THE DIRICHLET PROBLEM FOR THE FRACTIONAL LAPLACIAN

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ABSTRACT

THE DIRICHLET PROBLEM FOR THE FRACTIONAL LAPLACIAN

This thesis is an introduction to the fractional Sobolev spaces and the fractional Laplace operator. We define the fractional Sobolev spaces and give their properties by comparing them with the classical version of Sobolev spaces. After giving the motivation that comes from the random walk theory, we define the fractional Laplacian. We focus on the mean-value property of s-harmonic functions and get into details of extension and maximum principle of the weak solution of the Dirichlet problem for the fractional Laplacian. Afterall, we explain the regularity of the weak solution of the Dirichlet problem for the fractional Laplacian inside a domain and up to the boundary, respectively.

ÖZET

KESİRLİ LAPLASYAN İÇİN DIRICHLET PROBLEMİ

Bu tez kesirli Sobolev uzayları ve kesirli Laplas operatörü için bir tanıtımdır. Kesirli Sobolev uzayları tanımlanmış ve özellikleri, klasik Sobolev uzayları ile kıyaslanarak verilmiştir. Rassal yürüyüş teorisinden gelen motivasyon ile kesirli Laplasyan tanımlanmıştır. S-harmonik fonksiyonların ortalama-değer özelliği üzerinde durulmuş ve kesirli Laplasyan için Dirichlet probleminin zayıf çözümlerinin genişleme ve maksimum prensipleri detaylıca işlenmiştir. Tüm bu çalışmadan sonra, kesirli Laplasyan için Dirichlet probleminin zayıf çözümlerinin, sırasıyla tanım kümesinin iç kısmında ve kapanışındaki düzgünlüğü anlatılmıştır.

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CHAPTER 1

INTRODUCTION

The fractional version of the Laplace operator has become a very popular area in the partial differential equations recently. It appears in some probabilistic considerations, integro-differential equations, Levy process, conservation laws, finance, ultra-relativistic limits of quantum mechanics, multiple scattering, water waves, non-uniformly elliptic problems, anomalous diffusion, etc. We give an introduction to the fractional Sobolev spaces and the fractional Laplace operator in this thesis.

We study the fractional version of the Laplace operator by comparing it with the classical one. So, we start by introducing the properties of the weak solutions of Laplace's and Poisson's equations. We also give the definitions of Sobolev and Hölder spaces to understand the fractional Sobolev spaces and regularity of s-harmonic functions. We finish the preliminaries by giving some useful knowledge about real analysis and Fourier transform that we need when giving some results about the fractional Laplacian and the fractional Sobolev spaces.

In the next chapter, we start to construct the fractional Sobolev spaces and define the corresponding semi-norm and norm on a bounded domain respectively. The difference of the fractional version comes from the non-integer powers. So, when constructing the fractional Sobolev spaces and the norm we care about to coincide it with the classical version when the power is integer. After that we extend the space to $\mathbb{R}^n$ by using the similar process in the extension of the Sobolev spaces. Then we give some embedding and regularity results about the fractional Sobolev spaces. By realizing the magic choice $p = 2$ turns the space out to be a Hilbert space, we specialize the fractional Sobolev spaces as H^s and we continue by considering the functions in H^s as a Fourier transform. Then we point out that the results that come from the Fourier transform coincide with the basic definition of the fractional Sobolev spaces.

First aim of this study is constructing the fractional Laplacian and introduce the properties of the s-harmonic functions by comparing with the harmonic functions. Therefore, in Chapter 4, we make the fractional Laplacian appear with the Heuristic probabilistic motivation. After defining the fractional Laplace operator, we focus on the constant that appears in the definition and normalizes the integral to get the coincidence with the integer powers and in fact, to get the coincidence of the fractional and classical Laplace operators. Then we study the existence of the weak solution to the Dirichlet problem for the fractional Laplacian. After determining the conditions for the existence of the weak solution, we compare the results of the maximum and comparison principles, the mean value properties and Poisson kernel with the harmonic functions. Finally, we introduce the interior regularity of the weak solution in the given domain and then we extend this regularity up to the boundary.

CHAPTER 2

PRELIMINARIES

This chapter consists of some basic tools about the partial differential equations, Sobolev spaces, functional analysis, Fourier transform and distribution theory that we should know as a background to understand the given information in this thesis.

2.1. Laplace's Equation

We define, in this section, Laplace's and Poisson's equations and give their properties which will be compared with the fractional version in the last chapter. When defining Laplace's and Poisson's equations, Ω denotes an open set in $\mathbb{R}^n$, $x \in \Omega$ and the unknown function is $u : \overline{\Omega} \to \mathbb{R}$ with $u = u(x)$. The function $f : \Omega \to \mathbb{R}$ is given.

Now we can define Laplace's equation as

$$\Delta u = 0 \tag{2.1}$$

and Poisson's equation as

$$-\Delta u = f \tag{2.2}$$

where the *Laplacian* of u is $\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$.

Definition 2.1 *A C^2 function u satisfying Laplace's equation (2.1) is called a* harmonic *function.*

Definition 2.2 *The function*

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi} \log |x| & \textit{if } n = 2, \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & \textit{if } n \geq 3, \end{cases} \tag{2.3}$$

defined for $x \in \mathbb{R}^n$, $x \neq 0$ *is the* fundamental solution *of Laplace's equation, where* $\alpha(n)$ *denotes the volume of the unit ball in* $\mathbb{R}^n$.

Theorem 2.1 *(Evans, 2010) Define u by the convolution of the fundamental solution* Φ *of Laplace's equation and the given right hand side function* $f \in C_c^2(\mathbb{R}^n)$ *such that*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy, \tag{2.4}$$

then $u \in C^2(\mathbb{R}^n)$ *and u solves Poisson's equation in* $\mathbb{R}^n$.

Theorem 2.2 *(Evans, 2010)(Mean-value formulas for Laplace's equation) If* $u \in C^2(\Omega)$ *is harmonic, then*

$$u(x) = \frac{1}{\alpha(n)r^n} \int_{B(x,r)} u\, dy = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} u\, dS \tag{2.5}$$

for each ball $B(x,r) \subset \Omega$.

Theorem 2.3 *(Evans, 2010)(Maximum principle) Suppose* Ω *is an open and bounded domain in* $\mathbb{R}^n$ *and* $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *is harmonic within* Ω.

(i) *Then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u \tag{2.6}$$

(ii) *Furthermore, if* Ω *is connected and there exists a point* $x_0 \in \Omega$ *such that*

$$u(x_0) = \max_{\overline{\Omega}} u \tag{2.7}$$

then

$$u \text{ is constant within } \Omega. \tag{2.8}$$

Assertion (i) is the *maximum principle* for Laplace's equation and (ii) is the *strong* *maximum principle*. Replacing u by $-u$, we recover similar assertions with min replacing max.

Theorem 2.4 *(Evans, 2010)(Smoothness) If $u \in C(\Omega)$ satisfies the mean-value property (2.5) for each ball $B(x, r) \subset \Omega$, then*

$$u \in C^{\infty}(\Omega). \tag{2.9}$$

Note that u may not be smooth, or even continuous, up to the boundary.

Theorem 2.5 *(Evans, 2010)(Harnack's inequality) For each connected open set $V \subset\subset \Omega$, there exists a positive constant C, depending only on V, such that*

$$\sup_V u \leq C \inf_V u \tag{2.10}$$

for all nonnegative harmonic functions u in Ω.

Now consider the boundary-value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{2.11}$$

The solution of this boundary-value problem can be characterized as the minimizer of the *energy functional*

$$I[w] := \int_{\Omega} \frac{1}{2}|Dw|^2 - wf\, dx, \tag{2.12}$$

w belonging to the set

$$\mathcal{A} := \{w \in C^2(\overline{\Omega}) \,|\, w = g \text{ on } \partial\Omega\}, \tag{2.13}$$

with Ω is open and bounded and $\partial\Omega$ is of class C^1.

Theorem 2.6 *(Evans, 2010)(Uniqueness) There exists at most one solution $u \in C^2(\overline{\Omega})$ of the boundary-value problem (2.11).*

Theorem 2.7 *(Evans, 2010)(Dirichlet's principle) Assume $u \in C^2(\overline{\Omega})$ solves (2.11). Then*

$$I[u] = \min_{w \in \mathcal{A}} I[w]. \tag{2.14}$$

Conversely, if $u \in \mathcal{A}$ satisfies (2.14), then u solves the boundary-value problem (2.11).

2.2. Hölder Spaces

Hölder spaces play an important role when determining the regularity of the solutions of the partial differential equations. We assume Ω to be an open set in $\mathbb{R}^n$ in this section.

Definition 2.3 *Functions u satisfying*

$$|u(x) - u(y)| \leq C|x - y|^\gamma \tag{2.15}$$

for some constant C with $x, y \in \Omega$, are said to be Hölder continuous with exponent $\gamma \in (0, 1]$.

Definition 2.4 **(i)** *If $u : \Omega \to \mathbb{R}$ is bounded and continuous, we write*

$$||u||_{C(\overline{\Omega})} := \sup_{x \in \Omega} |u(x)| \tag{2.16}$$

(ii) *The γ^{th}-Hölder seminorm of $u : \Omega \to \mathbb{R}$ is*

$$[u]_{C^{0,\gamma}(\overline{\Omega})} := \sup_{\substack{x,y \in \Omega \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\gamma} \right\}, \tag{2.17}$$

and the γ^{th}-Hölder norm is

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} := \|u\|_{C(\overline{\Omega})} + [u]_{C^{0,\gamma}(\overline{\Omega})}. \tag{2.18}$$

Definition 2.5 *The Hölder space*

$$C^{k,\gamma}(\overline{\Omega}) \tag{2.19}$$

consists of all functions $u \in C^k(\overline{\Omega})$ for which the norm

$$\|u\|_{C^{k,\gamma}(\overline{\Omega})} := \sum_{|\alpha|\leq k} \|D^\alpha u\|_{C(\overline{\Omega})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\overline{\Omega})} \tag{2.20}$$

is finite.

2.3. Sobolev Spaces

Sobolev spaces play an important role in partial differential equations, especially when the solution is non-regular. In the second chapter, we will compare the following properties of Sobolev spaces with the fractional version.

2.3.1. Weak derivatives

Assume Ω to be an open set in $\mathbb{R}^n$ and remember that we call a function ϕ that belongs to $C_c^\infty(\Omega)$ a *test function*.

Definition 2.6 *Suppose $u, v \in L^1_{loc}(\Omega)$ and α is a multiindex. We say that v is the α^{th}-*weak partial derivative *of u, written*

$$D^\alpha u = v, \tag{2.21}$$

provided

$$\int_{\Omega} u D^{\alpha} \phi \, dx = (-1)^{|\alpha|} \int_{\Omega} v \phi \, dx \tag{2.22}$$

for all test functions $\phi \in C_c^{\infty}(\Omega)$.

Lemma 2.1 *(Evans, 2010) A weak α^{th}-partial derivative of u, if it exists, is uniquely defined up to a set of measure zero.*

2.3.2. Definition of Sobolev Spaces

Fix $1 \leq p \leq \infty$ and let $k \geq 0$ is an integer. Sobolev spaces are defined as certain function spaces, whose members have weak derivatives of some orders lying in some L^p spaces.

Definition 2.7 *The Sobolev space*

$$W^{k,p}(\Omega) \tag{2.23}$$

consists of all locally summable functions $u : \Omega \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^{\alpha} u$ exists in the weak sense and belongs to $L^p(\Omega)$.

Remark 2.1 **(i)** *If $p = 2$, we usually write*

$$H^k(\Omega) = W^{k,2}(\Omega) \tag{2.24}$$

Letter H is used since $H^k(\Omega)$ is a Hilbert space. Note also that $H^0(\Omega) = L^2(\Omega)$.

(ii) *We henceforth identify functions in $W^{k,p}(\Omega)$ which agree a.e.*

Definition 2.8 *If $u \in W^{k,p}(\Omega)$, we define its* norm *to be*

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^p \, dx\right)^{1/p} & \text{if } 1 \leq p < \infty \\ \sum_{|\alpha|\leq k} ess \sup_\Omega |D^\alpha u| & \text{if } p = \infty. \end{cases} \tag{2.25}$$

Definition 2.9 *We denote by*

$$W_0^{k,p}(\Omega) \tag{2.26}$$

the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

Remark 2.2 *It is customary to write*

$$H_0^k(\Omega) = W_0^{k,2}(\Omega). \tag{2.27}$$

2.3.3. Extensions

Theorem 2.8 *(Evans, 2010)(Extension Theorem) Suppose $1 \leq p \leq \infty$ and $\Omega \subset \mathbb{R}^n$ is bounded and of class C^1. Select a bounded open domain U such that $\Omega \subset\subset U$. Then there exists a bounded linear* extension *operator*

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n) \tag{2.28}$$

such that for each $u \in W^{1,p}(\Omega)$,

(i) *$Eu = u$ a.e. in Ω,*

(ii) *Eu has support within U,*

(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}$,

where the constant C depend on p, Ω and U.

2.3.4. Traces

Theorem 2.9 *(Evans, 2010)(Trace Theorem) Let* $1 \leq p < \infty$. *Assume* Ω *is bounded and* $\partial\Omega$ *is of class* C^1. *Then there exists a bounded linear* trace *operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega) \tag{2.29}$$

such that

(i) $Tu = u|_{\partial\Omega}$ *if* $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$,

(ii) $\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$,

for each $u \in W^{1,p}(\Omega)$. *with the constant* $C = C(p, \Omega)$.

Theorem 2.10 *(Evans, 2010)(Trace-zero functions in* $W^{1,p}$.*) Let* $1 \leq p < \infty$. *Assume* Ω *is bounded and* $\partial\Omega$ *is of class* C^1. *Suppose furthermore that* $u \in W^{1,p}(\Omega)$. *Then*

$$u \in W_0^{1,p}(\Omega) \quad \text{if and only if} \quad Tu = 0 \;\text{ on } \partial\Omega. \tag{2.30}$$

2.4. Fourier Transform

We define the Fourier transform and the Schwartz class in this section. This knowledge will be useful in defining H^s spaces in the next chapter.

Definition 2.10 *Given a complex-valued function* $f(x)$ *of a real variable in* $\mathbb{R}^n$, *define the* Fourier transform *of* f, *denoted* $\mathcal{F}f$ *or* $\hat{f}$, *by*

$$\hat{f}(\xi) := \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\, dx \tag{2.31}$$

and the inverse Fourier transform, *denoted* $\mathcal{F}^{-1}f$ *or* $\check{f}$, *by*

$$\check{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{ix\cdot\xi}\, dx \tag{2.32}$$

Remark 2.3 *There is not a general agreement on the constant* $(2\pi)^{-n}$ *and the minus sign in the definition (2.31). In some sources they could be defined in the inverse Fourier transform.*

Theorem 2.11 *(Strichartz, 1994)(Plancherel Formula)*

$$\int_{\mathbb{R}^n} |f(x)|^2\, dx = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\, d\xi. \tag{2.33}$$

Definition 2.11 *We say a function* f *is* rapidly decreasing *if there exist constants* M_N *such that*

$$|f(x)| \leq M_N |x|^{-N} \quad as \;\; x \to \infty, \tag{2.34}$$

for $N = 1, 2, ...$

In other words, a rapidly decreasing function $f(x)$ still goes to zero after multiplication by any polynomial $p(x)$ as $x \to \infty$.

Definition 2.12 *(Schwartz Class) If a function* $f \in C^{\infty}(\mathbb{R}^n)$ *and all its partial derivatives are rapidly decreasing then we say* f *is of Schwartz class. We denote the Schwartz class by* $\mathcal{S}(\mathbb{R}^n)$.

2.5. Measure Theory

It is not possible to measure every set, but if a set A in $\mathbb{R}^n$ is measurable then the measure of A need to be a nonnegative real number or ∞. We denote the measure of A by $\lambda(A)$ or $|A|$. We call it *Lebesque measure of* A. Notice that the word "measure" coincides with the words "length", "area" and "volume" for the spaces $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

If some property is valid everywhere except on a set of measure zero, then we say that this property is valid *almost everywhere*, abbreviated by a.e.

