{F_n!}\\
&= kE^{-kx}
\end{aligned}$$

□

3.4.1. Estimating the Number e_F

Fibonacci exponential function,

$$e_F^x \equiv \sum_{n=0}^{\infty} \frac{x^n}{F_n!}$$

determines Fibonacci analog of Euler number $e \equiv e^x \Big|_{x=1} \approx 2.718$.

To estimate it, first we have $e_F^x \Big|_{x=1} = e_F$ as the sum;

$$\begin{aligned} e_F = \sum_{n=0}^{\infty} \frac{1}{F_n!} &= \frac{1}{F_0!} + \frac{1}{F_1!} + \frac{1}{F_2!} + \frac{1}{F_3!} + \frac{1}{F_4!} + \frac{1}{F_5!} + \frac{1}{F_6!} + \sum_{n=7}^{\infty} \frac{1}{F_n!} \\ &= 1 + 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{30} + \frac{1}{240} + \sum_{n=7}^{\infty} \frac{1}{F_n!} \\ &\approx 3.70416666 + \sum_{n=7}^{\infty} \frac{1}{F_n!} \end{aligned}$$

This gives the lower bound,

$$\boxed{3.7041 < e_F}$$

To get the upper bound, we combine

$$\begin{aligned} e_F &= 3 + \frac{1}{F_3!} + \frac{1}{F_4!} + \frac{1}{F_1 \ldots F_4 \cdot F_5} + \frac{1}{F_1 \ldots F_4 \cdot F_5 \cdot F_6} \left(1 + \frac{1}{F_7} + \frac{1}{F_7 \cdot F_8} + \ldots \right) \\ &= 3.6666 + 0.0333 + \frac{1}{240} \left(1 + \frac{1}{F_7} + \frac{1}{F_7 \cdot F_8} + \ldots \right) \end{aligned}$$

Since,

$$F_n > 12 \quad \text{for all} \quad n > 7 \quad \Rightarrow \quad \frac{1}{F_n} < \frac{1}{12}$$

$$
\begin{aligned}
e_F &< 3.6666 + 0.0333 + \frac{1}{240}\left(1 + \frac{1}{12} + \frac{1}{12\cdot 12} + \frac{1}{12\cdot 12\cdot 12} + \dots\right)\\
e_F &< 3.6666 + 0.0333 + \frac{1}{240}\left(\frac{1}{1-\frac{1}{12}}\right)\\
e_F &< 3.6666 + 0.0333 + \frac{1}{240}\frac{12}{11}\\
e_F &< 3.6666 + 0.0333 + \frac{1}{220}
\end{aligned}
$$

Therefore, upper bound is obtained, as

$$\boxed{e_F < 3.7044}$$

Combining both, the lower and the upper bounds of this number, we get estimation;

$$3.7041 < e_F < 3.7044 \tag{3.48}$$

3.5. Golden Trigonometric and Hyperbolic Functions

Definition 3.5 *Fibonacci cosine and sine functions are defined by the power series,*

$$\cos_F(x) \equiv \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{F_{2n}!}, \tag{3.49}$$

$$\sin_F(x) \equiv \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{F_{2n+1}!}, \tag{3.50}$$

$$\cosh_F(x) \equiv \sum_{n=0}^{\infty} \frac{x^{2n}}{F_{2n}!}, \tag{3.51}$$

$$\sinh_F(x) \equiv \sum_{n=0}^{\infty} \frac{x^{2n+1}}{F_{2n+1}!}. \tag{3.52}$$

Proposition 3.5 *Since*

$$e_F^x = \sum_{n=0}^{\infty} \frac{x^n}{F_n!} \quad , \quad E_F^x = \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{x^n}{F_n!}$$

representations are valid, then;

$$\cos_F(x) = \frac{e_F^{ix} + e_F^{-ix}}{2} = \frac{E_F^x + E_F^{-x}}{2} \tag{3.53}$$
$$\sin_F(x) = \frac{e_F^{ix} - e_F^{-ix}}{2i} = \frac{E_F^x - E_F^{-x}}{2} \tag{3.54}$$
$$\cosh_F(x) = \frac{e_F^{x} + e_F^{-x}}{2} = \frac{E_F^{ix} + E_F^{-ix}}{2} \tag{3.55}$$
$$\sinh_F(x) = \frac{e_F^{x} - e_F^{-x}}{2} = \frac{E_F^{ix} - E_F^{-ix}}{2i} \tag{3.56}$$

are shown in Figures 3.5 and 3.6.

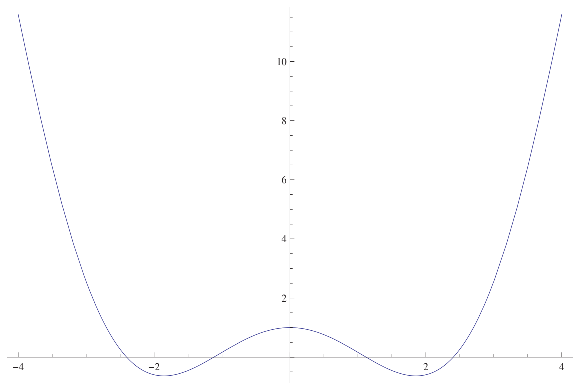

Figure 3.5. Graph of function $\cos_F(x)$

Since,

$$\cos_F(x) = \frac{E_F^x + E_F^{-x}}{2} = \frac{E_F^{-x} + E_F^{x}}{2} = \cos_F(-x),$$

it is even function and it is symmetric about y-axis.

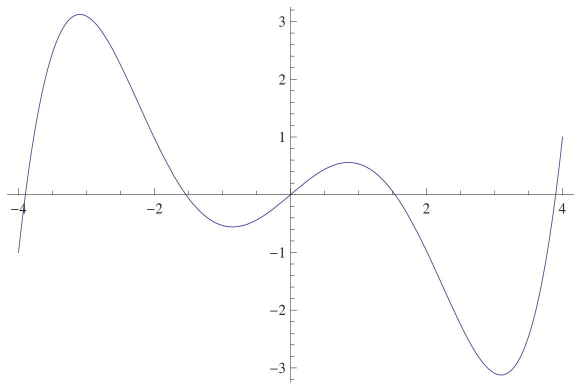

Figure 3.6. Graph of function $\sin_F(x)$

Since,

$$\sin_F(x) = \frac{E_F^x - E_F^{-x}}{2} = -\left(\frac{E_F^{-x} - E_F^x}{2}\right) = -\sin_F(-x),$$

$\sin_F(x)$ is odd function and it is symmetric about the origin.

Graphics of $\cosh_F(x)$ and $\sinh_F(x)$ are shown in Figure 3.7 and 3.8.

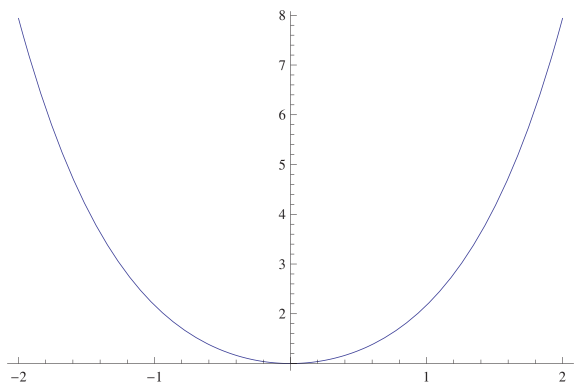

Figure 3.7. Graph of function $\cosh_F(x)$

Since,

$$\cosh_F(x) = \frac{e_F^x + e_F^{-x}}{2} = \frac{e_F^{-x} + e_F^x}{2} = \cosh_F(-x),$$

it is even function and it is symmetric about y-axis.

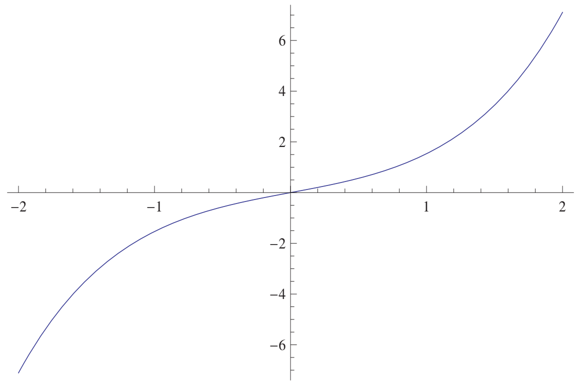

Figure 3.8. Graph of function $\sinh_F(x)$

Since,

$$\sinh_F(x) = \frac{e_F^x - e_F^{-x}}{2} = -\left(\frac{e_F^{-x} - e_F^x}{2}\right) = -\sinh_F(-x),$$

$\sinh_F(x)$ is odd function and it is symmetric about the origin.

Due to Proposition 3.5, we get analogues of Euler formula.

Proposition 3.6

$$e_F^{ix} = \cos_F(x) + i\sin_F(x) \tag{3.57}$$

$$E_F^{ix} = \cosh_F(x) + i\sinh_F(x) \tag{3.58}$$

$$e_F^{x} = \cosh_F(x) + \sinh_F(x) \tag{3.59}$$

$$E_F^{x} = \cos_F(x) + \sin_F(x) \tag{3.60}$$

We know that, hyperbolic and trigonometric functions are related,

$$\cosh(ix) = \frac{e^{ix} + e^{-ix}}{2} = \cos(x) \quad \text{and} \quad \sinh(ix) = \frac{e^{ix} - e^{-ix}}{2} = i\sin(x). \tag{3.61}$$

Similar relations exist between $\cosh_F(x)$ & $\cos_F(x)$ and $\sinh_F(x)$ & $\sin_F(x)$;

$$\cosh_F(ix) = \sum_{n=0}^{\infty} \frac{(ix)^{2n}}{F_{2n}!} = \sum_{n=0}^{\infty} \frac{i^{2n}x^{2n}}{F_{2n}!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{F_{2n}!} = \cos_F(x)$$

$$\Rightarrow \boxed{\cosh_F(ix) = \cos_F(x)}$$

$$\sinh_F(ix) = \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{F_{2n+1}!} = i\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{F_{2n+1}!} = i\sin_F(x)$$

$$\Rightarrow \boxed{\sinh_F(ix) = i\sin_F(x)}$$

3.5.1. Golden Oscillator

Lemma 3.1 *Golden derivatives of* $\cos_F(x)$ *and* $\sin_F(x)$ *functions are,*

$$D_F^x(\cos_F(x)) = -\sin_F(x) \tag{3.62}$$

$$D_F^x(\sin_F(x)) = \cos_F(x) \tag{3.63}$$

Proof

$$\begin{aligned} D_F^x(\cos_F(x)) &= \frac{1}{2}D_F^x(e_F^{ix}+e_F^{-ix}) = \frac{1}{2}(ie_F^{ix}-ie_F^{-ix}) = \frac{i}{2}(e_F^{ix}-e_F^{-ix}) = -\left(\frac{e_F^{ix}-e_F^{-ix}}{2i}\right) \\ &= -\sin_F(x) \\ D_F^x(\sin_F(x)) &= \frac{1}{2}D_F^x(e_F^{ix}-e_F^{-ix}) = \frac{(ie_F^{ix}+ie_F^{-ix})}{2i} = \frac{(e_F^{ix}+e_F^{-ix})}{2} = \cos_F(x) \end{aligned}$$

□

It can be generalized to arbitrary number k;

$$D_F^x(\cos_F(kx)) = -k\sin_F(kx) \tag{3.64}$$

$$D_F^x(\sin_F(kx)) = k\cos_F(kx) \tag{3.65}$$

By appliying the second derivative,

$$\Rightarrow (D_F^x)^2(\cos_F(kx)) = D_F^x(-k\sin_F(kx)) = -kk\cos_F(kx) = -k^2\cos_F(kx) \tag{3.66}$$

So, we have $(D_F^x)^2(\cos_F(kx)) = -k^2\cos_F(kx)$. Since $\cos_F(kx)$ satisfy this equation, it should also satisfy the following equation, which is called as Golden Oscillator equation;

$$\boxed{[(D_F^x)^2+k^2]\cos_F(kx) = 0} \tag{3.67}$$

Definition 3.6 *Golden oscillator equation is defined as,*

$$[(D_F^x)^2+k^2]\,y(x) = 0 \tag{3.68}$$

From another side, since $D_F^x(E_F^{kx}) = kE_F^{-kx}$, and if we replace $k \leftrightarrow -k$ we have,

$$D_F^x(E_F^{-kx}) = -kE_F^{kx} \tag{3.69}$$

Appliying it twice,

$$(D_F^x)^2(E_F^{kx}) = (D_F^x)(kE_F^{-kx}) \overset{(3.69)}{=} k(-k)E_F^{kx} = -k^2E_F^{kx} \Rightarrow (D_F^x)^2(E_F^{kx}) = -k^2E_F^{kx}$$

and we obtain,

$$\boxed{[(D_F^x)^2 + k^2]E_F^{kx} = 0} \tag{3.70}$$

Since E_F^{kx} and E_F^{-kx} from one side and $\cos_F(kx)$ from another side satisfy the same equations (3.67) and (3.70), they should be dependent. Their dependency can be seen from equation (3.53),

$$\cos_F(x) = \frac{E_F^x + E_F^{-x}}{2}. \tag{3.71}$$

Also, easy to check that $\sin_F(kx)$ satisfies the same equation as;

$$[(D_F^x)^2 + k^2](\sin_F(kx)) = 0 \tag{3.72}$$

Then, the general solution is written by superposition;

$$f(x) = A_1(x)\cos_F(kx) + A_2(x)\sin_F(kx), \tag{3.73}$$

where $A_1(x)$ and $A_2(x)$ are Golden periodic and even functions.

3.6. Golden Binomial $(x + y)_F^n$

In this Section we study Golden analogue of binomial. To introduce it, we need first define Fibonomials and their properties.

3.6.1. Fibonomial and Golden Pascal Triangle

Definition 3.7 *The product of Fibonacci numbers,*

$$F_1F_2\dots F_n = \prod_{i=1}^{n} F_i \equiv F_n! \tag{3.74}$$

is called the Fibonacci factorial. Another common name and notation for this number is the Fibonorial;

$$n!_F \equiv F_n! \tag{3.75}$$

Definition 3.8 *The Fibonacci-binomial coefficients are defined as,*

$$\begin{bmatrix} n \\ m \end{bmatrix}_F = \frac{F_nF_{n-1}\dots F_{n-m+1}}{F_mF_{m-1}\dots F_1} \equiv \frac{F_n!}{F_{n-m}!F_m!} \equiv \frac{[n]_F!}{[n-m]_F![m]_F!} \tag{3.76}$$

with n and m being nonnegative integers($n \geq m$). These coefficients are called Fibonomial coefficients.

We know that any number in the interior of Pascal's Triangle will be the sum of the two numbers appearing above it. Thus, we have formula;

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Golden analog of this formula exists. Let's begin with;

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_F &= \frac{F_n!}{F_{n-k}!F_k!} = \frac{F_n \ [n-1]_F!}{[k]_F! \ [n-k]_F \ [n-k-1]_F!} \\ &= \frac{F_n}{[n-k]_F} \ \frac{[n-1]_F!}{[k]_F! \ [n-k-1]_F!}. \end{aligned} \tag{3.77}$$

By using addition formula (2.13), we can write F_n;

$$F_n = F_{(n-k)+k} \overset{(2.13)}{=} \left(-\frac{1}{\varphi}\right)^k F_{n-k} + \varphi^{n-k} \ F_k$$

Then, the ratio $\frac{F_n}{F_{n-k}}$ becomes;

$$\frac{F_n}{F_{n-k}} = \left(-\frac{1}{\varphi}\right)^k + \varphi^{n-k} \ \frac{F_k}{F_{n-k}} \tag{3.78}$$

Substituting this into (3.77) gives,

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix}_F = \ldots \quad &= \left(\left(-\frac{1}{\varphi}\right)^k + \varphi^{n-k} \ \frac{F_k}{F_{n-k}}\right) \ \frac{[n-1]_F!}{[k]_F! \ [n-k-1]_F!} \\
&= \left(-\frac{1}{\varphi}\right)^k \frac{[n-1]_F!}{[k]_F! \ [n-k-1]_F!} + \varphi^{n-k} \ \frac{F_k}{F_{n-k}} \frac{[n-1]_F!}{[k]_F! \ [n-k-1]_F!} \\
&= \left(-\frac{1}{\varphi}\right)^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_F + \varphi^{n-k} \ \frac{[n-1]_F!}{[k-1]_F! \ F_{n-k} \ [n-k-1]_F!} \\
&= \left(-\frac{1}{\varphi}\right)^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_F + \varphi^{n-k} \ \frac{[n-1]_F!}{[k-1]_F! \ [n-k]_F!} \\
&= \left(-\frac{1}{\varphi}\right)^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_F + \varphi^{n-k} \ \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F .
\end{aligned}$$

So, we get the formula to construct the Golden Pascal Triangle,

$$\boxed{\begin{bmatrix} n \\ k \end{bmatrix}_F = \left(-\frac{1}{\varphi}\right)^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_F + \varphi^{n-k} \ \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F} \tag{3.79}$$

By using next property of Fibonomials,

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \frac{F_n!}{F_{n-k}!F_k!} = \frac{F_n!}{F_{n-(n-k)}!F_{n-k}!} = \begin{bmatrix} n \\ n-k \end{bmatrix}_F$$

we can derive equivalent rule to determine Golden Pascal Triangle.

If in (3.79) we replace $k \to n-k$,

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix}_F &= \left(-\frac{1}{\varphi}\right)^{n-k} \begin{bmatrix} n-1 \\ n-k \end{bmatrix}_F + \varphi^{n-(n-k)} \begin{bmatrix} n-1 \\ n-k-1 \end{bmatrix}_F \\
&= \left(-\frac{1}{\varphi}\right)^{n-k} \begin{bmatrix} n-1 \\ n-k \end{bmatrix}_F + \varphi^{k} \begin{bmatrix} n-1 \\ n-k-1 \end{bmatrix}_F \\
&= \left(-\frac{1}{\varphi}\right)^{n-k} \begin{bmatrix} n-1 \\ n-1-(n-k) \end{bmatrix}_F + \varphi^{k} \begin{bmatrix} n-1 \\ (n-1)-(n-k-1) \end{bmatrix}_F \\
&= \left(-\frac{1}{\varphi}\right)^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F + \varphi^{k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_F
\end{aligned}$$

Next equivalent rule also determine the Golden Pascal Triangle,

$$\boxed{\begin{bmatrix} n \\ k \end{bmatrix}_F = \left(-\frac{1}{\varphi}\right)^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F + \varphi^{k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_F} \tag{3.80}$$

where $1 \le k \le n-1$.

Thus, by using formula (3.79), we can construct the Golden Pascal Triangle as shown in Figure 3.9.

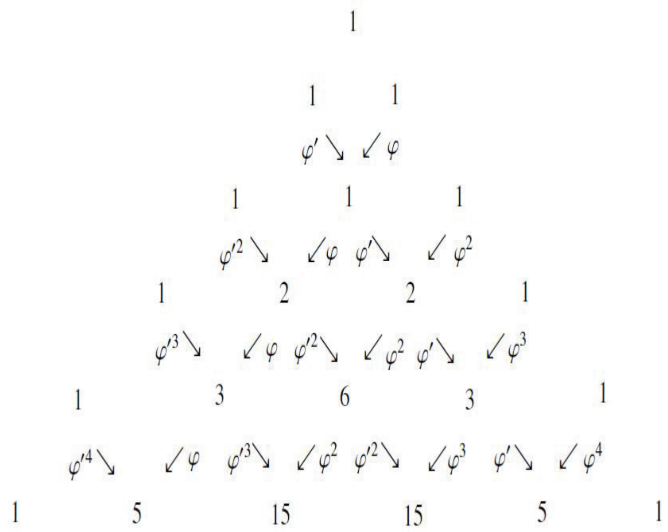

Figure 3.9. Golden Pascal Triangle

3.6.2. Golden Binomial

Definition 3.9 *The Golden Binomial is defined as;*

$$(x+y)_F^n \equiv \left(x+\varphi^{n-1}y\right)\left(x+\varphi^{n-2}\varphi' y\right)\dots\left(x+\varphi\varphi'^{n-2}y\right)\left(x+\varphi'^{n-1}y\right)$$

Since $\varphi\varphi' = -1$, Golden Binomial can be written in terms of just φ,

$$(x+y)_F^n \equiv \left(x+\varphi^{n-1}y\right)\left(x-\varphi^{n-3}y\right)\dots\left(x-\varphi'^{n-3}y\right)\left(x+(-1)^{n-1}\varphi^{-n+1}y\right) \tag{3.81}$$

For the Golden Binomial, the next expansion is also valid (Pashaev O.K. and Nalci S., 2012).

$$(x+y)_F^n \equiv \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_F (-1)^{\frac{k(k-1)}{2}} x^{n-k} y^k \tag{3.82}$$

In particular,

$$(x-1)_F^m = \left(x-\varphi^{m-1}\right)\left(x+\varphi^{m-3}\right)\dots\left(x-(-1)^{m-1}\varphi^{-m+1}\right) \tag{3.83}$$

First few binomial are,

$$\begin{aligned}
(x-1)_F^1 &= x-1 \\
(x-1)_F^2 &= (x-\varphi)(x-\varphi') \\
(x-1)_F^3 &= (x-\varphi^2)(x+1)(x-\varphi'^2) \\
(x-1)_F^4 &= (x-\varphi^3)(x+\varphi)(x+\varphi')(x-\varphi'^3)
\end{aligned}$$

and corresponding zeros are,

$$m=1 \quad\Rightarrow\quad x=1$$

$$
\begin{aligned}
m = 2 &\;\Rightarrow\; x = \varphi, x = \varphi' \\
m = 3 &\;\Rightarrow\; x = \varphi^2, x = -1, x = \varphi'^2 \\
m = 4 &\;\Rightarrow\; x = \varphi^3, x = -\varphi, x = -\varphi', x = \varphi'^3
\end{aligned} \tag{3.84}
$$

For arbitrary n, we have following zeros of Golden binomial.

$$
1)\, n = 2k \Rightarrow (x-1)_F^n = (x-1)_F^{2k} : \;\; \varphi^{n-1}, \varphi'^{n-1}, -\varphi^{n-3}, -\varphi'^{n-3}, \ldots, \pm\varphi, \pm\varphi' \tag{3.85}
$$

$$
2)\, n = 2k+1 \Rightarrow (x-1)_F^n = (x-1)_F^{2k+1} : \;\; \varphi^{n-1}, \varphi'^{n-1}, -\varphi^{n-3}, -\varphi'^{n-3}, \ldots, \pm 1 \tag{3.86}
$$

These zeros completely determine Golden binomials.

Lemma 3.2 *The application of the Golden derivative to the Golden Binomials gives;*

$$
D_F^x (x+y)_F^n = F_n (x+y)_F^{n-1}, \tag{3.87}
$$

$$
D_F^y (x+y)_F^n = F_n (x-y)_F^{n-1}, \tag{3.88}
$$

$$
D_F^y (x-y)_F^n = -F_n (x+y)_F^{n-1}. \tag{3.89}
$$

Proof For proof, see Appendix B.2. □

Appliying derivative several times, we get;

$$
(D_F^y)^{2k} (x+y)_F^{2k} = (-1)^k F_{2k}! \quad \text{and} \quad (D_F^y)^{2k+1} \, (x+y)_F^{2k+1} = (-1)^k F_{2k+1}!
$$

Proposition 3.7

$$
e_F^x \;\; E_F^y = e_F^x \;\; e_{-F}^y = e_F^{(x+y)_F}, \tag{3.90}
$$

where,

$$
e_F^{(x+y)_F} = e_F (x+y)_F \equiv \sum_{n=0}^{\infty} \frac{(x+y)_F^n}{F_n!}. \tag{3.91}
$$

Proof

$$
\begin{aligned}
e_F^x \;\; E_F^y &= \left(\sum_{n=0}^{\infty} \frac{x^n}{F_n!}\right)\left(\sum_{k=0}^{\infty}(-1)^{\frac{k(k-1)}{2}} \frac{y^k}{F_k!}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^n y^k}{F_n!F_k!}(-1)^{\frac{k(k-1)}{2}} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^n y^k F_N!}{F_n!F_k!F_N!}(-1)^{\frac{k(k-1)}{2}} \quad (Let\ n+k=N) \\
&= \sum_{N=0}^{\infty} \frac{1}{F_N!} \underbrace{\sum_{k=0}^{N} \frac{F_N!}{F_{N-k}!F_k!}(-1)^{\frac{k(k-1)}{2}} x^{N-k} y^k}_{(x+y)_F^N} \\
&= \sum_{N=0}^{\infty} \frac{(x+y)_F^N}{F_N!} \\
&= e_F(t+x)_F
\end{aligned}
$$

This function $e_F(t+x)_F$, we will use in Section 3.8. □

3.7. Remarkable Limit

From Golden Binomial expansion, we have;

$$
(x+y)_F^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_F (-1)^{\frac{k(k-1)}{2}} x^{n-k} y^k
$$

If we set $x = 1$ and $y \rightarrow \frac{y}{\varphi^n}$, then it gives us;

$$
\left(1 + \frac{y}{\varphi^n}\right)_F^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_F \; (-1)^{\frac{k(k-1)}{2}} \frac{y^k}{\varphi^{nk}}. \tag{3.92}
$$

By opening Fibonomials and taking the limit as n goes to infinity;

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{y}{\varphi^n}\right)_F^n &= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{F_n!}{F_{n-k}!\;F_k!}\;(-1)^{\frac{k(k-1)}{2}}\frac{y^k}{\varphi^{nk}}\\
&= \lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{F_nF_{n-1}\ldots F_{n-(k-1)}F_{n-k}!}{F_{n-k}!\;F_k!}\;(-1)^{\frac{k(k-1)}{2}}\frac{y^k}{\varphi^{nk}}\\
&= \lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{F_nF_{n-1}\ldots F_{n-(k-1)}}{F_k!}\;(-1)^{\frac{k(k-1)}{2}}\frac{y^k}{\varphi^{nk}}\\
&= \lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{(\varphi^n-\varphi'^n)\ldots(\varphi^{n-(k-1)}-\varphi'^{n-(k-1)})}{(\varphi-\varphi')^k\;F_k!}\;(-1)^{\frac{k(k-1)}{2}}\frac{y^k}{\varphi^{nk}}\\
&= \lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{\varphi^n\varphi^{n-1}\ldots\varphi^{n-(k-1)}}{(\varphi-\varphi')^k\;F_k!}\;(-1)^{\frac{k(k-1)}{2}}\frac{y^k}{\varphi^{nk}}\\
&= \lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{\varphi^{kn}\;\varphi^{-\left(\frac{k(k-1)}{2}\right)}}{(\varphi-\varphi')^k\;F_k!}\;(-1)^{\frac{k(k-1)}{2}}\frac{y^k}{\varphi^{nk}}\\
&= \sum_{k=0}^{\infty}\frac{1}{F_k!}\;(-1)^{\frac{k(k-1)}{2}}\frac{y^k}{(\varphi-\varphi')^k\;\varphi^{\frac{k(k-1)}{2}}}
\end{aligned}$$

Since, we have;

$$F_k=\frac{\varphi^k-\varphi'^k}{\varphi-\varphi'}=\frac{\varphi'^k\left(\left(\frac{\varphi}{\varphi'}\right)^k-1\right)}{\varphi'\left(\frac{\varphi}{\varphi'}-1\right)}=\varphi'^{k-1}\frac{\left(\left(\frac{\varphi}{\varphi'}\right)^k-1\right)}{\left(\frac{\varphi}{\varphi'}-1\right)}=\varphi'^{k-1}\frac{(-\varphi^2)^k-1}{(-\varphi^2)-1}=\varphi'^{k-1}[k]_{-\varphi^2}$$

$$\Rightarrow F_k=\varphi'^{k-1}[k]_{-\varphi^2} \tag{3.93}$$

Multiplying $F_kF_{k-1}\ldots F_1$ gives,

$$F_k!=\left(-\frac{1}{\varphi}\right)^{\frac{k(k-1)}{2}}[k]_{-\varphi^2}! \tag{3.94}$$

After substituting $F_k!$,

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{y}{\varphi^n}\right)_F^n = \ldots &= \sum_{k=0}^{\infty}\frac{1}{\left(-\frac{1}{\varphi}\right)^{\frac{k(k-1)}{2}}[k]_{-\varphi^2}!}\;(-1)^{\frac{k(k-1)}{2}}\frac{y^k}{(\varphi-\varphi')^k\;\varphi^{\frac{k(k-1)}{2}}}\\ &= \sum_{k=0}^{\infty}\frac{1}{[k]_{-\varphi^2}!}\;\left(\frac{y\varphi}{\varphi^2+1}\right)^k\\ &= \sum_{k=0}^{\infty}\frac{\left(\frac{y\varphi}{\varphi^2+1}\right)^k}{[k]_{-\varphi^2}!}\\ &= e_{-\varphi^2}\;\left(\frac{y\varphi}{\varphi^2+1}\right).\end{aligned}$$

Thus, it is obtained,

$$\lim_{n\to\infty}\left(1+\frac{y}{\varphi^n}\right)_F^n = e_{-\varphi^2}\;\left(\frac{y\varphi}{\varphi^2+1}\right), \tag{3.95}$$

$$\lim_{n\to\infty}\left(1+\frac{y}{\varphi^n}\right)_F^n = e_{-\varphi^2}\left(\frac{y}{\sqrt{5}}\right). \tag{3.96}$$

As a last step, after choosing $y = \sqrt{5}$, we obtain;

$$\lim_{n\to\infty}\left(1+\frac{\sqrt{5}}{\varphi^n}\right)_F^n = e_{-\varphi^2}(1) \tag{3.97}$$

Now, our aim is to calculate $e_{-\varphi^2}(1)$. From q-Calculus, we know the Jackson exponential function;

$$e_q(x) = \sum_{k=0}^{\infty}\frac{x^k}{[k]_q!} \tag{3.98}$$

By choosing $q = -\varphi^2$ and $x = 1$, we have;

$$e_{-\varphi^2}(1) = \sum_{k=0}^{\infty} \frac{1}{[k]_{-\varphi^2}!}.$$

From (3.94), $[k]_{-\varphi^2}! = (-1)^{\frac{k(k-1)}{2}} F_k!\, \varphi^{\frac{k(k-1)}{2}}$. By substituting this gives,

$$e_{-\varphi^2}(1) = \sum_{k=0}^{\infty} \frac{1}{(-1)^{\frac{k(k-1)}{2}} F_k!\, \varphi^{\frac{k(k-1)}{2}}} = \sum_{n=0}^{\infty} \frac{1}{F_k!\, (-\varphi)^{\frac{k(k-1)}{2}}}. \tag{3.99}$$

As a final result we can write,

$$\boxed{\lim_{n\to\infty} \left(1 + \frac{\sqrt{5}}{\varphi^n}\right)_F^n = \sum_{k=0}^{\infty} \frac{1}{F_k!\, (-\varphi)^{\frac{k(k-1)}{2}}} = \sum_{k=0}^{\infty} \frac{\varphi'^{\frac{k(k-1)}{2}}}{F_k!}} \tag{3.100}$$

3.8. Golden Heat Equation

Definition 3.10 *Fibonacci-exponential function of two arguments is defined as;*

$$e_F^{(t+x)_F} \equiv e_F(t+x)_F \equiv \sum_{n=0}^{\infty} \frac{(t+x)_F^n}{F_n!}. \tag{3.101}$$

Lemma 3.3 *By applying the D_F^t and D_F^x operators to $e_F(t+x)_F$ and $e_F(t-x)_F$, we obtain results,*

$$D_F^t(e_F(t+x)_F) = e_F(t+x)_F, \tag{3.102}$$
$$D_F^x(e_F(t+x)_F) = e_F(t-x)_F, \tag{3.103}$$
$$D_F^x(e_F(t-x)_F) = -e_F(t+x)_F. \tag{3.104}$$

Proof For proof, see Appendix B.3.1. □

3.8.1. Function $e_F(t+x)_F$ and Golden Heat Equation

It is known that,

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} \tag{3.105}$$

is the Heat Equation for temparature distribution $u(x,t)$. By choosing $\mu = -1$ and replacing partial derivatives with Golden derivatives as $\frac{\partial}{\partial t} \to D_F^t$ and $\frac{\partial^2}{\partial x^2} \to (D_F^x)^2$ gives the Golden Heat equation,

$$\boxed{[(D_F^x)^2 + D_F^t]u_F(x,t) = 0} \tag{3.106}$$

for unknown function $u_F(x,t)$. As we have seen,

$$D_F^t(e_F(t+x)_F) = e_F(t+x)_F$$

and as follows,

$$(D_F^x)^2(e_F(t+x)_F) = D_F^x(e_F(t-x)_F) = -e_F(t+x)_F.$$

Adding these equations gives,

$$[(D_F^x)^2 + D_F^t]e_F(t+x)_F = 0. \tag{3.107}$$

As a result, $e_F(t+x)_F$ is a solution of the Golden Heat equation.
This solution can be generalized for arbitrary number k. Assume that

$$u_F(x,t) = e_F(\omega t + kx)_F \equiv e_F^{(\omega t + kx)_F} \tag{3.108}$$

is a solution of Golden Heat equation (3.106).

