INJECTIVE MODULES AND THEIR GENERALIZATIONS

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ABSTRACT

INJECTIVE MODULES AND THEIR GENERALIZATIONS

The main goal of this thesis is to give a survey about some different generalizations of injective modules, namely, C1, C2, C3-conditions and the modules which satisfy the simple versions of these conditions. A right R-module *M* is called simple-direct-injective if every simple submodule which is isomorphic to a direct summand of M is itself a summand, or if the direct sum of any two simple summands whose intersection is zero is a direct summand of M. Firstly, various basic properties and some characterizations of these modules are presented. The relation between simple-direct-injective modules and C3-modules is exhibited. Also, we obtain the structure of simple-direct-injective modules over the ring of integers and over semilocal rings. It is shown that over a commutative ring every nonsingular module is simple-direct-injective.

ÖZET

İNJEKTİF MODÜLLER VE GENELLEŞTİRMELERİ

Bu tezin temel amacı C1,C2, C3 şartları ve bu şartların basit versiyonlarını sağlayan injektif modüllerin bazı farklı genelleştirmeleri hakkında araştırma yapmaktır. Bir M sağ R-modülünün direkt toplananına izomorf olan her basit alt modülü yine M'nin direkt toplananı ise ya da M'nin kesişimleri sıfır olan iki basit direkt toplananının direkt toplamı M'nin direkt toplananı ise, M'ye basit-direkt-injektif modül denir. Öncelikle bu modüllerin çeşitli temel özellikleri ve bazı karakterizasyonları verilmiştir. Basit-direkt-injektif modüllerle C3-modüller arasındaki ilişki gösterilmiştir. Ayrıca basit-direkt-injektif modüllerin tam sayılar ve yarıyerel halkalar üzerindeki yapısı elde edilmiştir. Değişmeli halkalar üzerinde her singuler olmayan modülün basit-direkt-injektif olduğu gösterilmiştir.

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LIST OF ABBREVIATIONS

R	an associative ring with unit unless otherwise stated
$\mathbb{Z}$, $\mathbb{Z}^+$	the ring of integers, the ring of positive integers
$\mathbb{Q}$	the field of rational numbers
Z_{p^∞}	the Prüfer p-group for any prime p
$Mod-R$	the category of *right* R-modules
$Hom_R(M,N)$	all R-module homomorphisms from M to N
$\oplus_{i\in I}M_i$	direct sum of R-modules M_i
$\prod_{i\in I} M_i$	direct product of R-modules M_i
$Kerf$	the kernel of the map f
Imf	the image of the map f
$SocM$	the socle of the R-module M
$RadM$	the radical of the R-module M
$E(M)$	the injective envelope (hull) of a module M
$Z(M)$	the singular submodule of a module M
$\ll$	small submodule
$\subseteq^{ess}$	essential submodule
$\subseteq^{\oplus}$	direct summand
$ann_r(X)$	the *right* annihilator of a subset X of a *right* R-module M
$\cong$	isomorphic

CHAPTER 1

INTRODUCTION

Throughout this thesis, rings are associative with unity and modules are unitary right R-modules. Let M be an R-module. If every submodule of M is essential in a direct summand of M, then M satisfies the C1-condition. If M is C1-module, then every complement submodule of M is a direct summand, so it is also called CS-module (Nicholson and Yousif, 2003). If every submodule which is isomorphic to a direct summand of M is itself a summand, then M satisfies the C2-condition and if, for direct summands A and B of M with $A \cap B = 0$, $A \oplus B \subseteq^{\oplus} M$, then M satisfies the C3-condition. If, for a submodule A of M, every map $f : A \to M$ extends to an endomorphism of M, then M is called quasi-injective. Every quasi-injective module satisfies the C1, C2 and C3-conditions. Also, C1, C2 or C3-modules need not be closed under direct sum (See Example 4.1) (Mohamed and Müller, 1990). (Nicholson and Yousif, 2003).

In this thesis, we study a recent generalization of C2-modules which is introduced and studied in (Camillo, Ibrahim, Yousif and Zhou, 2014). A right R-module M is called simple-direct-injective if every simple submodule which is isomorphic to a summand of M is a summand, or if the direct sum of any two simple summands whose intersection is zero is a summand of M. Our aim is to work on the concept of simple-direct-injective modules and investigate the rings and modules that can be characterized via these modules. The notions of these modules were introduced and studied by (Camillo, Ibrahim, Yousif and Zhou, 2014).

Chapter 2 deals with the definitions of some basic concepts and some of their properties which is needed for our further studies. In this chapter, the studies of (Hazewinkel, Gubareni and Kirichenko, 2004), (Nicholson and Yousif, 2003), (Anderson and Fuller, 1992) and (Rotman, 1979) is utilized.

In the third chapter, the definitions of M-injective modules and the injective hull are given. Also, we define quasi-injective modules and outline some of their characterizations. One of the most important characterizations of quasi-injective modules is that, a module M is quasi-injective if and only if $f(M) \subseteq M$ for every $f \in End(E(M))$, (see, (Mohamed and Müller, 1990)).

In chapter 4, we study the C1, C2 and C3-conditions and their properties. An R-module M is called continuous if it satisfies the C1 and C2-conditions, and is called quasi-continuous if it satisfies the C1 and C3-conditions. It is known that

$$injective \Rightarrow quasi - injective \Rightarrow continuous \Rightarrow quasi - continuous \Rightarrow CS - module$$

and that the opposite direction of this implications is not always true (Nicholson and Yousif, 2003).

In the final chapter, the simple versions of C2 and C3-conditions are presented (Min-C2 and Min-C3 (Nicholson and Yousif, 2003)). Some basic properties of these modules are studied. Several characterizations of these modules appear in (Camillo, Ibrahim, Yousif and Zhou, 2014). Although some characterizations of simple-direct-injective modules are known, there is not much about their structure over particular rings. We investigate the structure of the simple-direct-injective abelian groups. We show that every nonsingular module over a commutative ring is simple-direct-injective. Also, we give a characterization of simple-direct-injective modules over semilocal rings.

CHAPTER 2

PRELIMINARIES

In the first section, we start with necessary concepts. Also, by a ring, we mean associative ring with unity in our work.

2.1. Modules, Submodules and Module Homomorphisms

Definition 2.1 *An abelian group M is called* ***right R-module*** *if there is a map $f : MxR \to M$, by $(m, r) \mapsto mr$ where $m \in M$ and $r \in R$, satisfying the following axioms for all $1, r, r' \in R$ and for all $m, m' \in M$:*

1. $(m + m')r = mr + mr'$
2. $m(r + r') = mr + mr'$
3. $m(rr') = (mr)r'$
4. $m \cdot 1 = m$

and denoted by M_R.

In a similar way, we can define the concept of left R-module. In this thesis, our modules are always right R-modules and we will use "R-modules" or "modules" by right R-modules.

Example 2.1 ***1****- Any ring R is a module over itself.*
2- *Let I be a right ideal of the ring R; then I is a right R-module.*
3- *Let $M = M_{mn}(R)$ be the set of all mxn matrices with entries in R. Then M is an R-module with matrix addition and matrix multiplication.*
4-*If R is any ring, then R^n, the set of all n-tuples with components in R, is an R-module, with the usual addition and the scalar multiplication.*
5- *Every abelian group is a Z-module.*

A **submodule** K of an R-module M is a subgroup which is closed under scalar multiplication: $k \in K \Rightarrow kr \in K$ for all $r \in R$. Also, a nonzero submodule S of M is called **simple**

if it has only submodules 0 and S. If, for a nonzero submodule N of M and for any submodule N' with $N' \subseteq N$, either $N = N'$ or $N' = 0$, then N is called a **minimal** submodule of M. A proper submodule K of M is called **maximal** if, for a proper submodule K of M and for any submodule K' with $K \subseteq K'$, either $K = K'$ or $K' = M$.
For any right R-module M, the **right annihilator** of M is defined to be

$$ann_r(M) = \{r \in R : Mr = 0\}.$$

Proposition 2.1 *((Anderson and Fuller, 1992), Proposition 2.14) Let M be a left R-module and X be a subset of M. Then $ann_r(X)$ is a left ideal of R. Moreover, if X is a submodule of M, then $ann_r(X)$ is an ideal of R.*

Definition 2.2 *A **homomorphism** from a right R-module M to a right R-module N is a map $f : M \to N$ satisfying the following conditions:*

1. *$f(m_1 + m_2) = f(m_1) + f(m_2)$ for all $m_1, m_2 \in M$.*

2. *$f(mr) = f(m)r$ for all $m \in M$, $r \in R$.*

Let $f : M \to M'$ be an R-module homomorphism. If f is one-to-one, then f is called **monomorphism** and if f is onto, then f is called **epimorphism**. A one-to-one and onto module homomorphism is called **isomorphism**. The set

$$Ker f = \{m \in M \mid f(m) = 0\}$$

is a submodule of M and is called the **Kernel** of the homomorphism f. The **image** of the homomorphism f is the set

$$Im f = \{f(m) \mid m \in M\}$$

and it is a submodule of M'.
Let K be a submodule of an R-module M. Consider the set M/K of equivalence classes

$m + K, m \in M$. An R-module M/K is called **quotient module** of M defining by

$$\begin{aligned}(m+K)+(m'+K) &= (m+m')+K,\\ r(m+K) &= rm+K.\end{aligned}$$

Also, there is a natural map $\pi : M \to M/K$ by $\pi(m) = m+K$. Clearly, it is an epimorphism and is called **natural projection** of M onto M/K.
Let M and N be R-modules. All R-homomorphisms from M to N form an additive group and denoted by $Hom_R(M, N)$.

Proposition 2.2 *((Anderson and Fuller, 1992), Proposition 4.1) If M and N are left R-modules, then $Hom_R(M, N)$ is an abelian group with respect to the operation of addition $(f, g) \mapsto f + g$ defined by*

$$(f + g)(x) = f(x) + g(x)(x \in M).$$

Also, the endomorphisms of an R-module M is denoted by $End_R(M)$.

Definition 2.3 *Let M be an R-module. A submodule N of M is called **fully invariant** if $f(N) \subseteq N$ for every $f \in End_R(M)$.*

Corollary 2.1 *((Anderson and Fuller, 1992), Corollary 3.7) Let M and N be right R-modules.*

1. *If $f : M \to N$ is an epimorphism with $Kerf = K$, then there is a unique isomorphism $\eta : M/K \to N$ such that $\eta(m + K) = f(m)$ for all $m \in M$.*
2. *If $K \subseteq L \subseteq M$, then $M/L \cong (M/K)/(L/K)$.*
3. *If $H \subseteq M$ and $K \subseteq M$, then $(H + K)/K \cong H/(H \cap K)$*

Theorem 2.1 *((Anderson and Fuller, 1992), Theorem 3.6, The Factor Theorem) Let M, M', N and N' be R-modules and let $f : M \to N$ be an R-homomorphism.*

1. *If $g : M \to M'$ is an epimorphism with $Ker(g) \subseteq Ker(f)$, then there exists a unique homomorphism $h : M' \to N$ such that*

$$f = hg.$$

Moreover, $Kerh = g(Kerf)$ *and* $Im(h) = Im(f)$*, so that* h *is monic if and only if* $Ker(g) = Ker(f)$ *and* h *is epic if and only if* f *is epic.*

2. *If* $g : N' \to N$ *is a monomorphism with* $Imf \subseteq Img$*, then there exists a unique homomorphism* $h : M \to N'$ *such that*

$$f = gh.$$

Moreover, $Kerh = Kerf$ *and* $Im(h) = g^{\leftarrow}Im(f)$*, so that* h *is monic if and only if* f *is monic and* h *is epic if and only if* $Img = Imf$*.*

The following useful theorem is Modular Law.

