BOUNDARY CONTROLLER AND OBSERVER DESIGN FOR KORTEWEG-DE VRIES TYPE EQUATIONS

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ABSTRACT

BOUNDARY CONTROLLER AND OBSERVER DESIGN FOR KORTEWEG-DE VRIES TYPE EQUATIONS

This thesis studies the back-stepping boundary controllability of Korteweg-de Vries (KdV) type equations posed on a bounded interval. The results on the back-stepping controllability of the KdV equation obtained in Cerpa and Coron (2013) and Cerpa (2012) are reviewed and extended to the KdV-Burgers (KdVB) equation. The stability of the KdVB equation is boosted to any desired exponential rate for sufficiently small initial data with a boundary feedback controller acting on the Dirichlet boundary condition. Moreover, the case that there is no full access to the system is considered. For these kinds of systems, an observer is constructed assuming an appropriate boundary measurement is available. The ideas about designing output feedback control for the KdV equation presented in Marx and Cerpa (2016), and Hasan (2016) are reviewed and extended to the KdVB model.

ÖZET

KORTEWEG-DE VRIES TİPİNDEKİ DENKLEMLER İÇİN SINIR KONTROLÜ VE GÖZLEMCİ DİZAYNI

Bu tez, sonlu bir aralıkta düşünülen Korteweg-de Vries (KdV) tipi denklemlerin geri adım yöntemi ile sınırdan kontrol edilebilirliği üzerine bir çalışmadır. (Cerpa and Coron (2013)) ve (Cerpa (2012))'da bahsedilen KdV denklemi için geri adımlama tekniğinin sonuçları incelenmiş ve Korteweg-de Vries-Burgers (KdVB) denklemine genellenmiştir. Kararlılık, sol Dirichlet sınır koşulunda etkili olan sınır geri besleme kontrol girdisine sahip sistemler için başlangıç koşulunun yeterince küçük olduğu durumda sağlanmaktadır. Ayrıca dikkat edilmesi gereken nokta, üssel azalma hızının tercih edilen kadar büyük olmasıdır. Dahası, sisteme tam erişim olmayan durum düşünülmüştür. Bu tür sistemler için, uygun bir sınır ölçümü mevcutken bir gözlemci dizaynı oluşturulabilir. (Krstic (2009)), (Marx and Cerpa (2016)) ve (Hasan (2016))'da sunulan KdV denklemi için çıktı geri besleme kontrolünün tasarlanması ile ilgili fikirler üzerinde durulmuştur ve KdVB modeline aktarılmıştır. Buna ek olarak, kapalı döngü sistemlerinin üssel kararlılığı, gözlemci durumlarını da içeren Volterra dönüşümüne dayanan geri adım yöntemi kullanılarak kanıtlanmıştır.

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CHAPTER 1

INTRODUCTION

Control theory is used in many areas such as
– fluid flows in aerodynamics and propulsion applications
– flexible structures in civil engineering applications
– electromagnetic waves and quantum mechanical systems.

The back-stepping control is one of the methods developed for controlling PDEs from the boundary. The ideas behind it are associated with the feedback linearization. The nonlinearities in the system can have potential harms on stability. One method to deal with those nonlinearities is feedback linearization. On the other hand, a better method is the back-stepping method since it does require to cancel out the nonlinearities contrary to the feedback linearization.

The procedure of the back-stepping method for PDEs is as follows. At first, a desirably exponentially stable target system is chosen. The original plant is mapped into this target system via an integral transformation. This is called the forward transformation. Moreover, the original plant's boundary inputs are determined with respect to this transformation. Then, an inverse transformation is constructed and the target system is mapped back into the original plant. This is called the backward transformation. A combination of the forward and backward transformations gives the stability of the original plant subject to the boundary controllers determined by the forward transformation.

The back-stepping method has been applied to many PDEs in control design. For example, the back-stepping boundary controllers were constructed for some unstable parabolic, hyperbolic and also complex-valued PDEs (Krstic and Smyshlyaev (2008b), Cerpa (2012), Krstic and Smyshlyaev (2008a), Krstic (2009)). The Ginzburg-Landau equation (Aamo et al. (2005)), the Navier-Stokes equation (Vazquez and Krstic (2007)), and the Schrodinger equation (Krstic et al. (2011)) are some examples in the literature. In addition to these examples, oil well drilling problems (Di Meglio (2011)), kick problem (Hasan (2015)), and heave problem (Aamo (2013)) are some real life examples.

This thesis is devoted to the study of the boundary feedback controllability and observability of the Korteweg-de Vries (KdV) and Korteweg-de Vries Burgers (KdVB) equations posed on bounded intervals by using back-stepping technique.

KdV and KdVB equations are nonlinear partial differential equations of third order, which can describe approximately behaviour of long shallow water waves, see Korteweg and de Vries (1895). These equations have interesting mathematical quantities and several possible applications. The controllability and stabilization of KdV and KdVB equations have been deeply analysed because of their fascinating behaviours, (see e.g., Rosier (1997), Russell and Zhang (1996), Rosier and Zhang (2006), Rosier and Zhang (2009), Perla Menzala et al. (2002), Balogh and Krstic (2000), Jia and Zhang (2012), Krstic (1999), Tang and Krstic (2013), Gao and Deng (2007), Balogh et al. (2001), Cavalcanti et al. (2014), Hasan and Foss (2011), Qian et al. (2014), Crépeau and Prieur (2010), Liu and Krstić (2002)).