Lemma 3.4 *It is obtained that,*

$$D^t_F(e_F(\omega t + kx)_F) \quad = \quad \omega \, e_F(\omega t + kx)_F \tag{3.109}$$

$$(D^x_F)^2(e_F(\omega t + kx)_F) \quad = \quad -k^2 \, e_F(\omega t + kx)_F \tag{3.110}$$

Proof For proof, see Appendix B.3.2. □

By substituting $e_F(\omega t + kx)_F$ in the Golden Heat equation (3.106) gives,

$$[(D^x_F)^2 + D^t_F]e_F(\omega t + kx)_F = (-k^2 + \omega)e_F(\omega t + kx)_F = 0 \Rightarrow -k^2 + \omega = 0 \Rightarrow \boxed{\omega = k^2}$$

So, ω dependency in terms of k as dispersion relation $\omega = \omega(k)$ is obtained. Therefore, $e_F(k^2 t + kx)_F$ is one parametric solution for Golden Heat equation.

Since equation is linear, we can consider superposition of these functions as;

$$U_F(x,t) = \sum_k a_k \, e_F(k^2 t + kx)_F \tag{3.111}$$

with arbitrary coefficients a_k (more generally these are Golden periodic functions) as general solution of Golden heat equation.

3.9. Golden Wave Equation

In previous section, we studied Golden Heat equation and found its general solution. In this section, we search general solution of the Golden Wave equation.

The standard wave equation is known as,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{3.112}$$

By choosing $c = 1$ and replacing partial derivatives with Golden derivatives as $\frac{\partial^2}{\partial t^2} \rightarrow (D^t_F)^2$ and $\frac{\partial^2}{\partial x^2} \rightarrow (D^x_F)^2$ gives the Golden wave equation,

$$\boxed{[(D^x_F)^2 - (D^t_F)^2]u_F(x,t) = 0} \tag{3.113}$$

Let $e_F(\omega t + kx)_F$ be solution of Golden wave equation;

$$[(D_F^x)^2 - (D_F^t)^2]e_F(\omega t + kx)_F = 0. \tag{3.114}$$

By using Lemma 3.4 , we have;

$$\begin{aligned}
[(D_F^x)^2 - (D_F^t)^2]e_F(\omega t + kx)_F &= (D_F^x)^2(e_F(\omega t + kx)_F) - (D_F^t)^2(e_F(\omega t + kx)_F) \\
&= (-k^2)e_F(\omega t + kx)_F - \omega^2 e_F(\omega t + kx)_F \\
&= -[\omega^2 + k^2]\, e_F(\omega t + kx)_F = 0.
\end{aligned}$$

Then $\omega^2 + k^2 = 0 \;\Rightarrow\; \omega = \pm ik$ gives solutions for the Golden Wave equation as $e_F(ikt + kx)_F$ and $e_F(-ikt + kx)_F$. Since all linear combinations also become solution, the general solution is,

$$\boxed{U(x,t) = \sum_k a_k e_F(ikt + kx)_F + b_k e_F(-ikt + kx)_F} \tag{3.115}$$

where a_k's and b_k's are arbitrary constants(or more generally-Golden periodic functions).

CHAPTER 4

HIGHER ORDER FIBONACCI CALCULUS

4.1. Higher Order Fibonacci Derivatives

In this section, we introduce Higher Order Fibonacci derivative operators ${}_{(k)}D_F^x$.

Definition 4.1 *For arbitrary function $f(x)$,*

$$
{}_{(k)}D_F^x[f(x)] = \frac{f(\varphi^k x) - f(\varphi'^k x)}{(\varphi^k - \varphi'^k)\, x} \tag{4.1}
$$

where $k \in Z$.

The operator ${}_{(k)}D_F^x$ we call the Higher k^{th} order Golden derivative operator. It is the linear operator.

For the case $k = 1$, it coincides with Golden derivative.

$$
{}_{(1)}D_F^x[f(x)] = D_F^x[f(x)] \tag{4.2}
$$

Application of this derivative operator to function x^n produces Fibonacci numbers;

$$
{}_{(1)}D_F^x(x^n) = \frac{(\varphi x)^n - (\varphi' x)^n}{(\varphi - \varphi')x} = \frac{\varphi^n - \varphi'^n}{\varphi - \varphi'} x^{n-1} = F_n \;\; x^{n-1}.
$$

Now, by applying the Higher k^{th} Order Golden derivative ${}_{(k)}D_F^x$ to function x^n, we get the Higher order Fibonacci numbers $F_n^{(k)}$.

$$
{}_{(k)}D_F^x[x^n] = \frac{(\varphi^k x)^n - (\varphi'^k x)^n}{(\varphi^k - \varphi'^k)x} = \frac{(\varphi^k)^n - (\varphi'^k)^n}{\varphi^k - \varphi'^k} x^{n-1} = F_n^{(k)} x^{n-1},
$$

or

$$ {}_{(k)}D_F^x[x^n] = F_n^{(k)} x^{n-1}. \tag{4.3} $$

For negative values of k, this formula produces the numbers,

$$ F_n^{(-k)} = (-1)^{(n+1)k} F_n^{(k)} $$

according to (2.28). For Higher k^{th} Order Golden derivative the Leibnitz and quotient rules can be derived.

Proposition 4.1 *(The Leibnitz Rule)*

$$ {}_{(k)}D_F^x(f(x)g(x)) = {}_{(k)}D_F^x(f(x))\, g(\varphi^k x) + f\left(\varphi'^k x\right) {}_{(k)}D_F^x(g(x)) \tag{4.4} $$

Proposition 4.2 *(The Quotient Rule)*

$$ {}_{(k)}D_F^x\left(\frac{f(x)}{g(x)}\right) = \frac{{}_{(k)}D_F^x(f(x))\, g(\varphi^k x) - f(\varphi^k x)\, {}_{(k)}D_F^x(g(x))}{g(\varphi^k x)\, g\left(\varphi'^k x\right)} \tag{4.5} $$

Example 4.1 *We know that,*

$$ {}_{(k)}D_F^x(x^n) = F_n^{(k)} x^{n-1}. \tag{4.6} $$

Now, we calculate this in another way, by splitting the power;

$$ \begin{aligned} {}_{(k)}D_F^x(x^n) = {}_{(k)}D_F^x(x^m x^{n-m}) &\overset{(4.4)}{=} {}_{(k)}D_F^x(x^m)\,(\varphi^k x)^{n-m} + (\varphi'^k x)^m\, {}_{(k)}D_F^x(x^{n-m}) \\ &= F_m^{(k)} x^{m-1}(\varphi^k)^{n-m} x^{n-m} + (\varphi'^k)^m x^m F_{n-m}^{(k)} x^{n-m-1} \\ &= \left[F_m^{(k)}\,(\varphi^k)^{n-m} + F_{n-m}^{(k)}\,(\varphi'^k)^m\right] x^{n-1} \end{aligned} \tag{4.7} $$

Comparing the results (4.6) *and* (4.7) *gives next corollary.*

Corollary 4.1 *For any $m < n$,*

$$\boxed{F_n^{(k)} = F_m^{(k)} \, (\varphi^k)^{n-m} + F_{n-m}^{(k)} \, (\varphi'^k)^m} \tag{4.8}$$

This corollary allows us to formulate following proposition.

Proposition 4.3

$$F_n^{(k)} = F_{n-m}^{(k)} \, F_{m+1}^{(k)} + (-1)^{k+1} \, F_m^{(k)} \, F_{n-m-1}^{(k)} \tag{4.9}$$

For $k = 1$ this gives equation (2.15) *for Fibonacci numbers.*

$$F_n = F_{n-m} \, F_{m+1} + F_m \, F_{n-m-1}$$

Proof Due to corollary,

$$F_n^{(k)} = F_m^{(k)} \, (\varphi^k)^{n-m} + F_{n-m}^{(k)} \, (\varphi'^k)^m$$

by substituting (2.30) and (2.31), we get;

$$\begin{aligned}
F_n^{(k)} &= F_m^{(k)} \left(\varphi^k F_{n-m}^{(k)} + (-1)^{k+1} F_{n-m-1}^{(k)}\right) + F_{n-m}^{(k)} \left(\varphi'^k F_m^{(k)} + (-1)^{k+1} F_{m-1}^{(k)}\right) \\
&= F_m^{(k)} \, F_{n-m}^{(k)} \left(\varphi^k + \varphi'^k\right) + (-1)^{k+1} F_m^{(k)} \, F_{n-m-1}^{(k)} + (-1)^{k+1} F_{n-m}^{(k)} \, F_{m-1}^{(k)} \\
&= F_m^{(k)} \, F_{n-m}^{(k)} \, L_k + (-1)^{k+1} F_m^{(k)} \, F_{n-m-1}^{(k)} + (-1)^{k+1} F_{n-m}^{(k)} \, F_{m-1}^{(k)} \\
&= F_{n-m}^{(k)} \left(L_k \, F_m^{(k)} + (-1)^{k+1} \, F_{m-1}^{(k)}\right) + (-1)^{k+1} F_m^{(k)} \, F_{n-m-1}^{(k)} \\
&\overset{(2.24)}{=} F_{n-m}^{(k)} \, F_{m+1}^{(k)} + (-1)^{k+1} \, F_m^{(k)} \, F_{n-m-1}^{(k)}
\end{aligned}$$

□

Proposition 4.4 *Addition formula for higher order Fibonacci numbers is given by,*

$$F_{n+m}^{(k)} = F_m^{(k)} \, F_{n+1}^{(k)} + (-1)^{k+1} \, F_n^{(k)} \, F_{m-1}^{(k)}.$$

For $k = 1$, this gives standard addition formula (2.12).

4.1.1. Higher Golden Periodic Functions

Proposition 4.5 *Every Golden periodic function $D_F^x(f(x)) = 0 \quad (f(\varphi x) = f(\varphi' x))$ is also periodic for arbitrary Higher Order Golden derivatives, i.e;*

$$ {}_{(k)}D_F^x(f(x)) = 0 \Rightarrow {}_{(2)}D_F^x(f(x)) = 0,\ {}_{(3)}D_F^x(f(x)) = 0, \ldots,\ {}_{(k)}D_F^x(f(x)) = 0. \qquad (4.10)$$

It means that relation $f(\varphi x) = f(\varphi' x)$ implies;

$$f(\varphi^2 x) = f(\varphi'^2 x),\ f(\varphi^3 x) = f(\varphi'^3 x), \ldots,\ f(\varphi^k x) = f(\varphi'^k x), \qquad (4.11)$$

where $k = 2, 3, \ldots$

Proof Proof will be done by Principal of Mathematical induction. The statement is valid for $k = 1$. Let us show that it is valid also for $k = 2$.

$$f(\varphi^2 x) = f(\varphi(\varphi x)) = f(\varphi'(\varphi x)) = f(\varphi(\varphi' x)) = f(\varphi'(\varphi' x)) = f(\varphi'^2 x) \qquad (4.12)$$

Thus;

$$ {}_{(1)}D_F^x(f(x)) \Rightarrow {}_{(2)}D_F^x(f(x))$$

Now, let us suppose that the statement is valid for arbitrary $k-1$ and k:

$$ {}_{(k-1)}D_F^x(f(x)) = 0 \text{ and } {}_{(k)}D_F^x(f(x)) = 0$$

It means,

$$f(\varphi^{k-1} x) = f(\varphi'^{k-1} x), \qquad (4.13)$$

$$f(\varphi^k x) = f(\varphi'^k x). \qquad (4.14)$$

Let us show that it is valid also for $k+1$:

$$_{(k+1)}D_F^x(f(x)) = 0 \text{ and } f(\varphi^{k+1}x) = f(\varphi'^{k+1}x)$$

Calculating;

$$\begin{aligned} f(\varphi^{k+1}x) &= f(\varphi^k\varphi x) \overset{(4.14)}{=} f(\varphi'^k\varphi x) = f(-\varphi'^{k-1}x) \\ f(\varphi'^{k+1}x) &= f(\varphi'^k\varphi' x) \overset{(4.14)}{=} f(\varphi^k\varphi' x) = f(-\varphi^{k-1}x) \end{aligned}$$

for $k+1$ derivative;

$$_{(k+1)}D_F^x(f(x)) = \frac{f(\varphi^{k+1}x) - f(\varphi'^{k+1}x)}{(\varphi^{k+1} - \varphi'^{k+1})x},$$

we get,

$$\begin{aligned} _{(k+1)}D_F^x(f(x)) &= \frac{f(-\varphi'^{k-1}x) - f(-\varphi^{k-1}x)}{(\varphi^{k+1} - \varphi'^{k+1})x} = \frac{f(\varphi'^{k-1}(-x)) - f(\varphi^{k-1}(-x))}{(\varphi^{k+1} - \varphi'^{k+1})x} \\ &\overset{(4.13)}{=} \frac{f(\varphi^{k-1}(-x)) - f(\varphi^{k-1}(-x))}{(\varphi^{k+1} - \varphi'^{k+1})x} \\ &= 0. \end{aligned}$$

Therefore,

$$_{(k+1)}D_F^x(f(x)) = 0 \Rightarrow f(\varphi^{k+1}x) = f(\varphi'^{k+1}x)$$

□

Example 4.2 *We know that* $f(x) = \sin\left(\frac{\pi}{\ln\varphi} \ln|x|\right)$ *is Golden periodic function:*

$$D_F^x(f(x)) = 0$$

Due to previous proposition, it should be also periodic according to every Higher k^{th}

Order Golden derivative, which means;

$$_{(k)}D_F^x(f(x)) = 0 \Leftrightarrow f(\varphi^k x) = f(\varphi'^k x).$$

Indeed,

$$\begin{aligned}
f(\varphi^k x) &= \sin\left(\frac{\pi}{\ln\varphi}\ln|\varphi^k x|\right) = \sin\left(\frac{\pi}{\ln\varphi}\left(\ln|\varphi^k| + \ln|x|\right)\right) = \sin\left(\frac{\pi}{\ln\varphi}k\ln\varphi + \frac{\pi}{\ln\varphi}\ln|x|\right)\\
&= \sin\left(\pi k + \frac{\pi}{\ln\varphi}\ln|x|\right)\\
&= \sin(\pi k)\cos\left(\frac{\pi}{\ln\varphi}\ln|x|\right) + \sin\left(\frac{\pi}{\ln\varphi}\ln|x|\right)\cos(\pi k)\\
&= (-1)^k\sin\left(\frac{\pi}{\ln\varphi}\ln|x|\right)
\end{aligned}$$

and,

$$\begin{aligned}
f(\varphi'^k x) &= \sin\left(\frac{\pi}{\ln\varphi}\ln|\varphi'^k x|\right) = \sin\left(\frac{\pi}{\ln\varphi}\left(\ln|\varphi'^k| + \ln|x|\right)\right) = \sin\left(\frac{\pi}{\ln\varphi}k\ln|\varphi'| + \frac{\pi}{\ln\varphi}\ln|x|\right)\\
&= \sin\left(-\frac{\pi}{\ln\varphi}k\ln\varphi + \frac{\pi}{\ln\varphi}\ln|x|\right)\\
&= \sin\left(-\pi k + \frac{\pi}{\ln\varphi}\ln|x|\right)\\
&= -\sin\left(\pi k - \frac{\pi}{\ln\varphi}\ln|x|\right)\\
&= -\left[\sin(\pi k)\cos\left(\frac{\pi}{\ln\varphi}\ln|x|\right) - \sin\left(\frac{\pi}{\ln\varphi}\ln|x|\right)\cos(\pi k)\right]\\
&= (-1)^k\sin\left(\frac{\pi}{\ln\varphi}\ln|x|\right)
\end{aligned}$$

Therefore $f(\varphi^k x) = f(\varphi'^k x)$, and it is periodic according to Higher k^{th} Order Golden derivative.

As we have seen by Proposition 4.5, every Golden periodic function ($k = 1$) is periodic for arbitrary $k = 2, 3, 4, \ldots$

But opposite is not true in general. If function f(x) is k periodic,

$$_{(k)}D_F^x(f(x)) = 0 \Leftrightarrow f(\varphi^k x) = f(\varphi'^k x).$$

it is not necessarily $k = 1$ periodic. This can be seen from following example.

Example 4.3 *Function* $f(x) = \sin\left(\frac{\pi}{\ln\varphi^2}\ln|x|\right)$ *is Golden periodic function with* $k = 2$, *i.e* ${}_{(2)}D_F^x(f(x)) = 0$. *Let us calculate Golden derivative of this function.*

$$D_F^x(f(x)) = \frac{\sin\left(\frac{\pi}{\ln\varphi^2}\ln|\varphi x|\right) - \sin\left(\frac{\pi}{\ln\varphi^2}\ln|\varphi' x|\right)}{(\varphi - \varphi')x}.$$

In the numerator, we have;

$$\begin{aligned}\sin\left(\frac{\pi}{\ln\varphi^2}\ln|\varphi x|\right) &= \sin\left(\frac{\pi}{2\ln\varphi}\ln\varphi + \frac{\pi}{2\ln\varphi}\ln|x|\right) = \sin\left(\frac{\pi}{2} + \frac{\pi}{\ln(\varphi^2)}\ln|x|\right) \\ &= \cos\left(\frac{\pi}{\ln(\varphi^2)}\ln|x|\right),\end{aligned}$$

and

$$\begin{aligned}\sin\left(\frac{\pi}{\ln\varphi^2}\ln|\varphi' x|\right) &= \sin\left(\frac{\pi}{2\ln\varphi}\ln|\varphi'| + \frac{\pi}{2\ln\varphi}\ln|x|\right) = \sin\left(-\frac{\pi}{2} + \frac{\pi}{\ln(\varphi^2)}\ln|x|\right) \\ &= \sin\left(-\frac{\pi}{2} + \frac{\pi}{\ln(\varphi^2)}\ln|x|\right) = -\sin\left(\frac{\pi}{2} - \frac{\pi}{\ln(\varphi^2)}\ln|x|\right) \\ &= -\cos\left(\frac{\pi}{\ln(\varphi^2)}\ln|x|\right).\end{aligned}$$

Then, as we can see the derivative doesn't vanish;

$$D_F^x(f(x)) = 2\,\frac{\cos\left(\frac{\pi}{\ln(\varphi^2)}\ln|x|\right)}{(\varphi - \varphi')x} \neq 0$$

This is why this function is not the Golden periodic function.

4.2. Generating Function for Higher Order Fibonacci Numbers

Example 4.4 *We know that application of ${}_{(k)}D_F^x$ on x^n generates Higher order Fibonacci numbers;*

$$ {}_{(k)}D_F^x(x^n) = F_n^{(k)}\ x^{n-1} \tag{4.15} $$

So, these numbers can be represented also as,

$$ \boxed{F_n^{(k)} = \frac{{}_{(k)}D_F^x(x^n)}{x^{n-1}}} \tag{4.16} $$

Definition 4.2 *Function,*

$$ {}_{(k)}F(x) = \sum_{n=0}^{\infty} F_n^{(k)} x^n \tag{4.17} $$

is called the generating function of Higher order Fibonacci numbers $F_n^{(k)}$. According to Taylor formula;

$$ F_n^{(k)} = \frac{1}{n!}\frac{d^n}{dx^n}\ {}_{(k)}F(x)\bigg|_{x=0} \tag{4.18} $$

in a disk of analyticity around $x = 0$. Explicit form of the series is;

$$ {}_{(k)}F(x) = F_0^{(k)} + F_1^{(k)}x + F_2^{(k)}x^2 + F_3^{(k)}x^3 + F_4^{(k)}x^4 + F_5^{(k)}x^5 + \ldots \tag{4.19} $$

Proposition 4.6 *Generating function ${}_{(k)}F(x)$ in domain $|x| < \frac{1}{\varphi^k}$ has explicit representation;*

$$ {}_{(k)}F(x) = \sum_{n=0}^{\infty} F_n^{(k)} x^n = \frac{1}{1 - L_k x + (-1)^k x^2} \tag{4.20} $$

Proof To find the domain of convergency, we apply the ratio test:

$$\rho = \lim_{n\to\infty} \left| \frac{F_{n+1}^{(k)} x^{n+1}}{F_n^{(k)} x^n} \right| = \lim_{n\to\infty} \left| \frac{F_{n+1}^{(k)}}{F_n^{(k)}} \right| \lim_{n\to\infty} |x| = \varphi^k \, |x|$$

For convergency $\rho < 1$ implies,

$$|x| < \left(\frac{1}{\varphi}\right)^k$$

and as follows $|x| < \left(\frac{1}{\varphi}\right)^k < 1$ for any fixed positive value k. By using Golden derivative;

$$\begin{aligned}
{}_{(k)}F(x) &= \sum_{n=0}^{\infty} F_n^{(k)} x^n = F_0^{(k)} + \sum_{n=1}^{\infty} x F_n^{(k)} x^{n-1} = 0 + \sum_{n=1}^{\infty} x \, F_n^{(k)} \, x^{n-1} \overset{(4.15)}{=} \sum_{n=1}^{\infty} x \, {}_{(k)}D_F^x (x^n) \\
&= x \, {}_{(k)}D_F^x \sum_{n=1}^{\infty} x^n = x \, {}_{(k)}D_F^x \left(x + x^2 + x^3 + \ldots \right) \\
&= x \, {}_{(k)}D_F^x \, x \left(1 + x + x^2 + \ldots \right) \\
&\overset{|x|<1}{=} x \, D_F^x \left(x \, \frac{1}{1-x} \right) = x \, D_F^x \left(\frac{x}{1-x} \right) = x \, D_F^x \left(\frac{x-1+1}{1-x} \right) \\
&= x \left[{}_{(k)}D_F^x (-1) + {}_{(k)} \, D_F^x \left(\frac{1}{1-x} \right) \right] = x \left[{}_{(k)}D_F^x \left(\frac{1}{1-x} \right) \right] \\
&\overset{(4.1)}{=} x \, \frac{\left(\frac{1}{1-\varphi^k x} - \frac{1}{1-\varphi'^k x} \right)}{(\varphi^k - \varphi'^k) x} \\
&= \frac{x}{(1-\varphi^k x)(1-\varphi'^k x)} \\
&= \frac{x}{1 - (\varphi^k + \varphi'^k) x + (\varphi\varphi')^k \, x^2} \\
&= \frac{x}{1 - L_k \, x + (-1)^k \, x^2}
\end{aligned}$$

□

Corollary 4.2 *${}_{(k)}F(x)$ is the rational function with one zero at $x = 0$ and two single poles at,*

$$x = \frac{1}{\varphi^k}, \quad x = \frac{1}{\varphi'^k}.$$

Example 4.5 *If **k=1**, it reduces to generating function for Fibonacci numbers;*

$$_{(1)}F(x) = F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2} = \frac{x}{(1-\varphi x)(1-\varphi' x)} \quad (4.21)$$

*If **k=2**, it gives generating function for even " mod 2" Fibonacci numbers;*

$$_{(2)}F(x) = \sum_{n=0}^{\infty} F_{2n} x^n = \frac{x}{1-3x+x^2} = \frac{x}{(1-\varphi^2 x)(1-\varphi'^2 x)} \quad (4.22)$$

*If **k=3**, it is the generating function for " mod 3" Fibonacci numbers;*

$$_{(3)}F(x) = \frac{1}{2}\sum_{n=0}^{\infty} F_{3n} x^n = \frac{x}{1-4x-x^2} = \frac{x}{(1-\varphi^3 x)(1-\varphi'^3 x)} \quad (4.23)$$

For arbitrary k, it represents generating function for " mod k" Fibonacci numbers;

$$_{(k)}F(x) = \frac{1}{F_k}\sum_{n=0}^{\infty} F_{kn} x^n = \frac{x}{1-L_k x+(-1)^k x^2} = \frac{x}{(1-\varphi^k x)(1-\varphi'^k x)} \quad (4.24)$$

4.2.1. Entire Generating Function for Higher Order Fibonacci Numbers

Applying $_{(k)}D_F^x$ to e^x in power series form;

$$\begin{aligned} _{(k)}D_F^x(e^x) &= {}_{(k)}D_F^x\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right) = {}_{(k)}D_F^x\left(\frac{1}{0!}+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...\right) \\ &= \sum_{n=1}^{\infty} {}_{(k)}D_F^x\left(\frac{x^n}{n!}\right) = \sum_{n=1}^{\infty}\frac{_{(k)}D_F^x(x^n)}{n!} = \sum_{n=1}^{\infty}F_n^{(k)}\frac{x^{n-1}}{n!} = \sum_{n=0}^{\infty}F_{n+1}^{(k)}\frac{x^n}{(n+1)!} \end{aligned}$$

and so that,

$$ {}_{(k)}D_F^x(e^x) = \sum_{n=0}^{\infty} F_{n+1}^{(k)} \frac{x^n}{(n+1)!}. \tag{4.25} $$

This series converges for arbitrary x. By using definition (4.1) from another side,

$$ {}_{(k)}D_F^x(e^x) = \frac{e^{\varphi^k x} - e^{\varphi'^k x}}{(\varphi^k - \varphi'^k)x} = e^{\frac{\varphi^k}{2}x} e^{\frac{\varphi'^k}{2}x} \frac{e^{\left(\frac{\varphi^k - \varphi'^k}{2}\right)x} - e^{\left(\frac{\varphi'^k - \varphi^k}{2}\right)x}}{(\varphi^k - \varphi'^k)\, x} = e^{\frac{\varphi^k + \varphi'^k}{2}x} \frac{e^{\left(\frac{\varphi^k - \varphi'^k}{2}\right)x} - e^{-\left(\frac{\varphi^k - \varphi'^k}{2}\right)x}}{(\varphi^k - \varphi'^k)\, x} $$