Theorem 2.2 *((Hazewinkel, Gubareni and Kirichenko, 2004), Modular Law) Let* A*,* B *and* C *be submodules of* M *with* $B \subseteq A$*. Then* $A \cap (B + C) = B + (A \cap C)$*.*

2.2. Direct Product and Direct Sum

Now we introduce the concepts of direct product, direct sum and projection.

Definition 2.4 *Let* $\{M_i\}_{i\in A}$ *be a family of right* R*-modules. Then the cartesian product* $X_{i\in A}M_i$ *is a right module under the componentwise operations, that is, for* $(x_i), (y_i) \in X_{i\in A}M_i$ *and* $r \in R$

$$\begin{aligned} (x_i) + (y_i) &= (x_i + y_i), \\ (x_i)r &= (x_ir). \end{aligned}$$

The resulting module is called the ***direct product*** *of* $\{M_i\}_{i\in A}$ *and is denoted by* $\prod_A M_\alpha$*.*

If $M_\alpha = M$ for all $\alpha \in A$, then we use $M^A = \prod_A M$. The homomorphism $p_\alpha = \prod_A M_\alpha \to M_\alpha$ defined by $p_\alpha((x_\alpha)) = x_\alpha$ is the **projection** map on M_α.

Definition 2.5 *Let* M *be a right* R*-module and* $(M_\alpha)_{\alpha\in A}$ *a family of submodules of* M*. If* $M_\alpha \cap (\sum_{\beta\neq\alpha} M_\beta) = 0$ *for each* $\alpha \in A$*, then* $(M_\alpha)_{\alpha\in A}$ *is called* ***independent****. If* $(M_\alpha)_{\alpha\in A}$ *is an independent family of submodules of* M*, then we write* $\sum_A M_\alpha = \bigoplus_A M_\alpha$*. In addition, if* $M = \bigoplus_A M_\alpha$*, then* M *is called the* ***direct sum*** *of the family* $(M_\alpha)_{\alpha\in A}$*.*

Clearly, if A is a finite set, then $\bigoplus_A M_\alpha = \prod_A M_\alpha$.

Definition 2.6 *Let M be a nonzero R-module. M is called* ***indecomposable*** *if it cannot be written direct sum of nonzero two submodules of it.*

2.3. Exact Sequences and Functors

Now the next concept is exact sequence.

Definition 2.7 *Let $\{M_n \mid n \in \mathbb{Z}\}$ be a family of R-modules and the sequence*

$$\cdots \to M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \to \cdots$$

be R-homomorphisms. The sequence is said to be ***exact*** *at M_n if for all $n \in \mathbb{Z}$,*

$$Im(f_{n+1}) = Ker(f_n).$$

Proposition 2.3 *((Anderson and Fuller, 1992), Proposition 3.12) The sequence*

$$0 \to M \xrightarrow{f} N$$

is exact if and only if $Ker f = 0$; that is, if and only if f is a monomorphism. Also, the sequence

$$M \xrightarrow{g} N \to 0$$

is exact if and only if $Im g = N$; that is, if and only if g is an epimorphism.

Definition 2.8 *If the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact, that is f is monic and g is epic, such sequences are said to be* ***short exact sequences****. And, if there is a map $f' : B \to A$ such that $f'f = 1_A$, this sequence* ***splits****.*

Proposition 2.4 *((Hazewinkel, Gubareni and Kirichenko, 2004), Proposition 4.2.1) Let* $0 \to X \xrightarrow{f} M \xrightarrow{g} Y \to 0$ *be an exact sequence. Then the following statements are equivalent:*

1. *The sequence splits;*
2. *There exists a homomorphism* $g' : Y \to M$ *such that* $gg' = 1_Y$*;*
3. *There exists a homomorphism* $f' : M \to X$ *such that* $f'f = 1_X$*;*
4. $M \cong X \oplus Y$.

Recall that $Hom_R(A, B)$ forms an additive abelian group. Now we will give the definition of functor, and then we will show that $Hom_R(-, M)$ is a contravariant functor from the category **Mod-R** of right R-modules to the category **Ab** of abelian groups for each right R-module M,.

Definition 2.9 *If there are defined:*

1. *An* ***Objects Class*** *ObC, whose elements are called objects;*
2. *A* ***Morphisms Set*** *MorC, whose elements are called the morphisms;*
3. *for any morphism* $f \in MorC$ *there is an ordered pair of objects* (X, Y) *of the category C (* f *is a morphism from an object X to an object Y and write* $f : X \to Y$*);*
4. *for any ordered triple* $X, Y, Z \in ObC$ *and any pair of morphisms* $f : X \to Y$ *and* $g : Y \to Z$ *there is a uniquely defined morphism* $gf : X \to Z$*, which is called the composition or product of morphisms* f *and g;*
5. *composition of morphisms is associative,that is,* $h(gf) = (hg)f$ *for any morphisms* f*,g,h whose products are defined;*
6. *if* $X = X'$ *or* $Y = Y'$*, then* $Hom(X, Y)$ *and* $Hom(X', Y')$ *are disjoint sets;*
7. *for any object* $X \in ObC$ *there exists a morphism* $1_X \in Hom(X, X)$ *such that* $f \cdot 1_X = f$ *and* $1_X \cdot g = g$ *for any morphisms* $f : X \to Y$ *and* $g : Z \to X$*. (*1_X *is unique and is called the identity morphism of the object X).*

Example 2.2 ***Ab*** *is the category of Abelian groups. Ob**Ab** is the class of all Abelian groups.* $Hom(A, B)$ *is a set of all abelian group homomorphisms from A to B.* ***Mod-R*** *is the category of right R-modules. Ob**Mod-R** is the class of right R-modules.* $Mor_{Mod-R}(M, N)$ *is the set of all R-module homomorphisms from M to N.*

Definition 2.10 *(Hazewinkel, Gubareni and Kirichenko, 2004) Let M and N be categories. A **covariant functor** F is a pair of maps $F_{ob} : ObM \to ObN$ and $F_{mor} : MorM \to MorN$ satisfying the following conditions:*

1. *If $A, B \in ObM$, then to each morphism $f : A \to B$ in $MorM$ there corresponds a morphism $F_{mor}(f) : F_{ob}(A) \to F_{ob}(B)$ in $MorN$;*

2. *$F_{mor}(1_A) = 1_{Fob(A)}$ for all $A \in ObM$;*

3. *If the product of morphisms gf is defined in M, then*

$$F_{mor}(gf) = F_{mor}(g)F_{mor}(f).$$

*A **contravariant functor** F is a pair of maps $F_{ob} : ObM \to ObN$ and $F_{mor} : MorM \to MorN$ satisfying the following conditions:*

1. *If $A, B \in ObM$, then to each morphism $f : A \to B$ in $MorM$ there corresponds a morphism $F_{mor}(f) : F_{ob}(B) \to F_{ob}(A)$ in $MorN$;*

2. *$F_{mor}(1_A) = 1_{Fob(A)}$ for all $A \in ObM$;*

3. *If the product of morphisms gf is defined in M, then*

$$F_{mor}(gf) = F_{mor}(f)F_{mor}(g).$$

Definition 2.11 *A functor F is called **additive** if for any pair of morphisms $f_1 : A \to B$ and $f_2 : A \to B$ we have $F(f_1 + f_2) = F(f_1) + F(f_2)$.*

Definition 2.12 *Let F : **Mod-R** $\to$ **Ab** be a contravariant additive functor.*

1. *For any exact sequence of the form $\cdots \to 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \cdots$, if the sequence*

$$\cdots \to 0 \to F(C) \xrightarrow{F(g)} F(B) \xrightarrow{F(f)} F(A) \to 0 \cdots$$

*is exact, then F is called an **exact** functor.*

2. *For any exact sequence of the form* $\cdots \to 0 \to A \xrightarrow{f} B \xrightarrow{g} C$, *if the sequence*

$$\cdots \to 0 \to F(C) \xrightarrow{F(g)} F(B) \xrightarrow{F(f)} F(A)$$

is exact, then F *is called a* ***left exact*** *functor.*

3. *For any exact sequence of the form* $A \xrightarrow{f} B \xrightarrow{g} C \to 0 \cdots$, *if the sequence*

$$F(C) \xrightarrow{F(g)} F(B) \xrightarrow{F(f)} F(A) \to 0 \cdots$$

is exact, then F *is called a* ***right exact*** *functor.*

In a similar way, we can define exact functor for covariant functors.

To show that $Hom_R(-, M)$ is a contravariant functor from the category **Mod-R** to **Ab**, let A, B and C be right R-modules. If $f : A \to B$ is an R-homomorphism, then define $f^* = Hom(f, M) : Hom(B, M) \to Hom(A, M)$ by $f^*(g) = gf$ for any $g \in Hom(B, M)$. Thus, the composition is defined and $gf \in Hom(A, M)$. Since

$$f^*(g + h) = (g + h)(f) = gf + hf = f^*(g) + f^*(h),$$

where $g, h \in Hom(B, M)$, f^* is an homomorphism. If $h \in Hom(B, C)$, then

$$(fh)^*(g) = g(fh) = (f^*(g))h = h^*(f^*(g)) = h^* f^*(g).$$

Hence, $(fh)^* = h^* f^*$. Moreover, $(1_B)^* = 1_{Hom(B,M)}$. Also, since

$$(f + g)^*(h) = h(f + g) = hf + hg = f^*(h) + g^*(h),$$

where $h \in Hom(B, M)$, $Hom(-, M)$is a contravariant additive functor.

Proposition 2.5 *((Hazewinkel, Gubareni and Kirichenko, 2004), Proposition 4.3.3) The Hom functor is left exact in each variable.*

2.4. Essential and Small Submodules

Before going to the Injective Modules it is necessary to give definitions of essential and small submodules. Then we define a semisimple module and develop basic properties of it.