If $f(x) = g(x)$ in $\Omega \subset \mathbb{R}^n$ a.e. then there can exist a set $A \subset \Omega$ such that

$$A = \{x \;:\; f(x) \neq g(x) \;\; \text{for } x \in \Omega\} \quad \text{with } \lambda(A) = 0. \tag{2.35}$$

Hence clearly in the integration theory we can say that if $f = g$ in Ω a.e. then

$$\int_{\Omega} f = \int_{\Omega} g. \tag{2.36}$$

We can complete the section by giving the Lebesque Dominated Convergence Theorem(LDCT).

Theorem 2.12 *(Jones, 2001)(LDCT) Assume* $f_1, f_2, \ldots$ *are measurable functions on* $\mathbb{R}^n$. *Assume* $g \geq 0$, $g \in L^1(\mathbb{R}^n)$. *Assume*

$$\lim_{k\to\infty} f_k(x) \quad \textit{exists for all } x \in \mathbb{R}^n \tag{2.37}$$

and

$$|f_k(x)| \leq g(x) \quad \textit{for all } x \in \mathbb{R}^n. \tag{2.38}$$

Then $\lim_{k\to\infty} f_k \in L^1(\mathbb{R}^n)$ *and*

$$\int \left(\lim_{k\to\infty} f_k\right) d\lambda = \lim_{k\to\infty} \int f_k \, d\lambda. \tag{2.39}$$

CHAPTER 3

THE FRACTIONAL SOBOLEV SPACES

This chapter is a guide to the fractional Sobolev spaces $W^{s,p}$. One can see this spaces named as *Aronszajn, Gagliardo* or *Slobodeckij spaces*, but we use the most accepted name of it, i.e, the fractional Sobolev spaces. We give the main definition and some other versions of definition to make useful explanations in the trace theory. We deal with the regularity results, embeddings and extension domains by comparing the Sobolev spaces with the fractional version.

Let Ω be any smooth or non-smooth open set in $\mathbb{R}^n$. We define the fractional Sobolev space $W^{s,p}(\Omega)$ for any $0 < s < 1$ and $p \in [1, \infty)$ as follows:

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \quad : \quad \frac{|u(x) - u(y)|}{|x - y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\} \tag{3.1}$$

with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p \, dx \; + \; \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dxdy \right)^{1/p} . \tag{3.2}$$

$W^{s,p}(\Omega)$ is an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$ for $s \in (0, 1)$ and so-called Gagliardo seminorm of u appears in the definition of the norm such that

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dxdy \right)^{1/p} \tag{3.3}$$

Now let us consider the case $s > 1$. First of all one can easily realise that there is no need to consider the case $s = 1$, since $W^{1,p}(\Omega)$ is the classical Sobolev space. This is also valid for any $s \in \mathbb{Z}^+$. So in all our study, we consider s is a non-integer. We can seperate $s > 1$ into an integer and non-integer part such that for $m \in \mathbb{Z}^+$ and $\tilde{s} \in (0, 1)$, we write $s = m + \tilde{s}$. Then using the motivation that comes from the classical Sobolev spaces we define

$$W^{s,p}(\Omega) := \left\{ u \in W^{m,p}(\Omega) \; : \quad D^\alpha u \in W^{\tilde{s},p}(\Omega) \, , \quad \forall \alpha \text{ s.t } |\alpha| = m \right\} , \tag{3.4}$$

where α is a multiindex of order m.

Note that this is again a Banach space with respect to the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{W^{m,p}(\Omega)} + \sum_{|\alpha|=m} \|D^\alpha u\|^p_{W^{\tilde{s},p}(\Omega)} \right)^{1/p}. \tag{3.5}$$

For $s < 0$ and $p \in (1, \infty)$ we define $W^{s,p}(\Omega)$ as the dual space of $W_0^{-s,q}(\Omega)$ where $1/p + 1/q = 1$. Here, as motivated in the classical Sobolev spaces, $W_0^{s,p}(\Omega)$ denotes the closure of $C_c^\infty(\Omega)$ in the norm $\|\cdot\|_{W^{s,p}(\Omega)}$.

3.1. Extension to $\mathbb{R}^n$

We can extend any function $u \in W^{s,p}(\Omega)$ to a function, say $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$. The scientists have made that happen just by using the similar procedure in the classical one. Extension results play a remarkable role in some embedding theorems.

Definition 3.1 *(Extension domain) For any $s \in (0, 1)$ and $p \in [1, \infty)$, an open set $\Omega \subseteq \mathbb{R}^n$ is an* extension domain *for $W^{s,p}$ if there exists a positive constant C depending on n, s, p and Ω such that*

$$\forall u \in W^{s,p}(\Omega), \quad \exists \tilde{u} \in W^{s,p}(\mathbb{R}^n) \quad \text{with } \tilde{u}(x) = u(x) \quad \forall x \in \Omega \tag{3.6}$$

and

$$\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{s,p}(\Omega)}. \tag{3.7}$$

We will also show that any open set $\Omega \subseteq \mathbb{R}^n$ of class $C^{0,1}$ with bounded boundary is an extension domain for the fractional Sobolev spaces.

When constructing the extension of a function u, two method is used: when $\Omega = \mathbb{R}^n_+$ and when $u \equiv 0$ in a neighborhood of $\partial\Omega$.

Lemma 3.1 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let Ω be an open set in $\mathbb{R}^n$ and $u \in W^{s,p}(\Omega)$ with $s \in (0, 1)$ and $p \in [1, \infty)$. If there exists a compact subset $K \subset \Omega$ such* *that $u \equiv 0$ in Ω / K, then the extension function $\tilde{u}$ defined as*

$$\tilde{u}(x) = \begin{cases} u(x) & x \in \Omega \\ 0 & x \in \mathbb{R}^n / \Omega \end{cases} \tag{3.8}$$

belongs to $W^{s,p}(\mathbb{R}^n)$ and

$$\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{s,p}(\Omega)}, \tag{3.9}$$

where C is a suitable positive constant depending on n, s, p, K and Ω.

Proof First of all one can easily see that $\tilde{u} \in L^p(\mathbb{R}^n)$. Hence we just need to show that the Gagliardo semi-norm of $\tilde{u}$ in $\mathbb{R}^n$ is bounded by the one of u in Ω.

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x-y|^{n+sp}} dxdy &= \int_{\Omega} \int_{\Omega} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x-y|^{n+sp}} dxdy \\ &+ \int_{\mathbb{R}^n/\Omega} \int_{\mathbb{R}^n/\Omega} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x-y|^{n+sp}} dxdy \\ &+ \int_{\mathbb{R}^n/\Omega} \int_{\Omega} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x-y|^{n+sp}} dxdy \\ &+ \int_{\Omega} \int_{\mathbb{R}^n/\Omega} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x-y|^{n+sp}} dxdy \\ &= \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dxdy \\ &+2 \int_{\Omega} \int_{\mathbb{R}^n/\Omega} \frac{|u(x)|^p}{|x-y|^{n+sp}} dydx, \end{aligned}$$

by using the symmetry of the integral with respect to x and y. We know that $[u]_{W^{s,p}(\Omega)}$ is finite. For any $y \in \mathbb{R}^n / K$,

$$\frac{|u(x)|^p}{|x-y|^{n+sp}} = \frac{\chi_K(x)|u(x)|^p}{|x-y|^{n+sp}} \leq \chi_K(x)|u(x)|^p \sup_{x \in K} \frac{1}{|x-y|^{n+sp}} \tag{3.10}$$

and so

$$\int_{\Omega} \int_{\mathbb{R}^n - \Omega} \frac{|u(x)|^p}{|x-y|^{n+sp}} dydx \leq \|u\|^p_{L^p(\Omega)} \int_{\mathbb{R}^n - \Omega} \frac{1}{\operatorname{dist}(y, \partial K)^{n+sp}} dy < \infty, \tag{3.11}$$

since $\operatorname{dist}(\partial\Omega, \partial K) > 0$ and $n + sp > n$. Therefore we are done by choosing a constant C depending on n, s, p and K. □

Lemma 3.2 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let Ω be an open set in $\mathbb{R}^n$, symmetric with respect to the coordinate x_n, and consider the sets $\Omega_+ = \{x \in \Omega \ : \ x_n > 0\}$ and $\Omega_- = \{x \in \Omega \ : \ x_n \leq 0\}$. Let u be a function in $W^{s,p}(\Omega_+)$, with $s \in (0,1)$ and $p \in [1,\infty)$. Define*

$$\tilde{u}(x) = \begin{cases} u(x', x_n) & x_n \geq 0, \\ u(x', -x_n) & x_n < 0. \end{cases} \tag{3.12}$$

Then $\tilde{u}$ belongs to $W^{s,p}(\Omega)$ and

$$\|\tilde{u}\|_{W^{s,p}(\Omega)} \leq 4\|u\|_{W^{s,p}(\Omega_+)}. \tag{3.13}$$

Proof First of all notice that the domain Ω_+ is open. Now let us define $\bar{x} = (x', -x_n)$. And then we have

$$\begin{aligned} \|\tilde{u}\|^p_{L^p(\Omega)} &= \int_{\Omega_+} |u(x)|^p\, dx + \int_{\Omega_-} |u(x)|^p\, dx \\ &= \int_{\Omega_+} |u(x)|^p\, dx + \int_{\Omega_+} |u(\bar{x}', \bar{x}_n)|^p\, d\bar{x} \\ &= 2\|u\|^p_{L^p(\Omega_+)}. \end{aligned}$$

Also notice that if $x \in \mathbb{R}^n_+$ and $x \in \mathbb{R}^n/\mathbb{R}^n_+$ then $(x_n - y_n)^2 \geq (x_n + y_n)^2$ and hence

$$\begin{aligned} [\tilde{u}]^p_{W^{s,p}(\Omega)} &= \int_{\Omega_+}\int_{\Omega_+} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}}\, dxdy \\ &\quad +2\int_{\Omega_+}\int_{\Omega_-} \frac{|u(x) - u(y', -y_n)|^p}{|x-y|^{n+sp}}\, dxdy \\ &\quad +\int_{\Omega_-}\int_{\Omega_-} \frac{|u(x', -x_n) - u(y', -y_n)|^p}{|x-y|^{n+sp}}\, dxdy \\ &\leq 4\|u\|^p_{W^{s,p}(\Omega_+)}. \end{aligned}$$

This conclude the proof. □

Lemma 3.3 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let Ω be an open set in $\mathbb{R}^n$, $s \in (0,1)$ and $p \in [1,\infty)$. Let us consider $u \in W^{s,p}(\Omega)$ and $\psi \in C^{0,1}(\Omega)$, $0 \leq \psi \leq 1$. Then ψu belongs to $W^{s,p}(\Omega)$ and*

$$\|\psi u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)}. \tag{3.14}$$

where $C = C(n, p, s, \Omega)$.

Proof Since $|\psi| \leq 1$, we clearly say that $||\psi u||_{L^p(\Omega)} \leq ||u||_{L^p(\Omega)}$. Observe that for any $p \geq 1$, we have for any numbers a and b that

$$|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p). \tag{3.15}$$

And by adding and substracting the factor $\psi(x)u(y)$ we get

$$\begin{aligned}
[\psi u]^p_{W^{s,p}(\Omega)} &\leq 2^{p-1}\Big(\int_\Omega\int_\Omega \frac{|\psi(x)u(x)-\psi(x)u(y)|^p}{|x-y|^{n+sp}}\,dydx \\
&\quad + \int_\Omega\int_\Omega \frac{|\psi(x)u(y)-\psi(y)u(y)|^p}{|x-y|^{n+sp}}\,dydx\Big) \\
&\leq 2^{p-1}\Big(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dydx \\
&\quad + \int_\Omega\int_\Omega \frac{|u(x)|^p|\psi(x)-\psi(y)|^p}{|x-y|^{n+sp}}\,dydx\Big)
\end{aligned} \tag{3.16}$$

Since $\psi \in C^{0,1}(\Omega)$, we have

$$\begin{aligned}
\int_\Omega\int_\Omega \frac{|u(x)|^p|\psi(x)-\psi(y)|^p}{|x-y|^{n+sp}}\,dydx &\leq C_L\int_\Omega\int_{\Omega\cap|x-y|\leq 1} \frac{|u(x)|^p|x-y|^p}{|x-y|^{n+sp}}\,dydx \\
&\quad + \int_\Omega\int_{\Omega\cap|x-y|\geq 1} \frac{|u(x)|^p}{|x-y|^{n+sp}}\,dydx \\
&\leq \tilde{C}||u||^p_{L^p(\Omega)},
\end{aligned} \tag{3.17}$$

where C_L denotes the Lipschitz constant of ψ and $\tilde{C} > 0$ depends on n, p and s. By the way notice that the kernel $|x-y|^{-n+(1-s)p}$ is integrable with respect to y if $|x-y| \leq 1$ since $n+(s-1)p < n$. Notice also that the kernel $|x-y|^{-n-sp}$ is integrable when $|x-y| > 1$ since $n+sp > n$. Therefore the proof is done. □

These lemmas (3.1), (3.2) and (3.3) are the main steps of the proof of the theorem that states that every open Lipschitz set Ω with bounded boundary is an extension domain for the fractional Sobolev space.

Theorem 3.1 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let $p \in [1, \infty)$, $s \in (0,1)$ and $\Omega \subseteq \mathbb{R}^n$ be an open set of class $C^{0,1}$ with bounded boundary. Then $W^{s,p}(\Omega)$ is continuously embedded in $W^{s,p}(\mathbb{R}^n)$, namely for any $u \in W^{s,p}(\Omega)$ there exists $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$ such that*

$\tilde{u}|_{\Omega} = u$ and

$$\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{s,p}(\Omega)} \tag{3.18}$$

where $C = C(n, s, p, \Omega)$.

Proof We can cover $\partial\Omega$ with finitely many balls B_j such that $\partial\Omega \subset \bigcup_{j=1}^{k} B_j$, since the boundary $\partial\Omega$ is compact. So we can write $\mathbb{R}^n = \bigcup_{j=1}^{k} B_j \cup (\mathbb{R}^n/\partial\Omega)$. Then there exist a partition of unity related to this covering, i.e. there exist $k + 1$ smooth functions $\psi_0, \psi_1, ..., \psi_k$ such that supp$\psi_0 \subset \mathbb{R}^n - \partial\Omega$, supp$\psi_j \subset B_j$ for any $j \in \{1, 2, ...k\}, 0 \leq \psi_j \leq 1$ for any $j \in \{0, 1, ...k\}$ and $\sum_{j=0}^{k} \psi_j = 1$. Then first we have

$$u = \sum_{j=0}^{k} \psi_j u. \tag{3.19}$$

By lemma (3.3), we know that $\psi_0 u$ belongs to $W^{s,p}(\Omega)$. Moreover, we can extend it to the whole of $\mathbb{R}^n$, since $\psi_0 u \equiv 0$ near $\partial\Omega$, by setting

$$\tilde{\psi_0} u(x) = \begin{cases} \psi_0 u(x) & x \in \Omega, \\ 0 & x \in \mathbb{R}^n/\Omega \end{cases}$$

and $\widetilde{\psi_0 u} \in W^{s,p}(\mathbb{R}^n)$. And we have

$$\|\widetilde{\psi_0 u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|\psi_0 u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)}, \tag{3.20}$$

where $C = C(n, s, p, \Omega)$ is possibly different for each step.