Since $\varphi^k + \varphi'^k = L_k$ and $\varphi^k - \varphi'^k = F_k(\varphi - \varphi')$, we have;

$$ {}_{(k)}D_F^x(e^x) = 2e^{\frac{L_k}{2}x} \frac{\sinh\left(\frac{F_k}{2}(\varphi - \varphi')x\right)}{F_k(\varphi - \varphi')x} $$

Since $\varphi - \varphi' = \sqrt{5}$, finally we get;

$$ {}_{(k)}D_F^x(e^x) = e^{\frac{L_k}{2}x} \frac{\sinh\left(F_k \frac{\sqrt{5}}{2} x\right)}{\left(F_k \frac{\sqrt{5}}{2} x\right)} \tag{4.26} $$

Consequently, by equating both results (4.25) & (4.26), we obtain identity;

$$ \boxed{\sum_{n=0}^{\infty} \frac{F_{n+1}^{(k)}}{(n+1)!} x^n = e^{\frac{L_k}{2}x} \frac{\sinh\left(F_k \frac{\sqrt{5}}{2} x\right)}{\left(F_k \frac{\sqrt{5}}{2} x\right)}} \tag{4.27} $$

In particular case $k = 1$, it reduces to (3.18). This relation represents entire generating function for Higher Order Fibonacci numbers. For $x = 1$, it is;

$$ \sum_{n=0}^{\infty} \frac{F_n^{(k)}}{n!} = e^{\frac{L_k}{2}} \frac{\sinh\left(F_k \frac{\sqrt{5}}{2}\right)}{\left(F_k \frac{\sqrt{5}}{2}\right)} \tag{4.28} $$

or,

$$\sum_{n=0}^{\infty} \frac{F_{nk}}{n!} = e^{\frac{L_k}{2}} \frac{\sinh\left(F_k \frac{\sqrt{5}}{2}\right)}{\left(\frac{\sqrt{5}}{2}\right)}$$

In equation (4.27), after replacing $x \to ix$ we get,

$$\sum_{n=0}^{\infty} \frac{F_{n+1}^{(k)}}{(n+1)!}(i)^n x^n = e^{i\frac{L_k}{2}x} \frac{\sinh\left(F_k \frac{\sqrt{5}}{2} ix\right)}{\left(iF_k \frac{\sqrt{5}}{2} x\right)} \tag{4.29}$$

By using the identity $\sinh(ix) = i\sin(x)$ at the right hand side of equality, and splitting the sum at the left hand side to even and odd parts with $n = 2l$ and $n = 2l + 1$ gives;

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l x^{2l} + i\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l x^{2l+1} = e^{i\frac{L_k}{2}x} \frac{\sin\left(F_k \frac{\sqrt{5}}{2} x\right)}{F_k \frac{\sqrt{5}}{2} x} \tag{4.30}$$

Writing $e^{i\frac{L_k}{2}x} = \cos\left(\frac{L_k}{2}x\right) + i\sin\left(\frac{L_k}{2}x\right)$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l x^{2l} + i\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l x^{2l+1} = \cos\left(\frac{L_k}{2}x\right)\frac{\sin\left(F_k \frac{\sqrt{5}}{2} x\right)}{F_k \frac{\sqrt{5}}{2} x} + i\sin\left(\frac{L_k}{2}x\right)\frac{\sin\left(F_k \frac{\sqrt{5}}{2} x\right)}{F_k \frac{\sqrt{5}}{2} x}$$

By splitting to real and imaginary parts, we get generating functions for even and odd Higher order Fibonacci numbers as;

$$\boxed{\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l x^{2l} = \cos\left(\frac{L_k}{2}x\right) \frac{\sin\left(F_k \frac{\sqrt{5}}{2} x\right)}{F_k \frac{\sqrt{5}}{2} x}} \tag{4.31}$$

and,

$$\boxed{\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l x^{2l+1} = \sin\left(\frac{L_k}{2}x\right)\frac{\sin\left(F_k \frac{\sqrt{5}}{2} x\right)}{F_k \frac{\sqrt{5}}{2} x}} \tag{4.32}$$

From these entire functions, several identities follow. From (4.31) follow identities;

1) For $x = \pi$,

Since $\cos\left(\frac{L_k}{2}x\right) = \cos\left(\frac{\pi}{2}L_k\right)$, then;

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l \pi^{2l} = \cos\left(\frac{\pi}{2}L_k\right) \frac{\sin\left(F_k \frac{\sqrt{5}}{2}\pi\right)}{F_k \frac{\sqrt{5}}{2}\pi} \tag{4.33}$$

The right hand side vanishes for odd values of Lucas numbers L_k.

2) $x = \frac{2\pi}{\sqrt{5}}$,

Since $\sin(F_k\pi) = 0$, then;

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!} (-1)^l \frac{(2\pi)^{2l}}{5^l} = 0 \tag{4.34}$$

3) $x = \frac{\pi}{\sqrt{5}}$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l \frac{\pi^{2l}}{5^l} = \frac{2}{F_k\pi} \cos\left(L_k \frac{\pi}{2\sqrt{5}}\right) \sin\left(F_k \frac{\pi}{2}\right) \tag{4.35}$$

For even values of F_k the right hand side vanishes.

4) $x = 2\pi$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l (2\pi)^{2l} = \cos(L_k\pi) \frac{\sin\left(F_k \sqrt{5}\pi\right)}{F_k \sqrt{5}\pi} \tag{4.36}$$

5) $x = 1$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l = \cos\left(\frac{L_k}{2}\right) \frac{\sin\left(F_k \frac{\sqrt{5}}{2}\right)}{F_k \frac{\sqrt{5}}{2}} \tag{4.37}$$

6) $x = \frac{\pi}{L_k}$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l \left(\frac{\pi}{L_k}\right)^{2l} = 0 \tag{4.38}$$

7) $x = \frac{2\pi}{\sqrt{5}F_k}$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l \frac{(2\pi)^{2l}}{5^l (F_k)^{2l}} = 0 \tag{4.39}$$

In a similar way from (4.32) follow identities;

1) $x = \pi$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l (\pi)^{2l+1} = \sin\left(\frac{\pi}{2}L_k\right) \frac{\sin\left(F_k \frac{\sqrt{5}}{2}\pi\right)}{F_k \frac{\sqrt{5}}{2}\pi} \tag{4.40}$$

For even Lucas numbers, the right hand side is zero.

2) $x = \frac{2\pi}{\sqrt{5}}$,

Since $\sin(F_k \pi) = 0$, then;

$$\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!} (-1)^l \left(\frac{2\pi}{\sqrt{5}}\right)^{2l+1} = 0 \tag{4.41}$$

3) $x = \frac{\pi}{\sqrt{5}}$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l \left(\frac{\pi}{\sqrt{5}}\right)^{2l+1} = \frac{2}{F_k \pi} \sin\left(L_k \frac{\pi}{2\sqrt{5}}\right) \sin\left(F_k \frac{\pi}{2}\right) \tag{4.42}$$

For even Fibonacci numbers F_k the right hand side vanishes.

4) $x = 2\pi$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l (2\pi)^{2l+1} = 0 \tag{4.43}$$

5) $x = 1$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l = \sin\left(\frac{L_k}{2}\right) \frac{\sin\left(F_k \frac{\sqrt{5}}{2}\right)}{F_k \frac{\sqrt{5}}{2}} \tag{4.44}$$

6) $x = \frac{\pi}{L_k}$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l \left(\frac{\pi}{L_k}\right)^{2l+1} = \frac{\sin\left(\frac{F_k}{L_k}\frac{\sqrt{5}}{2}\pi\right)}{\frac{F_k}{L_k}\frac{\sqrt{5}}{2}\pi} \tag{4.45}$$

7) $x = \frac{2\pi}{L_k}$,
Since $\sin\left(\frac{L_k}{2}\frac{2\pi}{L_k}\right) = \sin(\pi) = 0$, then;

$$\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l \left(\frac{2\pi}{L_k}\right)^{2l+1} = 0 \tag{4.46}$$

From both (4.31) and (4.32) for $x = \frac{\pi}{\sqrt{5}F_k}$, we have;

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l \frac{\pi^{2l}}{5^l F_k^{2l}} = \frac{2}{\pi}\cos\left(\frac{L_k}{F_k}\frac{\pi}{2\sqrt{5}}\right) \tag{4.47}$$

and,

$$\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l \left(\frac{\pi}{\sqrt{5}F_k}\right)^{2l+1} = \frac{2}{\pi}\sin\left(\frac{L_k}{F_k}\frac{\pi}{2\sqrt{5}}\right) \tag{4.48}$$

4.3. Higher Fibonomials

Definition 4.3 *The product,*

$$k \cdot 2k \cdot 3k \ldots nk \equiv n!_{mod\ k} \tag{4.49}$$

is called mod k factorial. It is equal,

$$\prod_{i=1}^{n} ik = n!k^n \tag{4.50}$$

and for particular case it reduces to;

$$k = 1 \quad \Rightarrow \quad n!_{mod\ 1} = n!$$

Definition 4.4 *Product of Fibonacci numbers defined as,*

$$F_k F_{2k} \dots F_{nk} = \prod_{i=1}^{n} F_{ik} \equiv F_n!_{\ mod\ k}, \tag{4.51}$$

is called mod k Fibonacci factorial. For k $= 1$*, it gives the Fibonacci factorial;*

$$F_n!_{\ mod\ 1} = F_1 F_2 \dots F_n = F_n!. \tag{4.52}$$

For k $= 2$ *and n is even, it gives the double Fibonacci factorial;*

$$F_n!_{\ mod\ 2} = F_2 F_4 \dots F_{2n}. \tag{4.53}$$

Definition 4.5 *The product of Higher Fibonacci numbers,*

$$F_1^{(k)} F_2^{(k)} \dots F_n^{(k)} = \prod_{i=1}^{n} F_i^{(k)} \equiv F_n^{(k)}!, \tag{4.54}$$

is called the Higher Fibonacci factorial. This can be considered as the Higher Fibonorial or generalized Fibonorial. In particular case $k = 1$*, it reduces to Fibonacci factorial,* $F_n^{(1)}! = F_n!$*. For* $F_n^{(k)}!$ *we have next formula;*

$$F_n^{(k)}! = \frac{F_k F_{2k} F_{3k} \dots F_{nk}}{F_k F_k F_k \dots F_k} = \frac{F_k F_{2k} F_{3k} \dots F_{nk}}{(F_k)^n}, \tag{4.55}$$

or in terms of mod k Fibonacci factorial,

$$F_n^{(k)}! = \frac{F_n!_{\,mod\ k}}{(F_k)^n}. \tag{4.56}$$

Definition 4.6 *The higher order Fibonomial coefficients or shortly higher Fibonomial are defined as;*

$$_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F = \frac{F_1^{(k)} F_2^{(k)} \dots F_{n-m+1}^{(k)}}{F_1^{(k)} F_2^{(k)} \dots F_m^{(k)}} = \frac{F_n^{(k)}!}{F_m^{(k)}! F_{n-m}^{(k)}!}. \tag{4.57}$$

For $k = 1$*, it reduces to Fibonomials* (3.76)*,*

$$\begin{bmatrix} n \\ m \end{bmatrix}_F = \frac{F_n!}{F_{n-m}! F_m!}.$$

For arbitrary k it can be represented by mod k Fibonacci factorials (4.56)*;*

$$_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F = \frac{F_n!_{\,mod\ k}}{F_m!_{\,mod\ k}\, F_{n-m}!_{\,mod\ k}} \tag{4.58}$$

Similar way as for Fibonomials, it is possible to derive recursion formula for higher Fibonomials and interpretation of them in terms of Pascal type triangle. Higher Fibonomials can be used to define higher golden binomials.

4.4. Higher Golden Binomials

Definition 4.7 *The higher golden Binomial is the polynomial,*

$$_{(k)}(x-a)_F^n = \begin{cases} 1, & \text{if } n = 0; \\ \left(x - \varphi^{k(n-1)} a\right)\left(x - \varphi^{k(n-2)} \varphi'^k a\right) \dots \left(x - \varphi^k \varphi'^{k(n-2)} a\right)\left(x - \varphi'^{k(n-1)} a\right), & \text{if } n \geq 1. \end{cases}$$

For particular case $k = 1$, it reduces to Golden binomial (3.81). These polynomials satisfy the following formula.

Proposition 4.7 *(Factorization Property)*

$$
\begin{aligned}
{}_{(k)}(x-a)_F^{n+m} &= {}_{(k)}\left(x-\varphi^{km}a\right)_F^n \; {}_{(k)}\left(x-\varphi'^{kn}a\right)_F^m && (4.59)\\
&= {}_{(k)}\left(x-\varphi'^{km}a\right)_F^n \; {}_{(k)}\left(x-\varphi^{kn}a\right)_F^m && (4.60)
\end{aligned}
$$

Proof Sketch of the proof would be in Appendix B.3.3. □

Theorem 4.1 *Higher Golden binomial expansion is,*

$$
{}_{(k)}(x+y)_F^n = \sum_{m=0}^{n} {}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}} x^{n-m} y^m \tag{4.61}
$$

For particular case k = 1, *it reduces to standard one in* (3.82).

Proof Proof will be done by using the induction. Suppose for n, assumption in the theorem is true. For $n+1$, we will use the below factorization property such that,

$$
\begin{aligned}
{}_{(k)}(x+y)_F^{n+1} &= {}_{(k)}\left(x+\varphi'^k y\right)_F^n \; {}_{(k)}\left(x+\varphi^{kn}y\right)_F^1 = \left(x+\varphi^{kn}y\right) \; {}_{(k)}\left(x+\varphi'^k y\right)_F^n\\
&\overset{(4.61)}{=} \left(x+\varphi^{kn}y\right) \sum_{m=0}^{n} {}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}} x^{n-m} (\varphi'^k y)^m\\
&= \sum_{m=0}^{n} {}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}} x^{n-m+1} y^m \varphi'^{km}\\
&+ \sum_{m=0}^{n} {}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}} x^{n-m} y^{m+1} \varphi'^{km}\varphi^{kn}\\
&= \sum_{m=0}^{n} {}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}} x^{n-m+1} y^m \varphi'^{km}\\
&+ \sum_{m=0}^{n} {}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}} x^{n-m} y^{m+1} (-1)^{km} \varphi^{k(n-m)}
\end{aligned}
$$

In the second summation shifting $m \to m-1$ gives us,

$$
\begin{aligned}
{}_{(k)}(x+y)_F^{n+1} = \ldots &= \sum_{m=0}^{n} {}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}} x^{n-m+1} y^m \varphi'^{km}\\
&+ \sum_{m=1}^{n+1} {}_{(k)}\begin{bmatrix} n \\ m-1 \end{bmatrix}_F (-1)^{k\frac{(m-1)(m-2)}{2}} x^{n-m+1} y^m (-1)^{k(m-1)} \varphi^{k(n-m+1)}
\end{aligned}
$$

Here in the first summation for $m = n + 1$, term equal to zero. Because the coefficient ${}_{(k)}\begin{bmatrix} n \\ n+1 \end{bmatrix}_F = 0$. Also, in the second summation for $m = 0$, term equal to zero. Because the coefficient ${}_{(k)}\begin{bmatrix} n \\ -1 \end{bmatrix}_F = 0$. Thus, we can continue as;

$$\begin{aligned}
{}_{(k)}(x+y)_F^{n+1} = \ldots &= \sum_{m=0}^{n+1} {}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}} x^{n-m+1} y^m \varphi'^{km} \\
&+ \sum_{m=0}^{n+1} {}_{(k)}\begin{bmatrix} n \\ m-1 \end{bmatrix}_F (-1)^{k\frac{(m-1)(m-2)}{2}} (-1)^{k(m-1)} x^{n-m+1} y^m \varphi^{k(n-m+1)} \\
&= \sum_{m=0}^{n+1} {}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}} x^{n-m+1} y^m \varphi'^{km} \\
&+ \sum_{m=0}^{n+1} {}_{(k)}\begin{bmatrix} n \\ m-1 \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}} x^{n-m+1} y^m \varphi^{k(n-m+1)} \\
&= \sum_{m=0}^{n+1} \left({}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F \varphi'^{km} + {}_{(k)}\begin{bmatrix} n \\ m-1 \end{bmatrix}_F \varphi^{k(n-m+1)} \right) (-1)^{k\frac{m(m-1)}{2}} x^{n-m+1} y^m
\end{aligned}$$

It is easy to prove,

$$ {}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F = \varphi'^{km} {}_{(k)}\begin{bmatrix} n-1 \\ m \end{bmatrix}_F + \varphi^{k(n-m)} {}_{(k)}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_F \tag{4.62}$$

by following the steps to get the equation (3.79). Therefore, we have;

$$\begin{aligned}
{}_{(k)}(x+y)_F^{n+1} = \ldots &= \sum_{m=0}^{n+1} \left({}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F \varphi'^{km} + {}_{(k)}\begin{bmatrix} n \\ m-1 \end{bmatrix}_F \varphi^{k(n-m+1)} \right) (-1)^{k\frac{m(m-1)}{2}} x^{n-m+1} y^m \\
&\overset{(4.62)}{=} \sum_{m=0}^{n+1} {}_{(k)}\begin{bmatrix} n+1 \\ m \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}} x^{n-m+1} y^m
\end{aligned}$$

□

Corollary 4.3 *From this theorem, we obtain the identity,*

$$ {}_{(k)}(1+1)_F^n = \sum_{m=0}^{n} {}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}} \tag{4.63}$$

Lemma 4.1 *Higher order Fibonacci derivatives are acting on higher Golden binomials* *as,*

$$
\begin{aligned}
{}_{(k)}D_F^x \; {}_{(k)}(x+y)_F^n &= F_n^{(k)} \; {}_{(k)}(x+y)_F^{n-1} && (4.64)\\
{}_{(k)}D_F^y \; {}_{(k)}(x+y)_F^n &= F_n^{(k)} \; {}_{(k)}\left(x+(-1)^k y\right)_F^{n-1} && (4.65)\\
{}_{(k)}D_F^y \; {}_{(k)}(x-y)_F^n &= -F_n^{(k)} \; {}_{(k)}\left(x-(-1)^k y\right)_F^{n-1} && (4.66)
\end{aligned}
$$

For $k = 1$, these results give Lemma 3.2.

Proof To prove the first equality,

$$
\begin{aligned}
{}_{(k)}D_F^x \; {}_{(k)}(x+y)_F^n &\overset{(4.61)}{=} {}_{(k)}D_F^x \left[\sum_{m=0}^{n} {}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}} x^{n-m} y^m\right]\\
&= \sum_{m=0}^{n-1} \frac{F_n^{(k)}!}{F_{n-m}^{(k)}!\,F_m^{(k)}!} (-1)^{k\frac{m(m-1)}{2}} \; {}_{(k)}D_F^x\left(x^{n-m}\right) y^m\\
&= \sum_{m=0}^{n-1} \frac{F_n^{(k)}!}{F_{n-m}^{(k)}!\,F_m^{(k)}!} (-1)^{k\frac{m(m-1)}{2}} \; F_{n-m}^{(k)} \, x^{n-m-1} y^m\\
&= \sum_{m=0}^{n-1} \frac{F_n^{(k)}!}{F_{n-m-1}^{(k)}!\,F_m^{(k)}!} (-1)^{k\frac{m(m-1)}{2}} \; x^{n-m-1} y^m\\
&= F_n^{(k)} \sum_{m=0}^{n-1} \frac{F_{n-1}^{(k)}!}{F_{n-m-1}^{(k)}!\,F_m^{(k)}!} (-1)^{k\frac{m(m-1)}{2}} \; x^{n-m-1} y^m\\
&= F_n^{(k)} \; {}_{(k)}(x+y)_F^{n-1}
\end{aligned}
$$

To prove the second one,

$$
\begin{aligned}
{}_{(k)}D_F^y \; {}_{(k)}(x+y)_F^n &= {}_{(k)}D_F^y \left[\sum_{m=0}^{n} {}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}} x^{n-m} y^m\right]\\
&= \sum_{m=1}^{n} \frac{F_n^{(k)}!}{F_{n-m}^{(k)}!\,F_m^{(k)}!} (-1)^{k\frac{m(m-1)}{2}} \; x^{n-m} \; {}_{(k)}D_F^y\left(y^m\right)\\
&= \sum_{m=1}^{n} \frac{F_n^{(k)}!}{F_{n-m}^{(k)}!\,F_m^{(k)}!} (-1)^{k\frac{m(m-1)}{2}} \; x^{n-m} \, F_m^{(k)} \, y^{m-1}\\
&= \sum_{m=1}^{n} \frac{F_n^{(k)}!}{F_{n-m}^{(k)}!\,F_{m-1}^{(k)}!} (-1)^{k\frac{m(m-1)}{2}} \; x^{n-m} \, y^{m-1}\\
&\overset{(m\to m+1)}{=} \sum_{m=0}^{n-1} \frac{F_n^{(k)}!}{F_{(n-1)-m}^{(k)}!\,F_m^{(k)}!} (-1)^{k\frac{(m+1)m}{2}} \; x^{(n-1)-m} \, y^m
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{m=0}^{n-1} \frac{F_n^{(k)}!}{F_{(n-1)-m}^{(k)}!\, F_m^{(k)}!} (-1)^{k\frac{m(m-1)}{2}} (-1)^{km} x^{(n-1)-m} y^m \\
&= \sum_{m=0}^{n-1} \frac{F_n^{(k)} \cdot F_{n-1}^{(k)}!}{F_{(n-1)-m}^{(k)}!\, F_m^{(k)}!} (-1)^{k\frac{m(m-1)}{2}} x^{(n-1)-m} \left((-1)^k y\right)^m \\
&= F_n^{(k)} \sum_{m=0}^{n-1} \frac{F_{n-1}^{(k)}!}{F_{(n-1)-m}^{(k)}!\, F_m^{(k)}!} (-1)^{k\frac{m(m-1)}{2}} x^{(n-1)-m} \left((-1)^k y\right)^m \\
&= F_n^{(k)} \sum_{m=0}^{n-1} {}_{(k)}\begin{bmatrix} n-1 \\ m \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}} x^{(n-1)-m} \left((-1)^k y\right)^m \\
&= F_n^{(k)}\ {}_{(k)}\left(x + (-1)^k y\right)_F^{n-1}
\end{aligned}
$$

To prove the thirth equality,

$$
\begin{aligned}
{}_{(k)}D_F^y\ {}_{(k)}(x-y)_F^n &= {}_{(k)}D_F^y \left[\sum_{m=0}^{n} {}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}} x^{n-m} (-y)^m\right] \\
&= \sum_{m=1}^{n} \frac{F_n^{(k)}!}{F_{n-m}^{(k)}!\, F_m^{(k)}!} (-1)^{k\frac{m(m-1)}{2}} x^{n-m}\ {}_{(k)}D_F^y \left((-y)^m\right) \\
&= \sum_{m=1}^{n} \frac{F_n^{(k)}!}{F_{n-m}^{(k)}!\, F_m^{(k)}!} (-1)^{k\frac{m(m-1)}{2}} x^{n-m} (-1)^m F_m^{(k)} y^{m-1} \\
&= \sum_{m=1}^{n} \frac{F_n^{(k)}!}{F_{n-m}^{(k)}!\, F_{m-1}^{(k)}!} (-1)^{k\frac{m(m-1)}{2}} x^{n-m} (-1)^m y^{m-1} \\
&\overset{(m\to m+1)}{=} \sum_{m=0}^{n-1} \frac{F_n^{(k)}!}{F_{(n-1)-m}^{(k)}!\, F_m^{(k)}!} (-1)^{k\frac{(m+1)m}{2}} x^{(n-1)-m} (-1)^{m+1} y^m \\
&= -\sum_{m=0}^{n-1} \frac{F_n^{(k)}!}{F_{(n-1)-m}^{(k)}!\, F_m^{(k)}!} (-1)^{k\frac{m(m+1)}{2}} x^{(n-1)-m} (-1)^m y^m \\
&= -F_n^{(k)} \sum_{m=0}^{n-1} \frac{F_{n-1}^{(k)}!}{F_{(n-1)-m}^{(k)}!\, F_m^{(k)}!} (-1)^{k\frac{m(m+1)}{2}} x^{(n-1)-m} (-y)^m \\
&= -F_n^{(k)} \sum_{m=0}^{n-1} \frac{F_{n-1}^{(k)}!}{F_{(n-1)-m}^{(k)}!\, F_m^{(k)}!} (-1)^{k\frac{m(m-1)}{2}} (-1)^{km} x^{(n-1)-m} (-y)^m \\
&= -F_n^{(k)} \sum_{m=0}^{n-1} \frac{F_{n-1}^{(k)}!}{F_{(n-1)-m}^{(k)}!\, F_m^{(k)}!} (-1)^{k\frac{m(m-1)}{2}} x^{(n-1)-m} \left[(-1)^k \cdot (-y)\right]^m \\
&= -F_n^{(k)}\ {}_{(k)}\left(x - (-1)^k y\right)_F^{n-1} \qquad (4.67)
\end{aligned}
$$

□

As we have seen in (4.3),

$$_{(k)}D_F^x x^n = F_n^{(k)} x^{n-1}.$$

This implies to introduce monomials;

$$P_n^{(k)} \equiv \frac{x^n}{F_n^{(k)}!}, \tag{4.68}$$

such that

$$_{(k)}D_F^x \left(P_n^{(k)}\right) = P_{n-1}^{(k)}. \tag{4.69}$$

By these monomials we can derive the following Taylor expansion for arbitrary polynomials, according to Theorem (Kac, V. and Cheung, P., 2002).

Theorem 4.2 *(Higher Order Golden Taylor expansion)*

The derivative operator $_{(k)}D_F^x$ is a linear operator on the space of polynomials, and

$$P_n^{(k)}(x) \equiv \frac{x^n}{F_n^{(k)}!} \equiv \frac{x^n}{F_1^{(k)} \cdot F_2^{(k)} \ldots F_n^{(k)}}$$

satisfy the following conditions:
(i) $P_0^{(k)}(0) = 1$ and $P_n^{(k)}(0) = 0$ for any $n \geq 1$;
(ii) $deg(P_n^{(k)}) = n$;
(iii) $_{(k)}D_F^x(P_n^{(k)}(x)) = P_{n-1}^{(k)}(x)$ for any $n \geq 1$, and $_{(k)}D_F^x(1) = 0$.
Then, for any polynomial f(x) of degree N, one has the following Taylor formula;

$$f(x) = \sum_{n=0}^{N} ({}_{(k)}D_F^x)^n f\,(0) P_n^{(k)}(x) = \sum_{n=0}^{N} ({}_{(k)}D_F^x)^n f\,(0) \frac{x^n}{F_n^{(k)}!} \tag{4.70}$$

Example 4.6 *Let's expand function $f(x) = (x+1)^3$ in terms of the polynomials $P_n^{(k)}$,*

$k = 2$. The Taylor expansion becomes,

$$(x+1)^3 = \sum_{n=0}^{3} ({}_{(2)}D_F^x)^n f\ (0) \frac{x^n}{F_n^{(2)}!} \tag{4.71}$$

After expanding,

$$(x+1)^3 = ({}_{(2)}D_F^x)^3 f\ (0) \frac{x^3}{F_3^{(2)}!} + ({}_{(2)}D_F^x)^2 f\ (0) \frac{x^2}{F_2^{(2)}!} + ({}_{(2)}D_F^x) f\ (0) \frac{x}{F_1^{(2)}!} + f(0) \frac{1}{F_0^{(2)}!}$$

We should calculate the coefficients $({}_{(2)}D_F^x)^3 f, ({}_{(2)}D_F^x)^2 f, ({}_{(2)}D_F^x) f$ at $x = 0$. The derivatives are,

$$\begin{aligned}
({}_{(2)}D_F^x)\,(x+1)^3 &= ({}_{(2)}D_F^x)\left(x^3 + 3x^2 + 3x + 1\right) = F_3^{(2)} x^2 + 3F_2^{(2)} x + 3F_1^{(2)} \\
({}_{(2)}D_F^x)^2\,(x+1)^3 &= ({}_{(2)}D_F^x)\left(F_3^{(2)} x^2 + 3F_2^{(2)} x + 3F_1^{(2)}\right) = F_3^{(2)} F_2^{(2)} x + 3F_2^{(2)} F_1^{(2)} \\
({}_{(2)}D_F^x)^3\,(x+1)^3 &= ({}_{(2)}D_F^x)\left(F_3^{(2)} F_2^{(2)} x + 3F_2^{(2)} F_1^{(2)}\right) = F_3^{(2)} F_2^{(2)} F_1^{(2)}
\end{aligned}$$

At $x = 0$,

$$\begin{aligned}
({}_{(2)}D_F^x)\,(x+1)^3 \Big|_{x=0} &= 3F_1^{(2)}! \\
({}_{(2)}D_F^x)^2\,(x+1)^3 \Big|_{x=0} &= 3F_2^{(2)}! \\
({}_{(2)}D_F^x)^3\,(x+1)^3 \Big|_{x=0} &= F_3^{(2)}!
\end{aligned}$$

Finally, substituting them gives,

$$(x+1)^3 = F_3^{(2)}!\ P_3^{(2)}(x) + 3F_2^{(2)}!\ P_2^{(2)}(x) + 3F_1^{(2)}!\ P_1^{(2)}(x) + P_0^{(2)}(x) \tag{4.72}$$

Since $F_3^{(2)}! = 24,\ 3F_2^{(2)}! = 9,\ 3F_1^{(2)}! = 3$, we have;

$$(x+1)^3 = 24P_3^{(2)}(x) + 9P_2^{(2)}(x) + 3P_1^{(2)}(x) + P_0^{(2)}(x). \tag{4.73}$$

In the limit $N \to \infty$ (if it exists) the Taylor formula (4.70) determines expansion of function ${}_{(k)}\ f_F(x)$ in $P_n^{(k)}(x)$ polynomials.

$$
{}_{(k)}\ f_F(x) = \sum_{n=0}^{\infty} ({}_{(k)}D_F^x f)^n(0) \frac{x^n}{F_n^{(k)}!}
$$

Proposition 4.8 *Let,*

$$
f(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}
$$

is an entire complex valued function of complex variable z. Then exists complex function ${}_{(k)}\ f_F(z)$ determined by formula,

$$
{}_{(k)}\ f_F(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{F_n^{(k)}!}
$$

and this function is entire.