Definition 2.13 *Let K be a submodule of an R-module M. If $K \cap Y \neq 0$ for every nonzero submodule Y of M, then K is called* ***essential submodule*** *of M and denoted by $K \subseteq^{ess} M$. If $K + L \neq M$ for every proper submodule L of M, then K is called* ***small submodule*** *of M and denoted by $K \ll M$.*

Lemma 2.1 *((Nicholson and Yousif, 2003), Lemma 1.1) Let M be an R-module. The following conditions hold;*

1. *If $K \subseteq N \subseteq M$ then $K \subseteq^{ess} M$ if and only if $K \subseteq^{ess} N$ and $N \subseteq^{ess} M$.*
2. *$K \subseteq^{ess} N \subseteq M$ and $K' \subseteq^{ess} N' \subseteq M$ then $K \cap K' \subseteq^{ess} N \cap N'$.*
3. *If $f : M \to N$ is an R-homomorphism and $K \subseteq^{ess} N$, then $f^{-1}(K) \subseteq^{ess} M$.*
4. *Let $M = \oplus_{i \in I} M_i$ be a direct sum where $M_i \subseteq M$ for each i, and let $K_i \subseteq M_i$ for each i. Then $\oplus_{i \in I} K_i \subseteq^{ess} M$ if and only if $K_i \subseteq^{ess} M_i$ for each i.*

2.5. Socle and Radical of a Module

Let M be an R-module. The socle of the module M is characterized by

$$\begin{aligned} SocM &= \sum \{A \subseteq M : A \text{ is a simple submodule in } M\}, \\ &= \bigcap \{B \subseteq M : B \text{ is an essential submodule in } M\}. \end{aligned}$$

and the Jacobson radical of M is characterized by

$$\begin{aligned} RadM &= \bigcap \{B \subseteq M : B \text{ is a maximal submodule in } M\}, \\ &= \sum \{A \subseteq M : A \text{ is a small submodule in } M\} \end{aligned}$$

Proposition 2.6 *((Anderson and Fuller, 1992), Proposition 9.8) Let M and N be R-modules and $f : M \to N$ be an R-homomorphism. Then $f(SocM) \subseteq SocN$.*

Proposition 2.7 *((Anderson and Fuller, 1992), Proposition 9.14) Let M and N be R-modules and $f : M \to N$ be an R-homomorphism. Then $f(RadM) \subseteq RadN$.*

2.6. Finitely Generated and Finitely Cogenerated Modules

Definition 2.14 *An R-module M is called **finitely generated** in case for every set A of submodules of M that spans M, there is a finite set $F \subseteq A$ spans M, i.e., $\sum A = M \Rightarrow \sum F = M$ for some finite $F \subseteq A$. An R-module M is **finitely cogenerated** in case for every family of submodules $\{A_i : i \in I\}$ of M, $\bigcap_{i\in I} A_i \Rightarrow \bigcap_{i\in F} A_i$ for some finite subset $F \subseteq I$.*

Finitely generated and finitely cogenerated modules are determined by the radical and the socle,respectively.

Theorem 2.3 *((Anderson and Fuller, 1992), Theorem 10.4) Let M be a left R-module. Then*

1. *M is finitely generated if and only if $M/RadM$ is finitely generated and $RadM \ll M$.*
2. *M is finitely cogenerated if and only if $SocM$ is finitely cogenerated and $SocM \subseteq^{ess} M$.*

2.7. Semisimple Modules

Definition 2.15 *Let M be an R-module and $(S_\alpha)_{\alpha\in I}$ be an indexed set of simple submodules of M. If M is the direct sum of this set, then $M = \bigoplus_I A_\alpha$ is a semisimple decomposition of M. A module is called **semisimple** if it has a semisimple decomposition.*

Theorem 2.4 *((Rotman, 1979), Theorem 4.11) An R-module M is semisimple if and only if every submodule of M is a summand.*

Proposition 2.8 *((Anderson and Fuller, 1992), Proposition 10.15) Let M be an R-module. Then the following are equivalent:*

1. *$RadM = 0$ and M is artinian;*

2. *$RadM = 0$ and M is finitely cogenerated;*

3. *M is semisimple and finitely generated;*

4. *M is semisimple and noetherian;*

5. *M is the direct sum of a finite set of simple submodules.*

Corollary 2.2 *((Anderson and Fuller, 1992), Corollary 10.16) The following statements are equivalent for a semisimple module M:*

1. *M is artinian;*

2. *M is noetherian;*

3. *M is finitely generated;*

4. *M is finitely cogenerated.*

2.8. Noetherian and Artinian Modules

Now, we define noetherian and artinian modules.

Definition 2.16 *Let $\{A_i \mid i \in I\}$ be the family of submodules of an R-module M. If, for every chain $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$, there is an n such that $A_{n+i} = A_n$ $(i = 1, 2, \cdots)$, then M satisfies the ascending chain condition(ACC) and if, for every chain $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$, there is an n such that $A_{n+i} = A_n$ $(i = 1, 2, \cdots)$, then M satisfies the descending chain condition(DCC).*
*A module M is said to be **noetherian** if every family of submodules satisfies the ACC and a module is said to be **artinian** if every family of submodules satisfies the DCC.*

Proposition 2.9 *((Anderson and Fuller, 1992), Proposition 10.9) For a module M, the following statements are equivalent;*

1. *M is noetherian;*

2. *Every submodule of M is finitely generated;*

3. *Every non-empty set of submodules of M has maximal element.*

Proposition 2.10 *((Anderson and Fuller, 1992), Proposition 10.10) For a module M the following statements are equivalent;*

1. M is artinian;

2. Every factor module of M is finitely cogenerated;

3. Every non-empty set of submodules of M has minimal element.

2.9. Complement Submodules And Closure

Before ending this chapter, we will give some definitions and characterizations;

Definition 2.17 *Let A be a submodule of an R-module M. A submodule C of M is called a* ***complement*** *of A if C is maximal with respect to $A \cap C = 0$ (It exists by Zorn's Lemma). Also, a submodule C of M is called* ***closed*** *in M if C is complement of some submodule of M.*

Now, we continue with essential lemma, and then the characterization of closed submodules.

Lemma 2.2 *((Nicholson and Yousif, 2003), Lemma 1.7) Let $A \subseteq M$ and C be any complement of A. Then $A \oplus C \subseteq^{ess} M$.*

Proof Let B be a nonzero submodule of M. Our claim is $(A \oplus C) \cap B \neq 0$. If $B \subseteq C$, our claim holds. If $B \nsubseteq C$, $A \cap (B + C) \neq 0$ by maximality of C. Let $0 \neq a = b + c \in A \cap (B + C)$. Then $0 \neq a + (-c) = b$. Hence $0 \neq b \in (A \oplus C) \cap B$, where $A \cap C = 0$. □

Proposition 2.11 *((Nicholson and Yousif, 2003), Proposition 1.27) Let C be a submodule of an R-module M. The following are equivalent;*

1. C is closed in M.

2. If $C \subseteq^{ess} K \subseteq M$ then $C = K$.

3. If $C \subseteq K \subseteq^{ess} M$ then $K/C \subseteq^{ess} M/C$.

4. If N is any complement of C in M then C is a complement of N in M.

Proof (1) $\Rightarrow$ (2): Let C be a complement of A and $C \subseteq^{ess} K \subseteq M$. Our claim is $A \cap K = 0$ (By maximality of C, $C = K$). If $A \cap K \neq 0$, then $C \cap (A \cap K) \neq 0$ since $C \subseteq^{ess} K$. So $C \cap A \neq 0$ which is a contradiction by the definition of complement. Hence $A \cap K = 0$.
(2) $\Rightarrow$ (3): We will prove by contradiction. Let $C \subseteq K \subseteq^{ess} M$ and suppose $K/C \cap X/C =$

0, where $0 \neq X/C \subseteq M/C$. Since $X/C \neq 0$ and by our hypothesis, C is not essential in X. Now, let $C \cap X' = 0$ where $0 \neq X' \subseteq X$. Hence

$$K \cap X' = K \cap (X' \cap X) = (K \cap X) \cap X' = C \cap X' = 0.$$

But since $K \subseteq^{ess} M$, there is a contradiction. So $K/C \subseteq^{ess} M/C$.

(3) $\Rightarrow$ (4): Let N be a complement of C in M and $N \cap T = 0$, where $C \subseteq T$. Our claim is $C = T$. Since N is complement of C, $N \oplus C \subseteq^{ess} M$ by the Lemma 2.2. Then by our hypothesis, $(N \oplus C)/C \subseteq^{ess} M/C$. Now, we need to show that $(N \oplus C)/C \cap (T/C) = 0$. Let $n + C = t + C \in (N \oplus C)/C \cap (T/C)$, where $n \in N$ and $t \in T$. Then $n - t \in C \subseteq T$. Hence $n \in (N \cap T) = 0$ and so $n = 0$. Then $t \in C$. Therefore $C = T$.

(4) $\Rightarrow$ (1): Clear. $\square$

Let A be a proper submodule of an R-module M. There are maximal submodules C in M with respect to $A \subseteq^{ess} C \subseteq M$ by Zorn's Lemma and these maximal essential extensions are closed by the Proposition 2.11 and is called **closures** of A in M.

CHAPTER 3

INJECTIVES

Injective modules play a central role in this thesis. In this chapter, we consider injective modules and present their important characterizations.

3.1. Injective Modules

Definition 3.1 *A right R-module E is called* ***injective*** *if for every R-module monomorphism $f : K \to N$ and R-module homomorphism $g : K \to E$ there exists an R-homomorphism $h : N \to E$ such that $h \circ f = g$. In other words; the following diagram commutes.*

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "g"] & N \arrow[dl, dashed, "h"] \\ & E & \end{tikzcd}$$

Theorem 3.1 *((Rotman, 1979), Theorem 3.17) If $\{E_j : j \in J\}$ is a family of injective modules, then $\prod E_j$ is injective.*

Proof Let i_j and p_j be the injections and projections of the product $\prod E_j$, respectively. Consider the diagram

$$\begin{tikzcd} & \prod E_j \arrow[r, shift left, "p_j"] & E_j \arrow[l, shift left, "i_j"] \\ 0 \arrow[r] & A \arrow[u, "f"] \arrow[r, "\alpha"] & B \arrow[u, "g_j"'] \end{tikzcd}$$

Since E_j is injective, there is a map $g_j : B \to E_j$ with $g_j\alpha = p_j f$. Define $h : B \to \prod E_j$ by $b \mapsto (g_j b)$. Then

$$h\alpha a = (g_j\alpha a) = p_j f a = f a,$$

so that $h\alpha = f$ and $\prod E_j$ is injective. □

Theorem 3.2 *((Rotman, 1979), Theorem 3.18) Every direct summand K of an injective module is injective.*

Proof Consider the diagram

$$\begin{tikzcd}
 & K \arrow[r, shift left, "i"] & E \arrow[l, shift left, "p"] \\
0 \arrow[r] & A \arrow[u, "f"] \arrow[r, "\alpha"] & B \arrow[u, "g"']
\end{tikzcd}$$

where i and p are injection and projection, respectively. Since E is injective, there is a map $g : B \to E$ with $g\alpha = if$. Define $h : B \to K$ by $h = pg$. Then

$$h\alpha = pg\alpha = pif = f$$

since $pi = 1_K$. Thus, K is injective. □

It can be said that every direct summand of injective R-module is injective, and also a direct product of injective right R-modules is injective. But it is not true that every direct sum of injective modules is injective.