Since Ω is of class $C^{0,1}$, there exists an isomorphism $T_j : Q \to B_j$ which is defined in Appendix part. So for any $j \in \{1, 2, ..., k\}$, consider $u|_{B_j \cap \Omega}$ and set

$$v_j(y) := u(T_j(y)) \quad \forall y \in Q_+. \tag{3.21}$$

Now by setting $x = T_j(\tilde{x})$ we have

$$\begin{aligned}
\int_{Q_+}\int_{Q_+} \frac{|v_j(\tilde{x}) - v_j(\tilde{y})|^p}{|\tilde{x} - \tilde{y}|^{n+sp}} d\tilde{x}d\tilde{y} &= \int_{Q_+}\int_{Q_+} \frac{|u(T_j(\tilde{x})) - u(T_j(\tilde{y}))|^p}{|\tilde{x} - \tilde{y}|^{n+sp}} d\tilde{x}d\tilde{y} \\
&= \int_{B_j\cap\Omega}\int_{B_j\cap\Omega} \frac{|u(x) - u(y)|^p}{|T_j^{-1}(x) - T_j^{-1}(y)|^{n+sp}} \det(T_j^{-1}) dxdy \\
&\leq C\int_{B_j\cap\Omega}\int_{B_j\cap\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy,
\end{aligned}$$

since T_j is bi-Lipschitz. Moreover by Lemma (3.2) we can extend v_j to all Q so that the extension $\overline{v_j}$ belongs to $W^{s,p}(Q)$ and

$$||\overline{v_j}||_{W^{s,p}(Q)} \leq 4||v_j||_{W^{s,p}(Q_+)}. \tag{3.22}$$

Now set

$$w_j(x) := \overline{v_j}(T_j^{-1}(x)) \quad \forall x \in B_j. \tag{3.23}$$

Since T_j is bi-Lipschitz, we similarly get that $w_j \in W^{s,p}(B_j)$. Also notice that $w_j \equiv u$ on $B_j \cap \Omega$. By definition $\psi_j w_j$ has compact support in B_j and therefore we can consider the extension $\widetilde{\psi_j w_j}$ to all $\mathbb{R}^n$ in such a way that $\widetilde{\psi_j w_j} \in W^{s,p}(\mathbb{R}^n)$. Also using Lemmas (3.1), (3.2) and (3.3) we get

$$\begin{aligned}
||\widetilde{\psi_j w_j}||_{W^{s,p}(\mathbb{R}^n)} &\leq C||\psi_j w_j||_{W^{s,p}(B_j)} \\
&\leq C||w_j||_{W^{s,p}(B_j)} \\
&\leq C||\overline{v_j}||_{W^{s,p}(Q)} \\
&\leq C||v_j||_{W^{s,p}(Q_+)} \\
&\leq C||u||_{W^{s,p}(\Omega\cap B_j)},
\end{aligned}$$

where $C = C(n, s, p, \Omega)$ and it is possibly different at each step.

Finally, let

$$\tilde{u} = \widetilde{\psi_0 u} + \sum_{j=1}^{k} \widetilde{\psi_j w_j} \tag{3.24}$$

be the extension of u defined on all $\mathbb{R}^n$. By construction, it is clear that $\tilde{u}|_\Omega = u$ and

$$\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{s,p}(\Omega)}. \tag{3.25}$$

□

Corollary 3.1 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let* $p \in [1, \infty)$, $s \in (0, 1)$ *and* Ω *be an open set in* $\mathbb{R}^n$ *of class* $C^{0,1}$ *with bounded boundary. Then for any* $u \in W^{s,p}(\Omega)$, *there exists a sequence* $\{u_n\} \in C_c^\infty(\mathbb{R}^n)$ *such that* $u_n \to u$ *as* $n \to \infty$ *in* $W^{s,p}(\Omega)$, *i.e,*

$$\lim_{n\to\infty} \|u_n - u\|_{W^{s,p}(\Omega)} = 0. \tag{3.26}$$

Proof Let $\tilde{u}$ be defined as in Theorem (3.1).

$$\begin{aligned} \|u_n - u\|_{W^{s,p}(\Omega)} &\leq \|u_n - \tilde{u}\|_{W^{s,p}(\Omega)} + \|\tilde{u} - u\|_{W^{s,p}(\Omega)} \\ &\leq \|u_n - \tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} + \|\tilde{u} - u\|_{W^{s,p}(\Omega)} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

□

3.2. Fractional Sobolev Inequalities and Embeddings

In this section we show that the fractional Sobolev spaces can sometimes continuously, sometimes compactly be embedded in each other. We start with $s \in (0, 1)$ and go into details. All motivation comes from the classical type of Sobolev spaces.

First results point out that $W^{s',p}$ is continuously embedded in $W^{s,p}$ when $s \leq s'$.

Proposition 3.1 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let* $p \in [1, \infty)$ *and* $0 < s \leq s' < 1$. *Let* Ω *be an open set in* $\mathbb{R}^n$ *and* $u : \Omega \to \mathbb{R}$ *be a measurable function. Then*

$$\|u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{s',p}(\Omega)} \tag{3.27}$$

for some suitable constant $C = C(n, s, p) \geq 1$. *In particular,*

$$W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega). \tag{3.28}$$

Proof In order to use the fact that the kernel $|x-y|^{-n-sp}$ is integrable when $n + sp > n$, we need to seperate the domain.

$$\int_{\Omega}\int_{\Omega\cap\{|x-y|\geq 1\}} \frac{|u(x)|^p}{|x-y|^{n+sp}}\, dxdy \leq \int_{\Omega}\left(\int_{\{|z|\geq 1\}} \frac{1}{|z|^{n+sp}}\, dz\right)|u(x)|^p dx \leq C(n, s, p)\|u\|^p_{L^p(\Omega)}. \tag{3.29}$$

Hence,

$$\begin{aligned}\int_{\Omega}\int_{\Omega\cap\{|x-y|\geq 1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} &\leq 2^{p-1}\int_{\Omega}\int_{\Omega\cap\{|x-y|\geq 1\}} \frac{|u(x)|^p + |u(y)|^p}{|x-y|^{n+sp}} \\ &\leq 2^p C(n, s, p)\|u\|^p_{L^p(\Omega)}.\end{aligned} \tag{3.30}$$

On the other hand,

$$\int_{\Omega}\int_{\Omega\cap\{|x-y|<1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dxdy \leq \int_{\Omega}\int_{\Omega\cap\{|x-y|<1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+s'p}} dxdy \tag{3.31}$$

Therefore

$$\begin{aligned}\|u\|^p_{W^{s,p}(\Omega)} &\leq (2^p C(n, s, p) + 1)\|u\|^p_{L^p(\Omega)} + \int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+s'p}} dxdy \\ &\leq C(n, s, p)\|u\|_{W^{s',p}(\Omega)}.\end{aligned} \tag{3.32}$$

□

Now we show that the result in Proposition (3.1) holds also in the limit case, under some regularity properties of $\partial\Omega$.

Proposition 3.2 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let* $p \in [1, \infty)$ *and* $s \in (0, 1)$. *Let* Ω *be an open set in* $\mathbb{R}^n$ *of class* $C^{0,1}$ *with bounded boundary and* $u : \Omega \to \mathbb{R}$ *be a measurable function. Then*

$$\|u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)} \tag{3.33}$$

for some suitable constant $C = C(n,s,p) \geq 1$. In particular,

$$W^{1,p}(\Omega) \subseteq W^{s,p}(\Omega). \tag{3.34}$$

Proof We can extend the function $u \in W^{1,p}(\Omega)$ to a function $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$ such that $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$ and $\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}$ for a suitable constant C. Now by using Hölder inequality we get

$$\begin{aligned}
\int_\Omega \int_{\Omega\cap\{|x-y|<1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy &\leq \int_\Omega \int_{B_1} \frac{|u(x)-u(z+x)|^p}{|z|^{n+sp}}\,dzdx \\
&= \int_\Omega \int_{B_1} \frac{|u(x)-u(x+z)|^p}{|z|^p} \frac{1}{|z|^{n+(s-1)p}}\,dzdx \\
&\leq \int_\Omega \int_{B_1} \left(\int_0^1 \frac{|\nabla u(x+tz)|^p}{|z|^{\frac{n}{p}+s-1}}\,dt \right)^p dzdx \\
&\leq \int_{\mathbb{R}^n} \int_{B_1} \int_0^1 \frac{|\nabla \tilde{u}(x+tz)|^p}{|z|^{n+p(s-1)}}\,dtdzdx \\
&\leq \int_{B_1} \int_0^1 \frac{\|\nabla \tilde{u}\|_{L^p(\mathbb{R}^n)}}{|z|^{n+p(s-1)}}\,dtdz \\
&\leq C_1(n,s,p)\|\nabla \tilde{u}\|_{L^p(\mathbb{R}^n)} \\
&\leq C_2(n,s,p)\|\nabla \tilde{u}\|_{W^{1,p}(\Omega)}
\end{aligned}$$

Combining this result with the one in (3.30), we finish the proof. □

Similar results in Proposition (3.1) and Proposition (3.2) appear in the case $s > 1$.

Corollary 3.2 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let $p \in [1,\infty)$ and $s' \geq s > 1$. Let Ω be an open set in $\mathbb{R}^n$ of class $C^{0,1}$. Then we have*

$$W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega). \tag{3.35}$$

Proof Write $s = m + \delta$ and $s' = m' + \delta'$ with the integers m, m' and $\delta, \delta' \in (0,1)$. The proposition (3.1) is enough for the case $m = m'$, on the other hand if $m' \geq m+1$, then using the propositions (3.1) and (3.2) we have

$$W^{m'+\delta',p}(\Omega) \subseteq W^{m',p}(\Omega) \subseteq W^{m+1,p}(\Omega) \subseteq W^{m+\delta,p}(\Omega). \tag{3.36}$$

The proof is done. □

We need some preliminary results for the forthcoming theorems about embeddings. Therefore the following lemmas are to be used in constructing the embedding results. We give them without their proofs since they includes some details but the important thing for the main theorem is just the results of these lemmas.

Lemma 3.4 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Fix* $x \in \mathbb{R}^n$*. Let* $p \in [1, \infty)$*,* $s \in (0, 1)$ *and* $E \subset \mathbb{R}^n$ *be a measurable set with finite measure. Then,*

$$\int_{\mathbb{R}^n/E} \frac{dy}{|x-y|^{n+sp}} \geq C|E|^{-sp/n}, \tag{3.37}$$

for a suitable constant $C = C(n, s, p) > 0$*.*

Lemma 3.5 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let* $s \in (0, 1)$ *and* $p \in [1, \infty)$ *such that* $sp < n$*. Fix* $T > 1$*; let* $N \in \mathbb{Z}$ *and* a_k *be a bounded, nonnegative, decreasing sequence with* $a_k = 0$ *for any* $k \geq N$*. Then,*

$$\sum_{k\in\mathbb{Z}} a_k^{(n-sp)/n} T^k \leq C \sum_{k\in\mathbb{Z}_{a_k \neq 0}} a_{k+1} a_k^{-sp/n} T^k, \tag{3.38}$$

for a suitable constant $C = C(n, s, p, T) > 0$*, independent of* N*.*

Lemma 3.6 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let* $s \in (0, 1)$ *and* $p \in [1, \infty)$ *such that* $sp < n$*. Let* $f \in L^\infty(\mathbb{R}^n)$ *be compactly supported. For any* $k \in \mathbb{Z}$ *let*

$$a_k := |\{|f| > 2^k\}|. \tag{3.39}$$

Then,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dxdy \geq \sum_{k\in\mathbb{Z}_{a_k \neq 0}} a_{k+1} a_k^{-sp/n} T^k, \tag{3.40}$$

for a suitable constant $C = C(n, s, p) > 0$*.*

Lemma 3.7 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let* $q \in [1, \infty)$ *and* $f : \mathbb{R}^n \to \mathbb{R}$ *be a measurable function. For any* $N \in \mathbb{N}$*, let*

$$f_N(x) := \max\{\min\{f(x), N\}, -N\} \quad \forall x \in \mathbb{R}^n. \tag{3.41}$$

Then,

$$\lim_{N\to\infty} \|f_N\|_{L^q(\mathbb{R}^n)} = \|f\|_{L^q(\mathbb{R}^n)}. \tag{3.42}$$

Taking into account these lemmas, we are ready to state and prove some embedding theorems.

Theorem 3.2 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let $s \in (0,1)$ and $p \in [1,\infty)$ such that $sp < n$. Then there exists a positive constant $C = C(n,p,s)$ such that, for any measurable and compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$, we have*

$$\|f\|^p_{L^{p^*}(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dxdy, \tag{3.43}$$

where $p^* = p^*(n,s)$ is the so-called "fractional critical exponent" *and it is equal to $np/(n-sp)$.*

Consequently, the space $W^{s,p}(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [p, p^*]$.

Proof Notice first that if the right hand side of (3.43) is unbounded then the proof is automatically finished. So we may suppose that it is finite. Moreover we can suppose, without loss of generality, that $f \in L^\infty(\mathbb{R}^n)$. Since if the right hand side of (3.43) is bounded for f then it is also bounded for the function f_N, which is described in (3.41). Therefore by the Dominated Convergence Theorem together with Lemma (3.7) imply

$$\lim_{N\to\infty} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f_N(x)-f_N(y)|^p}{|x-y|^{n+sp}}\, dxdy = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dxdy. \tag{3.44}$$

We obtain the estimate (3.43) for the function f. Now define for any integer k,

$$A_k := \{|f| > 2^k\} \tag{3.45}$$

and

$$a_k := |A_k|. \tag{3.46}$$

Then we have

$$
\begin{aligned}
\|f\|^p_{L^{p^*}(\mathbb{R}^n)} &= \sum_{k\in\mathbb{Z}} \int_{A_k - A_{k+1}} |f(x)|^{p^*} dx \\
&\le \sum_{k\in\mathbb{Z}} \int_{A_k - A_{k+1}} (2^{k+1})^{p^*} dx \\
&\le \sum_{k\in\mathbb{Z}} (2^{(k+1)p^*} a_k \\
&\le 2^p \left(\sum_{k\in\mathbb{Z}} 2^{kp^*} a_k \right)^{p/p^*} \\
&\le 2^p \sum_{k\in\mathbb{Z}} 2^{kp} a_k^{(n-sp)/n}.
\end{aligned}
$$

Then by choosing $T = 2^p$, Lemma (3.5) yields for a suitable constant $C = C(n, s, p)$

$$
\|f\|^p_{L^{p^*}(\mathbb{R}^n)} \le C \sum_{\substack{k\in\mathbb{Z} \\ a_k \neq 0}} 2^{kp} a_{k+1} a_k^{-\frac{sp}{n}}. \tag{3.47}
$$

Finally applying Lemma (3.6), we obtain the desired result. Furthermore, the embedding for $q \in (p, p^*)$ follows from Hölder inequality. □

The above theorem does not usually hold for a smaller domain $\Omega \subset \mathbb{R}^n$, since the extension function requires some assumptions on the domain.

Theorem 3.3 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let $s \in (0, 1)$ and $p \in [1, \infty)$ such that $sp < n$. Let $\Omega \subseteq \mathbb{R}^n$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C = C(n, p, s, \Omega)$ such that, for any $f \in W^{s,p}(\Omega)$, we have*

$$
\|f\|_{L^q(\Omega)} \le C\|f\|_{W^{s,p}(\Omega)}, \tag{3.48}
$$

for any $q \in [p, p^*]$; i.e, the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [p, p^*]$.

If, in addition, Ω is bounded, then the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [1, p^*]$.

Proof Let $f \in W^{s,p}(\Omega)$. There exists a constant $C_1(n, s, p, \Omega) > 0$ with $\tilde{f}$ such that

$\tilde{f} = f$ in Ω and

$$\|\tilde{f}\|_{W^{s,p}(\mathbb{R}^n)} \leq C_1\|f\|_{W^{s,p}(\Omega)}, \tag{3.49}$$

since Ω is an extension domain.

By Theorem (3.2),

$$\|\tilde{f}\|_{L^q(\mathbb{R}^n)} \leq C_2\|\tilde{f}\|_{W^{s,p}(\mathbb{R}^n)}. \tag{3.50}$$

Combaining (3.49) and (3.50) we get for $C = C_1C_2$ that

$$\|f\|_{L^q(\Omega)} = \|\tilde{f}\|_{L^q(\Omega)} \leq \|\tilde{f}\|_{L^q(\mathbb{R}^n)} \leq C_2\|\tilde{f}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|f\|_{W^{s,p}(\Omega)}. \tag{3.51}$$

If Ω is bounded then the embedding for $q \in [1, p)$ follows from Hölder inequailty. □

Note that the critical exponent $p^* \to \infty$ as $sp \to n$ and so in this case f belongs to L^q for any q if $f \in W^{s,p}$ as stated in the following two theorems.

Theorem 3.4 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let $s \in (0, 1)$ and $p \in [1, \infty)$ such that $sp = n$. Then there exists a positive constant $C = C(n, p, s, \Omega)$ such that, for any measurable and compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$, we have*

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{W^{s,p}(\mathbb{R}^n)}, \tag{3.52}$$

for any $q \in [p, \infty)$; i.e, the space $W^{s,p}(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [p, \infty)$.

Theorem 3.5 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let $s \in (0, 1)$ and $p \in [1, \infty)$ such that $sp = n$. Let $\Omega \subseteq \mathbb{R}^n$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C = C(n, p, s, \Omega)$ such that, for any $f \in W^{s,p}(\Omega)$, we have*

$$\|f\|_{L^q(\Omega)} \leq C\|f\|_{W^{s,p}(\Omega)}, \tag{3.53}$$

for any $q \in [p, \infty)$; i.e, the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [p, \infty)$.