Proof To check convergency of these functions we apply the ratio test;

$$
\rho = |z| \lim_{n\to\infty} \left|\frac{1}{n+1}\right| \left|\frac{a_{n+1}}{a_n}\right| \tag{4.74}
$$

$$
\begin{aligned}
{}_{(k)}\rho_F &= |z| \lim_{n\to\infty} \left|\frac{1}{F_{n+1}^{(k)}}\right| \left|\frac{a_{n+1}}{a_n}\right| \qquad (4.75)\\
&= |z| \lim_{n\to\infty} \left|\frac{n+1}{F_{n+1}^{(k)}}\right| \left(\left|\frac{1}{n+1}\right| \left|\frac{a_{n+1}}{a_n}\right|\right)\\
&= \lim_{n\to\infty} \left|\frac{n+1}{F_{n+1}^{(k)}}\right| \rho
\end{aligned}
$$

Since $\lim_{n\to\infty} \left|\frac{n+1}{F_{n+1}^{(k)}}\right| = 0$ and for entire $f(z) \Rightarrow \rho = 0$, then ${}_{(k)}\rho_F = 0$ and ${}_{(k)}\ f_F(z)$ is entire. □

As an example we introduce higher order golden exponentials:

Definition 4.8 *(Higher Order Golden Exponentials)*

$$
\begin{aligned}
{}_{(k)}e_F^x &\equiv \sum_{n=0}^{\infty} \frac{x^n}{F_n^{(k)}!} && (4.76)\\
{}_{(k)}E_F^x &\equiv \sum_{n=0}^{\infty} (-1)^{k\frac{n(n-1)}{2}} \frac{x^n}{F_n^{(k)}!} && (4.77)
\end{aligned}
$$

where,

$$
F_n^{(k)}! = F_1^{(k)} \cdot F_2^{(k)} \cdot F_n^{(k)} \ldots F_n^{(k)} = \frac{F_k}{F_k} \cdot \frac{F_{2k}}{F_k} \cdot \frac{F_{3k}}{F_k} \cdots \frac{F_{nk}}{F_k} = \frac{F_k \cdot F_{2k} \cdot F_{3k} \ldots F_{nk}}{(F_k)^n} \quad (4.78)
$$

For the particular case if $k = 1$, it reduces to exponential functions in (3.41) *and* (3.42).

Proposition 4.9 *The Higher k^{th} order Golden derivative of these Higher order Golden exponentials is found as;*

$$
\begin{aligned}
{}_{(k)}D_F^x \left({}_{(k)}e_F^{\lambda x}\right) &= \lambda \; {}_{(k)}e_F^{\lambda x} && (4.79)\\
{}_{(k)}D_F^x \left({}_{(k)}E_F^{\lambda x}\right) &= \lambda \; {}_{(k)}E_F^{(-1)^k \lambda x} && (4.80)
\end{aligned}
$$

for an arbitrary k.

Proof

$$
\begin{aligned}
{}_{(k)}D_F^x \left({}_{(k)}e_F^{\lambda x}\right) &= {}_{(k)}D_F^x \left(\sum_{n=0}^{\infty} \frac{(\lambda x)^n}{F_n^{(k)}!} \right)\\
&= \sum_{n=1}^{\infty} \lambda^n \frac{{}_{(k)}D_F^x (x^n)}{F_n^{(k)}!}\\
&= \sum_{n=1}^{\infty} \frac{\lambda^n F_n^{(k)} x^{n-1}}{F_n^{(k)}!}\\
&= \sum_{n=1}^{\infty} \frac{\lambda^n x^{n-1}}{F_{n-1}^{(k)}!} = \sum_{n=0}^{\infty} \frac{\lambda^{n+1} x^n}{F_n^{(k)}!}\\
&= \lambda \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{F_n^{(k)}!} = \lambda \; {}_{(k)}e_F^{\lambda x}.
\end{aligned}
$$

$$
\begin{aligned}
{}_{(k)}D_F^x\left({}_{(k)}E_F^{\lambda x}\right) ={}_{(k)} D_F^x\left(\sum_{n=0}^{\infty}(-1)^{k\frac{n(n-1)}{2}}\frac{(\lambda x)^n}{F_n^{(k)}!}\right) \quad &= \sum_{n=1}^{\infty}(-1)^{k\frac{n(n-1)}{2}}\lambda^n\frac{{}_{(k)}D_F^x(x^n)}{F_n^{(k)}!}\\
&= \sum_{n=1}^{\infty}(-1)^{k\frac{n(n-1)}{2}}\lambda^n\frac{F_n^{(k)}x^{n-1}}{F_n^{(k)}!}\\
&= \sum_{n=1}^{\infty}(-1)^{k\frac{n(n-1)}{2}}\lambda^n\frac{x^{n-1}}{F_{n-1}^{(k)}!}\\
&\overset{(n\longrightarrow n+1)}{=} \sum_{n=0}^{\infty}(-1)^{k\frac{(n+1)n}{2}}\lambda^{n+1}\frac{x^n}{F_n^{(k)}!}\\
&= \lambda\sum_{n=0}^{\infty}(-1)^{k\frac{(n^2+n)}{2}}\lambda^n\frac{x^n}{F_n^{(k)}!}\\
&= \lambda\sum_{n=0}^{\infty}(-1)^{k\frac{(n^2+n)-n+n}{2}}\frac{(\lambda x)^n}{F_n^{(k)}!}\\
&= \lambda\sum_{n=0}^{\infty}(-1)^{k\frac{n(n-1)}{2}}(-1)^{kn}\frac{(\lambda x)^n}{F_n^{(k)}!}\\
&= \lambda\sum_{n=0}^{\infty}(-1)^{k\frac{n(n-1)}{2}}\frac{((-1)^k\lambda x)^n}{F_n^{(k)}!}\\
&= \lambda\ {}_{(k)}E_F^{(-1)^k\lambda x}
\end{aligned}
$$

□

CHAPTER 5

CARLITZ CHARACTERISTIC POLYNOMIALS AND GOLDEN BINOMIALS

5.1. Carlitz Polynomials

In Section 3.6, we have introduced the Golden binomials. In this Chapter, we are going to relate these binomials with characteristic equations for some matrices, constructed from binomial coefficients, which was derived by Carlitz (Carlitz, L., 1965).

Definition 5.1 *We define an $n+1 \times n+1$ matrix A_{n+1} with binomial coefficients,*

$$A_{n+1} = \left[\binom{r}{n-s} \right] \tag{5.1}$$

where $r, s = 0, 1, 2, ..., n$. Here,

$$\binom{n}{k} = \begin{cases} \frac{n!}{(n-k)!\,k!}, & \text{if } k \leq n; \\ 0, & k > n. \end{cases} \tag{5.2}$$

First few matrices are,

$$\textbf{n=0} \;\Rightarrow r = s = 0 \Rightarrow \; A_1 = \left[\binom{0}{0} \right] = (1)$$

$$\textbf{n=1} \;\Rightarrow r, s = 0, 1 \Rightarrow \; A_2 = \left[\binom{r}{1-s} \right] = \begin{pmatrix} \binom{0}{1} & \binom{0}{0} \\ \binom{1}{1} & \binom{1}{0} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\textbf{n=2} \Rightarrow r, s = 0, 1, 2 \Rightarrow A_3 = \left[\binom{r}{2-s} \right] = \begin{pmatrix} \binom{0}{2} & \binom{0}{1} & \binom{0}{0} \\ \binom{1}{2} & \binom{1}{1} & \binom{1}{0} \\ \binom{2}{2} & \binom{2}{1} & \binom{2}{0} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

Continuing, the general matrix A_{n+1} of order $(n+1)$ can be written as,

$$A_{n+1} = \begin{pmatrix} \ldots 0 & 0 & 0 & 0 & 1 \\ \ldots 0 & 0 & 0 & 1 & 1 \\ \ldots 0 & 0 & 1 & 2 & 1 \\ \ldots 0 & 1 & 3 & 3 & 1 \\ \ldots 1 & 4 & 6 & 4 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}_{(n+1)\times(n+1)}$$

We can notice that trace of first few matrices A_{n+1} give Fibonacci numbers. It would be shown in Theorem (5.2) equation (5.14) and it is valid for any n.

Definition 5.2 *Characteristic polynomial of matrix A_{n+1} is determined by,*

$$P_{n+1}(x) = det(xI - A_{n+1}) \tag{5.3}$$

Let's find first few polynomials;

n=0: $P_1(x) = 1 - x$

n=1: $P_2(x) = \det(xI - A_2) = \begin{vmatrix} x & -1 \\ -1 & x-1 \end{vmatrix} = x^2 - x - 1$

n=2: $P_3(x) = \det(xI - A_3) = \begin{vmatrix} x & 0 & -1 \\ 0 & x-1 & -1 \\ -1 & -2 & x-1 \end{vmatrix} = x^3 - 2x^2 - 2x + 1$

n=3:

$$\begin{aligned} P_4(x) = \det(xI - A_4) &= \begin{vmatrix} x & 0 & 0 & -1 \\ 0 & x & -1 & -1 \\ 0 & -1 & x-2 & -1 \\ -1 & -3 & -3 & x-1 \end{vmatrix} \\ &= -x^4 + 3x^3 + 6x^2 - 3x - 1 \end{aligned}$$

Corresponding eigenvalues are represented by powers of φ and φ';

n=0 $\Rightarrow$ $x_1 = 1$

n=1 $\Rightarrow$ $x_1 = \varphi, \quad x_2 = \varphi'$

n=2 $\Rightarrow$ $x_1 = \varphi^2, \quad x_2 = -1, \quad x_3 = \varphi'^2$

n=3 $\Rightarrow$ $x_1 = \varphi^3, x_2 = -\varphi, x_3 = -\varphi', x_4 = \varphi'^3$

Comparing zeros of first few characteristic polynomials, with zeros of Golden Binomial (3.84), we notice that they coincide. According to this, we have following.

Conjecture: The characteristic equation (5.3) of matrix A_{n+1} coincides with Golden Binomial;

$$P_{n+1}(x) = \det(xI - A_{n+1}) = (x-1)_F^{n+1}. \tag{5.4}$$

As a first step to prove this conjecture we represent Golden binomials in the product form.

Proposition 5.1 *The Golden binomial can be written as a product,*

$$(x-1)_F^{n+1} = \prod_{j=0}^{n} \left(x - \varphi^j \varphi'^{n-j}\right) \tag{5.5}$$

Proof We have Golden binomial in product representation as;

$$(x+y)_F^n \equiv \prod_{j=0}^{n-1} \left(x - (-1)^{j-1}\, \varphi^{n-1}\, \varphi^{-2j} y\right) \tag{5.6}$$

Since,

$$\varphi^{-2j} = \left(\frac{1}{\varphi}\right)^{2j} = \left(-\frac{1}{\varphi}\right)^{2j} = \varphi'^{2j}, \tag{5.7}$$

then, after choosing $y = -1$;

$$(x-1)_F^n \equiv \prod_{j=0}^{n-1} \left(x - (-1)^j\, \varphi^{n-1}\, \varphi'^{2j}\right)$$

Shifting $n \leftrightarrow n+1$ gives,

$$
\begin{aligned}
(x-1)_F^{n+1} &= \prod_{j=0}^{n}\left(x-(-1)^j\,\varphi^n\,\varphi'^{2j}\right)\\
&= \prod_{j=0}^{n}\left(x-(-1)^j\,\varphi^n\,\frac{(-1)^{2j}}{\varphi^j\varphi^j}\right)\\
&= \prod_{j=0}^{n}\left(x-\varphi^n\left(-\frac{1}{\varphi}\right)^j\,\frac{1}{\varphi^j}\right)\\
&= \prod_{j=0}^{n}\left(x-\varphi^{n-j}\varphi'^{j}\right)
\end{aligned}
$$

Here, at this step if we make $j = n - m$ substitution,

$$
(x-1)_F^{n+1} = \prod_{m=0}^{n}(x-\varphi^m\varphi'^{\,n-m})
$$

This formula explicitly shows that zeros of Golden binomial in (3.85) and (3.86) are determined by powers of φ and φ'. □

Corollary 5.1 *We can directly say that eigenvalues of the matrix A_{n+1} are the numbers,*

$$
\varphi^n, \varphi^{n-1}\varphi', \varphi^{n-2}\varphi'^2, \ldots, \varphi\,\varphi'^{n-1}, \varphi'^n \tag{5.8}
$$

As it was shown by Carlitz (Carlitz, L., 1965) this product formula is just characteristic equation (5.3) for matrix A_{n+1}. Since zeros of two polynomials $\det(A_{n+1} - xI)$ and $(x-1)_F^{n+1}$ coincide, then the conjecture is correct and we have following theorem.

Theorem 5.1 *Characteristic equation for combinatorial matrix A_{n+1} is given by Golden binomial:*

$$
P_{n+1}(x) = det(xI - A_{n+1}) = (x-1)_F^{n+1} \tag{5.9}
$$

5.2. Powers of Matrices A_{n+1}^k and Higher Fibonacci Numbers

Proposition 5.2 *Arbitrary n^{th} power of A_2 matrix is written in terms of Fibonacci numbers,*

$$A_2^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} \tag{5.10}$$

Proof Proof will be done by Principal of Mathematical induction. For $n = 1$,

$$A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} F_0 & F_1 \\ F_1 & F_2 \end{pmatrix}.$$

For $n = 2$,

$$A_2^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \\ F_2 & F_3 \end{pmatrix}.$$

Suppose for $n = k$,

$$A_2^k = \begin{pmatrix} F_{k-1} & F_k \\ F_k & F_{k+1} \end{pmatrix}$$

is true. Then for $n = k + 1$,

$$A_2^{k+1} = A_2^k A_2 = \begin{pmatrix} F_{k-1} & F_k \\ F_k & F_{k+1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} F_k & F_k + F_{k-1} \\ F_{k+1} & F_k + F_{k+1} \end{pmatrix} = \begin{pmatrix} F_k & F_{k+1} \\ F_{k+1} & F_{k+2} \end{pmatrix}.$$

This result can be understood from observation that eigenvalues of matrix A_2 are φ and φ', and eigenvalues of A_2^n are powers φ^n, φ'^n related with Fibonacci numbers.

□

As we have seen, eigenvalues of matrix A_3 are $\varphi^2, \varphi'^2, -1$. It implies that for A_3^n, eigenvalues are $\varphi^{2n}, \varphi'^{2n}, (-1)^n$, and this matrix can be expressed by $F_n^{(2)}$ Higher order Fibonacci numbers due to (2.30) and (2.31).

Proposition 5.3 *Arbitrary n^{th} power of A_3 matrix can be expressed in terms of Higher order Fibonacci numbers $F_n^{(2)}$,*

$$A_3^n = \frac{1}{5}\begin{pmatrix} (2F_n^{(2)} - 3F_{n-1}^{(2)} + 2(-1)^n) & (2F_n^{(2)} + 2F_{n-1}^{(2)} + 2(-1)^n) & (3F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n) \\ (F_n^{(2)} + F_{n-1}^{(2)} + (-1)^n) & (6F_n^{(2)} - 4F_{n-1}^{(2)} + (-1)^n) & (4F_n^{(2)} - F_{n-1}^{(2)} - (-1)^n) \\ (3F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n) & (8F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n) & (7F_n^{(2)} - 3F_{n-1}^{(2)} + 2(-1)^n) \end{pmatrix}$$

Proof Let's diagonalize the matrix A_3,

$$\phi_3 = \sigma_3^{-1} \; A_3 \; \sigma_3,$$

where ϕ_3 is the diagonalize matrix. Thus,

$$A_3 = \sigma_3 \, \phi_3 \, \sigma_3^{-1}.$$

Taking the n^{th} power of both sides gives;

$$A_3^n = (\sigma_3 \, \phi_3 \, \underbrace{\sigma_3^{-1}) \; (\sigma_3}_{I} \, \phi_3 \, \sigma_3^{-1}) \; \ldots \; (\sigma_3 \, \phi_3 \, \underbrace{\sigma_3^{-1}) \; (\sigma_3}_{I} \, \phi_3 \, \sigma_3^{-1})$$

Therefore, we obtain;

$$\boxed{A_3^n = \sigma_3 \, \phi_3^n \, \sigma_3^{-1}} \tag{5.11}$$

By using the diagonalizing principle σ_3 and σ_3^{-1} matrices can be obtained as,

$$\sigma_3 = \frac{1}{2}\begin{pmatrix} -\varphi' & \frac{4}{3} & -\varphi \\ 1 & \frac{2}{3} & 1 \\ \varphi & -\frac{4}{3} & \varphi' \end{pmatrix}$$

and,

$$\sigma_3^{-1} = \begin{pmatrix} \frac{2(\varphi'+2)}{5(\varphi-\varphi')} & -\frac{4(\varphi'+2)}{5\varphi'(\varphi-\varphi')} & \frac{2(2\varphi'-1)}{5\varphi'(\varphi-\varphi')} \\ \frac{3}{5} & \frac{3}{5} & -\frac{3}{5} \\ -\frac{2(\varphi+2)}{5(\varphi-\varphi')} & \frac{4(\varphi+2)}{5\varphi(\varphi-\varphi')} & \frac{2(1-2\varphi)}{5\varphi(\varphi-\varphi')} \end{pmatrix} = \frac{2}{5\sqrt{5}} \begin{pmatrix} \varphi'+2 & -2(1-2\varphi) & (2+\varphi) \\ \frac{3\sqrt{5}}{2} & \frac{3\sqrt{5}}{2} & -\frac{3\sqrt{5}}{2} \\ -(\varphi+2) & 2(1-2\varphi') & -(2+\varphi') \end{pmatrix}$$

Since eigenvalues of matrix A_3^n are $\varphi^2, -1, \varphi'^2$, the diagonal matrix ϕ_3 is,

$$\phi_3 = \begin{pmatrix} \varphi'^2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \varphi'^2 \end{pmatrix}, \tag{5.12}$$

and an arbitrary n^{th} power of this matrix is,

$$\phi_3^n = \begin{pmatrix} (\varphi'^2)^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (\varphi'^2)^n \end{pmatrix} \tag{5.13}$$

Now by using (5.11),

$$A_3^n = \frac{1}{5} \begin{pmatrix} (2F_n^{(2)} - 3F_{n-1}^{(2)} + 2(-1)^n) & (2F_n^{(2)} + 2F_{n-1}^{(2)} + 2(-1)^n) & (3F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n) \\ (F_n^{(2)} + F_{n-1}^{(2)} + (-1)^n) & (6F_n^{(2)} - 4F_{n-1}^{(2)} + (-1)^n) & (4F_n^{(2)} - F_{n-1}^{(2)} - (-1)^n) \\ (3F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n) & (8F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n) & (7F_n^{(2)} - 3F_{n-1}^{(2)} + 2(-1)^n) \end{pmatrix}$$

is obtained. □

We can expect that these results can be generalized to arbitrary matrix A_{n+1}. Since eigenvalues of A_{n+1} are powers φ^n,φ'^n, . . ., for A_{n+1}^N eigenvalues are φ^{nN},φ'^{nN}, . . . But these powers can be written in terms of higher Fibonacci numbers (2.30) and (2.31), and the matrix A_{n+1}^N itself can be represented by higher Fibonacci numbers $F_N^{(n)}$. For powers of matrix A_{n+1} the following identities hold.

Theorem 5.2 *Invariants of A^k_{n+1} matrix can be found as;*

$$Tr\left(A^k_{n+1}\right) = \frac{F_{kn+k}}{F_k} = F^{(k)}_{n+1} \tag{5.14}$$

$$det\left(A^k_{n+1}\right) = (-1)^{k\,\frac{n(n+1)}{2}} \tag{5.15}$$

For $k = 1$, it gives;

$$Tr\,(A_{n+1}) = F_{n+1},$$
$$det\,(A_{n+1}) = (-1)^{\frac{n(n+1)}{2}}$$

Proof Let's diagonalize the general matrix A_{n+1} as,

$$\phi_{n+1} = \sigma^{-1}_{n+1}\ A_{n+1}\ \sigma_{n+1}$$

where ϕ_{n+1} is diagonal. Thus,

$$A_{n+1} = \sigma_{n+1}\ \phi_{n+1}\ \sigma^{-1}_{n+1}$$

Taking the k^{th} power of both sides gives;

$$A^k_{n+1} = (\sigma_{n+1}\ \phi_{n+1}\ \underbrace{\sigma^{-1}_{n+1})\ (\sigma_{n+1}}_{I}\ \phi_{n+1}\ \sigma^{-1}_{n+1})\ \ldots\ (\sigma_{n+1}\ \phi_{n+1}\ \underbrace{\sigma^{-1}_{n+1})\ (\sigma_{n+1}}_{I}\ \phi_{n+1}\ \sigma^{-1}_{n+1})$$

and,

$$A^k_{n+1} = \sigma_{n+1}\ \phi^k_{n+1}\ \sigma^{-1}_{n+1}. \tag{5.16}$$

After taking trace of both sides and using the cyclic permutation property of trace;

$$Tr(A^k_{n+1}) = Tr\,(\sigma_{n+1}\ \phi^k_{n+1}\ \sigma^{-1}_{n+1}) = Tr\,(\sigma^{-1}_{n+1}\ \sigma_{n+1}\ \phi^k_{n+1}) = Tr(I\ \ \phi^k_{n+1}) = Tr\,(\phi^k_{n+1})$$

we get,

$$\boxed{Tr(A_{n+1}^k) = Tr\,(\phi_{n+1}^k)}$$

The eigenvalues of matrix A_{n+1} in (5.8), allow to construct the diagonal matrix ϕ_{n+1}.

$$Tr(A_{n+1}^k) = Tr \begin{pmatrix} \varphi^n & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \varphi^{n-1}\varphi' & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \varphi^{n-2}\varphi'^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \varphi^2\varphi'^{n-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \varphi\varphi'^{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \varphi'^n \end{pmatrix}^k$$

giving,

$$Tr(A_{n+1}^k) = Tr \begin{pmatrix} (\varphi^n)^k & 0 & 0 & . & 0 & 0 & 0 \\ 0 & (\varphi^{n-1}\varphi')^k & 0 & . & 0 & 0 & 0 \\ 0 & 0 & (\varphi^{n-2}\varphi'^2)^k & . & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & . & (\varphi^2\varphi'^{n-2})^k & 0 & 0 \\ 0 & 0 & 0 & . & 0 & (\varphi\varphi'^{n-1})^k & 0 \\ 0 & 0 & 0 & . & 0 & 0 & (\varphi'^n)^k \end{pmatrix} \quad (5.17)$$

Since trace is the addition of the diagonal elements of the matrix,

$$\begin{aligned} Tr(A_{n+1}^k) &= (\varphi^n)^k + (\varphi^{n-1}\varphi')^k + \ldots + (\varphi\varphi'^{n-1})^k + (\varphi'^n)^k \\ Tr(A_{n+1}^k) &= (\varphi^k)^n + (\varphi^k)^{n-1}\varphi'^k + \ldots + \varphi^k(\varphi'^k)^{n-1} + (\varphi'^k)^n \end{aligned}$$

The powers $(\varphi^k)^n$ and $(\varphi'^k)^n$ are known from equations (2.30) and (2.31), and substituting it gives;

$$\begin{aligned}
Tr(A^k_{n+1}) &= \left(\varphi^k\ F^{(k)}_n + (-1)^{k+1}\ F^{(k)}_{n-1}\right) + \left(\varphi^k\ F^{(k)}_{n-1} + (-1)^{k+1}\ F^{(k)}_{n-2}\right)\varphi'^k + \dots \\
&\quad + \left(\varphi^k\ F^{(k)}_1 + (-1)^{k+1}\ F^{(k)}_0\right)(\varphi'^k)^{n-1} + (\varphi'^k)^n \\
&= \varphi^k\left(F^{(k)}_n + F^{(k)}_{n-1}(\varphi'^k) + F^{(k)}_{n-2}(\varphi'^k)^2 + \dots + F^{(k)}_1(\varphi'^k)^{n-1}\right) \\
&\quad +(-1)^{k+1}\left(F^{(k)}_{n-1} + F^{(k)}_{n-2}(\varphi'^k) + F^{(k)}_{n-3}(\varphi'^k)^2 + \dots + F^{(k)}_0(\varphi'^k)^{n-1}\right) \\
&\quad +(\varphi'^k)^n \\
&= \varphi^k\left(\frac{F_{kn}}{F_k} + \frac{F_{(n-1)k}}{F_k}(\varphi'^k) + \frac{F_{(n-2)k}}{F_k}(\varphi'^k)^2 + \dots + \frac{F_k}{F_k}(\varphi'^k)^{n-1}\right) \\
&\quad +(-1)^{k+1}\left(\frac{F_{(n-1)k}}{F_k} + \frac{F_{(n-2)k}}{F_k}(\varphi'^k) + \frac{F_{(n-3)k}}{F_k}(\varphi'^k)^2 + \dots + \frac{F_0}{F_k}(\varphi'^k)^{n-1}\right) \\
&\quad +(\varphi'^k)^n \\
&\overset{(\varphi\varphi'=-1)}{=} \frac{F_{kn}}{F_k}\ \varphi^k + \frac{F_{(n-1)k}}{F_k}(-1)^k + \frac{F_{(n-2)k}}{F_k}(-1)^k(\varphi'^k) + \dots + \frac{F_k}{F_k}(\varphi^k)(\varphi'^k)^{n-1} \\
&\quad +\frac{F_{(n-1)k}}{F_k}(-1)^{k+1} + \frac{F_{(n-2)k}}{F_k}(-1)^{k+1}(\varphi'^k) + \frac{F_{(n-3)k}}{F_k}(-1)^{k+1}(\varphi'^k)^2 \\
&\quad + \dots + \frac{F_0}{F_k}(-1)^{k+1}\varphi'^{n-1} + (\varphi'^k)^n \\
&= \frac{F_{kn}}{F_k}\ \varphi^k + \frac{F_{(n-1)k}}{F_k}(-1)^k + \frac{F_{(n-2)k}}{F_k}(-1)^k(\varphi'^k) + \dots \\
&\quad +\frac{F_{(n-(n-1))k}}{F_k}(-1)^k(\varphi'^k)^{n-2} + \frac{F_{(n-1)k}}{F_k}(-1)^{k+1} + \frac{F_{(n-2)k}}{F_k}(-1)^{k+1}(\varphi'^k) \\
&\quad +\frac{F_{(n-3)k}}{F_k}(-1)^{k+1}(\varphi'^k)^2 + \dots + \frac{F_k}{F_k}(-1)^{k+1}(\varphi'^k)^{n-2} + (\varphi'^k)^n \\
&= \frac{F_{kn}}{F_k}\ \varphi^k + \frac{F_{(n-1)k}}{F_k}\left((-1)^k + (-1)^{k+1}\right) \\
&\quad +\frac{F_{(n-2)k}}{F_k}\left((-1)^k\varphi'^k + (-1)^{k+1}\varphi'^k\right) + \frac{F_{(n-3)k}}{F_k}\left((-1)^k(\varphi'^k)^2 + (-1)^{k+1}(\varphi'^k)^2\right) \\
&\quad + \dots + \frac{F_k}{F_k}\left((-1)^k(\varphi'^k)^{n-2} + (-1)^{k+1}(\varphi'^k)^{n-2}\right) + (\varphi'^k)^n \\
&= \frac{F_{kn}}{F_k}\ \varphi^k + \frac{F_{(n-1)k}}{F_k}(-1)^k(1 + (-1)) + \frac{F_{(n-2)k}}{F_k}(-1)^k\varphi'^k(1 + (-1)) \\
&\quad +\frac{F_{(n-3)k}}{F_k}(-1)^k(\varphi'^k)^2(1 + (-1)) + \dots + \frac{F_k}{F_k}(-1)^k(\varphi'^k)^{n+2}(1 + (-1)) \\
&\quad +(\varphi'^k)^n \\
&= \frac{F_{kn}}{F_k}\ \varphi^k + (\varphi'^k)^n \\
&\overset{(2.31)}{=} \frac{F_{kn}}{F_k}\ \varphi^k + \varphi'^k\ F^{(k)}_n + (-1)^{k+1}\ F^{(k)}_{n-1} \\
&= \frac{F_{kn}}{F_k}\ \varphi^k + \varphi'^k\ \frac{F_{kn}}{F_k} + (-1)^{k+1}\ \frac{F_{k(n-1)}}{F_k}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{F_k}\left(F_{kn}\varphi^k + \varphi'^k\, F_{kn} + (-1)^{k+1}\, F_{k(n-1)}\right) \\
&= \frac{1}{F_k}\frac{1}{\varphi-\varphi'}\left[\left(\varphi^{kn}-\varphi'^{kn}\right)\varphi^k + \varphi'^k\left(\varphi^{kn}-\varphi'^{kn}\right) + (-1)^{k+1}\left(\varphi^{(n-1)k}-\varphi'^{(n-1)k}\right)\right] \\
&= \frac{1}{F_k}\frac{1}{\varphi-\varphi'}\left[\varphi^{k(n+1)} - \varphi'^{kn}\varphi^k + \varphi'^k\varphi^{kn} - \varphi'^{k+kn} + (-1)^{k+1}\varphi^{(n-1)k} - (-1)^{k+1}\varphi'^{(n-1)k}\right] \\
&= \frac{1}{F_k}\frac{1}{\varphi-\varphi'}\left[\varphi^{k(n+1)} - \varphi'^{k(n+1)} - \left(-\frac{1}{\varphi}\right)^{kn}\varphi^k + \left(-\frac{1}{\varphi}\right)^{k}\varphi^{kn} + (-1)^{k+1}\varphi^{(n-1)k}\right. \\
&\quad \left. -(-1)^{k+1}\left(-\frac{1}{\varphi}\right)^{(n-1)k}\right] \\
&= \frac{1}{F_k}\frac{1}{\varphi-\varphi'}\left[\varphi^{k(n+1)} - \varphi'^{k(n+1)} - (-1)^{kn}\varphi^{k(1-n)} + (-1)^{k}\varphi^{k(n-1)} - (-1)^{k}\varphi^{k(n-1)}\right. \\
&\quad \left. +(-1)^{k}(-1)^{k(n-1)}\varphi^{k(1-n)}\right] \\
&= \frac{1}{F_k}\frac{1}{\varphi-\varphi'}\left[\varphi^{k(n+1)} - \varphi'^{k(n+1)} - (-1)^{kn}\varphi^{k(1-n)} + (-1)^{k}(-1)^{kn}(-1)^{-k}\varphi^{k(1-n)}\right] \\
&= \frac{1}{F_k}\frac{1}{\varphi-\varphi'}\left[\varphi^{k(n+1)} - \varphi'^{k(n+1)} - (-1)^{kn}\varphi^{k(1-n)} + (-1)^{kn}\varphi^{k(1-n)}\right] \\
&= \frac{1}{F_k}\;\frac{1}{\varphi-\varphi'}\left[\varphi^{k(n+1)} - \varphi'^{k(n+1)}\right] \\
&= \frac{1}{F_k}\;\frac{\varphi^{k(n+1)} - \varphi'^{k(n+1)}}{\varphi-\varphi'} \\
&= \frac{1}{F_k}F_{k(n+1)} \\
&= \frac{F_{k(n+1)}}{F_k}
\end{aligned}
$$