Proposition 3.1 *((Anderson and Fuller, 1992), Proposition 18.13) For a ring R, the following statements are equivalent:*

1. *Every direct sum of injective right R-modules is injective;*
2. *R is a right noetherian ring.*

Lemma 3.1 *((Nicholson and Yousif, 2003), Lemma 1.2) Let E be an R-module. Then E is injective if and only if, for $N \subseteq M$, every R-homomorphism $f : N \to E$ extends to an R-homomorphism $g : M \to E$.*

Proof Let $f : K \to M$ be an R-module monomorphism and $g : K \to E$ an R-module homomorphism. Then $f' : f(K) \to K$ is well defined by $f'(f(k)) = k$ for $k \in K$. By hypothesis, the map $gf' : f(K) \to E$ extends to $h : M \to E$ and $hf = g$. Conversely, it is clear by the definition of injective module. □

Lemma 3.2 *((Nicholson and Yousif, 2003), Lemma 1.4, Baer Criterion) A right R-module E is injective if and only if, whenever $T \subseteq R$ is a right ideal, every map $\gamma : T \to E$ extends to $R \to E$.*

Proof Necessity is clear. For sufficiency, let K be a submodule of an R-module M and $\beta : K \to E$ an R-homomorphism. Let $\Gamma = \{(K',\beta') : K \subseteq K' \subseteq M \text{ and } \beta'|_K = \beta\}$. Since $(K,\beta) \in \Gamma$, $\Gamma \neq 0$. Let $(K_i,\beta_i)_{i\in I}$ be a chain of Γ for an index set I. Let $L = \bigcup_I K_i$ and $\beta' : L \to E$ the map defined by $\beta'(k) = \beta_i(k)$ provided $k \in K_i$. Clearly, $\beta'|_K = \beta$. Therefore $(L,\beta') \in \Gamma$ is an upper bound for $(K_i,\beta_i)_{i\in I}$. Hence, by Zorn's Lemma, let (K'',β'') be a maximal element of Γ. Consider the diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d, "\beta"] & K'' \arrow[r] \arrow[dl, "\beta''"] & M \\ E & & \end{tikzcd}$$

We must show that $K'' = M$. If not, let $m \in M - K''$ and $T = \{r \in R : mr \in K''\}$, a right ideal, and define $\lambda : T \to E$ by $\lambda(r) = \beta''(mr)$. By hypothesis, there is a $\hat{\lambda} : R \to E$ that extends λ. Now define $\hat{\beta} : K'' + mR \to E$ by $\hat{\beta}(y + mr) = \beta''(y) + \hat{\lambda}(r)$, where $y \in K''$ and $r \in R$. $\hat{\beta}$ is well defined because $y + mr = 0$ implies that $mr \in K''$. Also, $\hat{\beta}|_{K''} = \beta''$. Therefore $(K'' + mR, \hat{\beta}) \in \Gamma$. This contradicts with the maximality of (K'',β'') in Γ. As a consequence $K'' = M$, and so E is injective. □

Theorem 3.3 *((Rotman, 1979), Theorem 3.16) A module E is injective if and only if $Hom(-,E)$ is exact.*

Proposition 3.2 *((Hazewinkel, Gubareni and Kirichenko, 2004), Proposition 5.2.3) Let E be an injective R-module. Then every exact sequence of R-modules*

$$0 \to E \to M \to N \to 0$$

splits.

Proof Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, "i"] \arrow[d, "1_E"] & M \arrow[r] & N \arrow[r] & 0 \\ & E & & & \end{tikzcd}$$

where i is a monomorphism. Since E is injective, there exists a homomorphism $f : M \to E$ such that $if = 1_E$. Then $M \cong E \oplus N$ and the sequence splits by Proposition 2.4. $\square$

Definition 3.2 *A right R-module M is called **divisible** if, for all r in R which are not zero divisors, $Mr = M$.*

Theorem 3.4 *((Rotman, 1979), Theorem 3.23) Every injective module is divisible.*

Proof Let $m \in E$ and $r_0 \in R$ a non-zero divisor. Define $f : Rr_0 \to E$ by $f(rr_0) = rm$; note that f is well defined because r_0 is not a zero divisor. Since E is injective, there is a map $g : R \to E$ extending f. In particular,

$$m = f(r_0) = g(r_0) = r_0 g(1),$$

so that m is divisible by r_0. $\square$

Theorem 3.5 *((Rotman, 1979), Theorem 3.24) If R is a principal ideal domain, then an R-module D is divisible if and only if it is injective.*

Proof By Baer Criterion, it suffices to extend every map $f : I \to D$ to R, where I is an ideal of R. Since R is a PID, we know $I = Rr_0$, $r_0 \in R$; clearly, we may assume $r_0 \neq 0$, and thus r_0 is not a zero divisor. Since D is divisible, there is an element $d \in D$ with $r_0 d = f(r_0)$. Define $g : R \to D$ by $r \mapsto rd$, and note that g extends f. $\square$

Example 3.1 *The additive group of the rational numbers $\mathbb{Q}$ and the Prüfer p-group Z_{p^∞} for any prime p are divisible $\mathbb{Z}$-modules. So they are injective by Theorem 3.5.*

Next lemma shows that divisible groups can be used to construct injective modules.

Lemma 3.3 *((Nicholson and Yousif, 2003), Lemma 1.5) The following conditions hold for any ring R;*

1. *If D is divisible group, then $Hom_{\mathbb{Z}}(R, D)$ is an injective right R-module.*
2. *Every right R-module M embeds in an injective right module.*

Proof

1. Write $Hom_{\mathbb{Z}}(R, D) = E_R$. If $f \in E$ and $a \in R$, E becomes a right R-module by $(f \cdot a)(r) = f(ar)$ for all $r \in R$. Let I be a right ideal of R and $g : I \to E_R$ an

R-homomorphism. Our claim is g extend to $R_R \to E_R$. Now, define $h : I \to D$ by $h(t) = [g(t)](1)$. Then h is a $\mathbb{Z}$-morphism since for $t_1, t_2 \in T$,

$$\begin{aligned} h(t_1 + t_2) &= [g(t_1 + t_2)](1) \\ &= [g(t_1) + g(t_2)](1) \\ &= [g(t_1)](1) + [g(t_2)](1) \\ &= h(t_1) + h(t_2). \end{aligned}$$

Since D is injective, there is an $\mathbb{Z}$-morphism $\hat{h} : R \to D$ extending h. Since $\hat{h} \in E_R$, define $\hat{g} : R \to E$ by $\hat{g}(a) = \hat{h} \cdot a$ for all $a \in R$. It is clear that $\hat{g}$ is an R-homomorphism. If $r \in R$, then

$$[\hat{g}(a)](r) = (\hat{h} \cdot a)(r) = \hat{h}(ar) = h(ar) = [g(ar)](1) = [g(a) \cdot r](1) = [g(a)](r)$$

since g is an isomorphism and $g(a) \in E_R$. Thus $\hat{g}$ extends g.

2. Let $\{X_\alpha\}_{\alpha \in I}$ be the set of generators for M. Then there is a group epimorphism $f : \mathbb{Z}^{(I)} \to M$. So ${}_{\mathbb{Z}}M \cong \mathbb{Z}^{(I)}/K \subseteq \mathbb{Q}^{(I)}/K$ where $K = Kerf$. Write $Q = \mathbb{Q}^{(I)}/K$ and note that Q is divisible. We have $M_R \cong Hom_R(R_R, M_R)$ by $m \mapsto m\cdot$,that is multiplication by an element $m \in M$. So

$$M_R \cong Hom_R(R_R, M_R) \subseteq Hom_{\mathbb{Z}}(R, M) \hookrightarrow Hom_{\mathbb{Z}}(R, Q).$$

Since $Hom_{\mathbb{Z}}(R, Q)$ is injective by (1), this proves (2).

□

Corollary 3.1 *((Hazewinkel, Gubareni and Kirichenko, 2004), Corollary 5.2.9) A module E is injective if and only if every exact sequence of the form*

$$0 \to E \to M \to N \to 0$$

splits.

Proof If E is injective, then the condition holds by Proposition 3.2. Conversely, let E be an R-module. By the Lemma 3.3, there exists an injective module M which contains the module E, so there is a split exact sequence

$$0 \to E \to M \to M/E \to 0$$

by hypothesis. So $M \cong E \oplus M/E$. By Theorem 3.2, E is injective. □

Corollary 3.2 *((Hazewinkel, Gubareni and Kirichenko, 2004), Corollary 5.2.10) A module E is injective if and only if it is a direct summand of every module which contains it.*

Proof Assume E is injective and E is a submodule of a module M, then there is a split exact sequence $0 \to E \to M \to M/E \to 0$ by the Corollary 3.1. Then E is a direct summand of M. Conversely, let E be an arbitrary R-module, then, by Lemma 3.3, there exists an injective module M containing E. Then, by hypothesis, E is a direct summand of M, and so E is injective by Theorem 3.2. □

Recall that a submodule N of M is called essential in M if it has nonzero intersection with every nonzero submodule of M. We also say that M is an essential extension of N.

Theorem 3.6 *((Hazewinkel, Gubareni and Kirichenko, 2004), Theorem 5.3.3) A module M is injective if and only if it has no proper essential extensions.*

Proof Let M be an injective module and E' an essential extension of it. So M is a direct summand of E' by Corollary 3.2, i.e., $E' = M \oplus N$ where $M \cap N = 0$. If $N \neq 0$, then M has no essential extension, so $N = 0$ and $M = E'$. Conversely, suppose M has no proper essential extensions. By Lemma 3.3, there exists an injective module E' containing M. Consider the set

$$A = \{S \subseteq E' : S \cap M = 0\}.$$

Since $0 \in A$, $A \neq 0$. It is a partially ordered set with respect to the relation of subset inclusion. Then, by Zorn's Lemma, there exists a maximal element $N \subset E'$ in A. Then $M \cap N = 0$ and $M + N \subseteq E'$. Our claim is $M + N = E'$. Suppose $M + N \neq E'$, then $(M + N)/N \subset E'/N$ and $(M + N)/N \neq E'/N$. Let K/N be a nonzero submodule of E'/N. Then $N \subset K$ and $N \neq K$. Since N is maximal in A, $M \cap K \neq 0$. Thus $M \cap K \nsubseteq N$. So $N \subset K \cap (M + N)$, which means that $K/N \cap (M + N)/N \neq 0$ and so E'/N is an essential extension of $(M+N)/N$. Since $M \cong M/(M \cap N) \cong (M+N)/N$, M is essential in E'/N. Since, by hypothesis, M has no proper essential extensions, $E'/N = (M+N)/N$ and this implies $E' = M + N$. Thus $E' = M \oplus N$. So M is an injective module by Theorem 3.2. □

Definition 3.3 *If an R-module E' is called* ***injective hull*** *(or* ***injective envelope****) of an R-module M if it is both an injective module and essential extension of M, and denoted by $E' = E(M)$.*

Theorem 3.7 *((Rotman, 1979), Theorem 3.30) Let E' be an R-module and $M \subseteq E'$. The following conditions are equivalent :*

1. *E' is a maximal essential extension of M. (i.e., no proper extension of E' is an essential extension of M);*

2. *E' is an essential extension of M and E' is injective;*

3. *E' is injective and there is no injective E'' with $M \subset E'' \subsetneq E'$.*

Moreover, such a module E' exists.