If, in addition, Ω is bounded, then the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [1, \infty)$.

We did not give the proofs of Theorems (3.4) and (3.5) since they can easily be obtained by combaining Theorems (3.2) and (3.3) with Proposition (3.1).

3.3. Hölder Regularity and Its Relation with $W^{s,\infty}$

We show the certain regularity results for functions in $W^{s,p}(\Omega)$ under some assumptions on the domain, dimension, fractional power, etc. We start by defining what an external cusp is. And then we give a lemma which is required to prove the main regularity theorem. After that we relate this regularity with the definition of $W^{s,\infty}$.

Lemma 3.8 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let $p \in [1,\infty)$ and $sp \in (n, n+p]$. Let $\Omega \subset \mathbb{R}^n$ be a domain with no external cusps and f be a function in $W^{s,p}(\Omega)$. Then, for any $x_0 \in \Omega$ and R, R', with $0 < R' < R < diam(\Omega)$, we have*

$$|\langle f\rangle_{B_R(x_0)\cap\Omega} - \langle f\rangle_{B_{R'}(x_0)\cap\Omega}| \leq c[f]_{p,sp}|B_R(x_0)\cap\Omega|^{(sp-n)/np} \tag{3.54}$$

where

$$[f]_{p,sp} := \left(\sup_{x_0\in\Omega,\rho>0} \rho^{-sp}\int_{B_\rho(x_0)\cap\Omega} |f(x) - \langle f\rangle_{B_\rho(x_0)\cap\Omega}|^p\,dx\right)^{1/p} \tag{3.55}$$

and

$$\langle f\rangle_{B_\rho(x_0)\cap\Omega} := \frac{1}{|B_\rho(x_0)\cap\Omega|}\int_{B_\rho(x_0)\cap\Omega} f(x)\,dx. \tag{3.56}$$

We don't give the proof of this lemma since it requires a detailed technique that we do not need. The importance of this lemma is being used in the proof of the following theorem, but one can find the proof in references.

Theorem 3.6 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let $\Omega \subseteq \mathbb{R}^n$ be an extension domain for $W^{s,p}$ with no external cusps and let $p \in [1,\infty)$, $s \in (0,1)$ such that $sp > n$. Then, there exists $C > 0$, depending on n, s, p and Ω, such that*

$$||f||_{C^{0,\alpha}(\Omega)} \leq C||f||_{W^{s,p}(\Omega)}, \tag{3.57}$$

for any $f \in L^p(\Omega)$, with $\alpha := (sp-n)/n$.

Proof We use C as suitable positive quantities that can differ in each step. And we assume $||f||_{W^{s,p}(\Omega)}$ is finite, otherwise the proof is so simple. Let $\tilde{f}$ be the extension of f.

Then Hölder inequality together with integratin over $B_r(x_0)$ yields

$$\int_{B_r(x_0)} |\tilde{f}(x) - \langle \tilde{f} \rangle_{B_r(x_0)}|^p \, dx \le \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \int_{B_r(x_0)} |\tilde{f}(x) - \tilde{f}(y)|^p \, dxdy. \tag{3.58}$$

Since $|x - y| \le 2r$ for any $x, y \in B_r x_0$, we get

$$\begin{aligned} \int_{B_r(x_0)} |\tilde{f}(x) - \langle \tilde{f} \rangle_{B_r(x_0)}|^p \, dx &\le \frac{(2r)^{n+sp}}{|B_r(x_0)|} \int_{B_r(x_0)} \int_{B_r(x_0)} \frac{|\tilde{f}(x) - \tilde{f}(y)|^p}{|x - y|^{n+sp}} \, dxdy \\ &\le \frac{2^{n+sp} r^{sp} C \|f\|^p_{W^{s,p}(\Omega)}}{|B_1|}, \end{aligned}$$

so

$$[f]^p_{p,sp} \le C \|f\|^p_{W^{s,p}(\Omega)}. \tag{3.59}$$

Now, we will show that f is Hölder continuous. Taking into account Lemma 3.8, it follows that the sequence of functions $x \to \langle f \rangle_{B_R(x) \cap \Omega}$ converges uniformly in $x \in \Omega$ when $R \to 0$. Also the limit function g will be continuous and the same holds for f, since by Lebesgue theorem theorem we have that

$$\lim_{R \to 0} \frac{1}{|B_R(x_0) \cap \Omega|} \int_{B_R(x) \cap \Omega} f(y) \, dy = f(x) \quad \text{for almost every } x \in \Omega. \tag{3.60}$$

Now, for any $x, y \in \Omega$ set $R = |x - y|$. We have

$$|f(x) - f(y)| \le |f(x) - \langle \tilde{f} \rangle_{B_{2R}(x)}| + |\langle \tilde{f} \rangle_{B_{2R}(x)} - \langle \tilde{f} \rangle_{B_{2R}(y)}| + |\langle \tilde{f} \rangle_{B_{2R}(y)} - f(y)|. \tag{3.61}$$

Getting the limit in Lemma 3.8 as $R' \to 0$ and writing $2R$ instead of R, we get for any $x \in \Omega$

$$|f(x) - \langle \tilde{f} \rangle_{B_{2R}(x)}| \le c[f]_{p,sp} |B_{2R}(x)|^{(sp-n)/np} \le C[f]_{p,sp} R^{(sp-n)/p}, \tag{3.62}$$

where the constant $C = c2^{(sp-n)/p}/|B_1|$.

We also have

$$|\langle \tilde{f} \rangle_{B_{2R}(x)} - \langle \tilde{f} \rangle_{B_{2R}(y)}| \le |f(z) - \langle \tilde{f} \rangle_{B_{2R}(x)}| + |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(y)}| \tag{3.63}$$

and so by integration on $z \in B_{2R}(x) \cap B_{2R}(y)$, we have

$$\begin{aligned} |\langle \tilde{f} \rangle_{B_{2R}(x)} - \langle \tilde{f} \rangle_{B_{2R}(y)}| &\leq \int_{B_{2R}(x) \cap B_{2R}(y)} |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(x)}| dz \\ &\quad + \int_{B_{2R}(x) \cap B_{2R}(y)} |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(y)}| dz \\ &\leq \int_{B_{2R}(x)} |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(x)}| dz \\ &\quad + \int_{B_{2R}(y)} |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(y)}| dz. \end{aligned}$$

Furthermore, since $B_R(x) \cup B_R(y) \subset (B_{2R}(x) \cap B_{2R}(y))$, we have

$$|B_R(x)| \leq |B_{2R}(x) \cap B_{2R}(y)| \quad \text{and} \quad |B_R(y)| \leq |B_{2R}(x) \cap B_{2R}(y)| \tag{3.64}$$

and so

$$\begin{aligned} |\langle \tilde{f} \rangle_{B_{2R}(x)} - \langle \tilde{f} \rangle_{B_{2R}(y)}| &\leq \frac{1}{|B_R(x)|} \int_{B_{2R}(x)} |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(x)}| dz \\ &\quad + \frac{1}{|B_R(y)|} \int_{B_{2R}(y)} |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(y)}| dz. \end{aligned}$$

On the other hand Hölder inequailty gives

$$\begin{aligned} \frac{1}{|B_R(x)|} \int_{B_{2R}(x)} |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(x)}| dz &\leq \frac{|B_{2R}(x)|^{(p-1)/p}}{|B_R(x)|} \left(\int_{B_{2R}(x)} |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(x)}|^p dz \right)^{1/p} \\ &\leq \frac{|B_{2R}(x)|^{(p-1)/p}}{|B_R(x)|} (2R)^s [f]_{p,sp} \\ &\leq C[f]_{p,sp} R^{(sp-n)/p}. \end{aligned} \tag{3.65}$$

And similarly we get

$$\frac{1}{|B_R(y)|} \int_{B_{2R}(y)} |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(y)}| dz \leq C[f]_{p,sp} R^{(sp-n)/p}. \tag{3.66}$$

Combining (3.65) with (3.66), we get

$$|f(x) - f(y)| \leq C[f]_{p,sp}|x-y|^{(sp-n)/p}, \tag{3.67}$$

up to relabeling the constant C.

Hence, by taking into account (3.59), we can conclude that for $\alpha = (sp-n)/p$, $f \in C^{0,\alpha}(\Omega)$. Finally taking R_0 <diam(Ω) with using Hölder inequality, we have for any $x \in \Omega$,

$$\begin{aligned} |f(x)| &\leq |\langle \tilde{f} \rangle_{B_R(x)}| + |f(x) - \langle \tilde{f} \rangle_{B_R(x)}| \\ &\leq \frac{C}{|B_{R_0}(x)|^{1/p}} \|f\|_{L^p(\Omega)} + c[f]_{p,sp}|B_{R_0}(x)|^{\alpha}. \end{aligned} \tag{3.68}$$

Hence, by (3.59), (3.67) and (3.68), we get

$$\begin{aligned} \|f\|_{C^{0,\alpha}(\Omega)} &= \|f\|_{L^\infty(\Omega)} + \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^{\alpha}} \\ &\leq C(\|f\|_{L^p(\Omega)} + [f]_{p,sp}) \\ &\leq C\|f\|_{W^{s,p}(\Omega)}. \end{aligned}$$

for a suitable constant C. □

Up to now, we have not defined the fractional Sobolev space for $p = \infty$, but now we observe from Theorem (3.6) that $W^{s,\infty}$ could be defined as the space of functions

$$\left\{ u \in L^\infty(\Omega) \quad : \quad \frac{|u(x) - u(y)|}{|x-y|^s} \in L^\infty(\Omega \times \Omega) \right\}, \tag{3.69}$$

but this space is nothing but $C^{0,s}(\Omega)$.

As a result the Hölder space $C^{0,s}(\Omega)$ is a characterization for the fractional Sobolev space $W^{s,\infty}(\Omega)$.

3.4. The Specialised Fractional Sobolev Space: H^s

When $p = 2$, the fractional Sobolev space $W^{s,2}(\mathbb{R}^n)$ turns out to be a Hilbert space. That is why we denote it by $H^s(\mathbb{R}^n)$. Similarly we denote $W_0^{s,2}(\mathbb{R}^n)$ by $H_0^s(\mathbb{R}^n)$. We can also define this special fractional Sobolev space by using the Fourier transform as follows:

$$H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^2 \, d\xi < \infty \right\} \tag{3.70}$$

for any $s > 0$, not need to be in $(0, 1)$. And for $s < 0$,

$$H^s(\mathbb{R}^n) := \left\{ u \in \mathcal{S}'(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\mathcal{F}u(\xi)|^2 \, d\xi < \infty \right\} \tag{3.71}$$

The two definitions of $H^s(\mathbb{R}^n)$, which are via the Gagliardo norm and via Fourier transform are equivalent.

Proposition 3.3 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let $s \in (0, 1)$. Then the fractional Sobolev space $H^s(\mathbb{R}^n)$ defined by Fourier transform and the Gagliardo norm coincide. In particular for any $u \in H^s(\mathbb{R}^n)$*

$$[u]^2_{H^s(\mathbb{R}^n)} = 2C^{-1} \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 \, d\xi. \tag{3.72}$$

where $C = C(n, s)$ is a constant.

Proof We use Plancherel Formula in this proof.

$$
\begin{aligned}
[u]^2_{H^s(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(z+y)-u(y)|^2}{|z|^{n+2s}}\,dzdy \\
&= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\left|\frac{u(z+y)-u(y)}{|z|^{n/2+s}}\right|^2 dy\right)dz \\
&= \int_{\mathbb{R}^n}\left\|\frac{u(z+\cdot)-u(\cdot)}{|z|^{n/2+s}}\right\|^2_{L^2(\mathbb{R}^n)} dz \\
&= \int_{\mathbb{R}^n}\left\|\mathcal{F}\left(\frac{u(z+\cdot)-u(\cdot)}{|z|^{n/2+s}}\right)\right\|^2_{L^2(\mathbb{R}^n)} dz \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|e^{i\xi\cdot z}-1|^2}{|z|^{n+2s}}|\mathcal{F}u(\xi)|^2 d\xi dz \\
&= 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(1-\cos\xi\cdot z)}{|z|^{n+2s}}|\mathcal{F}u(\xi)|^2 dz d\xi \\
&= 2C^{-1}\int_{\mathbb{R}^n} |\xi|^{2s}|\mathcal{F}u(\xi)|^2 d\xi
\end{aligned}
$$

□

Remark 3.1 *We will see that the constant C in the Propositon (3.3) is nothing but the normalization constant that will be defined in the definition of the fractional Laplacian operator.*

Remark 3.2 *The equivalence of the definitions of H^s relies on Pancherel's Formula. Unless $p = q = 2$, one cannot go forward and backward between L^p and L^q via Fourier transform. That is why the general fractional Sobolev space defined via Fourier transform for $1 < p < \infty$ does not coincide with the classical definition and this will not be discussed in this section.*

We can analyze the traces of the Sobolev functions by using the definition of $H^s(\mathbb{R}^n)$ via the Fourier transform. Let $\Omega \subseteq \mathbb{R}^n$ be an open set with continuous boundary $\partial\Omega$. We denote, as T, the *trace operator* which is the linear operator defined by the uniformly continuous extension of the operator of restriction to $\partial\Omega$ for the functions in the space $C_c^\infty(\mathbb{R}^n)$ restricted to $\overline{\Omega}$.

For any $x = (x', x_n) \in \mathbb{R}^n$ and for any $u \in \mathcal{S}(\mathbb{R}^n)$, we have

$$v(x') = u(x', 0) \quad \forall x' \in \mathbb{R}^{n-1}, \tag{3.73}$$

where $v \in \mathcal{S}(\mathbb{R}^{n-1})$ is the restriction of u on the hyperplane $x_n = 0$.

Consider now the Fourier transformation of v and the integration of the Fourier transform of u on the real line, respectively.

$$
\begin{aligned}
\mathcal{F}v(\xi') &= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-i\xi'\cdot x'} v(x')\, dx' \\
&= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-i\xi'\cdot x'} u(x', 0)\, dx'.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\int_{\mathbb{R}} \mathcal{F}u(\xi', \xi_n)\, d\xi_n &= \int_{\mathbb{R}} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i(\xi',\xi_n)\cdot(x',x_n)} u(x', x_n)\, dx'\, dx_n d\xi_n \\
&= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-i\xi'\cdot x'} \left[\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\xi_n\cdot x_n} u(x', x_n)\, dx_n d\xi_n \right] dx' \\
&= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-i\xi'\cdot x'} [u(x', 0)]\, dx'.
\end{aligned}
$$

So we see from these two calculations that

$$
\mathcal{F}v(\xi') = \int_{\mathbb{R}} \mathcal{F}u(\xi', \xi_n)\, d\xi_n \quad \forall \xi' \in \mathbb{R}^{n-1}. \tag{3.74}
$$

We can define the traces of the function in $H^s(\mathbb{R}^n)$ as the following proposition. We do not give the proof of it, since it needs some detailed technique but one can find it from the reference of it.

Proposition 3.4 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let $s > 1/2$, then any function $u \in H^s(\mathbb{R}^n)$ has a trace v on the hyperplane $\{x_n = 0\}$, such that $v \in H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$. Also, the trace operator T is surjective from $H^s(\mathbb{R}^n)$ onto $H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$.*

CHAPTER 4

FRACTIONAL LAPLACIAN

In this chapter we focus on the fractional Laplacian , i.e, $(-\Delta)^s$. We start by giving the definition of this operator and then continue with the motivation that introduces how the fractional Laplacian appeared. After all we study on the weak solutions and regularity properties of these solutions, respectively.

Definition 4.1 *The fractional Laplace operator is given by*

$$(-\Delta)^s u(x) := c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}}\, dy \tag{4.1}$$

where $s \in (0, 1)$ *is called the fractional power of the operator and* $c_{n,s} > 0$ *is the normalization constant depending on the dimension n and the fractional power s.*

First of all, one should notice that $(-\Delta)^s$ is a non-local operator. So it uses information about u far from x.

Notice also that the fractional Laplace operator has a singularity at $x = y$ so that it requires certain regularity of u in order to remove the singularity and evaluate the integral.

The fractional Laplace operator $(-\Delta)^s$ differentiates the function u in some sense because of the singularity at $x = y$. That is why it is known as an integro-differential operator.