To prove $det\left(A^k_{n+1}\right)$ relation, we take the determinant of both sides in (5.16),

$$
\det\left(A^k_{n+1}\right) = \det\left(\sigma_{n+1}\,\phi^k_{n+1}\,\sigma^{-1}_{n+1}\right) \tag{5.18}
$$

By using property of determinant,

$$
\det(AB) = \det(A)\det(B) \tag{5.19}
$$

we obtain,

$$
\det\left(A^k_{n+1}\right) \quad = \quad \det\left(\sigma_{n+1}\right)\;\det\left(\phi^k_{n+1}\right)\;\det\left(\sigma^{-1}_{n+1}\right)
$$

$$
\begin{aligned}
\det\left(A_{n+1}^{k}\right) &= \det\left(\sigma_{n+1}\right)\ \det\left(\sigma_{n+1}^{-1}\right)\ \det\left(\phi_{n+1}^{k}\right)\\
\det\left(A_{n+1}^{k}\right) &= \det\left(\sigma_{n+1}\ \sigma_{n+1}^{-1}\right)\ \det\left(\phi_{n+1}^{k}\right)\\
\det\left(A_{n+1}^{k}\right) &= \det\left(I\right)\ \det\left(\phi_{n+1}^{k}\right)\\
\det\left(A_{n+1}^{k}\right) &= 1.\ \det\left(\phi_{n+1}^{k}\right)\\
\det\left(A_{n+1}^{k}\right) &= \det\left(\phi_{n+1}^{k}\right)
\end{aligned}
$$

Since the matrix ϕ_{n+1}^{k} is known from (5.17), the above equation becomes;

$$
\begin{aligned}
\det\left(A_{n+1}^{k}\right) &= (\varphi^{n})^{k}\ (\varphi^{n-1}\varphi')^{k}\ (\varphi^{n-2}\varphi'^{2})^{k} \ldots (\varphi^{2}\varphi'^{n-2})^{k}\ (\varphi\varphi'^{n-1})^{k}\ (\varphi'^{n})^{k}\\
&= \left(\varphi^{nk}\ \varphi^{(n-1)k}\ \varphi^{(n-2)k} \ldots \varphi^{2k}\ \varphi^{k}\right)\left(\varphi'^{k}\ \varphi'^{2k} \ldots \varphi'^{(n-2)k}\ \varphi'^{(n-1)k}\ \varphi'^{nk}\right)\\
&= \left(\varphi^{nk+(n-1)k+(n-2)k+\ldots+2k+k}\right)\left(\varphi'^{\ k+2k+\ldots+(n-2)k+(n-1)k+nk}\right)\\
&= \varphi^{k[n+(n-1)+(n-2)+\ldots+2+1]}\ \varphi'^{\ k[1+2+\ldots+(n-2)+(n-1)+n]}\\
&= \varphi^{k\left(\frac{n(n+1)}{2}\right)}\ \varphi'^{k\left(\frac{n(n+1)}{2}\right)}\\
&= \left(\varphi^{\frac{n(n+1)}{2}}\right)^{k}\ \left(\varphi'^{\frac{n(n+1)}{2}}\right)^{k}\\
&= \left[(\varphi\varphi')^{\frac{n(n+1)}{2}}\right]^{k}\\
&\overset{(\varphi\varphi'=-1)}{=} (-1)^{k\ \frac{n\,(n+1)}{2}}
\end{aligned}
$$

□

CHAPTER 6

MOD 5 CONGRUENCE OF $F_N^{(2)} = F_{2N}$ FIBONACCI NUMBER SUPERPOSITIONS

In Section 5.1, we have seen the matrix,

$$A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \tag{6.1}$$

with integer valued elements. This means that an arbitrary power of this matrix A_3^n is also with integer valued elements. If we compare this with representation of matrix A_3^n given in Proposition (5.3), we observe that, elements of this matrix are combinations of $F_n^{(2)} = F_{2n}$-Fibonacci numbers with integer coefficients divided to 5. This result implies that 9- combinations of these even index Fibonacci numbers are divisible to 5. These are the mod 5 congruence relations.

To get the column matrix elements of matrix A_3^n, we use;

$$A_3^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2F_n^{(2)} - 3F_{n-1}^{(2)} + 2(-1)^n \\ F_n^{(2)} + F_{n-1}^{(2)} + (-1)^n \\ 3F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n \end{pmatrix}, \tag{6.2}$$

$$A_3^n \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2F_n^{(2)} + 2F_{n-1}^{(2)} + 2(-1)^n \\ 6F_n^{(2)} - 4F_{n-1}^{(2)} + (-1)^n \\ 8F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n \end{pmatrix}, \tag{6.3}$$

$$A_3^n \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n \\ 4F_n^{(2)} - F_{n-1}^{(2)} - (-1)^n \\ 7F_n^{(2)} - 3F_{n-1}^{(2)} + 2(-1)^n \end{pmatrix}. \tag{6.4}$$

6.1. Congruency of Fibonacci and Lucas Numbers

Matrix elements of A_3^n are all integer. Then, we should show their divisibility by 5, which is stated at the following proposition.

Proposition 6.1 *In the equality* (6.2) *all coefficients,*

$$\begin{aligned} a_n &= \frac{2F_n^{(2)} - 3F_{n-1}^{(2)} + 2(-1)^n}{5} = \frac{F_n^{(2)} + F_n^{(2)} - 3F_{n-1}^{(2)} + 2(-1)^n}{5} \overset{(2.24)}{=} \frac{F_n^{(2)} - F_{n-2}^{(2)} + 2(-1)^n}{5} \\ &= \frac{F_{2n} - F_{2(n-2)} + 2(-1)^n}{5}, \\ b_n &= \frac{F_n^{(2)} + F_{n-1}^{(2)} + (-1)^n}{5} = \frac{F_{2n} + F_{2(n-1)} + (-1)^n}{5}, \\ c_n &= \frac{3F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n}{5} = \frac{3F_n^{(2)} - F_{n-1}^{(2)} - F_{n-1}^{(2)} - 2(-1)^n}{5} \overset{(2.24)}{=} \frac{F_{n+1}^{(2)} - F_{n-1}^{(2)} - 2(-1)^n}{5} \\ &= \frac{F_{2(n+1)} - F_{2(n-1)} - 2(-1)^n}{5} \end{aligned}$$

are integer.

Here, since $c_n = a_{n+1}$, it is sufficient to prove only that b_n and c_n are integers. By using equation (2.2)*, b_n and a_n can be written as,*

$$b_n = \frac{L_{2n-1} + (-1)^n}{5}, \tag{6.5}$$

$$a_n = \frac{L_{2n-2} + 2(-1)^n}{5}. \tag{6.6}$$

To prove this, we should use following helpful proposition from (Koshy, T., 2001).

Proposition 6.2 *Lucas numbers,*

$$L_n \equiv (-1)^n \cdot 2^{n+1} \equiv 2 \cdot 3^n \quad (mod\ 5) \tag{6.7}$$

Proof (Proof of Proposition 6.1) Firstly, we should show that,

$$L_{2n-1} + (-1)^n \equiv 0 \pmod 5$$

If we choose $n \to 2n-1$ in the Proposition 6.2,

$$\begin{aligned} L_{2n-1} &\equiv (-1)^{2n-1} \cdot 2^{(2n-1)+1} \pmod 5 \\ &\equiv -(2^2)^n \pmod 5 \\ &\equiv -(5-1)^n \pmod 5. \end{aligned}$$

From Newton Binomial formula,

$$(5-1)^n = \sum_{k=0}^{n} \binom{n}{k} 5^{n-k} (-1)^k \equiv (-1)^n \pmod 5. \tag{6.8}$$

Then we can deduce that,

$$L_{2n-1} \equiv (-1)^{n+1} \pmod 5. \tag{6.9}$$

It says that b_n is integer. Secondly, to prove a_n is integer, we should show that;

$$L_{2n-2} \equiv 2 \cdot (-1)^{n+1} \pmod 5 \tag{6.10}$$

or,

$$L_{2m} \equiv 3 \cdot (-1)^{m+1} \pmod 5, \tag{6.11}$$

where $n-1=m$. To prove this, we replace $n \to 2m$ in the Proposition 6.2;

$$L_{2m} \equiv (-1)^{2m} \cdot 2^{2m+1} \pmod 5$$

$$
\begin{aligned}
&\equiv 2 \cdot 4^m \pmod 5 \\
&\overset{(6.8)}{\equiv} -3 \cdot (-1)^m \pmod 5 \\
&\equiv 3 \cdot (-1)^{m+1} \pmod 5 \\
&\equiv -2 \cdot (-1)^{m+1} \pmod 5 \\
&\equiv 2 \cdot (-1)^m \pmod 5.
\end{aligned}
$$

After shifting $m \to n - 1$, we get the desired equality,

$$L_{2n-2} \equiv 2 \cdot (-1)^{n+1} \pmod 5.$$

Therefore, it says that a_n is also integer. □

Our next goal is to prove that matrix elements in (6.3) and (6.4) are also integer.

Proposition 6.3 *Matrix elements given in* (6.3)*,*

$$
\frac{2F_n^{(2)} + 2F_{n-1}^{(2)} + 2(-1)^n}{5}, \quad
\frac{6F_n^{(2)} - 4F_{n-1}^{(2)} + (-1)^n}{5}, \quad
\frac{8F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n}{5}
$$

are integer. Or equivalently,

$$
\begin{aligned}
2F_n^{(2)} + 2F_{n-1}^{(2)} + 2(-1)^n &\equiv 0 \pmod 5 \\
6F_n^{(2)} - 4F_{n-1}^{(2)} + (-1)^n &\equiv 0 \pmod 5 \\
8F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n &\equiv 0 \pmod 5
\end{aligned}
$$

Proof We have for the first one,

$$
\begin{aligned}
2F_n^{(2)} + 2F_{n-1}^{(2)} + 2(-1)^n &\equiv 0 \pmod 5 \\
F_n^{(2)} + F_{n-1}^{(2)} + (-1)^n &\equiv 0 \pmod 5 \\
F_{2n} + F_{2n-2} + (-1)^n &\equiv 0 \pmod 5 \\
L_{2n-1} + (-1)^n &\equiv 0 \pmod 5
\end{aligned}
$$

$$L_{2n-1} \equiv (-1)^{n+1} \pmod 5.$$

Since it is just equation (6.9), then;

$$2F_n^{(2)} + 2F_{n-1}^{(2)} + 2(-1)^n \equiv 0 \pmod 5$$

is proved. Secondly,

$$\begin{aligned}
6F_n^{(2)} - 4F_{n-1}^{(2)} + (-1)^n &\equiv 0 \pmod 5 \\
6F_{2n} - 4F_{2(n-1)} + (-1)^n &\equiv 0 \pmod 5 \\
2F_{2n} + 4(F_{2n} - F_{2n-2}) + (-1)^n &\equiv 0 \pmod 5 \\
2F_{2n} + 4F_{2n-1} + (-1)^n &\equiv 0 \pmod 5 \\
2(F_{2n} + F_{2n-1}) + 2F_{2n-1} + (-1)^n &\equiv 0 \pmod 5 \\
2(F_{2n+1} + F_{2n-1}) + (-1)^n &\equiv 0 \pmod 5 \\
2L_{2n} + (-1)^n &\equiv 0 \pmod 5 \\
4L_{2n} + 2(-1)^n &\equiv 0 \pmod 5 \\
(-1) \cdot L_{2n} &\equiv -2(-1)^n \pmod 5 \\
L_{2n} &\equiv 2(-1)^n \pmod 5 \\
L_{2n} &\equiv 3(-1)^{n+1} \pmod 5
\end{aligned}$$

Since this is equation (6.11), then,

$$6F_n^{(2)} - 4F_{n-1}^{(2)} + (-1)^n \equiv 0 \pmod 5$$

is proved. Thirdly,

$$\begin{aligned}
8F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n &\equiv 0 \pmod 5 \\
8F_{2n} - 2F_{2n-2} - 2(-1)^n &\equiv 0 \pmod 5 \\
6F_{2n} + 2(F_{2n} - F_{2n-2}) - 2(-1)^n &\equiv 0 \pmod 5 \\
6F_{2n} + 2(F_{2n-1}) - 2(-1)^n &\equiv 0 \pmod 5
\end{aligned}$$

$$\begin{aligned}
4F_{2n} + 2\left(F_{2n} + F_{2n-1}\right) - 2(-1)^n &\equiv 0 \pmod 5 \\
4F_{2n} + 2F_{2n+1} - 2(-1)^n &\equiv 0 \pmod 5 \\
2F_{2n} + 2\left(F_{2n} + F_{2n+1}\right) - 2(-1)^n &\equiv 0 \pmod 5 \\
2\left(F_{2n} + F_{2n+2}\right) - 2(-1)^n &\equiv 0 \pmod 5 \\
2\left(F_{2n} + F_{2n+2} - (-1)^n\right) &\equiv 0 \pmod 5 \\
2\left(L_{2n+1} - (-1)^n\right) &\equiv 0 \pmod 5
\end{aligned}$$

We know that if;

$$c \cdot a \equiv c \cdot b \pmod n \tag{6.12}$$

then,

$$a \equiv b \quad \left(mod\ \frac{n}{d}\right), \tag{6.13}$$

where $d = \gcd(c, n)$. Therefore, we have;

$$\begin{aligned}
L_{2n+1} - (-1)^n &\equiv 0 \pmod 5 \\
L_{2n+1} &\equiv (-1)^n \pmod 5
\end{aligned}$$

By shifting $n \to n-1$,

$$L_{2n-1} \equiv (-1)^{n+1} \pmod 5$$

gives equation (6.9). So,

$$8F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n \equiv 0 \pmod 5$$

is proved. □

Proposition 6.4 *Matrix elements given in* (6.4),

$$\frac{3F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n}{5}$$
$$\frac{4F_n^{(2)} - F_{n-1}^{(2)} - (-1)^n}{5}$$
$$\frac{7F_n^{(2)} - 3F_{n-1}^{(2)} + 2(-1)^n}{5}$$

are integer. Or equivalently,

$$\begin{aligned}
3F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n &\equiv 0 \pmod 5\\
4F_n^{(2)} - F_{n-1}^{(2)} - (-1)^n &\equiv 0 \pmod 5\\
7F_n^{(2)} - 3F_{n-1}^{(2)} + 2(-1)^n &\equiv 0 \pmod 5
\end{aligned}$$

Proof For the first congruency,

$$\begin{aligned}
3F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n &\equiv 0 \pmod 5\\
3F_{2n} - 2F_{2n-2} - 2(-1)^n &\equiv 0 \pmod 5\\
F_{2n} + 2\left(F_{2n} - F_{2n-2}\right) - 2(-1)^n &\equiv 0 \pmod 5\\
F_{2n} + 2F_{2n-1} - 2(-1)^n &\equiv 0 \pmod 5\\
F_{2n} + F_{2n-1} + F_{2n-1} - 2(-1)^n &\equiv 0 \pmod 5\\
F_{2n+1} + F_{2n-1} - 2(-1)^n &\equiv 0 \pmod 5\\
L_{2n} &\equiv 2 \cdot (-1)^n \pmod 5\\
L_{2n} &\equiv (-3) \cdot (-1)^n \pmod 5\\
L_{2n} &\equiv 3 \cdot (-1)^{n+1} \pmod 5
\end{aligned}$$

Since this is just equation (6.11), then,

$$3F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n \equiv 0 \pmod 5$$

is proved. Secondly,

$$
\begin{aligned}
4F_n^{(2)} - F_{n-1}^{(2)} - (-1)^n &\equiv 0 \pmod 5 \\
4F_{2n} - F_{2n-2} - (-1)^n &\equiv 0 \pmod 5 \\
3F_{2n} + F_{2n} - F_{2n-2} - (-1)^n &\equiv 0 \pmod 5 \\
3F_{2n} + F_{2n-1} - (-1)^n &\equiv 0 \pmod 5 \\
2F_{2n} + F_{2n} + F_{2n-1} - (-1)^n &\equiv 0 \pmod 5 \\
2F_{2n} + F_{2n+1} - (-1)^n &\equiv 0 \pmod 5 \\
F_{2n} + F_{2n} + F_{2n+1} - (-1)^n &\equiv 0 \pmod 5 \\
F_{2n} + F_{2n+2} - (-1)^n &\equiv 0 \pmod 5 \\
L_{2n+1} - (-1)^n &\equiv 0 \pmod 5 \\
L_{2n+1} &\equiv (-1)^n \pmod 5
\end{aligned}
$$

Shifting $n \to n-1$ gives,

$$
\begin{aligned}
L_{2n-1} &\equiv (-1)^{n-1} \pmod 5 \\
L_{2n-1} &\equiv (-1)^{n+1} \pmod 5
\end{aligned}
$$

This is equation (6.9), then,

$$
4F_n^{(2)} - F_{n-1}^{(2)} - (-1)^n \equiv 0 \pmod 5
$$

is proved. To prove the third congruency,

$$
\begin{aligned}
7F_n^{(2)} - 3F_{n-1}^{(2)} + 2(-1)^n &\equiv 0 \pmod 5 \\
7F_{2n} - 3F_{2n-2} + 2(-1)^n &\equiv 0 \pmod 5 \\
4F_{2n} + 3\left(F_{2n} - F_{2n-2}\right) + 2(-1)^n &\equiv 0 \pmod 5 \\
4F_{2n} + 3F_{2n-1} + 2(-1)^n &\equiv 0 \pmod 5 \\
F_{2n} + 3\left(F_{2n} + F_{2n-1}\right) + 2(-1)^n &\equiv 0 \pmod 5 \\
F_{2n} + 3F_{2n+1} + 2(-1)^n &\equiv 0 \pmod 5
\end{aligned}
$$

$$
\begin{aligned}
2F_{2n+1} + F_{2n} + F_{2n+1} + 2(-1)^n &\equiv 0 \pmod 5 \\
2F_{2n+1} + F_{2n+2} + 2(-1)^n &\equiv 0 \pmod 5 \\
F_{2n+1} + F_{2n+1} + F_{2n+2} + 2(-1)^n &\equiv 0 \pmod 5 \\
F_{2n+1} + F_{2n+3} + 2(-1)^n &\equiv 0 \pmod 5 \\
L_{2n+2} + 2(-1)^n &\equiv 0 \pmod 5 \\
L_{2n+2} &\equiv 2(-1)^{n+1} \pmod 5 \\
L_{2(n+1)} &\equiv 2(-1)^{n+1} \pmod 5
\end{aligned}
$$

After shifting $n + 1 \to m$, it gives,

$$L_{2m} \equiv 2(-1)^m \pmod 5.$$

Since it is equation (6.11), then;

$$7F_n^{(2)} - 3F_{n-1}^{(2)} + 2(-1)^n \equiv 0 \pmod 5$$

is proved. □

CHAPTER 7

BERNOULLI FIBONACCI POLYNOMIALS

Definition 7.1 *The generating function for Bernoulli polynomials is defined by Taylor series expansion,*

$$\frac{z\ e^{zx}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}, \tag{7.1}$$

where $B_n(x)$ are the Bernoulli polynomials in x, for all $n > 0$.

Bernoulli numbers are a special values of the Bernoulli polynomials $B_n(x)$, $b_n = B_n(0)$. The generating function for Bernoulli numbers is;

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} b_n\frac{z^n}{n!}. \tag{7.2}$$

First few Bernoulli polynomials are;

$$\begin{aligned}
B_0(x) &= 1\\
B_1(x) &= x - \frac{1}{2}\\
B_2(x) &= x^2 - x + \frac{1}{6}\\
B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x\\
B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}\\
B_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x\\
B_6(x) &= x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}.
\end{aligned}$$

Corresponding Bernoulli numbers are;

$$b_0 = 1, \quad b_1 = -\frac{1}{2}, \quad b_2 = \frac{1}{6}, \quad b_3 = 0, \quad b_4 = -\frac{1}{30}, \quad b_5 = 0, \quad b_6 = \frac{1}{42}, \ldots$$

Following propositions are valid for Bernoulli numbers and polynomials (Kac, V. and Cheung, P., 2002).

Proposition 7.1 *It is known that for odd Bernoulli numbers;*

$$b_{2n+1} = 0 \tag{7.3}$$

where $n = 1, 2, \ldots$

Proposition 7.2 *Derivative of* n^{th} *Bernoulli polynom gives,*

$$\frac{d}{dx} B_n(x) = B_n^{'}(x) = n\ B_n(x) \tag{7.4}$$

Proposition 7.3 *For any* $n \geq 0$*, we have;*

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} b_j\ \ x^{n-j} \tag{7.5}$$

Proposition 7.4 *For any* $n \geq 1$*, we have;*

$$\sum_{j=0}^{n-1} \binom{n}{j} B_j(x) = n\ \ x^{n-1}$$

Corollary 7.1 *For any* $n \geq 2$*, from previous proposition if* $x = 0$*;*

$$\sum_{j=0}^{n-1} \binom{n}{j} b_j = 0$$

Proposition 7.5 *For any* $n \geq 2$*, we have;*

$$B_n(1) = b_n \tag{7.6}$$

Definition of Golden exponential function e_F^x in (3.41) suggests to introduce generating function and corresponding polynomials, similar to the Bernoulli polynomials.

Definition 7.2 *Generating function for Bernoulli-Fibonacci polynomials* $B_n^F(x)$ *is defined* *by series expansion,*

$$\frac{z\ e_F^{zx}}{e_F^z - 1} = \sum_{n=0}^{\infty} B_n^F(x)\frac{z^n}{F_n!}. \tag{7.7}$$

The Bernoulli-Fibonacci numbers are a special values of polynomials $B_n^F(x)$ such that,

$$b_n^F = B_n^F(0), \tag{7.8}$$

with generating function,

$$\frac{z}{e_F^z - 1} = \sum_{n=0}^{\infty} b_n^F \frac{z^n}{F_n!} \tag{7.9}$$

For the first few Bernoulli-Fibonacci polynomials we have,

$$\begin{aligned}
B_0^F(x) &= 1\\
B_1^F(x) &= \frac{x}{F_1!} - \frac{1}{F_2!}\\
B_2^F(x) &= x^2 - \frac{x}{F_1!} + \frac{1}{F_2!} - \frac{1}{F_3}\\
B_3^F(x) &= x^3 - x^2\frac{F_3!}{(F_2!)^2} + x\left(\frac{F_3!}{F_1!(F_2!)^2} - \frac{1}{F_1!}\right) + \frac{2}{F_2!} - \frac{1}{F_4} - F_3\\
B_4^F(x) &= x^4 - x^3\left(\frac{F_4!}{F_3!F_2!}\right) + x^2\left(\frac{F_4!}{(F_2!)^3} - \frac{F_4!}{F_2!F_3!}\right)\\
&+ x\left(-\frac{F_4!}{F_1!(F_2!)^3} + 2\frac{F_4!}{F_1!F_2!F_3!} - \frac{1}{F_1!}\right)\\
&+ \left(\frac{F_4!}{(F_2!)^4} - 3\frac{F_4!}{(F_2!)^2F_3!} + 2\frac{1}{F_2!} + \frac{F_4!}{(F_3!)^2} - \frac{1}{F_5}\right)\\
B_5^F(x) &= x^5 + x^4\left(-\frac{F_5!}{F_4!F_2!}\right) + x^3\left(-\frac{F_5!}{(F_3!)^2} + \frac{F_5!}{F_3!(F_2!)^2}\right)\\
&+ x^2\left(-\frac{F_5!}{F_2!F_4!} + 2\frac{F_5!}{(F_2!)^2F_3!} - \frac{F_5!}{(F_2!)^4}\right)\\
&+ x\left(-\frac{1}{F_1!} + \frac{F_5!}{F_1!(F_3!)^2} + 2\frac{F_5!}{F_1!F_2!F_4!} - 3\frac{F_5!}{F_1!(F_2!)^2F_3!} + \frac{F_5!}{F_1!(F_2!)^4}\right)\\
&+ \left(-\frac{F_5!}{(F_2!)^5} + 4\frac{F_5!}{(F_2!)^3F_3!} - 3\frac{F_5!}{(F_3!)^2F_2!} - 3\frac{F_5!}{(F_2!)^2F_4!} + 2\frac{F_5!}{F_3!F_4!} + \frac{2}{F_2!} - \frac{F_5!}{F_6!}\right)
\end{aligned}$$

In Figures 7.1, 7.2 and 7.3 we compare first three Bernoulli and Bernoulli-Fibonacci Polynomials on interval $-1 \leq x \leq 2$.

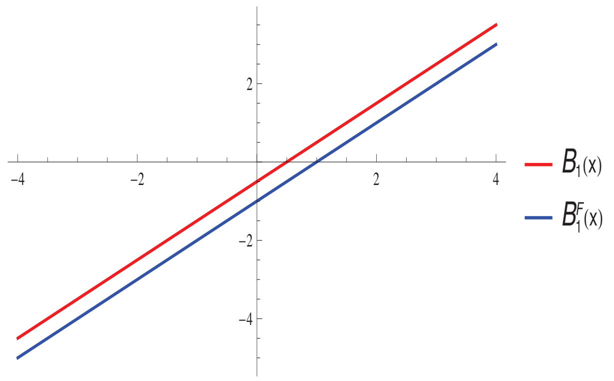

Figure 7.1. Graph of the polynomials $B_1(x)$ and $B_1^F(x)$

Since,

$$B_1(x) = B_1^F(x) + \frac{1}{2},$$

then we have the constant shift by $\frac{1}{2}$ in vertical direction.

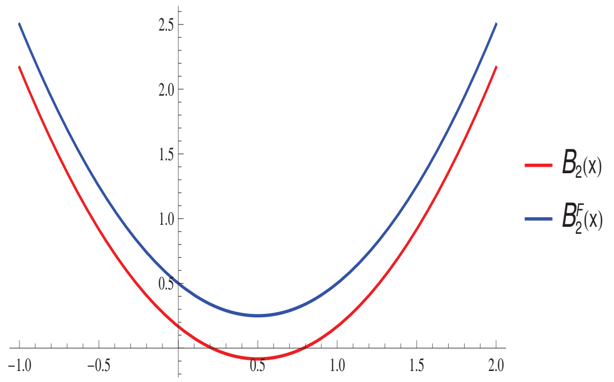

Figure 7.2. Graph of the polynomials $B_2(x)$ and $B_2^F(x)$

Since,

$$B_2(x) = B_2^F(x) - \frac{1}{3}$$

we have the constant shift by $-\frac{1}{3}$ in vertical direction.