3.2. Relative Injectivity

Definition 3.4 *Let E and M be right R-modules. E is called* ***M-injective*** *if for any submodule A of M, every right R-module homomorphism $f : A \to E$ can be extended to a right R-homomorphism $g : M \to E$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & M \arrow[dl, dashed, "g"] \\ E & \end{tikzcd}$$

commutes.

Note that a module E_R is injective in case it is M-injective for every module M. The following proposition gives some useful properties of M-injective modules with an R-module M.

Proposition 3.3 *((Nicholson and Yousif, 2003), Lemma 1.12) Let E be an M-injective module. If $K \subseteq M$, then E is K-injective and M/K-injective.*

Proof Let $N \subseteq K$ and $f : N \rightarrow E$ be an R-homomorphism. Given f extends to $g : M \rightarrow E$ since E is M-injective.

$$\begin{tikzcd} N \arrow[r, "i"] \arrow[d, "f"] & M \arrow[dl, dashed, "g"] \\ E & \end{tikzcd}$$

i.e $gi = f$.
Then the restriction $g|_K : K \rightarrow E$ extends f since for all $n \in N$, $g|_K(n) = g(n) = g(i(n)) = f(n)$. So E is K-injective.
To show that E is M/K-injective, let $N/K \subseteq M/K$ and $f : N/K \rightarrow E$ be an R-homomorphism where $K \subseteq N \subseteq M$. Let $p : N \rightarrow N/K$ and $p' : M \rightarrow M/K$ denote the natural epimorphisms. Since E is M-injective, there exist $\alpha : M \rightarrow E$ such that $\alpha|_N = fp$.

$$\begin{tikzcd} N \arrow[r] \arrow[d, "p"] & M \arrow[d, "p'"] \arrow[ddl, dashed, "\alpha"] \\ N/K \arrow[r] \arrow[d, "f"] & M/K \arrow[dl, dashed, "\beta"] \\ E & \end{tikzcd}$$

Since $\alpha(K) = fp(K) = f(0) = 0$, $Kerp' \subseteq Ker\alpha$. By Theorem 2.1, there exists $\beta : M/K \rightarrow E$ such that $\beta p' = \alpha$. Hence β extends f since for all $n \in N$,

$$\beta(n + K) = \beta(p'(n)) = \alpha(n) = fp(n) = f(n + K).$$

□

Lemma 3.4 *((Nicholson and Yousif, 2003), Lemma 1.11) Let $E = \prod_{i \in I} E_i$ be an R-module. Then E is M-injective if and only if E_i is M-injective for each $i \in I$.*

Proof Follows from Theorem 3.1 and 3.2. □

Proposition 3.4 *((Nicholson and Yousif, 2003), Lemma 1.13) Let E and $M = \bigoplus_{i \in I} M_i$ be R-modules. Then E is M-injective if and only if E is M_i-injective for each $i \in I$.*

Proof Only if part holds by Proposition 3.3. Conversely, suppose E is M_i-injective for each $i \in I$. Let $A \subseteq M$ and $f : A \to E$ be an R-homomorphism. Let Ω denote the set of pairs (A', f') with $A \subseteq A' \subseteq M$ and $f' : A' \to E$ extends f. Let (A'', f'') be maximal with $A \subseteq A'' \subseteq M$ and $f'' : A'' \to E$ extends f (i.e $f''|_A = f$) in Ω by Zorn's Lemma.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] & A'' \arrow[dl, dashed, "f''"] \\ E & \end{tikzcd}$$

Now we need to show that $A'' = M$. There exists $f_i = M_i \to E$ such that $f_i|_{M_i \cap A''} = f''|_{M_i \cap A''}$ by hypotesis.

$$\begin{tikzcd} M_i \cap A'' \arrow[r] \arrow[d, "f''|_{M_i \cap A''}"'] & M_i \arrow[dl, dashed, "f_i"] \\ E & \end{tikzcd}$$

Define $f_i' : M_i + A'' \to E$ by

$$f_i'(m_i + a) = f_i(m_i) + f''|_{M_i \cap A''}(a),$$

where $m_i \in M_i$ and $a \in A''$. Then f_i' is well defined : for $m_i, m_i' \in M_i$ and $a, a' \in A''$;

$$\begin{aligned} m_i + a &= m_i' + a' \\ m_i - m_i' &= a' - a \in M_i \cap A'' \\ f_i(m_i - m_i') &= f_i(a' - a) = f''|_{M_i \cap A''}(a' - a) \\ f_i(m_i) + f''|_{M_i \cap A''}(a) &= f_i(m_i') + f''|_{M_i \cap A''}(a') \\ f_i'(m_i + a) &= f_i'(m_i' + a') \end{aligned}$$

and also f_i' extends f : for all $a \in A$,

$$f_i'(a) = f_i'(0 + a) = f_i(0) + f''|_{M_i \cap A''}(a) = 0 + f(a) = f(a)$$

By the maximality of (A'', f''), $M_i + A'' = A''$ and so $M_i \subseteq A''$ for each i. Thus, $M = A''$. $\square$

The next lemma presents connection between M-injective modules and injective hull, and also it will give characterization of quasi-injective modules.

Lemma 3.5 *((Nicholson and Yousif, 2003), Lemma 1.14) Let M and N be R-modules. Then M is N-injective if and only if $\beta(N) \subseteq M$ for every $\beta : E(N) \to E(M)$.*

Proof If the condition holds, let $\alpha : K \to M$ be an R-homomorphism, where $K \subseteq N$.

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "\alpha"] & E(N) \arrow[ddl, dashed, "\beta"] \\
M \arrow[d, "i"] & \\
E(M) &
\end{tikzcd}$$

Since E(M) is injective, there exists $\beta : E(N) \to E(M)$ which extends α. By hypothesis, $\beta(N) \subseteq M$. Thus the restriction $\beta|_N : N \to M$ extends α. Hence, M is N-injective.
To show that only if part, let $\beta : E(N) \to E(M)$ be an R-homomorphism. Our claim is $\beta(N) \subseteq M$. Let $K = \{n \in N \mid \beta(n) \in M\}$.

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "\beta|_K"'] & N \arrow[r] \arrow[dl, dashed, "\alpha"] & E(N) \arrow[ddll, "\beta"] \\
M \arrow[d, "i"] & & \\
E(M) & &
\end{tikzcd}$$

Since M is N-injective, there exists $\alpha : N \to M$ extends $\beta|_K$. i.e $\alpha|_K = \beta|_K$.
Now we need to show that $(\beta - \alpha)(N) = 0$. Since M is essential in its injective hull, we can only show that $M \cap ((\beta - \alpha)(N)) = 0$. Let $m = (\beta - \alpha)(n)$ where $m \in M$ and $n \in N$. Then

$$\beta(n) = \alpha(n) + m \in M$$

since $\alpha(n) \in M$. Thus, $n \in K$. By the definition of α ,

$$\beta(n) = \alpha(n).$$

Lastly,

$$m = (\beta - \alpha)(n) = 0.$$

Therefore $\beta(N) \subseteq M$. □

Example 3.2 *Let B be an R-module. If B is semisimple, then every R-module A is B-injective.*

Proof Let K be a submodule of B and $f : K \to A$ be an R-homomorphism. By Theorem 2.8, K is a summand of B. Consider the projection $p : B \to K$ and the following diagram;

$$\begin{tikzcd} K \arrow[d, "f"] & B \arrow[l, "p"'] \\ A & \end{tikzcd}$$

So there exists $g : B \to A$ such that $g = fp$. Thus, if we take the inclusion map $i : K \hookrightarrow B$, clearly, g extends f. □

3.3. Quasi-Injective Modules

Definition 3.5 *Let M be a right R-module. M is said to be* ***quasi-injective*** *if M is M-injective, that is, for any submodule A of M if every map $f : A \to M$ extends to an endomorphisms of M.*

Corollary 3.3 *((Mohamed and Müller, 1990), Corollary 1.14) A module M is quasi-injective if and only if $f(M) \subseteq M$ for every $f \in End(E(M))$.*

Proof Follows from Lemma 3.5. □

Corollary 3.4 *((Mohamed and Müller, 1990), Corollary 1.15) Every module has a minimal quasi-injective extension, which is unique up to isomorphism.*

Example 3.3 1. *Every injective R-module is quasi injective.*

2. *Semisimple modules are quasi-injective.*

3. *Recall that $\mathbb{Z}_{p^\infty}$ is infinite p-group for any prime p whose subgroups are totally ordered by inclusion*

$$0 =< c_0 >\subset< c_1 >\subset< c_1 >\subset \cdots \subset< c_n >\subset \cdots$$

and every $< c_i >\cong \mathbb{Z}_{p^i}$ for each positive integer $i > 1$. $\mathbb{Z}_{p^n}$ is a quasi-injective module for any prime p and n an integer such that $n > 1$.

Proof *Let $< c_n >\subseteq \mathbb{Z}_{p^\infty}$ and $f \in End(\mathbb{Z}_{p^\infty})$ defined by $f(c_n) = a$. Then*

$$p^n a = p^n f(c_n) = f(p^n c_n) = f(0) = 0$$

Thus, $a \in< c_n >$. Hence $f(< c_n >) \subseteq< c_n >$. So $< c_n >\cong \mathbb{Z}_{p^n}$ is quasi injective. □

4. *By example (3), $\mathbb{Z}_{p^n}$ is a quasi injective module but not injective since it is not divisible $\mathbb{Z}$-module.*

Corollary 3.5 *((Nicholson and Yousif, 2003), Corollary 1.16) Let M be a quasi-injective module. If $E(M) = \oplus_{i\in I} A_i$, then $M = \oplus_{i\in I}(M \cap A_i)$.*

Now, we show that direct summands of quasi-injective modules are quasi-injective.

Lemma 3.6 *((Nicholson and Yousif, 2003), Lemma1.17) Let M be a quasi-injective module. Then every direct summand of M is also quasi-injective.*

Proof Suppose M is quasi-injective. Let $M = A \oplus B$ and $f : X \to A$ be R-homomorphism where $X \subseteq A$. Consider the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & M \arrow[ddl, dashed, "g"] \\
A \arrow[d, "i"] & \\
M &
\end{tikzcd}$$

Since M is quasi-injective, f extends to $g : M \to M$. Let $p : M \to A$ be the projection onto A with $Ker p = B$. Then $(pg)|_A$ extends f since for all $x \in X$,

$$(pg)|_A(x) = pg(x) = p(f(x)) = f(x).$$

So A is quasi-injective. □

Remark 3.1 *((Nicholson and Yousif, 2003)) The direct sum of quasi-injective modules need not be quasi-injective. For example; $\mathbb{Q}$ and $\mathbb{Z}_p$ are quasi-injective modules but $\mathbb{Q} \oplus \mathbb{Z}_p$ is not quasi-injective. Because if it is quasi-injective then $\mathbb{Q} \oplus \mathbb{Z}_p$ is $\mathbb{Q} \oplus \mathbb{Z}_p$-injective, and also $\mathbb{Q} \oplus \mathbb{Z}_p$ is $\mathbb{Q}$-injective by Proposition 3.4. Then, by Lemma 3.4, $\mathbb{Q}$ is $\mathbb{Z}_p$-injective but there is no nonzero map from $\mathbb{Q}$ to $\mathbb{Z}_p$.*

So next proposition states that there is a necessary condition for a direct sum $M = \oplus_{i \in I} M_i$ to be quasi-injective.