When $s \in (0, 1/2)$ the singularity is removable and the definition of the fractional Laplacian makes sense. But when $s \in [1/2, 1)$, the integral in the definition need to be understood in the principal value sense such that

$$(-\Delta)^s u(x) = c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}}\, dy = c_{n,s} \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^n / B_\epsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+2s}}\, dy \tag{4.2}$$

Remark 4.1 *In the case* $0 < s < 1/2$*, the integral in the definition of the fractional*

Laplace operator is not really singular near x. Indeed for any $u \in \mathcal{S}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|}{|x-y|^{n+2s}}\,dy &\leq C\int_{B_R}\frac{|x-y|}{|x-y|^{n+2s}}\,dy + \|u\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n/B_R}\frac{1}{|x-y|^{n+2s}}\,dy \\
&= C\left(\int_{B_R}\frac{1}{|x-y|^{n+2s-1}}\,dy + \int_{\mathbb{R}^n/B_R}\frac{1}{|x-y|^{n+2s}}\,dy\right) \\
&= C\left(\int_0^R \frac{1}{|\rho|^{2s}}d\rho + \int_R^\infty \frac{1}{|\rho|^{2s+1}}d\rho\right) < \infty
\end{aligned}$$

where C is a positive constant depending only on n and $\|u\|_{L^\infty(\mathbb{R}^n)}$.

In the literature, one can see the definition of the fractional Laplacian in some different ways. For example if we set $z = y - x$ then the singularity appears at the origin and we get the following definition for the fractional Laplacian:

$$(-\Delta)^s u(x) = c_{n,s}\int_{\mathbb{R}^n}\frac{u(x)-u(x+z)}{|z|^{n+2s}}\,dz \tag{4.3}$$

Another option to define $(-\Delta)^s$ is using the symmetry of $|z|^{-n-2s}$. By using some calculation tricks we get the following definition:

$$(-\Delta)^s u(x) = \frac{1}{2}c_{n,s}\int_{\mathbb{R}^n}\frac{2u(x)-u(x+z)-u(x-z)}{|z|^{n+2s}}\,dz \tag{4.4}$$

The beauty of this option is removing the principal value sense for $s \geq 1/2$ too, when some regularity properties about u is given, since it is a form of a weighted second-order differential quotient and for any smooth enough u, for example $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, a second-order Taylor expansion yields

$$\frac{2u(x)-u(x+z)-u(x-z)}{|z|^{n+2s}} \leq \frac{\|D^2u\|_{L^\infty(\mathbb{R}^n)}}{|z|^{n+2s-2}} \tag{4.5}$$

and clearly we see that the right-hand side is integrable near the origin.

Using this expression one can realise that if $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ then

$$|(-\Delta)^s u(x)| \leq C\int_{B_1}\|D^2u\|_{L^\infty(\mathbb{R}^n)}\,dz + C\int_{\mathbb{R}^n - B_1}\frac{4\|u\|_{L^\infty(\mathbb{R}^n)}}{|z|^{n+2s}}\,dz \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|D^2u\|_{L^\infty(\mathbb{R}^n)}). \tag{4.6}$$

4.1. Heuristic Probabilistic Motivation

The purpose of this section is to show the connection of the fractional Laplacian with the real-world phenomena. In general, singular integrals and nonlocal operators are now becoming fashionable because of this connection. This section illustrates how the fractional Laplace operator arises in long jump random walks.

Consider the function $K : \mathbb{R}^n \to [0, \infty)$ which is even, i.e, $K(y) = K(-y)$ for any $y \in \mathbb{R}^n$ and satisfying

$$\sum_{q \in \mathbb{Z}^n} K(q) = 1. \tag{4.7}$$

Give a small $h > 0$, we consider a random walk on the lattice $h\mathbb{Z}^n$. After any time $T > 0$, a particle jumps from a point of $h\mathbb{Z}^n$ to any other point. $K(q - \tilde{q})$ is the probability that the particle jumps from the point hq to the other point $h\tilde{q}$. Notice that the particle may experience arbitrarily long jumps, though with a small probability.

Let Ω be a bounded open set in $\mathbb{R}^n$ and $g : \mathbb{R}^n/\Omega \to \mathbb{R}$ be a given payoff function. We call $u(x)$ the expected payoff that the particle will get if it starts at $x \in h\mathbb{Z}^n$. Notice that $u(x) = g(x)$ in $\mathbb{R}^n/\Omega$. Moreover, if $x \in \Omega$, then the expected payoff equals the sum of all expected payoffs of all possible positions $x + hq$ weighted by the probability of jumping from x to $x + hq$,

$$u(x) = \sum_{q \in \mathbb{Z}^n} K(q)u(x + hq) \quad \text{if } x \in \Omega. \tag{4.8}$$

Multiplying both sides of (4.8) by (4.7) yields to write the identity

$$\sum_{q \in \mathbb{Z}^n} K(q)[u(x) - u(x + hq)] = 0. \tag{4.9}$$

For the identity (4.9), the most canonical and simple choice of the kernel is

$$K(y) = c|y|^{-n-2s} \tag{4.10}$$

for $y \neq 0$ with $s \in (0, 1)$, assuming $K(0) = 0$ to coincide the kernel with the probability function. Notice that c is chosen to satisfy (4.7).

With this choice of the kernel, the identity (4.9) turns out to be

$$\sum_{q \in \mathbb{Z}^n} \frac{u(x) - u(x + hq)}{|q|^{n+2s}} = 0. \tag{4.11}$$

Multiplying by an appropriate factor h^{2s}, we get

$$h^n \sum_{q\in\mathbb{Z}^n} \frac{u(x) - u(x + hq)}{|hq|^{n+2s}} = 0. \tag{4.12}$$

which is the approximate Riemann sum of

$$\int_{\mathbb{R}^n} \frac{u(x) - u(x + y)}{|y|^{n+2s}}\, dy = 0 \quad \text{for } x \in \Omega. \tag{4.13}$$

Therefore as $h \to 0$, the limiting stochastic process will satisfy the following situation:
The expected payoff function $u(x)$ solves the Dirichlet problem for the fractional Laplacian

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^n/\Omega. \end{cases} \tag{4.14}$$

If we consider running costs or expected exit times in case of local PDEs, we are led to Dirichlet problems of type

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n/\Omega. \end{cases} \tag{4.15}$$

Remark 4.2 *The boundary conditions are in $\mathbb{R}^n/\Omega$ instead of $\partial\Omega$ as in the classical version of the Laplace operator!*

4.2. Fourier Symbol and Asymptotics of The Normalization Constant

In this section we deal with the normalization constant $c_{n,s}$ which appears in the definition of the fractional Laplacian. We also look for the Fourier symbol(or multiplier) of $(-\Delta)^s$ which is a function $S : \mathbb{R}^n \to \mathbb{R}$ such that

$$(-\Delta)^s u = \mathcal{F}^{-1}(S(\mathcal{F} u)). \tag{4.16}$$

In view of the definition (4.4) we remind for $s \in (0,1)$ that

$$(-\Delta)^s u(x) = \frac{1}{2} c_{n,s} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+z) - u(x-z)}{|z|^{n+2s}}\, dz \tag{4.17}$$

and let us now define the normalization constant.

$$c_{n,s} = \left(\int_{\mathbb{R}^n} \frac{1 - \cos \eta_1}{|\eta|^{n+2s}}\, d\eta \right)^{-1} \tag{4.18}$$

Proposition 4.1 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let $s \in (0,1)$. Then for any $u \in \mathcal{S}$, the Fourier symbol S of $(-\Delta)^s$ is $|\xi|^{2s}$ for all $\xi \in \mathbb{R}^n$.*

Proof We want to check that

$$\mathcal{F}[(-\Delta)^s u](\xi) = |\xi|^{2s} \mathcal{F}[u](\xi). \tag{4.19}$$

Indeed, we have

$$\begin{aligned}
\mathcal{F}[(-\Delta)^s u](\xi) &= \mathcal{F}\left[\frac{1}{2} c_{n,s} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+z) - u(x-z)}{|z|^{n+2s}}\, dz \right] && (4.20)\\
&= \frac{1}{2} c_{n,s} \int_{\mathbb{R}^n} (2\mathcal{F}[u(x)] - \mathcal{F}[u(x+z)] - \mathcal{F}[u(x-z)]) \frac{dz}{|z|^{n+2s}} && (4.21)\\
&= \frac{1}{2} c_{n,s} \int_{\mathbb{R}^n} (2 - e^{i\xi\cdot z} - e^{-i\xi\cdot z}) \mathcal{F}[u](\xi) \frac{dz}{|z|^{n+2s}} && (4.22)\\
&= \left[c_{n,s} \int_{\mathbb{R}^n} (1 - \cos(\xi \cdot z)) \frac{dz}{|z|^{n+2s}} \right] \mathcal{F}[u](\xi). && (4.23)
\end{aligned}$$

Hence, in order to obtain (4.1), it suffices to show that

$$c_{n,s}\int_{\mathbb{R}^n}(1-\cos(\xi\cdot z))\frac{dz}{|z|^{n+2s}} = |\xi|^{n+2s}. \tag{4.24}$$

To check this, first observe that

$$\frac{1-\cos\eta_1}{|\eta|^{n+2s}} \leq \frac{|\eta_1|^2}{|\eta|^{n+2s}} \leq \frac{1}{|\eta|^{n-2+2s}} \tag{4.25}$$

near $\eta = 0$. Therefore,

$$\int_{\mathbb{R}^n}\frac{1-\cos\eta_1}{|\eta|^{n+2s}}\,d\eta \quad \text{is finite and positive.} \tag{4.26}$$

Now, construct the function $\mathcal{J} : \mathbb{R}^n \to \mathbb{R}$ as

$$\mathcal{J}(\xi) = \int_{\mathbb{R}^n}\frac{1-\cos(\xi\cdot z)}{|z|^{n+2s}}\,dz. \tag{4.27}$$

One can easily check that $\mathcal{J}$ is rotationally invariant, i.e,

$$\mathcal{J}(\xi) = \mathcal{J}(|\xi|e_1) \tag{4.28}$$

where e_1 denotes the first direction in $\mathbb{R}^n$. Therefore the substitution $\eta = |\xi|z$ gives that

$$\begin{aligned}
\mathcal{J}(\xi) &= \mathcal{J}(|\xi|e_1) && (4.29)\\
&= \int_{\mathbb{R}^n}\frac{1-\cos(|\xi|z_1)}{|z|^{n+2s}}\,dz && (4.30)\\
&= \frac{1}{|\xi|^n}\int_{\mathbb{R}^n}\frac{1-\cos\eta_1}{|\eta/|\xi||^{n+2s}}\,d\eta && (4.31)\\
&= c_{n,s}^{-1}|\xi|^{2s}. && (4.32)
\end{aligned}$$

Hence the proof is complete by the choice of $c_{n,s} = \left(\int_{\mathbb{R}^n}\frac{1-\cos\eta_1}{|\eta|^{n+2s}}\,d\eta\right)^{-1}$ as in (4.18). □

We consider now the limit case of the fractional Laplacian $(-\Delta)^s$ when $s \downarrow 0$ and

$s \uparrow 1$. If the weak solution u is a smooth function then we rightly expect that the operator behaves just like the function u itself and the classical Laplacian $-\Delta$, respectively. We start by a corollary, without its proof, which will be used when proving the claimed limit results.

Corollary 4.1 *(Di Nezza, & Palatucci, & Valdinoci, 2011) For any $n > 1$, let $c_{n,s}$ be the normalization constant of the fractional Laplacian, defined by (4.18). The following statements hold:*

(i) $\lim_{s\uparrow 1} \frac{c_{n,s}}{s(1-s)} = \frac{4n}{\alpha(n-1)}$,

(ii) $\lim_{s\downarrow 0} \frac{c_{n,s}}{s(1-s)} = \frac{2}{\alpha(n-1)}$,

where $\alpha(n-1)$ denotes the $(n-1)$-dimensional measure of the unit sphere S^{n-1}.

We are ready to define the limit cases now.

Proposition 4.2 *(Di Nezza, & Palatucci, & Valdinoci, 2011) Let $n > 1$. For any $u \in C_0^\infty(\mathbb{R}^n)$ the following statements hold:*

(i) $\lim_{s\uparrow 1}(-\Delta)^s u = -\Delta u$,

(ii) $\lim_{s\downarrow 0}(-\Delta)^s u = u$.

Proof Fix $x \in \mathbb{R}^n$, $R_0 > 0$ such that $\text{supp} u \subseteq B_{R_0}$ and set $R = R_0 + |x| + 1$. First of all,

$$\begin{aligned} \left| \int_{B_R} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy \right| &\leq \|u\|_{C^2(\mathbb{R}^n)} \int_{B_R} \frac{|y|^2}{|y|^{n+2s}} dy \\ &\leq \alpha(n-1)\|u\|_{C^2(\mathbb{R}^n)} \int_0^R \frac{1}{|\rho|^{2s-1}} d\rho \\ &= \frac{\alpha(n-1)\|u\|_{C^2(\mathbb{R}^n)} R^{2-2s}}{2(1-s)}. \end{aligned} \tag{4.33}$$

Observe also that $|x \pm y| \geq |y| - |x| \geq R - |x| > R_0$ and $u(x \pm y) = 0$ for $|y| \geq R$. Hence,

$$\begin{aligned} -\frac{1}{2} \int_{\mathbb{R}^n/B_R} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy &= u(x) \int_{\mathbb{R}^n/B_R} \frac{1}{|y|^{n+2s}} dy \\ &= \alpha(n-1)u(x) \int_R^\infty \frac{1}{|\rho|^{2s-1}} d\rho \\ &= \frac{\alpha(n-1)R^{-2s}}{2s} u(x). \end{aligned} \tag{4.34}$$

Then by (4.33) and Corollary 4.1, we have

$$\lim_{s\downarrow 0}\left[-\frac{c_{n,s}}{2}\int_{B_R}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy\right]=0 \tag{4.35}$$

and so we get

$$\begin{aligned}
\lim_{s\downarrow 0}(-\Delta)^s u &= \lim_{s\downarrow 0}\left[-\frac{c_{n,s}}{2}\int_{\mathbb{R}^n/B_R}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy\right]\\
&= \lim_{s\downarrow 0}\left[-\frac{c_{n,s}\alpha(n-1)R^{-2s}}{2s}u(x)\right]\\
&= u(x),
\end{aligned}$$

where the last identity follows from (4.34) and Corollary 4.1. So the first part of the proof is finished.