The surprising thing for KdV system is that the features of the control for this system varies depending on where the controls are put. See for example (Rosier (2004), Glass and Guerrero (2008)) for the control acting on the left boundary condition, (Rosier (1997)) for the control acting on the two right boundary conditions, and (Rosier (1997), Glass and Guerrero (2010)) for the control acting only on right-end point. In contrast to that, Coron and Crépeau (2004), Cerpa and Crépeau (2009), and Cerpa (2007) showed that the nonlinear KdV equation has the exact controllability for some spatial domains for which the linearized system cannot be controlled.

For example, consider the following linearized KdV model with the homogenous boundary conditions in $[0, 2\pi]$:

$$\begin{cases} u_t(t,x) + u_x(t,x) + u_{xxx}(t,x) = 0, \\ u(t,0) = 0, \quad u(t,2\pi) = 0, \quad u_x(t,2\pi) = 0. \end{cases} \tag{1.1}$$

One can easily show that the function $(1 - \cos(x))$ satisfies above system. Thus, one can also deduce that the solution is non-decreasing solution which implies that above system is not exponentially stable in the interval $[0, 2\pi]$.

The KdV-Burgers Equation (KdVB) also gives us distinctive properties. For example, one can deduce from Amick et al. (1989) that the Cauchy problem for KdVB equation has a solution on $\mathbb{R}$, which satisfies

$$||u(t)||_{L^2(\mathbb{R})} = O(t^{-\frac{1}{2}}). \tag{1.2}$$

It is mentioned in the same paper that there is no expectation to get a decay rate faster than this one.

Another example can be given for the case of bounded domains. Let us set the feedback controller $U \equiv 0$. Then, we multiply (1.8) by u and integrate over $(0, 1)$. By using Poincar*é* inequality and given boundary conditions the exponential decay of solutions can be obtained and u satisfies

$$||u(.,t)||_{L^2(0,1)} = O(e^{-t}). \tag{1.3}$$

On the other hand, let us consider the KdVB equation with a small viscocity coefficient $\epsilon > 0$:

$$u_t - \epsilon u_{xx} + u_{xxx} + uu_x = 0. \tag{1.4}$$

In that case, one can get a decay rate

$$\|u(t)\|_{L^2(0,1)} = O(e^{-\epsilon t}), \tag{1.5}$$

which is slower than above one.

Our main aim in this thesis is to answer the following stabilization problem:

" Is there a boundary feedback controller $U(t) = U(u(t,\cdot))$ *such that the solutions of* (1.6) *and* (1.8) *satisfy the exponential stability for any given* $\lambda > 0$*?"*

Chapter 2 is a review of Cerpa and Coron (2013). In this chapter, we study the boundary feedback controllability of the KdV equation posed on $[0, L]$

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, & \text{in } [0, L], \quad t > 0, \\ u(t,0) = U(t), \quad u(t,L) = 0, \quad u_x(t,L) = 0, & \text{in } \mathbb{R}_+, \\ u(0,x) = u_0(x), & \text{in } [0, L]. \end{cases} \tag{1.6}$$

In the above equation, $U(t)$ is called the feedback boundary controller, which is defined in terms of integral operator.

The goal is to choose a controller that steers the solution of (1.6) to zero as $t \to \infty$. Indeed, it is proven that given any positive λ, the solution of (1.6) with the boundary feedback controller $U(t)$ satisfies

$$\|u\|_{L^2_{(0,L)}} \leq e^{-\lambda t}\|u_0\|_{L^2_{(0,L)}}, \tag{1.7}$$

where u_0 is sufficiently small.

In Chapter 3, the ideas behind the boundary controllability of the KdV equation using the back-stepping method are extended to the following KdV-Burgers equation

$$
\begin{cases}
u_t - u_{xx} + u_{xxx} + uu_x = 0, & x \in [0,1], \quad t > 0, \\
u(0,t) = U(t), \quad u(1,t) = 0, \quad u_x(1,t) = 0, & \text{in } \mathbb{R}_+, \\
u(x,0) = u_0(x), & \text{in } [0,1].
\end{cases} \tag{1.8}
$$

The KdVB model with a feedback boundary controller $U(t)$ is settled in a bounded interval [0,1].

It is proven that if u_0 is sufficiently small, then the solution of (1.8) decays to zero as $t \to \infty$ with a positive rate λ which can be chosen as large as desired.