Proposition 3.5 *((Mohamed and Müller, 1990), Proposition 1.18) Let the direct sum $M = \oplus_{i \in I} M_i$ be an R-module. The following conditions are equivalent;*

1. *M is quasi-injective;*
2. *M_i is M_j-injective for all $i, j \in I$.*

CHAPTER 4

C1,C2 AND C3 CONDITIONS

In this chapter, we give three conditions and the definitions of continuous and quasi-continuous modules. After this, we look at some examples.
C1, C2 and C3 conditions for a right R-module M are as follows;

1. If every submodule of M is essential in a direct summand of M, then M satisfies the **C1-condition**.

2. If every submodule which is isomorphic to a summand of M is itself a direct summand, then M satisfies the **C2-condition**.

3. If $A \oplus B \subseteq^{\oplus} M$ for direct summands A and B of M with $A \cap B = 0$, then M satisfies the **C3-condition**.

The module is called C1-module (respectively, C2 or C3) if it satisfies the C1-condition (C2 or C3). A ring R is called right C1-ring (C2-ring or C3-ring,respectively) if the module R_R has the condition C1 (C2 or C3).

Proposition 4.1 *((Mohamed and Müller, 1990), Proposition 2.2) Let M be a right R-module. If M satisfies the C2-condition, then it satisfies the C3-condition.*

Proof Suppose M satisfies the C2-condition. Let A and B be direct summands of M with $A \cap B = 0$. Our claim is $A \oplus B \subseteq^{\oplus} M$. Say $M = A \oplus X$, where $X \subseteq M$, and let $p : M \to X$ be the projection onto X along A with $Kerp = A$. If $b \in B$ and $b = a + x$, where $a \in A$ and $x \in X$, then $p(b) = x$. So $b = a + p(b) \in A \oplus p(B)$. Hence

$$A \oplus B = A \oplus p(B).$$

Now, our new claim is $A \oplus p(B) \subseteq^{\oplus} M$. Consider the restriction $p|_B : B \to X$ of p. Since $Kerp|_B = B \cap Kerp = B \cap A = 0$, $p|_B$ is monic and $p|_B(B) = p(B) \cong B \subseteq^{\oplus} M$. So by hypothesis, $p(B) \subseteq^{\oplus} M$. Now,

$$X = X \cap M = X \cap (p(B) \oplus Y),$$

$Y \subseteq M$. By Theorem 2.2, $X = p(B) \oplus (X \cap Y)$. Hence

$$M = A \oplus X = A \oplus p(B) \oplus (X \cap Y).$$

$\square$

Remark 4.1 *(Nicholson and Yousif, 2003) Let M be an indecomposable right R-module.*

- *Clearly, M satisfies the C3-condition;*
- *M satisfies the C1-condition if and only if it is uniform;*

  Proof *If the condition holds, let $K \subseteq M$. Since M is indecomposable and uniform, $K \cap K' \neq 0$ for all nonzero $K' \subseteq M$. So K is essential in M. The converse is clear.* $\square$

- *M satisfies the C2-condition if and only if monomorphisms in End(M) are isomorphisms.*

  Proof *For only if part, let $f : M \to M$ be R-monic. So $M \cong f(M)$. By C2, $f(M) \subseteq^{\oplus} M$. Since M is indecomposable, $f(M) = M$. Conversely, if the condition holds, let A be a submodule that is isomorphic to a summand of M. Since M is indecomposable, $A \cong M$. Write $f : M \to A$ is an isomorphism. So $M \xrightarrow{f} A \xrightarrow{i} M$ where i is the inclusion. By hypothesis, if is an isomorphism. So i is epic. Hence $A = M$.* $\square$

Example 4.1 1. *$\mathbb{Z}$ satisfies the C1 and C3-conditions because $\mathbb{Z}$ is indecomposable and uniform as an abelian group. But it does not satisfy the C2-condition. For a submodule $2\mathbb{Z}$ of $\mathbb{Z}$, let $f : 2\mathbb{Z} \to \mathbb{Z}$ be an R-homomorphism defined by $f(2n) = n$, $n \in \mathbb{Z}$. It is clear that f is an isomorphism. Hence $2\mathbb{Z} \cong \mathbb{Z}$ but $2\mathbb{Z}$ is not a summand of M.*

2. *Direct sum of C1, C2 or C3-modules need not satisfy these conditions. For example, the $\mathbb{Z}$-module $\mathbb{Z}_2$ and $\mathbb{Z}_8$ are both satisfy the C1, C2 and C3-conditions. But the direct sum of $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ does not satisfy the C1 and C2-conditions.*

The following proposition is about injective modules.

Proposition 4.2 *((Nicholson and Yousif, 2003), Proposition 1.22) Quasi-injective modules satisfy C1 and C2-conditions.*

Proof Let M be a quasi-injective right R-module. For C1, let K be a submodule of M. Since $E(K)$ is injective, $E(M) = E(K) \oplus E(K')$ for some submodule K' of M. Since, by Corollary 3.5, $M = (M \cap E(K)) \oplus (M \cap E(K'))$. Then $K \subseteq^{ess} M \cap E(K)$. So M satisfies the C1-condition.

For C2, let $A \cong B \subseteq^{\oplus} M$. Then B is M-injective by Lemma 3.4, and hence A is M-injective. By the following diagram,

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "1_A"'] & M \arrow[dl, dashed, "f"] \\ A & \end{tikzcd}$$

the identity map 1_A extends to $f : M \to A$. Hence $M = \mathrm{Ker} f \oplus Imi = Kerf \oplus A$. So $A \subseteq^{\oplus} M$. $\square$

Definition 4.1 *An R-module is called* ***continuous*** *if it satisfies the C1 and C2-conditions, and an R-module is called* ***quasi-continuous*** *if it satisfies the C1 and C3-conditions.*

Example 4.2 *Recall that semisimple and injective modules are both quasi-injective, so they satisfy C1, C2 and C3-conditions. Hence, they are (quasi-) continuous modules. Since C2-condition implies C3-condition, every continuous module is quasi-continuous.*

Remark 4.2 *Let A be a direct summand of an R-module M and $A \subseteq^{ess} B \subseteq M$. Write $M = A \oplus A'$ where $A' \subseteq M$. Then*

$$M \cap B = (A \oplus A') \cap B = (B \cap A') \oplus A$$

by Theorem 2.2. Hence $B = (B \cap A') \oplus A$ and it follows that $A \subseteq^{\oplus} B$. Since A is a summand of B and it is essential in B, A = B. Therefore by Proposition 2.11 A is closed in M. Hence every direct summand of M is closed in M.

Let M satisfy the C1-condition and C be a closed submodule of M. Then C is essential in a summand of M by C1, and by Proposition 2.11, C is a summand of M. Conversely, let every closed submodule be a direct summand of M. Our claim is M holds C1-condition. Let A be a submodule of M and C be the closure of A in M. So C is closed and so a summand of M by hypothesis.(i.e $A \subseteq^{ess} C \subseteq^{\oplus} M$). Thus M satisfies C1. Hence a module which satisfies the C1-condition are called as ***CS-module****.*

Recall that direct sum of C1, C2 or C3-modules need not satisfy C1, C2 or C3-conditions, respectively. But direct summands of C1, C2 or C3-modules satisfy C1, C2 or C3-conditions, respectively. Now, we will show it in the following proposition.

Proposition 4.3 *Direct summands of a (quasi-) continuous modules are (quasi-) continuous.*

Proof Let M be a (quasi-) continuous module and $M' \subseteq^{\oplus} M$. Our claim is that M' satisfies the C1, C2 and C3-conditions. Note that M' is closed in M. Let A be closed in M'. Thus A is closed in M since closure is transitive. Hence A is a summand of M by C1, and also a summand of M'. So M' holds C1. The rest of the proof is clear. □

Now the following theorem gives the characterization of quasi-continuous modules.

Theorem 4.1 *((Nicholson and Yousif, 2003), Theorem 1.31) Let M be an R-module. The following conditions are equivalent;*

1. *M is quasi-continuous.*
2. *If C and N are complements of each other then $M = C \oplus N$.*
3. *$f(M) \subseteq M$ for every $f^2 = f \in End(E(M))$.*
4. *If $E(M) = \oplus_{i \in I} E_i$ then $M = \oplus_{i \in I}(M \cap E_i)$.*

The other important theorem is the following ;

Theorem 4.2 *((Nicholson and Yousif, 2003), Theorem 1.33) Let M be a direct sum of submodules M_i for $i = 1, 2, ..., n$. The following conditions are equivalent;*

1. *M is quasi-continuous.*
2. *Each M_i is quasi-continuous and M_i is M_j-injective for all $i \neq j$.*

In conclusion, we have this implication;

$$injective \Rightarrow quasi-injective \Rightarrow continuous \Rightarrow quasi-continuous \Rightarrow CS-module$$

But the opposite direction of this implication is not always true. Now, we have some examples about it.

Example 4.3 1. *$\mathbb{Z}_{p^n}$ is a quasi-injective module but not injective for any prime p and n an integer such that $n > 1$.*

2. *Recall that the integers $\mathbb{Z}$ satisfies the C1 and C3-conditions but not C2. So it is quasi-continuous but not continuous.*

3. *Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ where F is a field. Then R is right CS-ring but not right C2-ring so not continuous.*

   Proof $\begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, $\begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$, $\begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$ *and* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *are right ideals of R. The Jacobson radical $J(R)$ of R is $\begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} \cong e_{12}R \subseteq^{\oplus} R$, where e_{12} is the matrix unit. But $J(R)$ is not a direct summand of R, and so R is not right C2-ring. Now, let I be a nonzero right ideal of R. The $SocR$ of R is $\begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ and if $I \nsubseteq SocR$, then $I = e_{11}R$ or $I = R$. So $I = e_{11}R \subseteq^{\oplus} R$ or $I = R$. Thus I is a summand. If $I = SocR$, then I is essential in R since R is right artinian. Now, assume that $dim_F(I) = 1$. Let $I = aR$ where $a \in SocR$. If $a^2 = a \neq 0$, then $I = aR$ is a summand of R. Now, if $a \in J(R)$, then $I = aR = J(R)$ and $I = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} \subseteq^{ess} e_{11}R \subseteq^{\oplus} R$. So R is right C1-ring and so right CS-ring.* □

CHAPTER 5

SIMPLE-DIRECT-INJECTIVE MODULES

In this chapter, firstly, we give the definition of "mininjective module" and some properties of it, then we investigate the modules which satisfy the simple versions of C2 and C3-conditions.