Now consider the case $s\uparrow 1$ in a similar fashion.

$$\begin{aligned}
\left|\int_{\mathbb{R}^n/B_1}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy\right| &\le 4\|u\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n/B_1}\frac{1}{|y|^{n+2s}}dy\\
&\le 4\alpha(n-1)\|u\|_{L^\infty(\mathbb{R}^n)}\int_1^\infty\frac{1}{|\rho|^{2s+1}}d\rho\\
&= \frac{2\alpha(n-1)}{s}\|u\|_{L^\infty(\mathbb{R}^n)}.
\end{aligned}$$

Also we have

$$\lim_{s\uparrow 1}\left[-\frac{c_{n,s}}{2}\int_{\mathbb{R}^n-B_1}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy\right]=0 \tag{4.36}$$

On the other hand, we have

$$\begin{aligned}
\left|\int_{B_1}\frac{u(x+y)+u(x-y)-2u(x)-D^2u(x)y\cdot y}{|y|^{n+2s}}dy\right| &\le \|u\|_{C^3(\mathbb{R}^n)}\int_{B_1}\frac{|y|^3}{|y|^{n+2s}}dy\\
&\le \alpha(n-1)\|u\|_{C^3(\mathbb{R}^n)}\int_0^1\frac{1}{|\rho|^{2s-2}}d\rho\\
&= \frac{\alpha(n-1)\|u\|_{C^3(\mathbb{R}^n)}}{3-2s}
\end{aligned}$$

and this implies that

$$\lim_{s\uparrow 1}\left[-\frac{c_{n,s}}{2}\int_{B_1}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy\right]=\lim_{s\uparrow 1}\left[-\frac{c_{n,s}}{2}\int_{B_1}\frac{D^2u(x)y\cdot y}{|y|^{n+2s}}dy\right]. \quad (4.37)$$

Notice that if $i \neq j$ then

$$\int_{B_1}\partial^2_{ij}u(x)y_i\cdot y_j\,dy=-\int_{B_1}\partial^2_{ij}u(x)\tilde{y}_i\cdot\tilde{y}_j\,d\tilde{y}, \quad (4.38)$$

where $\tilde{y}_k = y_k$ for any $k \neq j$ and $\tilde{y}_j = -y_j$, and thus

$$\int_{B_1}\partial^2_{ij}u(x)y_i\cdot y_j\,dy=0. \quad (4.39)$$

Also for any fixed i, we get

$$\begin{aligned}\int_{B_1}\frac{\partial^2_{ij}u(x)y_i^2}{|y|^{n+2s}}dy &= \partial^2_{ij}u(x)\int_{B_1}\frac{y_i^2}{|y|^{n+2s}}dy\\ &= \partial^2_{ij}u(x)\int_{B_1}\frac{y_1^2}{|y|^{n+2s}}dy\\ &= \frac{\partial^2_{ij}u(x)}{n}\sum_{j=1}^{n}\int_{B_1}\frac{y_j^2}{|y|^{n+2s}}dy\\ &= \frac{\partial^2_{ij}u(x)}{n}\int_{B_1}\frac{|y|^2}{|y|^{n+2s}}dy\\ &= \frac{\partial^2_{ij}u(x)\alpha(n-1)}{2n(1-s)}. \end{aligned} \quad (4.40)$$

Finally, combining (4.36), (4.37), (4.39), (4.40) and Corollary 4.1, we conclude that

$$\begin{aligned}
\lim_{s\uparrow 1}(-\Delta)^s u &= \lim_{s\uparrow 1}\left[-\frac{c_{n,s}}{2}\int_{B_1}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy\right]\\
&= \lim_{s\uparrow 1}\left[-\frac{c_{n,s}}{2}\int_{B_1}\frac{D^2u(x)y\cdot y}{|y|^{n+2s}}dy\right]\\
&= \lim_{s\uparrow 1}\left[-\frac{c_{n,s}}{2}\sum_{i=1}^{n}\int_{B_1}\frac{\partial^2_{ij}u(x)y_i^2}{|y|^{n+2s}}dy\right]\\
&= \lim_{s\uparrow 1}\left[-\frac{c_{n,s}\alpha(n-1)}{4n(1-s)}\sum_{i=1}^{n}\partial^2_{ij}u(x)\right]\\
&= -\Delta u(x).
\end{aligned}$$

□

4.3. Existence of The Weak Solutions

Definition 4.2 *We say that $u \in H^s(\mathbb{R}^n)$ is a weak solution to the problem*

$$\begin{cases} (-\Delta)^s = f & in\ \Omega \\ u = g & in\ \mathbb{R}^n/\Omega, \end{cases} \tag{4.41}$$

if $\forall\varphi \in H^s(\mathbb{R}^n)$ with $\varphi \equiv 0$ in $\mathbb{R}^n/\Omega$,

$$\frac{1}{2}c_{n,s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{[u(x)-u(y)][\varphi(x)-\varphi(y)]}{|x-y|^{n+2s}}\,dxdy = \int_\Omega f\varphi. \tag{4.42}$$

Notice that if $u \in C^2(\mathbb{R}^n)$ the we can define the weak solution to (4.41) as satisfying

$$\int_\Omega (-\Delta)^s u\,\varphi = \int_\Omega f\varphi \qquad \forall\varphi \in C_c^\infty(\Omega), \tag{4.43}$$

since

$$\begin{aligned}
\int_\Omega (-\Delta)^s u(x)\varphi(x)\,dx &= \int_\Omega \left[c_{n,s}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy\right]\varphi(x)\,dx && (4.44)\\
&= c_{n,s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{[u(x)-u(y)]\varphi(x)}{|x-y|^{n+2s}}\,dydx && (4.45)\\
&= \frac{1}{2}c_{n,s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{[u(x)-u(y)][\varphi(x)-\varphi(y)]}{|x-y|^{n+2s}}\,dxdy. && (4.46)
\end{aligned}$$

Moreover if $u, \varphi \in H^s(\mathbb{R}^n)$ with $\varphi \equiv 0$ in $\mathbb{R}^n/\Omega$ and $u \equiv 0$ *a.e* in $\mathbb{R}^n/\Omega$, then we can change the definition as satisfying

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2} u\,(-\Delta)^{s/2}\varphi\,dx = \int_\Omega f\,\varphi\,dx. \qquad (4.47)$$

This definition comes from the equality

$$\begin{aligned}
\int_{\mathbb{R}^n} (-\Delta)^s u\,\varphi &= \int_{\mathbb{R}^n} |\xi|^{2s}\,\hat{u}\,\hat{\varphi} && (4.48)\\
&= \int_{\mathbb{R}^n} (|\xi|^s\,\hat{u})\,(|\xi|^s\,\hat{\varphi}) && (4.49)\\
&= \int_{\mathbb{R}^n} (-\Delta)^{s/2} u\,(-\Delta)^{s/2}\varphi && (4.50)
\end{aligned}$$

basically in Fourier side but it can also be derived from the main definition of the fractional Laplacian.

One should notice now that in all types of definitions that we derived above $\varphi \equiv 0$ in $\mathbb{R}^n/\Omega$. So we have for $D := (\mathbb{R}^n \times \mathbb{R}^n) - (\Omega^c \times \Omega^c)$ the following:

$$\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{[u(x)-u(y)][\varphi(x)-\varphi(y)]}{|x-y|^{n+2s}}\,dxdy = \frac{c_{n,s}}{2}\int\int_D \frac{[u(x)-u(y)][\varphi(x)-\varphi(y)]}{|x-y|^{n+2s}}\,dxdy. \qquad (4.51)$$

Consequently, it suffices to satisfy that

$$\frac{c_{n,s}}{2}\int\int_D \frac{[u(x)-u(y)][\varphi(x)-\varphi(y)]}{|x-y|^{n+2s}}\,dxdy < \infty \qquad (4.52)$$

for the weak solution u instead of satisfying $u \in H^s(\mathbb{R}^n)$. The importance of this change appears when g is not regular in $\mathbb{R}^n/\Omega$ or when it does not vanish at infinity. But if g is regular outside the domain Ω then the definitions coincide.

We started by the basic definition of the weak solutions and after that we considered the extra assumptions on u and also we tried to decrease the assumptions to give the best definition of the weak solution. Now let us define the last and the most general form of the weak solution of (4.41).

Definition 4.3 *We say that u is a weak solution of (4.41) if*

$$\frac{c_{n,s}}{2}\int\int_D \frac{[u(x)-u(y)]^2}{|x-y|^{n+2s}}\,dxdy < \infty \tag{4.53}$$

and

$$\frac{c_{n,s}}{2}\int\int_D \frac{[u(x)-u(y)][\varphi(x)-\varphi(y)]}{|x-y|^{n+2s}}\,dxdy = \int_\Omega f\varphi \tag{4.54}$$

for all $\varphi \in H^s(\mathbb{R}^n)$ *with* $\varphi \equiv 0$ *in* $\mathbb{R}^n/\Omega$.

Remember that we use also the energy functionals to find the weak solution to the classical Laplacian. This motivates that we can use the similar strategy for the fractional Laplacian. The energy functional associated to the problem (4.41) is

$$\mathcal{I}(u) = \frac{c_{n,s}}{4}\int\int_D \frac{[u(x)-u(y)]^2}{|x-y|^{n+2s}}\,dxdy - \int_\Omega f\,u \tag{4.55}$$

for functions u satisfying $u = g$ in $\mathbb{R}^n/\Omega$.

When g satisfies

$$\int_{\Omega^c}\int_{\Omega^c} \frac{[g(x)-g(y)]^2}{|x-y|^{n+2s}}\,dxdy < \infty, \tag{4.56}$$

then we could take the energy functional as

$$\mathcal{I}(u) = \frac{c_{n,s}}{4}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{[u(x)-u(y)]^2}{|x-y|^{n+2s}}\,dxdy - \int_\Omega f\,u \tag{4.57}$$

since the only difference between two functionals would be a constant.

Proposition 4.3 *(Ros-Oton, 2016) If u minimizes the energy functional* $\mathcal{I}(u)$ *among the* *functions u satisfying $u = g$ in $\mathbb{R}^n/\Omega$, then it is a weak solution of the problem (4.41).*

Proof If u is a minimizer then for all $\varphi \in H^s(\mathbb{R}^n)$ such that $\varphi \equiv 0$ in $\mathbb{R}^n/\Omega$, we have

$$\mathcal{I}(u + \epsilon\varphi) \geq \mathcal{I}(u) \quad \forall\epsilon > 0. \tag{4.58}$$

Thus,

$$\frac{d\mathcal{I}(u + \epsilon\varphi)}{d\epsilon}|_{\epsilon=0} = 0. \tag{4.59}$$

Hence,

$$\begin{aligned} 0 &= \frac{d\mathcal{I}(u + \epsilon\varphi)}{d\epsilon}|_{\epsilon=0} \\ &= \frac{c_{n,s}}{2}\int\int_D \frac{[u(x) - u(y)][\varphi(x) - \varphi(y)]}{|x - y|^{n+2s}}\, dxdy - \int_\Omega f\,\varphi. \end{aligned}$$

This means that u is a weak solution to the problem (4.41). □

We focus on the case $g \equiv 0$ in $\mathbb{R}^n/\Omega$, i.e,

$$\begin{cases} (-\Delta)^s = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n/\Omega. \end{cases} \tag{4.60}$$

Observe first that if g is not identically zero, but regular enough, then we can extend u to a nice function $\bar{u} = u - g$ and we get the similar Dirichlet problem (4.60) for the fractional Laplacian with f turning out to be $\bar{f} = f - (-\Delta)^s g$. This means that we will give the theorem for $g \equiv 0$ but it also holds for any regular g.

Theorem 4.1 *(Ros-Oton, 2016) Given $f \in L^2(\Omega)$, there exists a unique weak solution $u \in H^s(\mathbb{R}^n)$ of the Dirichlet problem (4.60) for the fractional Laplacian.*

Proof Define the set

$$X := \{u \in H^s(\mathbb{R}^n) \ : \ u \equiv 0 \text{ in } \mathbb{R}^n/\Omega\} \subseteq H^s(\mathbb{R}^n). \tag{4.61}$$

By Poincare' inequality, X is a Hilbert space with the scalar product

$$(u, v)_X = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{[u(x) - u(y)][v(x) - v(y)]}{|x - y|^{n+2s}} dxdy. \tag{4.62}$$

Then the weak formulation is

$$(u, \varphi)_X = \int_\Omega f \varphi \quad \forall \varphi \in X. \tag{4.63}$$

The existence and uniqueness of the weak solution follows from Riesz representation theorem. □

4.4. Some Explicit Solutions

In this section we see some examples of the Dirichlet problem for the fractional Laplacian and the corresponding explicit solutions.

Example 4.1 *For any* $r > 0$ *and* $x_0 \in \mathbb{R}^n$*,*

$$u(x) = \frac{2^{-2s}\Gamma(n/2)}{\Gamma(\frac{n+2s}{2})\Gamma(1+s)}(r^2 - |x - x_0|^2)^s \quad \textit{in } B_r(x_0) \tag{4.64}$$

is an explicit solution to the problem

$$\begin{cases} (-\Delta)^s u = 1 & \textit{in } B_r(x_0), \\ u = 0 & \textit{in } \mathbb{R}^n / B_r(x_0). \end{cases} \tag{4.65}$$

Example 4.2 *The function*

$$(x_+)^s = \begin{cases} x^s & \textit{if } x > 0 \\ 0 & \textit{if } x \leq 0 \end{cases} \tag{4.66}$$

satisfies $(-\Delta)^s (x_+)^s = 0$ *in* $(0, \infty)$.

Example 4.3 *The function*

$$u_0(x) = c_s(1 - x^2)_+^s \tag{4.67}$$

satisfies

$$\begin{cases} (-\Delta)^s u_0 = 1 & in\ (-1,1) \\ u_0 = 0 & in\ (-\infty,-1] \cup [1,\infty) \end{cases} \tag{4.68}$$

4.5. Maximum and Comparison Principles

Proposition 4.4 *(Ros-Oton, 2016)(Maximum Principle) Assume that* $u \in C^2(\mathbb{R}^n)$ *solves* $(-\Delta)^s u = 0$ *in* $\Omega \subset \mathbb{R}^n$.
Then, u cannot attain a global maximum *inside* Ω *unless u is constant in all of* $\mathbb{R}^n$.
In other words,

$$\max_{\mathbb{R}^n} u = \max_{\mathbb{R}^n/\Omega} u. \tag{4.69}$$

Proof Assume $x_0 \in \Omega$ and $u(x_0) \geq u(z)$ for all $z \in \mathbb{R}^n$, then

$$(-\Delta)^s u(x_0) = c_{n,s} \int_{\mathbb{R}^n} [u(x_0) - u(x_0 + y)] \frac{dy}{|y|^{n+2s}} \geq 0, \tag{4.70}$$

with equality if and only if $u(x_o) = u(z)$ a.e. in $\mathbb{R}^n$. Hence u is constant in $\mathbb{R}^n$. □

Proposition 4.5 *(Ros-Oton, 2016) Assume that* $u \in C^2(\mathbb{R}^n)$ *satisfies*

$$\begin{cases} (-\Delta)^s u \leq 0 & in\ \Omega \\ u \leq 0 & in\ \mathbb{R}^n/\Omega \end{cases} \tag{4.71}$$

Then,

$$u \leq 0 \quad in\ \Omega \tag{4.72}$$

Proof If u attains a positive value in Ω then it has a maximum element in Ω. So let $x_o \in \Omega$ be that element. Then $u(x_0) \geq u(z)$ for any $z \in \mathbb{R}^n$. Then, we get (4.70) again. So u is s-harmonic. By the maximum principle, u must contain its maximum in $\mathbb{R}^n/\Omega$. Since $u \leq 0$ in $\mathbb{R}^n/\Omega$, the proof is completed. □

Proposition 4.6 *(Ros-Oton, 2016) Let u be the weak solution of (4.41). Then,*

$$\begin{cases} f \geq 0 & in\ \Omega \\ g \geq 0 & in\ \mathbb{R}^n/\Omega \end{cases} \tag{4.73}$$

implies

$$u \geq 0 \quad in\ \Omega \tag{4.74}$$

Proof Recall that if u solves (4.41) then

$$\frac{c_{n,s}}{2}\int\int_D \frac{[u(x)-u(y)][\varphi(x)-\varphi(y)]}{|x-y|^{n+2s}}\,dxdy = \int_\Omega f\,\varphi. \tag{4.75}$$

for all $\varphi \in H^s(\mathbb{R}^n)$ with $\varphi \equiv 0$ in $\mathbb{R}^n/\Omega$. Now let $u = u^+ - u^-$, where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. Take $\varphi = u^-$ by assuming u^- is not identically zero as the test function. Then since $f, \varphi \geq 0$ we have

$$\int_\Omega f\,\varphi \geq 0. \tag{4.76}$$

On the other hand,

$$\begin{aligned} \int\int_D \frac{[u(x)-u(y)][\varphi(x)-\varphi(y)]}{|x-y|^{n+2s}}\,dxdy \;&=\; \int_\Omega\int_\Omega \frac{[u(x)-u(y)][u^-(x)-u^-(y)]}{|x-y|^{n+2s}}\,dxdy \\ &\quad +2\int_\Omega\int_{\mathbb{R}^n/\Omega} \frac{[u(x)-g(y)]u^-(x)}{|x-y|^{n+2s}}\,dydx \end{aligned}$$

Moreover notice that $[u^+(x) - u^+(y)][u^-(x)u^-(y)] \leq 0$ and so that

$$\int_\Omega\int_\Omega \frac{[u(x)-u(y)][u^-(x)-u^-(y)]}{|x-y|^{n+2s}}\,dxdy \leq -\int_\Omega\int_\Omega \frac{[u^-(x)-u^-(y)]^2}{|x-y|^{n+2s}}\,dxdy < 0. \tag{4.77}$$

Note that strict inequality comes from assuming $u^- \neq 0$ identically.