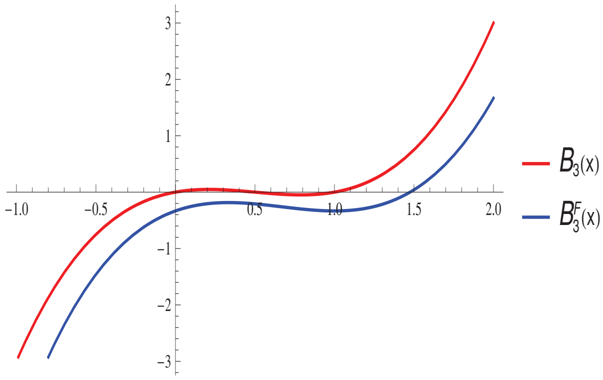

Figure 7.3. Graph of the polynomials $B_3(x)$ and $B_3^F(x)$

Since,

$$B_3(x) = B_3^F(x) + \left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{3}\right),$$

then we have the parabolic shift in vertical direction. For example, at $x = 0$, we have shifting on $\frac{1}{3}$ units in vertical direction.

We notice that all coefficients of Bernoulli-Fibonacci polynomials are Fibonacci rational.

Proposition 7.6 *(Compare with Proposition* (7.2)*)*

Golden derivative application to Bernoulli-Fibonacci polynomials $B_n^F(x)$ gives;

$$D_F^x(B_n^F(x)) = F_n\, B_{n-1}^F(x) \tag{7.10}$$

Proof Taking Golden derivative of both sides in the equation (7.7),

$$\begin{aligned} D_F^x\left(\frac{z\ e_F^{zx}}{e_F^z-1}\right) &= D_F^x\left(\sum_{n=0}^{\infty} B_n^F(x)\frac{z^n}{F_n!}\right) \\ \frac{z\ D_F^x(e_F^{zx})}{e_F^z-1} &= D_F^x\left(B_0^F(x)+B_1^F(x)\frac{z}{F_1!}+B_2^F(x)\frac{z^2}{F_2!}+\ldots\right) \end{aligned}$$

For the left hand side, Golden derivative can be calculated from equation (3.46). For the right hand side, it is clear that;

$$D_F^x(B_0^F(x)) = D_F^x(1)=0$$

Then,

$$\begin{aligned} z\cdot\frac{ze_F^{zx}}{e_F^z-1} &= \sum_{k=1}^{\infty} D_F^x\left(B_k^F(x)\right)\frac{z^k}{F_k!} \\ z\sum_{n=0}^{\infty} B_n^F(x)\frac{z^n}{F_n!} &= \sum_{k=1}^{\infty} D_F^x\left(B_k^F(x)\right)\frac{z^k}{F_k!} \\ \sum_{n=0}^{\infty} B_n^F(x)\frac{z^{n+1}}{F_n!} &= \sum_{k=1}^{\infty} D_F^x\left(B_k^F(x)\right)\frac{z^k}{F_k!} \end{aligned}$$

In the right hand side, after denoting $k-1=n$;

$$\begin{aligned} \sum_{n=0}^{\infty} B_n^F(x)\frac{z^{n+1}}{F_n!} &= \sum_{n=0}^{\infty} D_F^x\left(B_{n+1}^F(x)\right)\frac{z^{n+1}}{F_{n+1}!} \\ \sum_{n=0}^{\infty} B_n^F(x)\frac{z^{n+1}}{F_n!} &= \sum_{n=0}^{\infty} D_F^x\left(B_{n+1}^F(x)\right)\frac{1}{F_{n+1}}\frac{z^{n+1}}{F_n!} \end{aligned}$$

From equality of these two series, we have;

$$B_n^F(x)=\frac{D_F^x\left(B_{n+1}^F(x)\right)}{F_{n+1}}$$

□

Since $b_n^F = B_n^F(0)$, we can find Bernoulli-Fibonacci numbers from Bernoulli-Fibonacci polynomials as;

$$b_0^F = 1,\ b_1^F = -1,\ b_2^F = \frac{1}{2},\ b_3^F = -\frac{1}{3},\ b_4^F = \frac{3}{10},\ b_5^F = -\frac{5}{8},\ b_6^F = \frac{101}{39}$$

Proposition 7.7 *(Compare with Proposition* (7.3)*)*

Another representation for Bernoulli Fibonacci Polynomials is;

$$B_n^F(x) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F \ x^{n-j} \tag{7.11}$$

Proof We show that if (7.11) is valid then $B_n^F(x)$ are satisfying equation (7.10).

$$\begin{aligned}
D_F^x\left[B_n^F(x)\right] &\overset{(7.11)}{=} D_F^x\left[\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F \ x^{n-j}\right] = D_F^x\left[\begin{bmatrix} n \\ 0 \end{bmatrix}_F b_0^F \ x^n + \begin{bmatrix} n \\ 1 \end{bmatrix}_F b_1^F \ x^{n-1} + \ldots + \begin{bmatrix} n \\ n \end{bmatrix}_F b_n^F\right] \\
&= D_F^x\left[\begin{bmatrix} n \\ 0 \end{bmatrix}_F b_0^F \ x^n + \begin{bmatrix} n \\ 1 \end{bmatrix}_F b_1^F \ x^{n-1} + \ldots + \begin{bmatrix} n \\ n \end{bmatrix}_F b_n^F\right] \\
&= D_F^x\left[\sum_{j=0}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F \ x^{n-j}\right] \\
&= \sum_{j=0}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F \ D_F^x\left(x^{n-j}\right) \\
&= \sum_{j=0}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F \ F_{n-j} \ x^{n-j-1} \\
&= \sum_{j=0}^{n-1} \frac{F_n!}{F_{n-j}! \ F_j!} b_j^F \ F_{n-j} \ x^{n-j-1} \\
&= \sum_{j=0}^{n-1} \frac{F_n!}{F_{n-j-1}! \ F_j!} b_j^F \ x^{n-j-1} \\
&= F_n \sum_{j=0}^{n-1} \frac{F_{n-1}!}{F_{n-j-1}! \ F_j!} b_j^F \ x^{n-j-1} \\
&\overset{(7.11)}{=} F_n \ B_{n-1}^F(x)
\end{aligned}$$

□

Proposition 7.8 *(Compare with Proposition* (7.4)*)*

For $n \geq 1$, Bernoulli-Fibonacci polynomials can be calculated recursively by:

$$\sum_{l=0}^{n-1} \begin{bmatrix} n \\ l \end{bmatrix}_F B_l^F(x) = F_n \ \ x^{n-1}$$

Proof We know the generating function,

$$\frac{z \ \ e_F^{zx}}{e_F^z - 1} = \sum_{n=0}^{\infty} B_n^F(x) \frac{z^n}{F_n!}.$$

Multiplying this by e_F^z;

$$\frac{z \ \ e_F^{zx} \ \ e_F^z}{e_F^z - 1} = \sum_{n=0}^{\infty} B_n^F(x) \ \ e_F^z \frac{z^n}{F_n!},$$

and taking the difference, one gets;

$$\begin{aligned}
\frac{z \ \ e_F^{zx}}{e_F^z - 1}\left(e_F^z - 1\right) &= \sum_{n=0}^{\infty} \left(B_n^F(x) \ \ e_F^z - B_n^F(x)\right) \frac{z^n}{F_n!} \\
z \ \ e_F^{zx} &= \sum_{n=0}^{\infty} \left(B_n^F(x) \ \ e_F^z - B_n^F(x)\right) \frac{z^n}{F_n!} \\
D_F^x\left(e_F^{zx}\right) &= \sum_{n=0}^{\infty} \left(B_n^F(x) \ \ e_F^z - B_n^F(x)\right) \frac{z^n}{F_n!} \\
D_F^x\left(\sum_{n=0}^{\infty} \frac{(zx)^n}{F_n!}\right) &= \sum_{n=0}^{\infty} \left(B_n^F(x) \ \ e_F^z - B_n^F(x)\right) \frac{z^n}{F_n!} \\
D_F^x\left(\sum_{n=1}^{\infty} \frac{z^n \ x^n}{F_n!}\right) &= \sum_{n=0}^{\infty} \left(B_n^F(x) \ \ e_F^z - B_n^F(x)\right) \frac{z^n}{F_n!} \\
\sum_{n=1}^{\infty} \frac{z^n \ D_F^x \ \ (x^n)}{F_n!} &= \sum_{n=0}^{\infty} \left(B_n^F(x) \ \ e_F^z - B_n^F(x)\right) \frac{z^n}{F_n!} \\
\sum_{n=1}^{\infty} \frac{z^n \ F_n \ x^{n-1}}{F_n!} &= \sum_{l=0}^{\infty} B_l^F(x) \ \ e_F^z \frac{z^l}{F_l!} - \sum_{n=0}^{\infty} B_n^F(x) \frac{z^n}{F_n!}.
\end{aligned}$$

In the right hand side of this equation;

$$\sum_{l=0}^{\infty} B_l^F(x) \ \ e_F^z \frac{z^l}{F_l!} \quad = \quad \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} B_l^F(x) \frac{z^k}{F_k!} \frac{z^l}{F_l!} = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} B_l^F(x) \frac{z^{k+l}}{F_k! F_l!}$$

$$
\begin{aligned}
&= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} B_l^F(x)\frac{z^{k+l}}{F_k!F_l!}\\
&\overset{k+l=n}{=} \sum_{n=0}^{\infty}\frac{1}{F_n!}\sum_{l=0}^{n} B_l^F(x)\frac{z^n\ F_n!}{F_{n-l}!F_l!}\\
&= \sum_{n=0}^{\infty}\frac{z^n}{F_n!}\left(\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_F B_l^F(x)\right)
\end{aligned}
$$

After substituting this,

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{z^n\ F_n\ x^{n-1}}{F_n!} &= \sum_{n=0}^{\infty}\frac{z^n}{F_n!}\left(\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_F B_l^F(x)\right)-\sum_{n=0}^{\infty}B_n^F(x)\frac{z^n}{F_n!}\\
\sum_{n=1}^{\infty}\frac{z^n\ F_n\ x^{n-1}}{F_n!} &= B_0^F(x)\begin{bmatrix} 0\\ 0\end{bmatrix}_F+\sum_{n=1}^{\infty}\frac{z^n}{F_n!}\left(\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_F B_l^F(x)\right)-B_0^F(x)-\sum_{n=1}^{\infty}B_n^F(x)\frac{z^n}{F_n!}\\
\sum_{n=1}^{\infty}\frac{z^n}{F_n!}\ F_n\ x^{n-1} &= \sum_{n=1}^{\infty}\frac{z^n}{F_n!}\left(\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_F B_l^F(x)\right)-\sum_{n=1}^{\infty}B_n^F(x)\frac{z^n}{F_n!}
\end{aligned}
$$

By equating the series, we have;

$$
\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_F B_l^F(x)-B_n^F(x)=F_n\ x^{n-1},
$$

where $n \geq 1$. Expanding n^{th} term in the summation,

$$
\sum_{l=0}^{n-1}\begin{bmatrix} n\\ l\end{bmatrix}_F B_l^F(x)+B_n^F(x)\ \begin{bmatrix} n\\ n\end{bmatrix}_F-B_n^F(x)=F_n\ x^{n-1},
$$

finally, we get the desired equality,

$$
\sum_{l=0}^{n-1}\begin{bmatrix} n\\ l\end{bmatrix}_F B_l^F(x)=F_n\ x^{n-1}.
$$

□

Corollary 7.2 *(Compare with Corollary* (7.1)*)*

From previous proposition if $x = 0$, the formula allows us to compute the Bernoulli *numbers inductively,*

$$\sum_{j=0}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F = 0 \tag{7.12}$$

where $n \geq 2$.

Proof Proof will be done by equating expansion of $B_n^F(x) + F_n x^{n-1}$, obtained in two ways. On the one hand, we obtain;

$$B_n^F(x) + F_n x^{n-1} = x^n + \sum_{j=2}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F x^{n-j}. \tag{7.13}$$

On the other hand, we found that;

$$B_n^F(x) + F_n x^{n-1} = H_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_F B_{n-k}^F(x) \tag{7.14}$$

These equations are derived in Appendix C. Equating these two equations gives;

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_F B_{n-k}^F(x) = x^n + \sum_{j=2}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F x^{n-j}.$$

After expanding $k = n$ case in the left hand side,

$$\sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_F B_{n-k}^F(x) + B_0^F(x) = x^n + \sum_{j=2}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F x^{n-j}$$

$$x^n - 1 = \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_F B_{n-k}^F(x) - \sum_{j=2}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F x^{n-j}$$

$$x^n - 1 = \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_F B_{n-k}^F(x) - \left(\begin{bmatrix} n \\ 2 \end{bmatrix}_F b_2^F x^{n-2} + \ldots + \begin{bmatrix} n \\ n-1 \end{bmatrix}_F b_{n-1}^F x + \begin{bmatrix} n \\ n \end{bmatrix}_F b_n^F \right)$$

Now putting $x = 0$ gives,

$$-1 = \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_F b_{n-k}^F - b_n^F.$$

Firstly, expanding the sum for $k = 0$ in the right hand side,

$$-1 = \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_F b_{n-k}^F + \begin{bmatrix} n \\ 0 \end{bmatrix}_F b_n^F - b_n^F,$$

thus, we get;

$$\sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_F b_{n-k}^F + 1 = 0.$$

By using symmetry property of Fibonomial coefficients, we can rewrite it as,

$$\sum_{k=1}^{n-1} \begin{bmatrix} n \\ n-k \end{bmatrix}_F b_{n-k}^F + 1 = 0.$$

After denoting $n - k = j$ it gives,

$$\sum_{j=n-1}^{1} \begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F + 1 = 0.$$

Since $b_0^F = 1$;

$$\sum_{j=1}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F + \begin{bmatrix} n \\ 0 \end{bmatrix}_F b_0^F = 0$$

Therefore, we obtained the desired result;

$$\sum_{j=0}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F = 0$$

□

Proposition 7.9 *(Compare with Proposition* (7.5)*)*

Bernoulli-Fibonacci polynomials and corresponding Bernoulli-Fibonacci numbers satisfy the following equation;

$$B_n^F(1) = b_n^F \tag{7.15}$$

where $n = 2, 3, \ldots$

Proof Starting with,

$$\begin{aligned}
\frac{e_F^{zx} - 1}{e_F^z - 1} &= \left(\frac{e_F^{zx}}{e_F^z - 1} - \frac{1}{e_F^z - 1} \right) = \left(\frac{e_F^{zx}}{e_F^z - 1} - \frac{1}{e_F^z - 1} \right) \frac{z}{z} = \left(\frac{z\ e_F^{zx}}{e_F^z - 1} - \frac{z}{e_F^z - 1} \right) \frac{1}{z} \\
&\overset{(7.7)}{=} \left(\sum_{n=0}^{\infty} B_n^F(x) \frac{z^n}{F_n!} - \frac{z}{e_F^z - 1} \right) \frac{1}{z} \\
&\overset{(7.9)}{=} \left(\sum_{n=0}^{\infty} B_n^F(x) \frac{z^n}{F_n!} - \sum_{n=0}^{\infty} b_n^F \frac{z^n}{F_n!} \right) \frac{1}{z} \\
&= \sum_{n=0}^{\infty} \left(B_n^F(x) - b_n^F \right) \frac{z^{n-1}}{F_n!}
\end{aligned}$$

Thus, we have,

$$\begin{aligned}
\frac{e_F^{zx} - 1}{e_F^z - 1} &= \left(B_0^F(x) - b_0^F \right) \frac{z^{-1}}{F_0!} + \left(B_1^F(x) - b_1^F \right) \frac{1}{F_1!} + \sum_{n=2}^{\infty} \left(B_n^F(x) - b_n^F \right) \frac{z^{n-1}}{F_n!} \\
\frac{e_F^{zx} - 1}{e_F^z - 1} &= (1 - 1) \frac{z^{-1}}{F_0!} + (x - 1 - (-1)) \frac{1}{F_1!} + \sum_{n=2}^{\infty} \left(B_n^F(x) - b_n^F \right) \frac{z^{n-1}}{F_n!} \\
\frac{e_F^{zx} - 1}{e_F^z - 1} &= x + \sum_{n=2}^{\infty} \left(B_n^F(x) - b_n^F \right) \frac{z^{n-1}}{F_n!} \qquad (7.16)
\end{aligned}$$

From this expansion, if we put $x = 1$;

$$\frac{e_F^z - 1}{e_F^z - 1} = 1 + \sum_{n=2}^{\infty} \left(B_n^F(1) - b_n^F \right) \frac{z^{n-1}}{F_n!}$$

$$\sum_{n=2}^{\infty} \left(B_n^F(1) - b_n^F \right) \frac{z^{n-1}}{F_n!} = 0.$$

Since this infinite sum is zero for any z, the coefficients at every power of z are also zero. Therefore,

$$B_n^F(1) - b_n^F = 0 \tag{7.17}$$

for $n = 2, 3, 4, \ldots$ □

CHAPTER 8

FIBONACCI MEETS APOLLONIOUS

8.1. Apollonious Gaskets

The following theorem is assigned to ancient Greek mathematician Apollonious from Perga (BC 240):

”Given three fixed circles, find a circle that (kisses) touches each of them.”

It is possible to prove that if the given circles are mutually tangential, then there exists two circles satisfying this property. (Simplest proof is based on inversion of circles.)

Let r_1, r_2, and r_3 denote the radiuses of the given circles, and R and r are radiuses of external and internal circles, respectively. Assume $r_1 < r_2 < r_3$ and $r < R$.

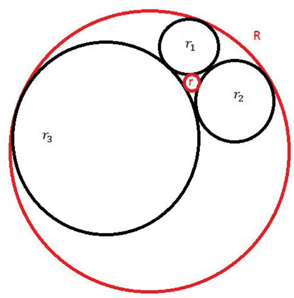

Figure 8.1. Solution of Apollonious Problem

In above figure, circles with radiuses r_1, r_2, r_3, r and r_1, r_2, r_3, R are kissing each other at six distinct points. These circles are mutually tangential to each other.

Theorem 8.1 *(Descartes Theorem)*

In plane geometry, if four circles with radiuses r_1, r_2, r_3, r_4 are mutually tangential to each other at six distinct points, then the circles' curvatures $\kappa_i = \frac{1}{r_i}$ $(i = 1, ..., 4)$ are *connected by quadratic relation:*

$$\boxed{(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)^2 = 2\,({\kappa_1}^2 + {\kappa_2}^2 + {\kappa_3}^2 + {\kappa_4}^2)} \tag{8.1}$$

Applied to Apollonious problem, the Descartes formula relates radiuses(or curvatures) of four circles:

$$\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}, \frac{1}{r} \;\&\; \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}, \frac{1}{R}$$

Example 8.1 *We choose the kissing circles' radiuses as;*

$$r_1 = \frac{1}{2}, \quad r_2 = \frac{1}{2}, \quad r_3 = \frac{1}{3}.$$

Now, by putting them into Descartes Theorem, formula (8.1)*, we can find radiuses of the circles, touching these three circles as,*

$$(2 + 2 + 3 + \kappa_4)^2 = 2\left(2^2 + 2^2 + 3^2 + {\kappa_4}^2\right)$$

$$\kappa_4 = 15 \;\; \textit{and} \;\; \kappa_4 = -1 \quad \Rightarrow \quad r_4 = \frac{1}{15} \;\; \textit{and} \;\; r_4 = 1$$

The positive and negative sign of curvature in these formulas is related with so called signed curvature in plane (positive or negative direction of rotation).

By choosing new sets of three kissing circles, one can derive the recursion process for the so called Apollonious Gasket. In figure 4.2*, example of integer Apollonious gasket is shown.*

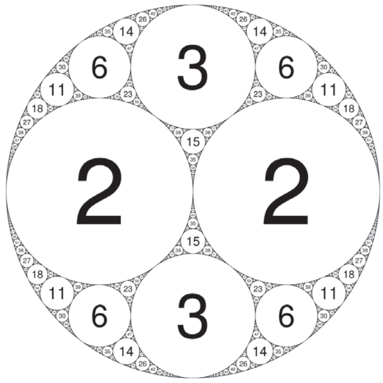

Figure 8.2. Apollonious Gaskets

Proposition 8.1 *For three kissing circles with arbitrary radiuses r_1, r_2, r_3, the corresponding radiuses of mutually kissing circles are;*

$$r = \frac{r_1\ r_2\ r_3}{r_1 r_2 + r_1 r_3 + r_2 r_3 + 2\sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)}} \tag{8.2}$$

$$R = \frac{r_1\ r_2\ r_3}{r_1 r_2 + r_1 r_3 + r_2 r_3 - 2\sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)}} \tag{8.3}$$

Proof Let κ be the curvature of the mutually kissing circles. Our goal is to find κ from the Descartes formula,

$$2\,({\kappa_1}^2 + {\kappa_2}^2 + {\kappa_3}^2 + \kappa^2) = (\kappa_1 + \kappa_2 + \kappa_3 + \kappa)^2.$$

One can reduce this equation to quadratic one in κ,

$$\kappa^2 - 2\,(\kappa_1 + \kappa_2 + \kappa_3)\,\kappa + {\kappa_1}^2 + {\kappa_2}^2 + {\kappa_3}^2 - 2\,(\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3) = 0$$

Let κ_+ and κ_- be the solutions of this quadratic equation,i.e, κ_+ and κ_- be the curvatures of the circles with radiuses r and R, respectively,

$$\kappa_+ = \frac{1}{r} \ \text{ and } \ \kappa_- = \frac{1}{R}.$$

Then,

$$\kappa_+, \kappa_- = \frac{2(\kappa_1 + \kappa_2 + \kappa_3) \pm \sqrt{16(\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3)}}{2}$$

and,

$$\begin{aligned}
\kappa_+ &= (\kappa_1 + \kappa_2 + \kappa_3) + 2\sqrt{(\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3)} \\
\kappa_- &= (\kappa_1 + \kappa_2 + \kappa_3) - 2\sqrt{(\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3)}
\end{aligned}$$

Substituting corresponding curvatures $\kappa_i = \frac{1}{r_i}$, where $i = 1, 2, 3$, gives;

$$\begin{aligned}
\frac{1}{r} &= \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) + 2\sqrt{\left(\frac{1}{r_1}\frac{1}{r_2} + \frac{1}{r_1}\frac{1}{r_3} + \frac{1}{r_2}\frac{1}{r_3}\right)} \\
\frac{1}{R} &= \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) - 2\sqrt{\left(\frac{1}{r_1}\frac{1}{r_2} + \frac{1}{r_1}\frac{1}{r_3} + \frac{1}{r_2}\frac{1}{r_3}\right)}
\end{aligned}$$

So, we easily obtain;

$$\begin{aligned}
r &= \frac{r_1\, r_2\, r_3}{r_1r_2 + r_1r_3 + r_2r_3 + 2\sqrt{r_1r_2r_3(r_1 + r_2 + r_3)}} \\
R &= \frac{r_1\, r_2\, r_3}{r_1r_2 + r_1r_3 + r_2r_3 - 2\sqrt{r_1r_2r_3(r_1 + r_2 + r_3)}}
\end{aligned}$$

□

8.2. Fibonacci Apollonious Gaskets

Here we consider the special case of three kissing circles with integer radiuses,

$$r_{1_n} = F_nF_{n+1}, \quad r_{2_n} = F_nF_{n+2}, \quad r_{3_n} = F_{n+1}F_{n+3} \tag{8.4}$$

satisfying inequality $r_{1_n} < r_{2_n} < r_{3_n}$ (Koshy, T., 2001).

Proposition 8.2 *Triangle with vertices at centers of kissing circles of radiuses* $r_{1_n}, r_{2_n}, r_{3_n}$ *given by* (8.4) *is a Pythagorean Triangle.*

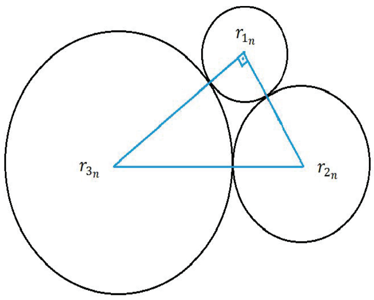

Figure 8.3. Pythagorean Triangle

Proof Sides of this triangle are $r_{1_n} + r_{2_n}$, $r_{1_n} + r_{3_n}$, $r_{2_n} + r_{3_n}$. Let's denote these sides as;

$$\begin{aligned} r_{1_n} + r_{2_n} &= F_n F_{n+3} \equiv a_n \\ r_{1_n} + r_{3_n} &= F_{n+1}(F_n + F_{n+3}) \equiv b_n \\ r_{2_n} + r_{3_n} &= F_n F_{n+2} + F_{n+1} F_{n+3} \equiv c_n. \end{aligned}$$

We have to prove that $c_n^2 = b_n^2 + a_n^2$, or

$$a_n^2 = c_n^2 - b_n^2 \quad \Rightarrow \quad a_n^2 = (c_n - b_n)(c_n + b_n).$$

By calculating right hand side,

$$\begin{aligned} (c_n - b_n)(c_n + b_n) &= \Big(F_n F_{n+2} + F_{n+1} F_{n+3} - F_n F_{n+1} - F_{n+1} F_{n+3}\Big) \cdot \\ &\quad \Big(F_n F_{n+2} + F_{n+1} F_{n+3} + F_n F_{n+1} + F_{n+1} F_{n+3}\Big) \\ &= (F_n(F_{n+2} - F_{n+1})) \ (F_n(F_{n+1} + F_{n+2}) + 2F_{n+1} F_{n+3}) \\ &= F_n^2 \ (F_{n+3}(F_n + 2F_{n+1})) \\ &= F_n^2 \ (F_{n+3}(F_n + F_{n+1} + F_{n+1})) \\ &= F_n^2 \ (F_{n+3}(F_{n+2} + F_{n+1})) \end{aligned}$$

$$= \quad F_n^2\, F_{n+3}^2 = a_n^2$$

□

Corollary 8.1 *Three numbers a_n, b_n and c_n written in terms of Fibonacci numbers,*

$$\begin{aligned} a_n &= F_n F_{n+3} \\ b_n &= 2F_{n+1}F_{n+2} \\ c_n &= F_{n+1}^2 + F_{n+2}^2 \end{aligned}$$

are Pythagorean triples(see the following figure).