5.1. Mininjective Modules

Definition 5.1 *A right ideal I of the ring R is called* ***extensive*** *if every R-homomorphism $\alpha : I \to R_R$ is extended to $\beta : R_R \to R_R$; that is, $\alpha = a\cdot$ is left multiplication by an element $a \in R$.*

Definition 5.2 *(Nicholson and Yousif, 2003) Let M be a right R-module. M is called* ***mininjective*** *if, for every simple right ideal I of the ring R, every R-homomorphism $f : I \to M_R$ extends to $g : R \to M_R$; that is, $f = m\cdot$ is multiplication by some $m \in M$.*

Clearly, R_R is right mininjective if and only if every simple right ideal I of R is extensive.

Definition 5.3 *(Camillo, Ibrahim, Yousif and Zhou, 2014) Let M and N be right R-modules. If for any simple submodule K of N, every R-homomorphism $f : K \to M$ extends to $g : N \to M$, then M is called* ***min-N-injective****. If M is min-M-injective, then it is called* ***min-quasi-injective****. M is called* ***mininjective*** *if it is min-R-injective.*

Lemma 5.1 *((Nicholson and Yousif, 2003), Lemma 2.1) Let R be a ring. Then the following statements are equivalent :*

1. *R is right mininjective.*
2. *If aR is simple for $a \in R$, then $lr(a) = Ra$.*
3. *If aR is simple for $a \in R$ and $r(a) \subseteq r(b)$ for $b \in R$, then $Rb \subseteq Ra$.*
4. *If aR is simple for $a \in R$ and $f : aR \to R$ is R-homomorphism, then $f(a) \in Ra$.*

Proof (1) $\Rightarrow$ (2): Suppose R is right mininjective. We will show that $lr(a) \subseteq Ra$. Let $a' \in lr(a)$. Then $a'r(a) = 0$ and so $r(a) \subseteq r(a')$. Thus $f : aR \to R$ is well defined by $f(ar) = a'r$. By our hypothesis, $f = c\cdot$ for some $c \in R$. Then $a' = f(a) = ca \in Ra$. Hence $lr(a) \subseteq Ra$. The other inclusion is always true.
(2) $\Rightarrow$ (3): Suppose (2) holds. Since $r(a) \subseteq r(b)$, $b \in lr(a) = Ra$. Thus $Rb \subseteq Ra$.
(3) $\Rightarrow$ (4): Given (3), let $f : aR \to R$ be an R-homomorphism. If $f(a) = b$, then $r(a) \subseteq r(b)$. So $b \in Ra$ by (3).
(4) $\Rightarrow$ (1): Let $f : aR \to R$ be an R-homomorphism. By our hypothesis, $f(a) \in Ra$. So $f(a) = ka$ where $k \in R$. Thus $f = k\cdot$ is left multiplication by some $k \in R$. □

The following proposition gives that "right mininjective rings" satisfy min-C2 and min-C3-conditions.

Proposition 5.1 *((Nicholson and Yousif, 2003), Proposition 2.18) Let R be a right mininjective ring.*

1. *(Min-C2) If I is a simple right ideal and I is isomorphic to a summand of R, then I is a summand of R.*
2. *(Min-C3) If I and I' are simple summands of R with $I \neq I'$, then $I \oplus I'$ is also a summand of R.*

Proof

1. Let $f : I \to eR$ be an isomorphism. Since R is right mininjective, $f = a\cdot$, $a \in R$. Then $aI = eR \nsubseteq J(R)$. Hence $I^2 \neq 0$ and I is a summand of R since I is simple.
2. Note that $eR \oplus fR = eR \oplus (1-e)fR$. If $(1-e)fR = 0$, then (2) holds. If not, then $(1-e)fR \cong fR$. By (1), $(1-e)fR = gR$, where $g^2 = g$. Now

$$eg = e(1-e)f = ef - e^2 f = ef - ef = 0$$

and

$$(e + g - ge)(e + g - ge) = e + g - eg.$$

Say $e + g - ge = h$ is an idempotent with $eh = e = he$ and $gh = g = hg$. So $eR \oplus fR = eR \oplus gR = hR \subseteq^{\oplus} R$.

□

5.2. Simple-Direct-Injective Modules

In this section, all of notions can be found in (Camillo, Ibrahim, Yousif and Zhou, 2014).

Definition 5.4 *A right R-module M is called* ***simple-direct-injective*** *if every simple submodule which is isomorphic to a summand of M is a summand, or equivalently, if the direct sum of any two simple summands whose intersection is zero is a summand of M.*

Example 5.1

1. *Every indecomposable module is simple-direct-injective, in particular $\mathbb{Z}_{\mathbb{Z}}$ is simple-direct-injective.*
2. *Every min-quasi-injective module is simple-direct-injective.*

   Proof *Follows from Proposition 4.2.* □

Proposition 5.2 *(Camillo, Ibrahim, Yousif and Zhou, 2014) Let M be a right R-module. The following statements are equivalent :*

1. *For any simple submodule K and simple summand N of M with $K \cong N$, $K \subseteq^{\oplus} M$.*
2. *For any simple summands K and N of M with $K \cap N = 0$, $K \oplus N \subseteq^{\oplus} M$.*
3. *If $M = K \oplus N$, where K is simple and $f : K \to N$ is an R-homomorphism, then $Imf \subseteq^{\oplus} N$.*

Proof (1) $\Rightarrow$ (2): By Proposition 4.1.
(2) $\Rightarrow$ (3): Assume (2) holds. Without loss of generality, we may assume that $f \neq 0$. So f is an R-monomorphism. Let $X = \{k + f(k) : k \in K\}$ be a submodule of M. Our claim is $M = X \oplus N$. For all $a \in M$, $a = k + n$, where $k \in K$ and $n \in N$. Then

$$a = k + f(k) - f(k) + n \in X + N.$$

Thus, $M = X + N$. Now we need to show that $X \cap N = 0$. Let $x \in X \cap N$. Then $x = k + f(k)$ for some $k \in K$, and so $k = x - f(k) \in K \cap N = 0$. Hence $x = 0$. Therefore $M = X \oplus N$. Now our new claim is $K \cap X = 0$. Let $y \in K \cap X$. Then $y = k' + f(k')$ for some $k' \in K$ and $y - k' = f(k') \in K \cap N = 0$. Since f is monomorphism, $k' = 0$, and so $y = 0$. Thus $K \cap X = 0$. Since $X \cong M/N \cong K$, X is simple. By (2), $K \oplus X \subseteq^{\oplus} M$. Our last claim is

$K \oplus X = K \oplus Imf$. Let $n \in Imf$, then $n = f(k)$ for some $k \in K$. So

$$n = -k + k + f(k) \in K + X.$$

Hence $K \oplus X = K \oplus Imf$, and so $Imf \subseteq^{\oplus} M$. Thus $Imf \subseteq^{\oplus} N$.

(3) $\Rightarrow$ (1): Suppose (3) holds. Let $f : K \to N$ be the isomorphism for a simple submodule K and a simple summand N of M. If $K \cap N \neq 0$, we are done. Otherwise, suppose that $K \cap N = 0$. Let $M = N \oplus X$ for some $X \subseteq M$ and $p : M \to X$ be the projection onto X. If, for all $k \in K$, $k = n + x$, where $n \in N$ and $x \in X$, then $p(k) = x$. So $k = n + p(k) \in N \oplus p(K)$. Thus $N \oplus K = N \oplus p(K)$. Consider the restriction $p|_K : K \to X$ of p. Since $K \cong p|_K(K) = p(K)$ and K is simple, $p(K)$ is simple. Then the composition map $p|_K f^{-1} : N \to X$ is a monomorphism since N is simple. Now

$$Im(p|_K f^{-1}) = p|_K(f^{-1}(N)) = p|_K(K) = p(K).$$

So $Im(p|_K f^{-1}) = p(K) \subseteq^{\oplus} X$ by hypothesis. Hence

$$M = N \oplus X = N \oplus p(K) \oplus Y = N \oplus K \oplus Y$$

for some $Y \subseteq M$. Therefore $K \subseteq^{\oplus} M$. □

Lemma 5.2 *(Camillo, Ibrahim, Yousif and Zhou, 2014) Let M be a simple-direct-injective module. Then:*

1. $\sum_{i=1}^{n} A_i \subseteq^{\oplus} M$ *for any finite set* $\{A_1, A_2, \cdots, A_n\}$ *of simple summands of* M.
2. *The sum of all simple summands of M is fully invariant in M.*

Proposition 5.3 *(Camillo, Ibrahim, Yousif and Zhou, 2014) Let M be a finitely generated module. If M is simple-direct-injective, then, for any semisimple submodules K and N of M with* $K \cong N \subseteq^{\oplus} M$, $K \subseteq^{\oplus} M$.

Proof K and N are both finitely generated for any semisimple submodule K, N of M with $K \cong N \subseteq^{\oplus} M$. So $K = \sum_{i=1}^{k} K_i$ and $N = \sum_{i=1}^{k} N_i$, where K_i and N_i are simple for each i. Since N is a summand of M and N_i is simple for each i, N_i is also a summand of M for each i. Write an isomorphism $f : K \to N$. Consider the restriction $f|_{K_i}$ of f. Since

$$Kerf|_{K_i} = Kerf \cap K_i = 0 \cap K_i = 0,$$

$f|_{K_i}$ is a monomorphism, and so $f|_{K_i}(K_i) = f(K_i) \cong K_i$, where K_i simple for each i. Hence $f(K_i) = N_i$ for some $i \in I$. Thus $K_i \cong N_i$. By Proposition 5.2, each K_i is a summand of M and by Lemma 5.2, K is a summand of M. □

Proposition 5.4 *(Camillo, Ibrahim, Yousif and Zhou, 2014) Let M be a module such that the sum of all simple summands is essential in a summand of M. Then M is simple-direct-injective if and only if $M = K \oplus N$ where $soc(K) \cap rad(K) = 0$, $soc(K)$ is fully invariant in M, and $soc(N) \subseteq rad(N)$.*

Definition 5.5 *Let M be an R-module. The direct sum $\bigoplus_{i \in I} A_i$ of simple submodules of M is called* ***locally simple summand*** *of M if there is a finite subset F of I such that $\bigoplus_{i \in F} A_i$ is a summand of M.*

Proposition 5.5 *(Camillo, Ibrahim, Yousif and Zhou, 2014) Suppose every locally simple summand of M is a summand. Then M is simple-direct-injective if and only if $M = K \oplus N$ where K is fully invariant semisimple submodule of M and $soc(N) \subseteq rad(N)$.*

Remark 5.1 *(Camillo, Ibrahim, Yousif and Zhou, 2014) Let M be an R-module and satisfy ascending chain condition(ACC) on summands. Then every locally simple summand of M is a direct sum of finitely many simple summands, thus it is a summand of M.*

The next corollary follows from the remark 5.1 and Proposition 5.5.