Also by nonnegativity of g we have

$$\int_\Omega\int_{\mathbb{R}^n/\Omega} \frac{[u(x)-g(y)]u^-(x)}{|x-y|^{n+2s}}\,dydx = -\int_\Omega\int_{\mathbb{R}^n/\Omega} \frac{[(u^-(x))^2-g(y)u^-(x)]}{|x-y|^{n+2s}}\,dydx \leq 0 \tag{4.78}$$

Hence we get

$$\frac{c_{n,s}}{2}\int\int_D \frac{[u(x)-u(y)][\varphi(x)-\varphi(y)]}{|x-y|^{n+2s}}\,dxdy < 0 \tag{4.79}$$

and this contradicts with (4.76). □

Corollary 4.2 *(Ros-Oton, 2016)(Comparison Principle) Assume that* $u_1 \in C^2(\mathbb{R}^n)$ *and* $u_1 \in C^2(\mathbb{R}^n)$ *satisfy*

$$\begin{cases} (-\Delta)^s u_1 = f_1 & \text{in } \Omega \\ u_1 = g_1 & \text{in } \mathbb{R}^n/\Omega \end{cases} \tag{4.80}$$

and

$$\begin{cases} (-\Delta)^s u_2 = f_2 & \text{in } \Omega \\ u_2 = g_2 & \text{in } \mathbb{R}^n/\Omega \end{cases} \tag{4.81}$$

respectively. Then,

$$f_1 \geq f_2 \tag{4.82}$$

$$g_1 \geq g_2 \tag{4.83}$$

implies

$$u_1 \geq u_2 \tag{4.84}$$

Proof Just use Proposition (4.6) with $u = u_1 - u_2$. □

4.6. Poisson Kernel and Its Applications

In the view of Laplace operator, Poisson kernel for a ball yields to the mean value property. Also we know that harmonic functions are infinitely differentiable as an application of Poisson kernel. In this section we see what happens for the fractional Laplacian.

Theorem 4.2 *(Ros-Oton, 2016)(Poisson kernel for $(-\Delta)^s$ in a ball)*
Consider the system

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B_1 \\ u = g & \text{in } \mathbb{R}^n/B_1. \end{cases} \tag{4.85}$$

Then,

$$u(x) = c\int_{\mathbb{R}^n/B_1} \frac{g(y)(1-|x|^2)^s}{(|y|^2-1)^s|x-y|^n}\,dy \tag{4.86}$$

We will not give the proof because it requires very detailed computations. We will use this argument to show some properties of s-harmonic functions, like analyticity or mean-value property, etc.

Corollary 4.3 *(Ros-Oton, 2016) Assume u is s-harmonic in B_1. Then, u is C^∞ inside B_1.*

Proof The dependence on x in the right hand side of the representation (4.86) is only on the term $\frac{(1-|x|^2)^s}{|x-y|^n}$ and this term is infinitely differentiable inside B_1, since $|y| \geq 1$. Therefore we can differentiate under integral infinitely many times. □

Corollary 4.4 *(Ros-Oton, 2016) Assume u is s-harmonic in B_1. Then,*

$$|D^k u(0)| \leq C^k k!\, \|u\|_{L^\infty(\mathbb{R}^n)}. \tag{4.87}$$

In particular u is analytic.

Proof

$$\begin{aligned} |D^k u(0)| &= \left| c\int_{\mathbb{R}^n/B_1} \frac{u(y)}{(|y|^2-1)^s} D^k\left(\frac{(1-|x|^2)^s}{|x-y|^n}\right)(0)dy \right| \\ &\leq c\|u\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n/B_1} \frac{1}{(|y|^2-1)^s}\frac{c^k k!}{|y|^{-n}}dy \\ &\leq c^k k! \|u\|_{L^\infty(\mathbb{R}^n)}. \end{aligned}$$

□

Corollary 4.5 *(Ros-Oton, 2016) Assume u is s-harmonic in $\Omega \subset \mathbb{R}^n$. Then, u is C^∞ inside Ω.*

Proof For any $B_r(x_0) \subset \Omega$ we get $u \in C^\infty(B_r(x_0))$ by translating and rotating Corollary 4.3. Since it can be done for any such ball, $u \in C^\infty(\Omega)$. □

Remark 4.3 *By rescaling the Poisson kernel in B_1, we find the Poisson kernel in B_r:*

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B_r \\ u = g & \text{in } \mathbb{R}^n / B_r. \end{cases} \tag{4.88}$$

Then,

$$u(x) = c \int_{\mathbb{R}^n / B_r} \frac{g(y)(r^2 - |x|^2)^s}{(|y|^2 - r^2)^s |x - y|^n} \, dy. \tag{4.89}$$

Using this Poisson kernel in B_r, we find that if u is s-harmonic then

$$u(0) = c \int_{\mathbb{R}^n / B_r} \frac{u(y) r^{2s}}{(|y|^2 - r^2)^s |y|^n} \, dy. \tag{4.90}$$

In particular we have the following proposition which is the analogous of the mean-value property for harmonic functions.

Proposition 4.7 *(Ros-Oton, 2016)(Mean value property for s-harmonic functions) If u is s-harmonic in Ω then for every $B_r \subset \Omega$, we have*

$$u(0) = c \int_{\mathbb{R}^n / B_r} \frac{u(y) r^{2s}}{(|y|^2 - r^2)^s |y|^n} \, dy. \tag{4.91}$$

Corollary 4.6 *(Ros-Oton, 2016) There exists a function $w_s(y)$ such that, if u is s-harmonic in B_1 then*

$$u(0) = \int_{\mathbb{R}^n} u(y) w_s(y) \, dy. \tag{4.92}$$

Moreover, the function $w_s(y)$ satisfies

$$\frac{C^{-1}}{1 + |y|^{n+2s}} \leq w_s(y) \leq \frac{C}{1 + |y|^{n+2s}}. \tag{4.93}$$

Proof

$$
\begin{aligned}
u(0) &= n\int_0^1 r^{n-1}u(0)\,dr \\
&= c\int_0^1\int_{\mathbb{R}^n-B_r}\frac{r^{n+2s-1}u(y)}{(|y|^2-r^2)^s|y|^n}dydr, \quad \text{by mean value property} \\
&= \underbrace{c\int_0^1\int_{|y|\geq 1}\frac{r^{n+2s-1}u(y)}{(|y|^2-r^2)^s|y|^n}dydr}_{I_1}+\underbrace{c\int_{|y|<1}\int_0^{|y|}\frac{r^{n+2s-1}u(y)}{(|y|^2-r^2)^s|y|^n}drdy}_{I_2}.
\end{aligned}
$$

We separated $u(0)$ into two integrals as I_1 and I_2. We can consider each integral respectively now. By changing variable as $r = |y|t$ we have

$$
\begin{aligned}
I_1 &= \int_{|y|\geq 1}\frac{u(y)}{|y|^n}\left(\int_0^1\frac{r^{n+2s-1}}{(|y|^2-r^2)^s}\,dr\right)dy \\
&= \int_{|y|\geq 1}u(y)\left(\int_0^{1/|y|}\frac{t^{n+2s-1}}{(1-t^2)^s}\,dt\right)dy.
\end{aligned}
$$

Similarly for I_2 with the same change of variable $r = |y|t$, we have

$$
\begin{aligned}
I_2 &= \int_{|y|<1}\frac{u(y)}{|y|^n}\left(\int_0^{|y|}\frac{r^{n+2s-1}}{(|y|^2-r^2)^s}\,dr\right)dy \\
&= \int_{|y|<1}u(y)\left(\int_0^1\frac{t^{n+2s-1}}{(1-t^2)^s}\,dt\right)dy.
\end{aligned}
$$

Hence we have shown that

$$
u(0) = \int_{\mathbb{R}^n}u(y)w_s(y)\,dy, \tag{4.94}
$$

for the function w_s as

$$
w_s(y) := \begin{cases} \int_0^1 \frac{t^{n+2s-1}}{(1-t^2)^s}\,dt, & |y| < 1 \\ \int_0^{1/|y|} \frac{t^{n+2s-1}}{(1-t^2)^s}\,dt, & |y| \geq 1. \end{cases} \tag{4.95}
$$

Now, we can clearly show that the function $w_s(y)$ is comparable with $(1 + |y|^{n+2s})^{-1}$, by using Cauchy-Schwartz inequality and the mean value theorem. □

Corollary 4.7 *(Ros-Oton, 2016)(Harnack inequality) If $u \geq 0$ in $\mathbb{R}^n$ and u is s-harmonic in B_1, then*

$$\sup_{B_{1/2}} u \leq C \inf_{B_{1/2}} u. \tag{4.96}$$

Moreover, both quantities $\sup_{B_{1/2}} u$ and $\inf_{B_{1/2}} u$ are comparable to

$$\int_{\mathbb{R}^n} \frac{u(y)}{1 + |y|^{n+2s}} \, dy. \tag{4.97}$$

4.7. Interior Regularity

We see what the regularity of u is inside the domain Ω, in this section. We start by defining Riesz potential and the fundamental solution for the fractional Laplacian. After giving the definition of the Riesz potential, we continue with the interior regularity of the weak solution u.

Definition 4.4 *(Riesz Potential) For the equation $(-\Delta)^s u = f$ we have*

$$u = (-\Delta)^{-s} f(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2s}} dy \tag{4.98}$$

for $n \geq 2$ and it is called the Riesz potential *of order $2s$. And it is denoted by $I_{2s}(f)$.*

Remark 4.4 *The function $|z|^{-n+2s}$ is the fundamental solution of the equation $(-\Delta)^s u = f$.*

Before giving the interior regularity results we need to remember that, the general embedding theorem for the Riesz potential in $\mathbb{R}^n$ yields the following:

Theorem 4.3 *(Ros-Oton, 2016) Let u be the weak solution of $(-\Delta)^s u = f$. Then for non-integers α and $\alpha + 2s$ we have*

$$||u||_{C^{2s+\alpha}(\mathbb{R}^n)} \leq C \left(||f||_{C^{\alpha}(\mathbb{R}^n)} + ||u||_{L^{\infty}(\mathbb{R}^n)} \right). \tag{4.99}$$

Now we are ready to consider regularity of the weak solution inside the domain. We consider the assumptions on the right hand side f, like boundedness or Hölder continuity.

Theorem 4.4 *(Ros-Oton, 2010) Let* $u \in L^\infty(\mathbb{R}^n)$ *be a solution to* $(-\Delta)^s u = f$ *in* B_1*. Then,*

$$\|u\|_{C^{\alpha+2s}(B_{1/2})} \leq C(\|f\|_{C^\alpha(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}), \tag{4.100}$$

whenever α *and* $\alpha + 2s$ *are not integers.*

Proof Let $F \in C^\alpha(\mathbb{R}^n)$ with supp$f \subseteq B_2$ such that $F = f$ in Ω and

$$\|F\|_{C^\alpha(\mathbb{R}^n)} \leq c\|f\|_{C^\alpha(B_1)}. \tag{4.101}$$

Let $w := I_{2s}(F)$. By Theorem 4.3 we have

$$\|w\|_{C^{2s+\alpha}(\mathbb{R}^n)} \leq C\left(\|F\|_{C^\alpha(\mathbb{R}^n)} + \|w\|_{L^\infty(\mathbb{R}^n)}\right). \tag{4.102}$$

Since F has support in B_2, we get

$$|w(x)| \leq C \int_{B_2} \frac{\|F\|_{L^\infty(B_2)}}{|x-y|^{n-2s}} dy \leq c\|f\|_{L^\infty(B_1)}, \tag{4.103}$$

which implies

$$\|w\|_{L^\infty(\mathbb{R}^n)} \leq c\|f\|_{L^\infty(B_1)}. \tag{4.104}$$

Combining (4.101) and (4.104) we find

$$\|w\|_{C^{2s+\alpha}(\mathbb{R}^n)} \leq c\|f\|_{C^\alpha(B_1)}. \tag{4.105}$$

Define now a s-harmonic function v in B_1 such that $v = u - w$. Notice that

$$\|v\|_{L^\infty(\mathbb{R}^n)} \leq \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1)}. \tag{4.106}$$

Therefore $v \in L^\infty(\mathbb{R}^n)$ is a s-harmonic function and thus it satisfies

$$\|v\|_{C^\beta(B_{1/2})} \leq C\|v\|_{L^\infty(\mathbb{R}^n)} \tag{4.107}$$

for any $\beta > 0$. Taking $\beta = \alpha + 2s$ completes the proof. □

After the main regularity estimate for the fractional Laplacian, now consider what happens if f is not $C^\alpha(B_1)$ but only $L^\infty(B_1)$. In that case the previous argument yields $u \in C^{2s-\epsilon}(B_{1/2})$ for all $\epsilon > 0$.

We next consider the case that the weak solution u is not bounded. For this we denote

$$\|u\|_{L^1_w(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}}\, dx. \tag{4.108}$$

Corollary 4.8 *(Ros-Oton, 2016) Let $u \in L^1_w(\mathbb{R}^n)$ be a solution to $(-\Delta)^s u = f$ in B_1. Then,*

$$\|u\|_{C^{\alpha+2s}(B_{1/2})} \leq C(\|f\|_{C^\alpha(B_1)} + \|u\|_{L^\infty(B_2)} + \|u\|_{L^1_w(\mathbb{R}^n)}) \tag{4.109}$$

provided that α and $\alpha + 2s$ are not integers.

Proof Consider $\tilde{u} := u\chi_{B_2}$ and notice that

$$(-\Delta)^s \tilde{u} = (-\Delta)^s u - (-\Delta)^s (u\chi_{\mathbb{R}^n/B_2}) = f - (-\Delta)^s(u\chi_{\mathbb{R}^n/B_2}) := f - h := \tilde{f}. \tag{4.110}$$

Our aim is to see that $\tilde{f} \in C^\alpha(B_1)$ in order to apply the previous theorem. For this, we need to prove that $h \in C^\alpha(B_1)$, namely

$$\|h\|_{C^\alpha(B_1)} \leq C\|u\|_{L^1_w(\mathbb{R}^n)}. \tag{4.111}$$

So take $x, \tilde{x} \in B_1$, then

$$h(x) = c_{n,s}\int_{\mathbb{R}^n} \frac{(u\chi_{\mathbb{R}^n/B_2})(x) - (u\chi_{\mathbb{R}^n/B_2})(y)}{|x-y|^{n+2s}} dy = -c_{n,s}\int_{\mathbb{R}^n/B_2} \frac{u(y)}{|x-y|^{n+2s}} dy \tag{4.112}$$

and thus

$$|h(x) - h(\tilde{x})| \leq c \int_{\mathbb{R}^n/B_2} u(y) \left(\frac{1}{|x-y|^{n+2s}} - \frac{1}{|\tilde{x}-y|^{n+2s}} \right) dy \tag{4.113}$$

Then we get

$$|D^k h(x)| = c \int_{\mathbb{R}^n/B_2} \frac{|u(y)|}{|x-y|^{n+2s+k}} dy \leq c \int_{\mathbb{R}^n/B_2} \frac{|u(y)|}{1+|y|^{n+2s+k}} dy \leq c\|u\|_{L^1_w(\mathbb{R}^n)}. \tag{4.114}$$

In particular, getting

$$\|h\|_{C^\alpha(B_1)} \leq C\|u\|_{L^1_w(\mathbb{R}^n)} \tag{4.115}$$

completes the proof. □

Our next aim is to show the boundedness of the weak solution u to the problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^n/\Omega. \end{cases} \tag{4.116}$$

Proposition 4.8 *(Ros-Oton, 2016) Let Ω be any bounded domain and u be the weak solution of (4.116) with $f \in L^\infty(\Omega)$ and $g \in L^\infty(\mathbb{R}^n)$. Then,*

$$\|u\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\mathbb{R}^n)} + C\|f\|_{L^\infty(\Omega)}. \tag{4.117}$$

Proof Let B_R be a large ball in $\mathbb{R}^n$ such that $\Omega \subset B_R$ and let $\eta \in C_c^\infty(B_R)$ be a function such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ in Ω. Then for any $x \in \Omega$, $\eta(x) = \max_{\mathbb{R}^n} \eta$, and therefore

$$\begin{aligned} (-\Delta)^s \eta(x) &= c_{n,s} \int_{\mathbb{R}^n} [\eta(x) - \eta(y)] \frac{dy}{|x-y|^{n+2s}} \\ &\geq c_{n,s} \int_{\mathbb{R}^n/B_R} \frac{dy}{|x-y|^{n+2s}} \\ &= c \int_{\mathbb{R}^n/B_R(x)} \frac{dz}{|z|^{n+2s}} \\ &\geq c \int_{\mathbb{R}^n/B_{2R}} \frac{dz}{|z|^{n+2s}} \\ &= c_0 > 0 \end{aligned}$$

Hence, η satisfies

(i) $(-\Delta)^s \eta \geq c_0 > 0 \quad \text{in } \Omega$,

(ii) $\eta = 1 \quad \text{in } \Omega$,

(iii) $\eta \geq 0 \quad \text{in } \mathbb{R}^n$.