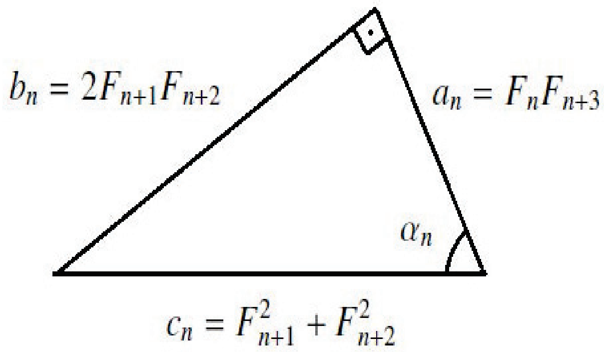

Figure 8.4. Pythagorean triples

Example 8.2 *For $n = 1$, sides of the Pythagorean triangle becomes $3, 4$ and 5. Also for $n = 2$, sides become $5,12$ and 13.*

By using angle α_n, we can write,

$$\begin{aligned} \cos\alpha_n &= \frac{a_n}{c_n} = \frac{F_nF_{n+3}}{F_{n+1}^2 + F_{n+2}^2} \\ \sin\alpha_n &= \frac{b_n}{c_n} = \frac{2F_{n+1}F_{n+2}}{F_{n+1}^2 + F_{n+2}^2} \end{aligned}$$

$$\tan\alpha_n = \frac{b_n}{a_n} = \frac{2F_{n+1}F_{n+2}}{F_nF_{n+3}}$$

Proposition 8.3 *Limit of these results is written in terms of Golden ratio φ,*

$$\lim_{n\to\infty} \cos\alpha_n \equiv \cos\alpha = \frac{\varphi}{\varphi+2} \tag{8.5}$$

$$\lim_{n\to\infty} \sin\alpha_n \equiv \sin\alpha = \frac{2\varphi}{\varphi+2} \tag{8.6}$$

$$\lim_{n\to\infty} \tan\alpha_n \equiv \tan\alpha = 2 \tag{8.7}$$

Proof Firstly,

$$\begin{aligned}\cos\alpha \equiv \lim_{n\to\infty} \cos\alpha_n &= \lim_{n\to\infty} \frac{F_nF_{n+3}}{F_{n+1}^2 + F_{n+2}^2} = \lim_{n\to\infty} \frac{(\varphi^n - \varphi'^n)\left(\varphi^{n+3} - \varphi'^{n+3}\right)}{(\varphi^{n+1} - \varphi'^{n+1})^2 + (\varphi^{n+3} - \varphi'^{n+3})^2} \\ &= \frac{\varphi^n\varphi^{n+3}}{\varphi^{2(n+1)} + \varphi^{2(n+2)}} = \frac{\varphi^3}{\varphi^2 + \varphi^4} = \frac{\varphi}{1+\varphi^2} = \frac{\varphi}{\varphi+2}\end{aligned}$$

Secondly,

$$\begin{aligned}\sin\alpha \equiv \lim_{n\to\infty} \sin\alpha_n &= \lim_{n\to\infty} \frac{2F_{n+1}F_{n+2}}{F_{n+1}^2 + F_{n+2}^2} = \lim_{n\to\infty} 2\frac{\left(\varphi^{n+1} - \varphi'^{n+1}\right)\left(\varphi^{n+2} - \varphi'^{n+2}\right)}{(\varphi^{n+1} - \varphi'^{n+1})^2 + (\varphi^{n+2} - \varphi'^{n+2})^2} \\ &= 2\frac{\varphi^{n+1}\varphi^{n+2}}{\varphi^{2(n+1)} + \varphi^{2(n+2)}} = \frac{2\varphi}{1+\varphi^2} = \frac{2\varphi}{1+\varphi^2} = \frac{2\varphi}{\varphi+2}\end{aligned}$$

Thirdly,

$$\tan\alpha \equiv \lim_{n\to\infty} \tan\alpha_n = \lim_{n\to\infty} \frac{\sin\alpha_n}{\cos\alpha_n} = \frac{\lim_{n\to\infty}\sin\alpha_n}{\lim_{n\to\infty}\cos\alpha_n} = \frac{\frac{2\varphi}{\varphi+2}}{\frac{\varphi}{\varphi+2}} = 2$$

□

Proposition 8.4 *By using* (8.5) *and* (8.6), *we have trigonometric identity;*

$$\cos^2\alpha + \sin^2\alpha = 1 \tag{8.8}$$

Proof

$$\cos^2\alpha + \sin^2\alpha \;=\; \frac{\varphi^2}{(\varphi+2)^2} + \frac{4\varphi^2}{(\varphi+2)^2} = \frac{5\varphi^2}{\varphi^2+4\varphi+4} \overset{(2.8)}{=} \frac{5(\varphi+1)}{\varphi+1+4\varphi+4} = 1$$

□

In the limiting case, when $n \to \infty$, our Phytagorean Triangle becomes the "Golden Phytagorean Triangle". Because all sides of this triangle are written by using number $''\varphi''$: $\varphi, 2\varphi, \varphi+2$. (See Figure 8.5)

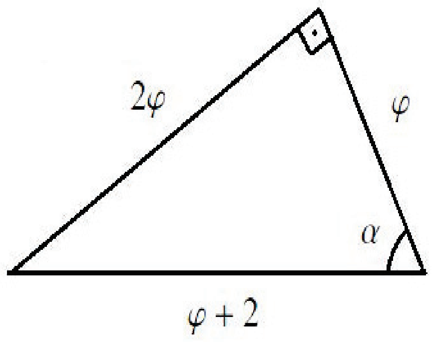

Figure 8.5. Golden Phytagorean Triangle

Proposition 8.5 *For three kissing circles with radiuses,*

$$r_{1_n} = F_n F_{n+1}, \;\; r_{2_n} = F_n F_{n+2}, \;\; r_{3_n} = F_{n+1} F_{n+3}$$

the corresponding radiuses of mutually kissing circles are:

$$R_n = F_{n+2}F_{n+3} \;\;, \quad r_n = \frac{F_n F_{n+1} F_{n+2} F_{n+3}}{4F_{n+2}F_{n+3} - F_n F_{n+1}} \tag{8.9}$$

Proof By using Proposition (8.1), we calculate r_n and R_n.

$$
\begin{aligned}
r_{1_n}\, r_{2_n}\, r_{3_n} &= F_n F_{n+1} F_n F_{n+2} F_{n+1} F_{n+3} = F_n^2 F_{n+1}^2 F_{n+2} F_{n+3} \\
r_{1_n} r_{2_n} + r_{1_n} r_{3_n} + r_{2_n} r_{3_n} &= F_n F_{n+1} F_n F_{n+2} + F_n F_{n+1} F_{n+1} F_{n+3} + F_n F_{n+2} F_{n+1} F_{n+3} \\
&= F_n^2 F_{n+1} F_{n+2} + F_n F_{n+1}^2 F_{n+3} + F_n F_{n+1} F_{n+2} F_{n+3} \\
r_{1_n} r_{2_n} r_{3_n} \left(r_{1_n} + r_{2_n} + r_{3_n}\right) &= F_n^2 F_{n+1}^2 F_{n+2} F_{n+3} \left(F_n F_{n+1} + F_n F_{n+2} + F_{n+1} F_{n+3}\right) \\
&= F_n^2 F_{n+1}^2 F_{n+2} F_{n+3} \left(F_n F_{n+3} + F_{n+1} F_{n+3}\right) \\
&= F_n^2 F_{n+1}^2 F_{n+2} F_{n+3} \left(F_{n+2} F_{n+3}\right) = F_n^2 F_{n+1}^2 F_{n+2}^2 F_{n+3}^2
\end{aligned}
$$

Starting from r_n;

$$
\begin{aligned}
r_n &= \frac{F_n^2 F_{n+1}^2 F_{n+2} F_{n+3}}{F_n^2 F_{n+1} F_{n+2} + F_n F_{n+1}^2 F_{n+3} + F_n F_{n+1} F_{n+2} F_{n+3} + 2\sqrt{F_n^2 F_{n+1}^2 F_{n+2}^2 F_{n+3}^2}} \\
&= \frac{F_n^2 F_{n+1}^2 F_{n+2} F_{n+3}}{F_n^2 F_{n+1} F_{n+2} + F_n F_{n+1}^2 F_{n+3} + F_n F_{n+1} F_{n+2} F_{n+3} + 2F_n F_{n+1} F_{n+2} F_{n+3}} \\
&= \frac{F_n F_{n+1} F_{n+2} F_{n+3}}{F_n F_{n+2} + F_{n+1} F_{n+3} + 3F_{n+2} F_{n+3}} = \frac{F_n F_{n+1} F_{n+2} F_{n+3}}{F_n F_{n+2} + \left(F_{n+2} - F_n\right) F_{n+3} + 3F_{n+2} F_{n+3}} \\
&= \frac{F_n F_{n+1} F_{n+2} F_{n+3}}{F_n F_{n+2} + F_{n+2} F_{n+3} - F_n F_{n+3} + 3F_{n+2} F_{n+3}} = \frac{F_n F_{n+1} F_{n+2} F_{n+3}}{4F_{n+2} F_{n+3} - F_n \left(F_{n+3} - F_{n+2}\right)} \\
&= \frac{F_n F_{n+1} F_{n+2} F_{n+3}}{4F_{n+2} F_{n+3} - F_n F_{n+1}}.
\end{aligned}
$$

Calculating R_n;

$$
\begin{aligned}
R_n &= \frac{F_n^2 F_{n+1}^2 F_{n+2} F_{n+3}}{F_n^2 F_{n+1} F_{n+2} + F_n F_{n+1}^2 F_{n+3} + F_n F_{n+1} F_{n+2} F_{n+3} - 2\sqrt{F_n^2 F_{n+1}^2 F_{n+2}^2 F_{n+3}^2}} \\
&= \frac{F_n F_{n+1} F_{n+2} F_{n+3}}{F_n F_{n+2} + F_{n+1} F_{n+3} - F_{n+2} F_{n+3}} = \frac{F_n F_{n+1} F_{n+2} F_{n+3}}{F_n F_{n+2} + F_{n+3} \left(F_{n+1} - F_{n+2}\right)} \\
&= \frac{F_n F_{n+1} F_{n+2} F_{n+3}}{F_n F_{n+2} - F_{n+3} F_n} = \frac{F_n F_{n+1} F_{n+2} F_{n+3}}{F_n \left(F_{n+2} - F_{n+3}\right)} = -\frac{F_n F_{n+1} F_{n+2} F_{n+3}}{F_n F_{n+1}} \\
&= -F_{n+2} F_{n+3}.
\end{aligned}
$$

Here R_n comes with negative sign, due to signed curvature. □

Proposition 8.6 *The limit of the internal and external radiuses r_n and R_n, given by* (8.9) *is finite and equal to:*

$$\lim_{n\to\infty}\frac{r_n}{R_n}=\frac{1}{12\varphi+7} \tag{8.10}$$

Proof

$$\lim_{n\to\infty}\frac{r_n}{R_n}=\lim_{n\to\infty}\frac{F_nF_{n+1}}{4F_{n+2}F_{n+3}-F_nF_{n+1}}=\lim_{n\to\infty}\frac{1}{4\frac{F_{n+2}}{F_{n+1}}\frac{F_{n+3}}{F_n}-1}\overset{(2.6)}{=}\frac{1}{4\,\varphi^4-1}=\frac{1}{12\varphi+7}$$

□

Example 8.3 *From equation* (8.4)*, if $n=1$, the kissing circles' radiuses are;*

$$r_{1_1}=F_1\,F_2=1,\quad r_{2_1}=F_1\,F_3=2,\quad r_{3_1}=F_2\,F_4=3$$

Substituting them into Descartes formula gives;

$$\left(1+\frac{1}{2}+\frac{1}{3}+\kappa_4\right)^2=2\left(1+\frac{1}{4}+\frac{1}{9}+\kappa_4{}^2\right)$$

$$\kappa_4=-\frac{1}{6}\quad and\quad \kappa_4=\frac{23}{6}$$

Corresponding radiuses become;

$$\Rightarrow\quad R_1=F_3\,F_4=6\quad and\quad r_1=\frac{F_1\,F_2\,F_3\,F_4}{4F_3\,F_4-F_1\,F_2}=\frac{6}{23}$$

Corollary 8.2 *The ratio of two kissing circles' radiuses, when n goes to infinity is related with Golden Ratio;*

$$\lim_{n\to\infty}\frac{r_{1_n}}{r_{2_n}}=\frac{1}{\varphi},\quad \lim_{n\to\infty}\frac{r_{1_n}}{r_{3_n}}=\frac{1}{\varphi^3},\quad \lim_{n\to\infty}\frac{r_{2_n}}{r_{3_n}}=\frac{1}{\varphi^2}.$$

It means that side a_n in the limit $n\to\infty$ is divided in Golden ratio φ. Side b_n in the limit $n\to\infty$ is divided in qubic Golden ratio $\varphi^3=2\varphi+1$. Side c_n in the limit $n\to\infty$ is divided in square Golden ratio $\varphi^2=\varphi+1$.

Theorem 8.2 *Since centers of* r_{1_n}, r_{2_n}, r_{3_n} *given in* (8.4) *form the vertices of a Pythagorean triangle, the area of this Pythagorean triangle is;*

$$\boxed{A_n = F_n F_{n+1} F_{n+2} F_{n+3}} \tag{8.11}$$

Proof From Figure 4.2, Pythagorean triangle's area is expressed as;

$$\begin{aligned}
A_n &= \frac{(r_{1_n} + r_{2_n})(r_{1_n} + r_{3_n})}{2} = \frac{(F_n F_{n+1} + F_n F_{n+2})(F_n F_{n+1} + F_{n+1} F_{n+3})}{2} \\
&= \frac{F_n (F_{n+1} + F_{n+2})(F_n F_{n+1} + F_{n+1} F_{n+3})}{2} = \frac{F_n F_{n+3} (F_n F_{n+1} + F_{n+1} F_{n+3})}{2} \\
&= \frac{F_n^2 F_{n+1} F_{n+3} + F_n F_{n+1} F_{n+3}^2}{2} = \frac{F_n F_{n+1} F_{n+3} (F_n + F_{n+3})}{2} \\
&= \frac{F_n F_{n+1} F_{n+3} (F_n + F_{n+3})}{2} = \frac{2 F_n F_{n+1} F_{n+3} F_{n+2}}{2} \\
&= F_n F_{n+1} F_{n+2} F_{n+3}
\end{aligned}$$

□

Example 8.4 *For different values of n, areas can be calculated,*

$$\begin{aligned}
A_1 &= F_1 F_2 F_3 F_4 = 1 \cdot 1 \cdot 2 \cdot 3 = 6 \\
A_2 &= F_2 F_3 F_4 F_5 = 1 \cdot 2 \cdot 3 \cdot 5 = 30 \\
A_3 &= F_3 F_4 F_5 F_6 = 2 \cdot 3 \cdot 5 \cdot 8 = 240
\end{aligned}$$

Corollary 8.3 *The limit of the ratio of two consecutive areas,*

$$\lim_{n \to \infty} \frac{A_{n+1}}{A_n} = \varphi^4$$

Definition 8.1 *Suppose we have three mutually tangential kissing circles. Circle which is passing through the intersection points of these three circles is called the dual circle.(See Figure 8.6)*

Proposition 8.7 *The radius of the dual circle, to the ones with* r_{1_n}, r_{2_n}, r_{3_n} *given by* (8.4) *is found as;*

$$r_{dual} = r_{1_n} = F_n F_{n+1} \tag{8.12}$$

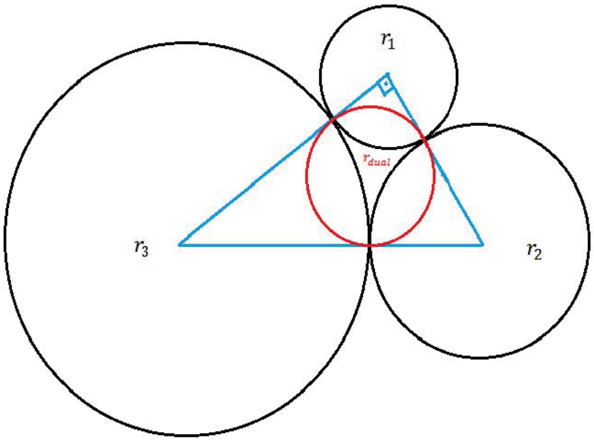

Figure 8.6. Radius of dual circle

Proof The area of the Pythagorean triangle in (8.11) is written;

$$\begin{aligned}
F_n F_{n+1} F_{n+2} F_{n+3} &= \frac{r_{dual}(r_1 + r_2)}{2} + \frac{r_{dual}(r_1 + r_3)}{2} + \frac{r_{dual}(r_2 + r_3)}{2} \\
F_n F_{n+1} F_{n+2} F_{n+3} &= \frac{1}{2} r_{dual}[2(r_1 + r_2 + r_3)] \\
&= r_{dual}\ (F_n F_{n+1} + F_n F_{n+2} + F_{n+1} F_{n+3}) \\
&= r_{dual}\ (F_n F_{n+3} + F_{n+1} F_{n+3}) \\
&= r_{dual}\ F_{n+2} F_{n+3}
\end{aligned}$$

Thus, the radius r_{dual} is;

$$r_{dual} = r_{1_n} = F_n F_{n+1}.$$

□

Proposition 8.8 *The radius of the circle, circumscribed around the triangle with sides a,b,c is given by,*

$$R_S = \frac{abc}{\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}} \quad (8.13)$$

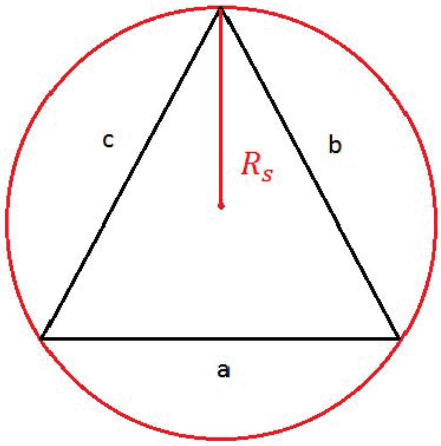

Figure 8.7. Radius of R_S

Proposition 8.9 *The radius of circumscribed circle around the triangle with sides* $r_{1_n}+r_{2_n}$, $r_{1_n}+r_{3_n}$, $r_{2_n}+r_{3_n}$ *given by* (8.4) *is found as,*

$$\frac{F_{n+1}^2 + F_{n+2}^2}{2} \quad (8.14)$$

Proof By using the previous proposition, we can denote,

$$a \equiv r_{1_n} + r_{2_n}$$
$$b \equiv r_{1_n} + r_{3_n}$$
$$c \equiv r_{2_n} + r_{3_n}$$

Then,

$$\begin{aligned}
a+b+c &= 2\left(r_{1_n}+r_{2_n}+r_{3_n}\right)=2\left(F_nF_{n+1}+F_nF_{n+2}+F_{n+1}F_{n+3}\right)=2F_{n+2}F_{n+3}\\
b+c-a &= a+b+c-2a=2F_{n+2}F_{n+3}-2F_nF_{n+3}=2F_{n+3}\left(F_{n+2}-F_n\right)=2F_{n+1}F_{n+3}\\
c+a-b &= a+b+c-2b=2F_{n+2}F_{n+3}-2F_nF_{n+1}-2F_{n+1}F_{n+3}=2F_nF_{n+2}\\
a+b-c &= a+b+c-2c=2F_{n+2}F_{n+3}-2F_nF_{n+2}-2F_{n+1}F_{n+3}=2F_nF_{n+1}
\end{aligned}$$

Therefore R_S becomes,

$$\begin{aligned}
R_S &= \frac{F_nF_{n+3}\left(F_{n+1}(F_n+F_{n+3})\right)\left(F_nF_{n+2}+F_{n+1}F_{n+3}\right)}{\sqrt{(2F_{n+2}F_{n+3})\left(2F_{n+1}F_{n+3}\right)\left(2F_nF_{n+2}\right)\left(2F_nF_{n+1}\right)}}\\
&= \frac{F_nF_{n+3}F_{n+1}(F_n+F_{n+3})\left(F_nF_{n+2}+F_{n+1}F_{n+3}\right)}{4F_nF_{n+1}F_{n+2}F_{n+3}}\\
&= \frac{(F_n+F_{n+3})\left(F_nF_{n+2}+F_{n+1}F_{n+3}\right)}{4F_{n+2}}\\
&= \frac{(F_n+F_{n+3})\left(F_nF_{n+2}+F_{n+1}F_{n+2}+F_{n+1}F_{n+1}\right)}{4F_{n+2}}\\
&= \frac{(F_n+F_{n+3})\left(F_{n+2}F_{n+2}+F_{n+1}F_{n+1}\right)}{4F_{n+2}}\\
&= \frac{(F_n+F_{n+2}+F_{n+1})\left(F_{n+2}^2+F_{n+1}^2\right)}{4F_{n+2}}\\
&= \frac{2F_{n+2}\left(F_{n+2}^2+F_{n+1}^2\right)}{4F_{n+2}}\\
&= \frac{F_{n+1}^2+F_{n+2}^2}{2}
\end{aligned}$$

□

By using kissing circles with radiuses $r_{1_n}, r_{2_n}, r_{3_n}$, from Descartes formula, we have obtained r_n and R_n respectively, given by (8.9). Here, by applying Descartes Formula iteratively for the new set of kissing circles, we can find the radiuses $r_{13_n}, r_{12_n}, r_{23_n}$ which are given in the following Figure 8.8.

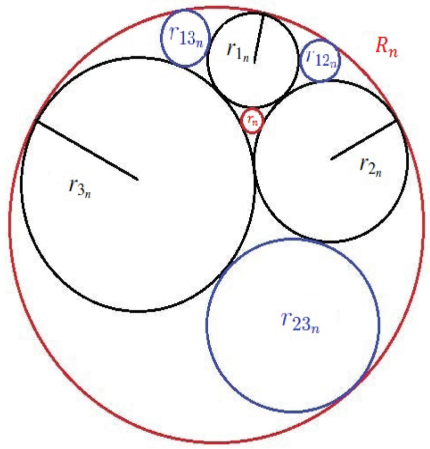

Figure 8.8. Forming Fibonacci-Apollonious Gaskets

Here r_{13_n} is kissing r_{1_n} and r_{3_n}. r_{12_n} is kissing r_{1_n} and r_{2_n} and r_{23_n} is kissing r_{2_n} and r_{3_n}. All of them are kissing external circle R_n. By this way, we get the construction of Fibonacci-Apollonious gaskets in plane.

Proposition 8.10 *Radiuses of the gasket are obtained as;*

$$r_{12_n} = \frac{F_n^2 F_{n+1} F_{n+2} F_{n+3}}{F_{n+2} F_{n+3}^2 - 4F_{n+1}^3}, \quad r_{13_n} = \frac{F_n F_{n+1}^2 F_{n+2} F_{n+3}}{F_{n+2}^2 F_{n+3} - F_n^3}, \quad r_{23_n} = \frac{F_n F_{n+1} F_{n+2}^2 F_{n+3}}{F_{n+1}^2 F_{n+3} + F_n^3} \tag{8.15}$$

Proof To get r_{12_n}, we use the Descartes formula and the radiuses r_{1_n}, r_{2_n}, R_n. By substituting them into Descartes formula,

$$2\left(\frac{1}{r_{1_n}^2} + \frac{1}{r_{2_n}^2} + \left(-\frac{1}{R_n}\right)^2 + \kappa\right) = \left(\frac{1}{r_{1_n}} + \frac{1}{r_{2_n}} - \frac{1}{R_n} + \kappa\right)^2$$

$$2\left(\frac{1}{F_n^2 F_{n+1}^2} + \frac{1}{F_n^2 F_{n+2}^2} + \frac{1}{F_{n+2}^2 F_{n+3}^2} + \kappa\right) = \left(\frac{1}{F_n^2 F_{n+1}^2} + \frac{1}{F_n^2 F_{n+2}^2} - \frac{1}{F_{n+2}^2 F_{n+3}^2} + \kappa\right)^2$$

By solving this quadratic equation using the recursion formula for Fibonacci numbers, we can get the radiuses of circles,

$$r_{12_n} = \frac{F_n^2 F_{n+1} F_{n+2} F_{n+3}}{F_{n+2} F_{n+3}^2 - 4F_{n+1}^3}$$

$$r_{3_n} = F_{n+1} F_{n+3}$$

which are kissing the circles with radiuses r_{1_n}, r_{2_n}, R_n. With the similar logic r_{13_n} and r_{23_n} can be obtained. □

8.3. Lucas-Apollonious Gaskets

In this Section, we choose the radiuses of kissing circles as Lucas numbers in the form;

$$r_{1_n} = L_n L_{n+1}, \quad r_{2_n} = L_n L_{n+2}, \quad r_{3_n} = L_{n+1} L_{n+3} \tag{8.16}$$

After appliying Descartes formula, we obtain kissing circles' radiuses in the form similar to the case of Fibonacci-Apollonious Gaskets.

$$R_n = L_{n+2} L_{n+3} \ , \quad r_n = \frac{L_n L_{n+1} L_{n+2} L_{n+3}}{4 L_{n+2} L_{n+3} - L_n L_{n+1}}. \tag{8.17}$$

This result follows easily from observation that recursion formulas for Fibonacci and Lucas numbers are the same, and in the proof we use only these recurrence relations.

In addition, by applying the Descartes Formula (8.1) we find radiuses the Lucas-Apollonious Gaskets $r_{13_n}, r_{12_n}, r_{23_n}$ as;

$$r_{12_n} = \frac{L_n^2 L_{n+1} L_{n+2} L_{n+3}}{L_{n+2} L_{n+3}^2 - 4 L_{n+1}^3}, \quad r_{13_n} = \frac{L_n L_{n+1}^2 L_{n+2} L_{n+3}}{L_{n+2}^2 L_{n+3} - L_n^3}, \quad r_{23_n} = \frac{L_n L_{n+1} L_{n+2}^2 L_{n+3}}{L_{n+1}^2 L_{n+3} + L_n^3} \tag{8.18}$$

8.4. More General Family of Apollonious Gaskets

In this section, generalizing previous results, we can choose the kissing circles radiuses as;

$$r_1 = G_n G_{n+1}, \quad r_2 = G_n G_{n+2}, \quad r_3 = G_{n+1} G_{n+3}, \tag{8.19}$$

where G_n is the Generalized Fibonacci sequence (Definition(2.2)) with recursion relation;

$$G_{n+1} = G_n + G_{n-1}$$

and initial values G_0 and G_1. This sequence can be described as addition of two Fibonacci sequences:

$$G_n = G_1 F_n + G_0 F_{n-1}.$$

By using (8.2) and (8.3), we obtain the kissing circles' radiuses as;

$$r_n = \frac{G_n G_{n+1} G_{n+2} G_{n+3}}{4G_{n+2}G_{n+3} - G_n G_{n+1}}, \tag{8.20}$$
$$R_n = G_{n+2}G_{n+3}. \tag{8.21}$$

By substituting $G_n = G_1F_n + G_0F_{n-1}$, one finds these radiuses in terms of Fibonacci numbers and the initial values as,

$$r_n = \frac{(G_1F_n + G_0F_{n-1})(G_1F_{n+1} + G_0F_n)(G_1F_{n+2} + G_0F_{n+1})(G_1F_{n+3} + G_0F_{n+2})}{4(G_1F_{n+3} + G_0F_{n+2})(G_1F_{n+2} + G_0F_{n+1}) - (G_1F_{n+1} + G_0F_n)(G_1F_n + G_0F_{n-1})}$$

and,

$$R_n = (G_1F_{n+2} + G_0F_{n+1})(G_1F_{n+3} + G_0F_{n+2}). \tag{8.22}$$

These results determine the Generalized Fibonacci Apollonious gasket with kissing radiuses;

$$r_{12_n} = \frac{G_n^2\, G_{n+1}\, G_{n+2}\, G_{n+3}}{G_{n+2}\, G_{n+3}^2 - 4G_{n+1}^3}$$
$$r_{13_n} = \frac{G_n\, G_{n+1}^2\, G_{n+2}\, G_{n+3}}{G_{n+2}^2\, G_{n+3} - G_n^3}$$

$$r_{23_n} = \frac{G_n\, G_{n+1}\, G_{n+2}^2\, G_{n+3}}{G_{n+1}^2\, G_{n+3} + G_n^3}$$

By substituting G_n's it gives;

$$r_{12_n} = \frac{(G_1F_n + G_0F_{n-1})^2\,(G_1F_{n+1} + G_0F_n)\,(G_1F_{n+2} + G_0F_{n+1})\,(G_1F_{n+3} + G_0F_{n+2})}{(G_1F_{n+2} + G_0F_{n+1})\,(G_1F_{n+3} + G_0F_{n+2})^2 - 4\,(G_1F_{n+1} + G_0F_n)^3}$$

$$r_{13_n} = \frac{(G_1F_n + G_0F_{n-1})^2\,(G_1F_{n+1} + G_0F_n)^2\,(G_1F_{n+2} + G_0F_{n+1})\,(G_1F_{n+3} + G_0F_{n+2})}{(G_1F_{n+2} + G_0F_{n+1})^2\,(G_1F_{n+3} + G_0F_{n+2}) - (G_1F_{n+1} + G_0F_{n-1})^3}$$

$$r_{23_n} = \frac{(G_1F_n + G_0F_{n-1})\,(G_1F_{n+1} + G_0F_n)\,(G_1F_{n+2} + G_0F_{n+1})^2\,(G_1F_{n+3} + G_0F_{n+2})}{(G_1F_{n+1} + G_0F_n)^2\,(G_1F_{n+3} + G_0F_{n+2}) + (G_1F_n + G_0F_{n-1})^3}$$

As a particular cases we get,

- $G_0 = 0$ and $G_1 = 1$ $\Rightarrow$ Fibonacci-Apollonious Gasket
- $G_0 = 2$ and $G_1 = 1$ $\Rightarrow$ Lucas-Apollonious Gasket

CHAPTER 9

CONCLUSION

In conclusions, we emphasize main results obtained in this thesis. By introducing Golden-Fibonacci calculus in terms of finite difference operator with bases φ and φ', we constructed generating functions for the Fibonacci numbers. The entire generating functions for Fibonacci numbers were derived as Golden exponential functions.

In terms of these functions, the Golden trigonometric and hyperbolic functions for Golden oscillator were derived. By the Golden binomial, the Golden-heat and the Golden-wave equations and corresponding solutions were obtained.

The Golden calculus was generalized to higher order Golden Fibonacci calculus by introducing higher order Golden Fibonacci derivatives. By using these derivatives, we found generating function for higher order Fibonacci numbers, higher Fibonomials and higher Golden binomials. As we proved, the higher Golden binomials are equivalent to Carlitz's characteristic polynomials for combinatorial matrices.

The congruency of Fibonacci and Lucas numbers as combinations of $k = 2$ higher order Fibonacci numbers of mod 5 integer numbers were constructed.

By using Golden exponential function e_F^x, the generating function for new type of polynomials, which we called Bernoulli-Fibonacci polynomials was derived. Properties of these polynomials and corresponding numbers, similar to usual ones were studied.

As a geometrical application, the Apollonious gasket of kissing circles was derived and the set of Fibonacci, Lucas and General family of Apollonious gaskets were obtained.

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APPENDIX A

INTRODUCTION

A.1. Another representation of Binet Formula

Derivation of formula (2.9)

$$F_n = 2^{n-1} \sum_{k=0}^{n-1} (-1)^k \cos^{n-k-1}\left(\frac{\pi}{5}\right) \sin^k\left(\frac{\pi}{10}\right)$$

will be done.

Since we know from (Koshy, T., 2001) that,

$$\varphi = 2\cos\left(\frac{\pi}{5}\right) \text{ and } \varphi' = -2\sin\left(\frac{\pi}{10}\right), \tag{A.1}$$

we can employ these trigonometric values of φ and φ' to develop a trigonometric summation formula for F_n.