Corollary 5.1 *(Camillo, Ibrahim, Yousif and Zhou, 2014) Let M be an R-module and satisfy ascending chain condition(ACC) on summands. Then M is simple-direct-injective if and only if $M = K \oplus N$, where K is fully invariant semisimple submodule of M and $soc(N) \subseteq rad(N)$.*

Corollary 5.2 *(Camillo, Ibrahim, Yousif and Zhou, 2014) Let M be an R-module and satisfy ascending chain condition(ACC) on summands. If M is simple-direct-injective, then for any semisimple submodules K and N of M with $K \cong N \subseteq^{\oplus} M$, $K \subseteq^{\oplus} M$.*

Lemma 5.3 *(Camillo, Ibrahim, Yousif and Zhou, 2014) Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of right R-modules. Then a right R-module E is min-M-injective if and only if E is min-M_i-injective for each $i \in I$.*

Lemma 5.4 *(Camillo, Ibrahim, Yousif and Zhou, 2014) If M is simple-direct-injective right R-module, then every simple summand of M is min-M-injective.*

Proof Let K be a simple summand of M and $f : N \to K$ an R-homomorphism, where N is a simple submodule of M. We need to show that f extends to M. If $f = 0$, we are done. Otherwise, $N \cong K$. Since M is simple-direct-injective, N is also a summand of M. Let $p : M \to N$ be the projection onto N. Consider the diagram

$$\begin{tikzcd} N \arrow[d, "f"'] & M \arrow[l, "p"'] \arrow[dl, dashed, "g"] \\ K & \end{tikzcd}$$

so there is $g : M \to K$ such that $fp = g$. Also, it extends f. Thus K is min-M-injective. □

5.2.1. When do Simple-Direct-Injective Modules satisfy C3-Condition?

Now, we continue with the rings of which simple-direct-injective modules satisfy the C3-condition. We begin with the following lemma.

Lemma 5.5 *((Camillo, Ibrahim, Yousif and Zhou, 2014), Lemma 3.1) Any direct sum of injective modules is simple-direct-injective.*

Proof Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of injective modules and $K \cong N \subseteq^{\oplus} M$, where K and N are simple submodules of M. We need to show that $K \subseteq^{\oplus} M$. Since N is simple, $N \subseteq^{\oplus} (\bigoplus_{i \in F} M_i)$ for a finite subset F of I. Thus, N is injective. Since $K \cong N$, K is also injective and $K \subseteq^{\oplus} M$. □

Lemma 5.6 *((Camillo, Ibrahim, Yousif and Zhou, 2014), Lemma 3.2)*

1. *If $M = K \oplus N$ is a C3-module and $f : K \to N$ is an R-monomorphism, then $Imf \subseteq^{\oplus} N$.*

2. *If $M \oplus M$ is a C3-module, then M is a C2-module.*

Lemma 5.7 *If M is indecomposable module which is not simple, then $M \oplus E(M)$ is simple-direct-injective.*

Before giving the characterization of rings whose simple-direct-injective modules satisfy C3-condition, we need to give the definitions of uniserial modules and rings.

Definition 5.6 *A module is called **uniserial** if the lattice of its submodules is totally ordered under inclusion. A ring R is called left **uniserial** if ${}_RR$ is a uniserial module. A ring is called **serial** if both modules ${}_RR$ and R_R are direct sums of uniserial modules.*

Theorem 5.1 *((Camillo, Ibrahim, Yousif and Zhou, 2014), Theorem 3.4) Let R be a ring. The following conditions are equivalent:*

1. *Every simple-direct-injective right R-module is a C3-module.*
2. *Every simple-direct-injective right R-module is quasi-injective.*
3. *Every right R-module is a direct sum of a semisimple module and a family of injective uniserial modules of length 2.*
4. *Every right R-module is a direct sum of a semisimple module and an injective module.*
5. *R is an artinian serial ring with $J(R)^2 = 0$.*

5.2.2. When is Every Right R-Module Simple-Direct-Injective?

Definition 5.7 *A ring R is called right V**-ring** if every simple right R-module is injective.*

In the following proposition, the rings all of whose right modules simple-direct-injective are characterized.

Proposition 5.6 *((Camillo, Ibrahim, Yousif and Zhou, 2014), Proposition 4.1) Let R be a ring. The following statements are equivalent:*

1. *R is right V-ring.*
2. *Every right R-module is simple-direct-injective.*
3. *Every finitely cogenerated right R-module is simple-direct-injective.*
4. *Direct sums of simple-direct-injective modules are simple-direct-injective.*
5. *Every 2-genereted right R-module is simple-direct-injective.*

Definition 5.8 *A ring R is called* **von Neumann regular** *if for any $a \in R$, there exists $x \in R$ such that $a = axa$.*

From now on, by a regular ring, we mean a von Neumann regular ring.

Theorem 5.2 *((Camillo, Ibrahim, Yousif and Zhou, 2014), Theorem 4.4) A regular ring R is right V-ring if and only if every cyclic right R-module is simple-direct-injective.*

5.3. On the Structure of Simple-Direct-Injective Modules

The results that we have presented so far are related with the properties of simple-direct-injective modules over arbitrary rings. In this section we characterize simple-direct-injective modules over some particular rings. First, we shall give a characterization of simple-direct-injective modules over the ring of integers.

Definition 5.9 *An element a of a group G is* **divisible by n**, *denoted by $n|a$, if there is an element x in G such that $nx = a$ for integer n, or equivalently, if a belongs to nG.*

Definition 5.10 *A group G is called* **bounded** *if the orders of the elements of G remain under a fixed finite bound n (i.e. $nG = 0$ for integer n).*

Definition 5.11 *A subgroup S of a group G is called* **pure** *if the equation $nx = a \in S$ is solvable in S whenever it has a solution in G for every natural number n; that is, if $n|a$ in G implies $n|a$ in S.*

It is more convenient to express purity in the form of an equation: S is pure if and only if $nS = S \cap nG$ for every natural number n. (Fuchs, 1970)

Theorem 5.3 *((Fuchs, 1970), Theorem 27.5) A bounded pure subgroup is a direct summand.*

Lemma 5.8 *((Fuchs, 1970), Lemma 26.1) Let A, B be subgroups of G such that $A \subseteq B \subseteq G$. If A is pure in B and B is pure in G , then A is pure in G.*

Now let define the following; for an R-module M,

$$S' := \sum \{U \subseteq M \mid U \text{ is simple and } U \subseteq^{\oplus} M\}.$$

Theorem 5.4 *For an abelian group G, the following are equivalent:*

1. *G is simple-direct-injective.*

2. *S' is a fully invariant subgroup of G.*

3. *S' is a pure subgroup of G.*

Proof (1) ⇒ (2): Follows from Lemma 5.2 (1)
(2) ⇒ (3): Let U_1 and U_2 be simple subgroups of S'. So

$$G = U_1 \oplus K = U_2 \oplus K',$$

where $K, K' \subseteq G$. Our claim is $U_1 \oplus U_2 \subseteq^{\oplus} G$. Let $U_2 =< a >$, $a \in U_2$, and $a = b + c$ where $b \in U_1$ and $c \in K$. Let $p : G \to K$ be the projection onto K along U_1. Then $p(a) = c \in SocK$ by Proposition 2.6. Since S' is fully invariant in G, $p(U_2) =< c >\subsetneq S'$, where $< c >$ is simple. So $p(U_2)$ is a summand of G. Also, $U_1 \oplus U_2 = U_1 \oplus p(U_2)$. Hence we show that $U_1 \oplus p(U_2) \subseteq^{\oplus} G$. Since

$$\begin{aligned} K &= K \cap G \\ &= K \cap (p(U_2) \oplus X) \\ &= p(U_2) \oplus (K \cap X) \end{aligned}$$

by Modular Law, where $X \subseteq G$, $p(U_2) \subseteq^{\oplus} K$. Therefore $U_1 \oplus U_2 \subseteq^{\oplus} G$.
By induction every finitely generated subgroup of S' is a direct summand of G. To prove S' is a pure subgroup of G, let $n \in \mathbb{Z}^+$ and $x \in S' \cap nG$. Then there is a finitely generated submodule L of S' such that $x \in L$. So L is a direct summand of G. Therefore $x \in L \cap nG = nL \subseteq nS'$. Hence S' is a pure subgroup of G.
(3) ⇒ (1): Let A and B be simple direct summands of G with $A \cap B = 0$. So $A, B \subseteq S'$. We need to show that $A \oplus B \subseteq^{\oplus} G$. Since $A \oplus B$ is pure in S' and S' is pure in G, so $A \oplus B$ is pure in G by Lemma 5.8. Since $A \oplus B$ is pure and bounded, it is a direct summand of G by Theorem 5.3. □

An immediate consequence of Theorem 5.4 is the following.

Corollary 5.3 *A finitely generated abelian group G is simple-direct-injective if and only* *if*

$$G \cong \mathbb{Z}_{p_0} \oplus \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_n} \oplus \mathbb{Z}_{q_0^{k_0}} \oplus \cdots \oplus \mathbb{Z}_{q_t^{k_t}} \oplus \mathbb{Z}^m,$$

where $p_0, \cdots, p_n, q_0, \cdots q_t$ *are distinct prime integers,* $k_i = 0$ *or* $k_i \geq 2$ *for* $i = 0, \cdots, t$, *and* n, t, m *are positive integers.*

Definition 5.12 *A ring R is called* ***semilocal*** *if* $R/RadR$ *is a left artinian ring,or, equivalently* $R/RadR$ *is semisimple ring.*

By similar arguments as in the proof of Theorem 5.4, we have the following.

Theorem 5.5 *Let R be a semilocal ring. For a right R-module M, the following are equivalent:*

1. *M is simple-direct-injective.*
2. S' *is a fully invariant submodule of* M.
3. S' *is fully invariant pure submodule of* M.

Definition 5.13 *Let M be a right R-module. An element m in M is called* ***singular element*** *of M if the right ideal* $ann(m)$ *is essential in R. The set of all singular elements of M is called the* ***singular submodule*** *of M and denoted by* $Z(M)$. M_R *is called* ***singular****(respectively,* ***nonsingular****) module if* $Z(M) = M$ *(respectively,* $Z(M) = 0$*). (Lam, 1999)*

Note that a simple right module is either singular or projective. This fact will be used in the following theorem. The following theorem shows that, over commutative rings, every nonsingular module is simple-direct-injective.

Theorem 5.6 *Let R be a commutative ring. Then every nonsingular module is simple-direct-injective.*

Proof Let M be a nonsingular module. Let A and B be simple submodules of M such that A is a direct summand of M and $A \cong B$. Since M is nonsingular, A is nonsingular too. Thus A is projective, and so B is projective. Then B is injective by ((Ware, 1971), Lemma 2.6). Hence B is a direct summand of M. This proves that M is simple-direct-injective. □

CHAPTER 6

CONCLUSION

In this thesis, we study a recent generalization of C2-modules which is introduced and studied in (Camillo, Ibrahim, Yousif and Zhou, 2014). A right R-module *M* is called simple-direct-injective if every simple submodule which is isomorphic to a summand of *M* is a summand, or if the direct sum of any two simple summands whose intersection is zero is a summand of M. Although some characterizations of simple-direct-injective modules are known, there is not much about their structure over particular rings. We consider the structure of the simple-direct-injective abelian groups. The relation between simple-direct-injective modules and C3-modules is exhibited. We show that every nonsingular module over a commutative ring is simple-direct-injective. Also, we give a characterization of simple-direct-injective modules over semilocal rings.

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