Let now $v(x) = \|g\|_{L^\infty(\mathbb{R}^n)} + \frac{1}{c_0}\|f\|_{L^\infty(\Omega)}\eta(x)$. Then,

$$\begin{cases} (-\Delta)^s v \geq \|f\|_{L^\infty(\Omega)} & \text{in } \Omega, \\ v \geq \|g\|_{L^\infty(\mathbb{R}^n)} & \text{in } \mathbb{R}^n/\Omega. \end{cases} \tag{4.118}$$

In particular,

$$\begin{cases} (-\Delta)^s v \geq (-\Delta)^s u & \text{in } \Omega, \\ v \geq u & \text{in } \mathbb{R}^n/\Omega. \end{cases} \tag{4.119}$$

Hence from Comparison Principle we get $v \geq u$. Therefore

$$\|u\|_{L^\infty(\Omega)} \leq \|v\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\mathbb{R}^n)} + \frac{1}{c_0}\|f\|_{L^\infty(\Omega)}. \tag{4.120}$$

□

4.8. Regularity up to the boundary

We focus on the Dirichlet problem for the fractional Laplacian, i.e,

$$\begin{cases} (-\Delta)^s u = g & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n/\Omega \end{cases} \tag{4.121}$$

for some $s \in (0, 1)$ and $g \in L^\infty(\Omega)$. And our goal is to show that

$$u \in C^s(\mathbb{R}^n). \tag{4.122}$$

Notice that Hölder regularity of the weak solution u is not local. In this section we assume Ω to be a bounded Lipschitz domain satisfying the exterior ball condition. We first analyze that u is C^β in Ω, for all $\beta \in (0, 2s)$. And then we continue by the help of the sharp bounds for the corresponding seminorms near the boundary, we reach our goal.

Remember that we have given some different types of definitions of the weak solution. In here, we use the following one:

Definition 4.5 *We say that u is a weak solution of (4.121) if $u \in H^s(\mathbb{R}^n)$, $u \equiv 0$ a.e. in $\mathbb{R}^n/\Omega$ and*

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2} u \, (-\Delta)^{s/2} v \, dx = \int_\Omega g \, v \, dx \tag{4.123}$$

for all $v \in H^s(\mathbb{R}^n)$ such that $v \equiv 0$ in $\mathbb{R}^n/\Omega$.

Now we give some statements that we will use when proving the regularity of the weak solution. They are somehow about the interior regularity.

Proposition 4.9 *(Ros-Oton, & Serra, 2012) Assume that $w \in C^\infty(\mathbb{R}^n)$ solves $(-\Delta)^s w = h$ in B_1 and that neither β nor $\beta + 2s$ is an integer. Then,*

$$||w||_{C^{\beta+2s}(\overline{B_{1/2}})} \leq C(||w||_{C^\beta(\mathbb{R}^n)} + ||h||_{C^\beta(\overline{B_1})}), \tag{4.124}$$

where C is a constant depending only on n, s and β.

Proposition 4.10 *(Ros-Oton, & Serra, 2012) Assume that $w \in C^\infty(\mathbb{R}^n)$ solves $(-\Delta)^s w = h$ in B_1. Then, for every $\beta \in (0, 2s)$,*

$$||w||_{C^\beta(\overline{B_{1/2}})} \leq C(||w||_{L^\infty(\mathbb{R}^n)} + ||h||_{L^\infty(B_1)}), \tag{4.125}$$

where C is a constant depending only on n, s and β.

Corollary 4.9 *(Ros-Oton, & Serra, 2012) Assume that $w \in C^\infty(\mathbb{R}^n)$ solves $(-\Delta)^s w = h$ in B_2 and that neither β nor $\beta + 2s$ is an integer. Then,*

$$||w||_{C^{\beta+2s}(\overline{B_{1/2}})} \leq C(||(1 + |x|)^{-n-2s} w(x)||_{L^1(\mathbb{R}^n)} + ||w||_{C^\beta(\overline{B_2})} + ||h||_{C^\beta(\overline{B_2})}), \tag{4.126}$$

where C is a constant depending only on n, s and β.

Proof Let $\eta \in C^\infty(\mathbb{R}^n)$ be such that $\eta \equiv 0$ in $\mathbb{R}^n/B_2$ and $\eta \equiv 1$ in $B_{3/2}$. Then $\tilde{w} := w\eta \in C^\infty(\mathbb{R}^n)$ and $(-\Delta)^s\tilde{w} = \tilde{h} := h - (-\Delta)^s(w(1-\eta))$. Note that for $x \in B_{3/2}$ we have

$$(-\Delta)^s(w(1-\eta))(x) = c_{n,s}\int_{\mathbb{R}^n/B_{3/2}} \frac{-(w(1-\eta))(y)}{|x-y|^{n+2s}}dy. \tag{4.127}$$

So we have

$$\|(-\Delta)^s(w(1-\eta))\|_{L^\infty(B_1)} \leq C\|(1+|y|)^{-n-2s}w(y)\|_{L_1(\mathbb{R}^n)} \tag{4.128}$$

and for all $\gamma \in (0,\beta]$,

$$\begin{aligned}
[(-\Delta)^s(w(1-\eta))]_{C^\gamma(\overline{B_1})} &\leq C\|(1+|y|)^{-n-2s-\gamma}w(y)\|_{L_1(\mathbb{R}^n)}\\
&\leq C\|(1+|y|)^{-n-2s}w(y)\|_{L_1(\mathbb{R}^n)}
\end{aligned}$$

for some constant C depending only on n, s, β and η. Therefore

$$\|\tilde{h}\|_{C^\beta(\overline{B_1})} \leq C(\|h\|_{C^\beta(\overline{B_2})} + \|(1+|x|)^{-n-2s}w(x)\|_{L^1(\mathbb{R}^n)}), \tag{4.129}$$

while we also have

$$\|\tilde{w}\|_{C^\beta(\mathbb{R}^n)} \leq C\|w\|_{C^\beta(\overline{B_2})}. \tag{4.130}$$

Now the proof is completed by applying Proposition 4.9 with w replaced by $\tilde{w}$. □

Corollary 4.10 *(Ros-Oton, & Serra, 2012) Assume that $w \in C^\infty(\mathbb{R}^n)$ solves $(-\Delta)^s w = h$ in B_2. Then, for every $\beta \in (0, 2s)$,*

$$\|w\|_{C^\beta(\overline{B_{1/2}})} \leq C(\|(1+|x|)^{-n-2s}w(x)\|_{L^1(\mathbb{R}^n)} + \|w\|_{L^\infty(B_2)} + \|h\|_{L^\infty(B_2)}), \tag{4.131}$$

where C is a constant depending only on n, s and β.

Proof Analog to the proof of Corollary 4.9. □

Now we will find an explicit upper barrier for $|u|$. This is the first step to obtain C^s regularity of the weak solution u to the problem (4.121). We need the following lemma first to construct this barrier. One can find its proof from the references.

Lemma 4.1 *(Ros-Oton, & Serra, 2012)(Supersolution) There exist* $C_1 > 0$ *and a radial continuous function* $\varphi_1 \in H^s_{loc}(\mathbb{R}^n)$ *satisfying*

$$\begin{cases} (-\Delta)^s \varphi_1 \geq 1 & \text{in } B_4/B_1, \\ \varphi_1 = 0 & \text{in } B_1, \\ 0 \leq \varphi_1 \leq C_1(|x|-1)^s & \text{in } B_4/B_1, \\ 1 \leq \varphi_1 \leq C_1 & \text{in } \mathbb{R}^n/B_4. \end{cases} \tag{4.132}$$

We construct the upper barrier for $|u|$ by scaling and translating the supersolution from Lemma 4.1.

Lemma 4.2 *(Ros-Oton, & Serra, 2012) Let* Ω *be a bounded domain satisfying the exterior ball condition and let* $g \in L^\infty(\Omega)$*. Let* u *be the solution of (4.121). Then,*

$$|u(x)| \leq C\|g\|_{L^\infty(\Omega)} d^s(x) \quad \forall x \in \Omega, \tag{4.133}$$

where C *is a constant depending only on* Ω *and* s*,* $d =$*dist*$(x, \partial\Omega)$*.*

Proof The domain Ω can be contained in a large ball of radius diamΩ. Then by scaling the explicit solution for the ball (4.64) we obtain

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C(\text{diam}\Omega)^{2s}\|g\|_{L^\infty(\Omega)}, \tag{4.134}$$

where u and g is taken from (4.121).

Also there exists $\rho_0 > 0$ such that every point of $\partial\Omega$ can be touched from outside by a ball of radius ρ_0. Then by scaling and translating φ_1, for each of this exterior tangent balls B_{ρ_0} we find an upper barrier in $B_{2\rho_0}/B_{\rho_0}$ vanishing in $\overline{B_{\rho_0}}$. This yields the bound $u \leq Cd^s$ in a ρ_0-neighborhood of $\partial\Omega$. By using (4.134) we have the same bound in all of $\overline{\Omega}$. Repeating the same argument with $-u$, we find $|u| \leq C\|g\|_{L^\infty(\Omega)} d^s$. □

We give the interior estimates for u now and after that we finish with the main claim, C^s regularity of the weak solution u to the problem (4.121).

Lemma 4.3 *(Ros-Oton, & Serra, 2012) Let* Ω *be a bounded domain satisfying the exterior ball condition,* $g \in L^\infty(\Omega)$ *and* u *be the weak solution of (4.121). Then,* $u \in C^\beta(\Omega)$ *for all* $\beta \in (0, 2s)$ *and for all* $x_0 \in \Omega$ *we have the following semi-norm estimate in*

$B_R(x_0) = B_{d(x_0)/2}(x_0)$*:*

$$[u]_{C^\beta(\overline{B_R(x_0)})} \leq CR^{s-\beta}\|g\|_{L^\infty(\Omega)}, \tag{4.135}$$

where $C = C(s, \Omega, \beta)$ *is a constant.*

Proof Without loss of generality we can assume u is smooth. Because if not then by using the standard mollifier we can regularize u and pass to the limit.

Note that $B_R(x_0) \subset B_{2R}(x_0) \subset \Omega$. Let $\tilde{u} = u(x_0 + Ry)$. Then we have

$$(-\Delta)^s \tilde{u}(y) = R^{2s} g(x_0 + Ry) \quad \text{in } B_1. \tag{4.136}$$

Also using the property $|u| \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)})d^s$ in Ω we obtain

$$\|\tilde{u}\|_{L^\infty(B_1)} \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)})R^s \tag{4.137}$$

by Lemma4.2 and observing that $|\tilde{u}(y)| \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)})R^s(1 + |y|^s)$ in all of $\mathbb{R}^n$,

$$\|(1 + |y|)^{-n-2s}\tilde{u}(y)\|_{L^1(\mathbb{R}^n)} \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)})R^s, \tag{4.138}$$

with C depending only on n and s.

Next using Corollary 4.10 with (4.136), (4.137) and (4.138), we have

$$\|\tilde{u}\|_{C^\beta(\overline{B_{1/4}})} \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)})R^s \tag{4.139}$$

for all $\beta \in (0, 2s)$, where $C = C(\Omega, s, \beta)$. Finally we observe that

$$[u]_{C^\beta(\overline{B_{R/4}(x_0)})} = R^{-\beta}[\tilde{u}]_{C^\beta(\overline{B_{1/4}})}. \tag{4.140}$$

Hence, by a standard covering argument we can complete the proof. □

Finally, we are ready to prove the C^s regularity of u.

Proposition 4.11 *(Ros-Oton, & Serra, 2012) Let* Ω *be a bounded Lipschitz domain satisfying the exterior ball condition,* $g \in L^\infty(\Omega)$ *and* u *be the weak solution of (4.121). Then,*

$u \in C^s(\mathbb{R}^n)$ *and*

$$\|u\|_{C^s(\mathbb{R}^n)} \leq C\|g\|_{L^\infty(\Omega)}, \tag{4.141}$$

where C is a constant depending only on s and Ω*.*

Proof By Lemma 4.3, taking $\beta = s$ we obtain

$$\frac{|u(x) - u(y)|}{|x - y|^s} \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)}) \tag{4.142}$$

for all x, y such that $y \in B_R(x)$ with $R = d(x)/2$. Observe also that after a Lipschitz change of coordinates, the bound (4.142) remains the same except the constant C. Hence we can flatten the boundary near $x_0 \in \partial\Omega$ to assume that $\Omega \cap B_{\rho_0}(x_0) = \{x_n > 0\} \cap B_1$. Now (4.142) holds for all x, y satisfying $|x - y| \leq \lambda x_n$ for some $\lambda = \lambda(\Omega) \in (0, 1)$ depending on the Lipschitz map. Next, let $z = (z', z_n)$ and $w = (w', w_n)$ be two points in $\{x_n > 0\} \cap B_{1/4}$, and $r = |z - w|$. Let us define $\bar{z} = (z', z_n + r), \bar{w} = (w', w_n + r)$ and $z_k = (1 - \lambda^k)z + \lambda^k\bar{z}, w_k = \lambda^k w + (1 - \lambda^k)\bar{w}, k \geq 0$. Then using that bound (4.142) holds whenever $|x - y| \leq \lambda x_n$, we have

$$|u(z_{k+1}) - u(z_k)| \leq C|z_{k+1} - z_k|^s = C|\lambda^k(z - \bar{z})(\lambda - 1)|^s \leq C|z - \bar{z}|^s. \tag{4.143}$$

Moreover, since $x_n > r$ in all segment joining $\bar{z}$ and $\bar{w}$, splitting this segment into a bounded number of segments of length less that λr, we obtain

$$|u(\bar{z}) - u(\bar{w})| \leq C|\bar{z} - \bar{w}|^s \leq Cr^s. \tag{4.144}$$

Therefore,

$$\begin{aligned}
|u(z) - u(w)| &\leq |u(z) - u(\bar{z})| + |u(\bar{z}) - u(\bar{w})| + |u(\bar{w}) - u(w)| \\
&\leq \sum_{k\geq 0} |u(z_{k+1}) - u(z_k)| + |u(\bar{z}) - u(\bar{w})| + \sum_{k\geq 0} |u(w_{k+1}) - u(w_k)| \\
&\leq \left(C\sum_{k\geq 0}(\lambda^k r)^s + Cr^s\right)(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)}) \\
&\leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)})|z - w|^s,
\end{aligned}$$

as wanted. ∎

Note that this C^s regularity is optimal, i.e the weak solution u of the problem (4.121) is not in general C^α for $\alpha > s$. This can be seen by looking the solution (4.64) to the problem (4.65).

CHAPTER 5

CONCLUSION

We defined the fractional Sobolev spaces and the corresponding norm to consider the weak solution of the fractional version of Laplace's equation. We gave the embedding theorems and regularity results in the fractional Sobolev space $W^{s,p}$. After that we constructed the extension and trace theorems which are very important in the analysis of the partial differential equations. We showed that $W^{s,2} = H^s$ is a Hilbert space and so we studied the space H^s in the view of Fourier transformation.

We got motivated by the probability theory about the long jump random walk and defined the fractional Laplacian $(-\Delta)^s$. We showed, as expected, that $(-\Delta)^s$ behaves as the identity operator and the classical Laplace operator as $s \downarrow 0$ and $s \uparrow 1$, respectively. We gave the mean-value property of s-harmonic functions. Motivation of the maximum principle of the harmonic functions provided us to show that a s-harmonic function cannot attain its maximum inside its domain and this property lead to construct the comparison principle. Then we gave the explicit solutions of some example to express the topic in a better way. After all, we gave the interior regularity results of the Dirichlet problem for the fractional Laplacian. Finally, we extended this regularity up to the boundary and showed that the weak solution of the Dirichlet problem for the fractional Laplacian has an optimal C^s regularity in the closure of its domain.

There were a lot of papers about the fractional Laplacian in the literature. So, we tried to collect all the useful knowledge and prepare a thesis that can be a handbook about the fractional Laplacian. Our method was comparing the results with the classical version. We need to emphasize that one important difference of the Dirichlet problem for the fractional Laplacian is constructing the problem for the domain and outside the domain. Remember that in the classical version we work on the domain and on its boundary. It is also worth-noticing that the fractional Laplacian is a non-local operator, i.e. it needs the information away from any fixed point. Finally, this operator is constructed to coincide with the classical version for the integer powers owing to the normalization constant.

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