By Binet Formula, we have;

$$\begin{aligned} F_n = \frac{\varphi^n - \varphi'^n}{\varphi - \varphi'} &= \frac{(\varphi - \varphi')(\varphi^{n-1} + \varphi^{n-2}\varphi' + \varphi^{n-3}\varphi'^2 + \ldots + \varphi\varphi'^{n-2} + \varphi'^{n-1})}{\varphi - \varphi'} \\ &= \varphi^{n-1} + \varphi^{n-2}\varphi' + \varphi^{n-3}\varphi'^2 + \ldots + \varphi\varphi'^{n-2} + \varphi'^{n-1} \\ &= \sum_{k=0}^{n-1} \varphi^{n-k-1}\varphi'^k \text{ after substituting (A.1) gives us,} \\ &= \sum_{k=0}^{n-1} \left[2\cos\left(\frac{\pi}{5}\right)\right]^{n-k-1} \left[-2\sin\left(\frac{\pi}{10}\right)\right]^k \\ &= \sum_{k=0}^{n-1} 2^{n-k-1} \cos^{n-k-1}\left(\frac{\pi}{5}\right)(-2)^k \sin^k\left(\frac{\pi}{10}\right) \\ &= 2^{n-1} \sum_{k=0}^{n-1} 2^{-k} \cos^{n-k-1}\left(\frac{\pi}{5}\right)(-1)^k(2)^k \sin^k\left(\frac{\pi}{10}\right) \end{aligned}$$

$$= \quad 2^{n-1} \sum_{k=0}^{n-1} (-1)^k \cos^{n-k-1}\left(\frac{\pi}{5}\right) \sin^k\left(\frac{\pi}{10}\right)$$

So,

$$\boxed{F_n = 2^{n-1} \sum_{k=0}^{n-1} (-1)^k \cos^{n-k-1}\left(\frac{\pi}{5}\right) \sin^k\left(\frac{\pi}{10}\right)} \tag{A.2}$$

is obtained.

APPENDIX B

FIBONACCI CALCULUS

B.1. Showing the Golden periodicity of function $A(x) = \sin\left(\frac{\pi}{\ln\varphi}\ln|x|\right)$

Let's show the Golden periodicity of the function;

$$A(x) = \sin\left(\frac{\pi}{\ln\varphi}\ln|x|\right) \tag{B.1}$$

To say that it is Golden periodic function, we should show that $A(\varphi x) = A\left(-\frac{x}{\varphi}\right)$.

$$\begin{aligned} A(\varphi x) &= \sin\left(\frac{\pi}{\ln\varphi}\ln|\varphi x|\right) = \sin\left(\frac{\pi}{\ln\varphi}\ln|\varphi| + \frac{\pi}{\ln\varphi}\ln|x|\right) = \sin\left(\pi + \frac{\pi}{\ln\varphi}\ln|x|\right) \\ &= \sin(\pi)\cos\left(\frac{\pi}{\ln\varphi}\ln|x|\right) + \sin\left(\frac{\pi}{\ln\varphi}\ln|x|\right)\cos(\pi) \\ &= -\sin\left(\frac{\pi}{\ln\varphi}\ln|x|\right) \end{aligned} \tag{B.2}$$

$$\begin{aligned} A\left(-\frac{x}{\varphi}\right) &= \sin\left(\frac{\pi}{\ln\varphi}\ln\left|-\frac{x}{\varphi}\right|\right) = \sin\left(\frac{\pi}{\ln\varphi}(\ln|x| - \ln|-\varphi|)\right) \\ &= \sin\left(\frac{\pi}{\ln\varphi}\ln|x| - \frac{\pi}{\ln\varphi}\ln|-\varphi|\right) \\ &= \sin\left(\frac{\pi}{\ln\varphi}\ln|x| - \frac{\pi}{\ln\varphi}\ln\varphi\right) \\ &= \sin\left(\frac{\pi}{\ln\varphi}\ln|x| - \pi\right) \\ &= \sin\left(\frac{\pi}{\ln\varphi}\ln|x|\right)\cos(\pi) - \sin(\pi)\cos\left(\frac{\pi}{\ln\varphi}\ln|x|\right) \\ &= -\sin\left(\frac{\pi}{\ln\varphi}\ln|x|\right) \end{aligned} \tag{B.3}$$

Thus, we got the same results from both sides of equality. So,

$$A(x) = \sin\left(\frac{\pi}{\ln\varphi}\ln|x|\right) \tag{B.4}$$

is Golden periodic function.

B.2. Showing Golden derivative applications to Golden Binomials

The application of the Golden derivative D_F^x to the Golden Binomial $(x+y)_F^n$ gives;

$$\begin{aligned}
D_F^x(x+y)_F^n &= D_F^x\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_F (-1)^{\frac{k(k-1)}{2}} x^{n-k}y^k\right)\\
&= D_F^x\left(\sum_{k=0}^{n}\frac{F_n!}{F_{n-k}!F_k!}\ (-1)^{\frac{k(k-1)}{2}} x^{n-k}y^k\right)\\
&= D_F^x\left(\frac{F_n!}{F_n!F_0!}x^n + \frac{F_n!}{F_{n-1}!F_1!}x^{n-1}y + \ldots + \frac{F_n!}{F_0!F_n!}(-1)^{\frac{n(n-1)}{2}}y^n\right)\\
&= D_F^x\left(\sum_{k=0}^{n-1}\frac{F_n!}{F_{n-k}!F_k!}\ (-1)^{\frac{k(k-1)}{2}} x^{n-k}y^k\right)\\
&= \sum_{k=0}^{n-1}\frac{F_n!}{F_{n-k}!F_k!}\ (-1)^{\frac{k(k-1)}{2}}\ D_F^x(x^{n-k})\ y^k\\
&= \sum_{k=0}^{n-1}\frac{F_n!}{F_{n-k}!F_k!}\ (-1)^{\frac{k(k-1)}{2}}(F_{n-k}\ x^{n-k-1})\ y^k\\
&= \sum_{k=0}^{n-1}\frac{F_n!}{F_{n-k-1}!F_k!}\ (-1)^{\frac{k(k-1)}{2}}\ x^{n-k-1}\ y^k\\
&= \sum_{k=0}^{n-1}\frac{F_nF_{n-1}!}{F_{n-1-k}!F_k!}\ (-1)^{\frac{k(k-1)}{2}}\ x^{(n-1)-k}\ y^k\\
&= F_n\ \sum_{k=0}^{n-1}\frac{F_{n-1}!}{F_{n-1-k}!F_k!}\ (-1)^{\frac{k(k-1)}{2}}\ x^{(n-1)-k}\ y^k\\
&= F_n\ (x+y)_F^{n-1}
\end{aligned} \tag{B.5}$$

So, it is proved that $D_F^x(x+y)_F^n = F_n(x+y)_F^{n-1}$.

The application of the Golden derivative D_F^y to the Golden Binomial $(x+y)_F^n$ gives;

$$
\begin{aligned}
D_F^y(x+y)_F^n &= D_F^y\left(\sum_{k=0}^{n} \frac{F_n!}{F_{n-k}!F_k!}\ (-1)^{\frac{k(k-1)}{2}} x^{n-k}y^k\right) \\
&= D_F^y\left(\frac{F_n!}{F_n!F_0!}x^n + \frac{F_n!}{F_{n-1}!F_1!}x^{n-1}y + \ldots + \frac{F_n!}{F_0!F_n!}(-1)^{\frac{n(n-1)}{2}}y^n\right) \\
&= D_F^y\left(\sum_{k=1}^{n} \frac{F_n!}{F_{n-k}!F_k!}\ (-1)^{\frac{k(k-1)}{2}} x^{n-k}y^k\right) \\
&= \sum_{k=1}^{n} \frac{F_n!}{F_{n-k}!F_k!}\ (-1)^{\frac{k(k-1)}{2}} x^{n-k} D_F^y(y^k) \\
&= \sum_{k=1}^{n} \frac{F_n!}{F_{n-k}!F_k!}\ (-1)^{\frac{k(k-1)}{2}} x^{n-k}(F_k\ \ y^{k-1}) \\
&= \sum_{k=0}^{n-1} \frac{F_n!}{F_{n-(k+1)}!F_{k+1}!}\ (-1)^{\frac{(k+1)(k+1-1)}{2}} x^{n-(k+1)}(F_{k+1}\ \ y^{(k+1)-1}) \\
&= \sum_{k=0}^{n-1} \frac{F_n!}{F_{(n-1)-k}!F_{k+1}!}\ (-1)^{\frac{(k^2+k)}{2}} x^{(n-1)-k}(F_{k+1}\ \ y^{k}) \\
&= \sum_{k=0}^{n-1} \frac{F_n!}{F_{(n-1)-k}!F_{k}!}\ (-1)^{\frac{(k^2+(k-k)+k)}{2}} x^{(n-1)-k}\ \ y^{k} \\
&= \sum_{k=0}^{n-1} \frac{F_n!}{F_{(n-1)-k}!F_{k}!}\ (-1)^{\frac{k(k-1)}{2}}\ \ x^{(n-1)-k}\ \ (-y)^{k} \\
&= F_n \sum_{k=0}^{n-1} \frac{F_{n-1}!}{F_{(n-1)-k}!F_{k}!}\ (-1)^{\frac{k(k-1)}{2}}\ \ x^{(n-1)-k}\ \ (-y)^{k} \\
&= F_n\ \ (x-y)_F^{n-1}
\end{aligned}
\tag{B.6}
$$

So, it is proved that,

$$D_F^y(x+y)_F^n = F_n(x-y)_F^{n-1}.$$

With another type of approach, also we can prove,

$$D_F^y(x-y)_F^n = -F_n(x+y)_F^{n-1} \tag{B.7}$$

$$
\begin{aligned}
D_F^y(x-y)_F^n &= \frac{(x-\varphi y)_F^n - (x-\varphi' y)_F^n}{(\varphi-\varphi')y} \\
&= \frac{(x-\varphi^n y)\ldots(x-(-1)^{n-1}\varphi^{-n+2}y) - (x+\varphi^{n-2}y)\ldots(x-(-1)^n\varphi^{-n}y)}{(\varphi-\varphi')y} \\
&= \frac{(x+\varphi^{n-2}y)\ldots(x-(-1)^{n-1}\varphi^{-n+2}y)\left[x-\varphi^n y - x + (-1)^n\varphi^{-n}y\right]}{(\varphi-\varphi')y} \\
&= (x+y)_F^{n-1}\left(-\frac{(\varphi^n-\varphi'^n)y}{(\varphi-\varphi')y}\right) \\
&= -F_n(x+y)_F^{n-1} \qquad \text{(B.8)}
\end{aligned}
$$

Therefore, we proved that

$$
D_F^y(x-y)_F^n = -F_n(x+y)_F^{n-1}. \qquad \text{(B.9)}
$$

B.3. Golden Heat Equation

B.3.1. Golden derivative applications to the function $e_F(t+x)_F$

After application of D_F^t,

$$
\begin{aligned}
D_F^t(e_F(t+x)_F) &= D_F^t\left(\sum_{n=0}^{\infty}\frac{(t+x)_F^n}{F_n!}\right) \\
&= D_F^t\left(\frac{1}{F_0!} + \frac{(t+x)_F^1}{F_1!} + \frac{(t+x)_F^2}{F_2!} + \ldots\right) \\
&= \sum_{n=1}^{\infty}\frac{D_F^t(t+x)_F^n}{F_n!} = \sum_{n=1}^{\infty}\frac{F_n(t+x)_F^{n-1}}{F_n!} \\
&= \sum_{n=1}^{\infty}\frac{(t+x)_F^{n-1}}{F_{n-1}!} = \sum_{k=0}^{\infty}\frac{(t+x)_F^k}{F_k!} \\
&= e_F(t+x)_F
\end{aligned}
$$

We obtain the equality $D_F^t \ (e_F(t+x)_F) = e_F(t+x)_F$. Application of D_F^x gives,

$$\begin{aligned}
D_F^x \ e_F(t+x)_F &= D_F^x\left(\sum_{n=0}^{\infty}\frac{(t+x)_F^n}{F_n!}\right)\\
&= D_F^t\left(\frac{1}{F_0!}+\frac{(t+x)_F^1}{F_1!}+\frac{(t+x)_F^2}{F_2!}+\dots\right)\\
&= \sum_{n=1}^{\infty}\frac{D_F^x(t+x)_F^n}{F_n!}=\sum_{n=1}^{\infty}\frac{F_n(t-x)_F^{n-1}}{F_n!}\\
&= \sum_{n=1}^{\infty}\frac{(t-x)_F^{n-1}}{F_{n-1}!}=\sum_{k=0}^{\infty}\frac{(t-x)_F^k}{F_k!}\\
&= e_F(t-x)_F
\end{aligned}$$

Thus, it is obtained $D_F^x \ (e_F(t+x)_F) = e_F(t-x)_F$. Application of D_F^x to the $e_F(t-x)_F$,

$$\begin{aligned}
D_F^x \ e_F(t-x)_F &= D_F^x\left(\sum_{n=0}^{\infty}\frac{(t-x)_F^n}{F_n!}\right)\\
&= D_F^t\left(\frac{1}{F_0!}+\frac{(t-x)_F^1}{F_1!}+\frac{(t-x)_F^2}{F_2!}+\dots\right)\\
&= \sum_{n=1}^{\infty}\frac{D_F^x(t-x)_F^n}{F_n!}=\sum_{n=1}^{\infty}\frac{-F_n(t+x)_F^{n-1}}{F_n!}\\
&= \sum_{n=1}^{\infty}-\frac{(t+x)_F^{n-1}}{F_{n-1}!}=-\sum_{k=0}^{\infty}\frac{(t+x)_F^k}{F_k!}\\
&= -e_F(t+x)_F
\end{aligned}$$

So, we proved that,

$$D_F^x \ (e_F(t-x)_F) = -e_F(t+x)_F \tag{B.10}$$

B.3.2. Golden derivative applications to the function $e_F(\omega t + kx)_F$

After application D_F^t gives,

$$\begin{aligned} D_F^t(e_F(\omega t + kx)_F) &= D_F^t\left(\sum_{n=0}^{\infty} \frac{(\omega t + kx)_F^n}{F_n!}\right) \\ &= D_F^t\left(\frac{1}{F_0!} + \frac{(\omega t + kx)_F^1}{F_1!} + \frac{(\omega t + kx)_F^2}{F_2!} + \ldots\right) \\ &= \sum_{n=1}^{\infty} \frac{D_F^t(\omega t + kx)_F^n}{F_n!} \end{aligned}$$

To calculate $D_F^t(\omega t + kx)_F^n$;

$$D_F^x(kx + y)_F^n = \frac{(k\varphi x + y)_F^n - (k\varphi' x + y)_F^n}{(\varphi - \varphi')x} = k^n \frac{\left(\varphi x + \frac{y}{k}\right)_F^n - \left(\varphi' x + \frac{y}{k}\right)_F^n}{(\varphi - \varphi')x} \tag{B.11}$$

Also we have,

$$D_F^z(z + \omega)_F^n = \frac{(z\varphi + \omega)_F^n - (z\varphi' + \omega)_F^n}{(\varphi - \varphi')z} = F_n\,(z + \omega)_F^{n-1} \tag{B.12}$$

Comparing results (B.11) and (B.12) says that if we choose $z = x$ and $\omega = \frac{y}{k}$, equation (B.11) becomes;

$$D_F^x(kx + y)_F^n = k^n D_F^x\left(x + \frac{y}{k}\right)_F^n = k^n\,F_n\,\left(x + \frac{y}{k}\right)_F^{n-1} = k\,F_n\,(kx + y)_F^{n-1} \tag{B.13}$$

Thus,

$$D_F^x(kx + y)_F^n = k\,F_n\,(kx + y)_F^{n-1} \tag{B.14}$$

So, we can conclude that,

$$\boxed{D_F^t(\omega t + kx)_F^n = \omega\,F_n\,(\omega t + kx)_F^{n-1}} \tag{B.15}$$

$$
\begin{aligned}
D_F^t(e_F(\omega t+kx)_F) = \ldots &= \sum_{n=1}^{\infty} \frac{D_F^t(\omega t+kx)_F^n}{F_n!} \quad \text{(by using the result (B.15))} \\
&= \sum_{n=1}^{\infty} \frac{\omega\, F_n\, (\omega t+kx)_F^{n-1}}{F_n!} \\
&= \omega \sum_{n=1}^{\infty} \frac{(\omega t+kx)_F^{n-1}}{F_{n-1}!} \quad \text{(denoting n-1=m)} \\
&= \omega \sum_{m=0}^{\infty} \frac{(\omega t+kx)_F^{m}}{F_{m}!} \\
&= \omega\, e_F(\omega t+kx)_F
\end{aligned} \tag{B.16}
$$

Therefore, the desired result came as;

$$
\boxed{D_F^t(e_F(\omega t+kx)_F) = \omega\, e_F(\omega t+kx)_F} \tag{B.17}
$$

With the similar logic, it can be shown that,

$$
(D_F^x)^2(e_F(\omega t+kx)_F) = -k^2\, e_F(\omega t+kx)_F \tag{B.18}
$$

Application of D_F^x gives,

$$
\begin{aligned}
D_F^x(e_F(\omega t+kx)_F) &= D_F^x\left(\sum_{n=0}^{\infty} \frac{(\omega t+kx)_F^n}{F_n!}\right) \\
&= D_F^x\left(\frac{1}{F_0!} + \frac{(\omega t+kx)_F^1}{F_1!} + \frac{(\omega t+kx)_F^2}{F_2!} + \ldots\right) \\
&= \sum_{n=1}^{\infty} \frac{D_F^x(\omega t+kx)_F^n}{F_n!}
\end{aligned}
$$

Let's calculate $D_F^x(\omega t+kx)_F^n$.

$$
D_F^y(x+\omega y)_F^n = \frac{(x+\omega\varphi y)_F^n - (x+\omega\varphi' y)_F^n}{(\varphi-\varphi')y} = \omega^n\, \frac{\left(\frac{x}{\omega}+\varphi y\right)_F^n - \left(\frac{x}{\omega}+\varphi' y\right)_F^n}{(\varphi-\varphi')y} \tag{B.19}
$$

Also we have,

$$D_F^z(a+z)_F^n = \frac{(a+\varphi z)_F^n - (a+\varphi' z)_F^n}{(\varphi-\varphi')z} = F_n\ (a-z)_F^{n-1} \tag{B.20}$$

Comparing results (B.19) and (B.20) says that if we choose $z = y$ and $a = \frac{x}{\omega}$, equation (B.19) becomes;

$$D_F^y(x+\omega y)_F^n = \omega^n D_F^y\left(\frac{x}{\omega}+y\right)_F^n = \omega^n\ F_n\ \left(\frac{x}{\omega}-y\right)_F^{n-1} = \omega\ F_n\ (x-\omega y)_F^{n-1} \tag{B.21}$$

Thus,

$$D_F^y(x+\omega y)_F^n = \omega\ F_n\ (x-\omega y)_F^{n-1} \tag{B.22}$$

So, we can conclude that,

$$\boxed{D_F^x(\omega t+kx)_F^n = k\ F_n\ (\omega t-kx)_F^{n-1}} \tag{B.23}$$

$$\begin{aligned}
D_F^x(e_F(\omega t+kx)_F) = \ldots &= \sum_{n=1}^{\infty}\frac{D_F^x(\omega t+kx)_F^n}{F_n!} \quad \text{(by using the result (B.23))}\\
&= \sum_{n=1}^{\infty}\frac{k\ F_n\ (\omega t-kx)_F^{n-1}}{F_n!}\\
&= k\sum_{n=1}^{\infty}\frac{(\omega t-kx)_F^{n-1}}{F_{n-1}!} \quad \text{(denoting n-1=p)}\\
&= k\sum_{p=0}^{\infty}\frac{(\omega t-kx)_F^{p}}{F_p!}\\
&= k\ e_F(\omega t-kx)_F
\end{aligned} \tag{B.24}$$

So, we can obtain;

$$\boxed{D_F^x(e_F(\omega t+kx)_F) = k\ e_F(\omega t-kx)_F} \tag{B.25}$$

Then,

$$(D_F^x)^2(e_F(\omega t + kx)_F) = D_F^x(ke_F(\omega t - kx)_F) = k(-k)\ e_F(\omega t + kx)_F$$

$$\boxed{(D_F^x)^2(e_F(\omega t + kx)_F) = -k^2\ e_F(\omega t + kx)_F}$$

B.3.3. Proof of Factorization property

We will prove the factorization property of Higher Golden binomials;

$$\begin{aligned} {}_{(k)}(x-a)_F^{n+m} &= {}_{(k)}\left(x-\varphi^{km}a\right)_F^n\ {}_{(k)}\left(x-\varphi'^{kn}a\right)_F^m & \text{(B.26)} \\ &= {}_{(k)}\left(x-\varphi'^{km}a\right)_F^n\ {}_{(k)}\left(x-\varphi^{kn}a\right)_F^m & \text{(B.27)} \end{aligned}$$

By using the definition,

$${}_{(k)}(x-a)_F^n = \begin{cases} 1, & \text{if } n = 0; \\ \left(x-\varphi^{k(n-1)}a\right)\left(x-\varphi^{k(n-2)}\varphi'^{k}a\right)\ldots\left(x-\varphi^{k}\varphi'^{k(n-2)}a\right)\left(x-\varphi'^{k(n-1)}a\right), & \text{if } n \geq 1. \end{cases}$$

We can write,

$${}_{(k)}(x-a)_F^N = \left(x-\varphi^{k(N-1)}a\right)\left(x-\varphi^{k(N-2)}\varphi'^{k}a\right)\ldots\left(x-\varphi^{k}\varphi'^{k(N-2)}a\right)\left(x-\varphi'^{k(N-1)}a\right) \quad \text{(B.28)}$$

After denoting $N = n + m$,

$$\begin{aligned} {}_{(k)}(x-a)_F^{n+m} &= \left(x-\varphi^{k(n+m-1)}a\right)\left(x-\varphi^{k(n+m-2)}\varphi'^{k}a\right)\ldots\left(x-\varphi^{k}\varphi'^{k(n+m-2)}a\right)\left(x-\varphi'^{k(n+m-1)}a\right) \\ &= \left(x-\varphi^{k(n+m-1)}a\right)\left(x-\varphi^{k(n+m-2)}\varphi'^{k}a\right)\left(x-\varphi^{k(n+m-3)}\varphi'^{2k}a\right)\ldots \\ &\quad \left(x-\varphi^{k(n+m-n)}\varphi'^{(n-1)k}a\right)\cdot\left(x-\varphi^{k(n+m-(n+1))}\varphi'^{(kn)}a\right)\left(x-\varphi^{k(n+m-(n+2))}\varphi'^{(n+1)k}a\right) \\ &\quad \ldots\left(x-\varphi^{k(n+m-(n+m-2))}\varphi'^{(n+m-3)k}a\right)\left(x-\varphi^{k(n+m-(n+m-1))}\varphi'^{k(n+m-2)}a\right) \\ &\quad \left(x-\varphi^{k(n+m-(n+m))}\varphi'^{k(n+m-1)}a\right) \end{aligned}$$

$$\begin{aligned}
&= \left(x-\varphi^{k(n-1)}\left(\varphi^{km}a\right)\right)\left(x-\varphi^{k(n-2)}\varphi'^{k}\left(\varphi^{km}a\right)\right)\left(x-\varphi^{k(n-3)}\varphi'^{2k}\left(\varphi^{km}a\right)\right)\\
&\quad \ldots\left(x-\varphi^{k(n-n)}\varphi'^{(n-1)k}\left(\varphi^{km}a\right)\right)\cdot\left(x-\varphi^{k(m-1)}\left(\varphi'^{kn}a\right)\right)\left(x-\varphi^{k(m-2)}\varphi'^{k}\left(\varphi'^{kn}a\right)\right)\\
&\quad \ldots\left(x-\varphi^{2k}\varphi'^{k(m-3)}\left(\varphi'^{kn}a\right)\right)\left(x-\varphi^{k}\varphi'^{k(m-2)}\left(\varphi'^{kn}a\right)\right)\left(x-\varphi'^{k(m-1)}\left(\varphi'^{kn}a\right)\right)\\
&\overset{(B.28)}{=} {}_{(k)}\left(x-\varphi^{km}a\right)_F^n\cdot\left(x-\varphi^{k(m-1)}\left(\varphi'^{kn}a\right)\right)\left(x-\varphi^{k(m-2)}\varphi'^{k}\left(\varphi'^{kn}a\right)\right)\\
&\quad \ldots\left(x-\varphi^{2k}\varphi'^{k(m-3)}\left(\varphi'^{kn}a\right)\right)\left(x-\varphi^{k}\varphi'^{k(m-2)}\left(\varphi'^{kn}a\right)\right)\left(x-\varphi'^{k(m-1)}\left(\varphi'^{kn}a\right)\right)\\
&\overset{(B.28)}{=} {}_{(k)}\left(x-\varphi^{km}a\right)_F^n\cdot{}_{(k)}\left(x-\varphi'^{kn}a\right)_F^m
\end{aligned}$$

After changing $n \leftrightarrow m$ it gives the another result as,

$$ {}_{(k)}(x-a)_F^{m+n} = {}_{(k)}\left(x-\varphi'^{km}a\right)_F^n \; {}_{(k)}\left(x-\varphi^{kn}a\right)_F^m $$

APPENDIX C

BERNOULLI FIBONACCI POLYNOMIALS

C.1. Getting $B_n^F(x) + F_n x^{n-1}$ in two ways

Our aim is to get the equation;

$$B_n^F(x) + F_n x^{n-1} = H_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_F B_{n-k}^F(x) \tag{C.1}$$

Starting with,

$$\begin{aligned}
B_n^F(x) + F_n x^{n-1} &= H_n(x) \\
\sum_{n=0}^{\infty} B_n^F(x)\frac{z^n}{F_n!} + \sum_{n=0}^{\infty} F_n x^{n-1}\frac{z^n}{F_n!} &= \sum_{n=0}^{\infty} H_n(x)\frac{z^n}{F_n!} \\
\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{F_n!} - \sum_{n=0}^{\infty} B_n^F(x)\frac{z^n}{F_n!} &= \sum_{n=0}^{\infty} F_n x^{n-1}\frac{z^n}{F_n!}
\end{aligned}$$

For the right hand side, we get;

$$z\, e_F^{zx} = D_F^x(e_F^{zx}) = D_F^x\left(\sum_{n=1}^{\infty} \frac{x^n z^n}{F_n!}\right) = \sum_{n=1}^{\infty} \frac{F_n\ x^{n-1} z^n}{F_n!} = \sum_{n=0}^{\infty} \frac{F_n x^{n-1} z^n}{F_n!}$$

Then, we have;

$$\begin{aligned}
\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{F_n!} - \sum_{n=0}^{\infty} B_n^F(x)\frac{z^n}{F_n!} &= z\, e_F^{zx} \\
\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{F_n!} &= z\, e_F^{zx} + \sum_{n=0}^{\infty} B_n^F(x)\frac{z^n}{F_n!}
\end{aligned}$$

$$
\begin{aligned}
\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{F_n!} &= z\, e_F^{zx} + \frac{z\, e_F^{zx}}{e_F^z - 1} \\
\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{F_n!} &= z\, e_F^{zx}\left(1 + \frac{1}{e_F^z - 1}\right) = z\, e_F^{zx}\frac{e_F^z}{e_F^z - 1} \\
\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{F_n!} &= \sum_{n=0}^{\infty} B_n^F(x)\frac{z^n}{F_n!} \cdot e_F^z \\
\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{F_n!} &= \sum_{n=0}^{\infty} B_n^F(x)\frac{z^n}{F_n!} \cdot \sum_{k=0}^{\infty}\frac{z^k}{F_k!} \\
\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{F_n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{B_n^F(x)}{F_n!}\frac{z^{n+k}}{F_k!} \\
\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{F_n!} &\overset{(n+k=N)}{=} \sum_{N=0}^{\infty}\sum_{k=0}^{N}\frac{B_{N-k}^F(x)}{F_{N-k}!}\frac{z^N}{F_k!}\left(\frac{F_N!}{F_N!}\right) \\
\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{F_n!} &= \sum_{N=0}^{\infty}\frac{z_N}{F_N!}\sum_{k=0}^{N}\frac{F_N!}{F_{N-k}!F_k!}B_{N-k}^F(x) \\
\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{F_n!} &= \sum_{N=0}^{\infty}\frac{z_N}{F_N!}\sum_{k=0}^{N}\begin{bmatrix} N \\ k \end{bmatrix}_F B_{N-k}^F(x) \\
\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{F_n!} &\overset{(N=n)}{=} \sum_{n=0}^{\infty}\frac{z_n}{F_n!}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_F B_{n-k}^F(x)
\end{aligned}
$$

By equating two series, we get;

$$
H_n(x) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_F B_{n-k}^F(x) \tag{C.2}
$$

Following in another way we show that,

$$
B_n^F(x) + F_n x^{n-1} = H_n(x) = x^n + \sum_{j=2}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F x^{n-j}
$$

Starting with,

$$
\begin{aligned}
B_n^F(x) + F_n x^{n-1} &\overset{(7.11)}{=} \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F x^{n-j} + F_n x^{n-1} \\
&= \begin{bmatrix} n \\ 0 \end{bmatrix}_F b_0^F x^n + \begin{bmatrix} n \\ 1 \end{bmatrix}_F b_1^F x^{n-1} + \sum_{j=2}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F x^{n-j} + F_n x^{n-1}
\end{aligned}
$$

$$
\begin{aligned}
&= x^n - \frac{F_n!}{F_{n-1}!F_1!}x^{n-1} + \sum_{j=2}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F x^{n-j} + F_n x^{n-1} \\
&= x^n + \sum_{j=2}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F x^{n-j}
\end{aligned}
$$

Thus, we obtained;

$$
B_n^F(x) + F_n x^{n-1} = H_n(x) = x^n + \sum_{j=2}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_F b_j^F x^{n-j} \tag{C.3}
$$