Chapter 4 reviews the observer design presented in Marx and Cerpa (2016) and Hasan (2016) for the KdV equation introduced in (1.6). The observer system is defined by

$$
\begin{cases}
\hat{u}_t(x,t) - \hat{u}_{xx}(x,t) + \hat{u}_{xxx}(x,t) + p_1(x)[y(t) - \hat{u}_{xx}(1,t)] = 0, \\
\hat{u}(0,t) = U(t), \ \hat{u}(1,t) = \hat{u}_x(1,t) = 0,
\end{cases} \tag{1.9}
$$

with the error $\tilde{u} := u - \hat{u}$ satisfying the error system

$$
\begin{cases}
\tilde{u}_t + \tilde{u}_x + \tilde{u}_{xxx} - p_1(x)\tilde{u}_{xx}(L,t) = 0, \\
\tilde{u}(0,t) = \tilde{u}(L,t) = \tilde{u}_x(L,t) = 0,
\end{cases} \tag{1.10}
$$

where $y = u_{xx}(L,t)$ is a measurement to extract some information when there is no access to the full state of the system. This measurement is used to construct an observer and apply the back-stepping technique to design an output feedback control making the closed-loop system exponentially stable. Local exponential H^3 stability of the state and of the error can be shown via Lyapunov operator. That is, for any positive λ there exists a positive constant C such that the solution of the coupled system (1.6)-(1.9) satisfies

$$
||u(.,t)||_{H^3_{(0,L)}} + ||\hat{u}(.,t)||_{L^2_{(0,L)}} \quad \leq \quad Ce^{-\lambda t}||u_0||_{H^3_{(0,L)}}. \tag{1.11}
$$

Finally, in Chapter 5, we design an observer design for the KdVB equation based on the observer model of the KdV equation in the previous chapter. More precisely, at first an observer with a boundary measurement $y = u_{xx}(1,t)$ can be defined by

$$\begin{cases} \hat{u}_t(x,t) - \hat{u}_{xx}(x,t) + \hat{u}_{xxx}(x,t) + p_1(x)[y(t) - \hat{u}_{xx}(1,t)] = 0, \\ \hat{u}(0,t) = U(t),\ \hat{u}(1,t) = \hat{u}_x(1,t) = 0. \end{cases} \tag{1.12}$$

with the error $\tilde{u} := u - \hat{u}$ satisfying the error system

$$\begin{cases} \tilde{u}_t(x,t) - \tilde{u}_{xx}(x,t) + \tilde{u}_{xxx}(x,t) - p_1(x)\tilde{u}_{xx}(1,t) = 0, \\ \tilde{u}(0,t) = \tilde{u}(1,t) = \tilde{u}_x(1,t) = 0. \end{cases} \tag{1.13}$$

After applying the back-stepping technique , it is proven that the solution of the coupled system (1.8)-(1.12) tends to zero as t goes to infinity with positive decay rate λ. Moreover, it can be proven in the same way as the KdV that the system and the observer error have local exponential H^3 stability.

CHAPTER 2

RAPID STABILIZATION FOR THE KORTOWEG-DE VRIES EQUATION FROM THE LEFT DIRICHLET BOUNDARY CONDITION

In this chapter, we will review a stabilization result for the KdV equation, which was recently proved in Cerpa and Coron (2013). In their paper, Cerpa and Coron considered the following KdV equation posed on the bounded interval [0, L] for a given $L > 0$

$$\begin{cases} u_t(t,x) + u_x(t,x) + u_{xxx}(t,x) + u(t,x)u_x(t,x) = 0, \\ u(t,0) = U(t), \quad u(t,L) = 0, \quad u_x(t,L) = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{2.1}$$

where $U(t)$ is a feedback type controller at the left end point of the boundary.

The aim is to find a boundary feedback controller which enforces the exponential decay of the system with the decay rate as large as desired.

In order to obtain such a control law, Cerpa and Coron used the back-stepping technique. See for example Cerpa and Coron (2013);Tang and Krstic (2013) for this method. At first, the exponential stabilization of the corresponding linearized system is studied. Then, the same result is extended to the nonlinear system under assumption that the initial datum is small.

Theorem 2.1 *Let $\lambda > 0$. Then, there exists $r > 0$, $D > 0$, a function $k = k(x,y)$ and a control law $U(t) = U(u(t,.)) = \int_0^L k(0,y)u(t,y)dy$ associated with the gain kernel function $k(x,y)$ such that*

$$\|u(t,.)\|_{L^2(0,L)} \leq De^{-\lambda t}\|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0 \tag{2.2}$$

for any solution of (2.1) satisfying $\|u(0,.)\|_{L^2(0,L)} \leq r$.

2.1. Control Design

Based on the linear part of the equation, we consider the system linearized around the origin

$$\begin{cases} u_t(t,x) + u_x(t,x) + u_{xxx}(t,x) = 0, & \text{in } [0,L], \quad t > 0, \\ u(t,0) = U(t), \quad u(t,L) = 0, \quad u_x(t,L) = 0, & \text{in } \mathbb{R}_+, \\ u(0,x) = u_0(x), & \text{in } [0,L]. \end{cases} \tag{2.3}$$

We use a transformation $\Pi : L^2(0,L) \to L^2(0,L)$ defined by

$$w(t,x) = \Pi(u(x)) := u(t,x) - \int_x^L k(x,y)u(t,y)dy. \tag{2.4}$$

Here, $k(x,y)$ is an unknown kernel function, and our purpose is to find out the kernel k so that if $u(x,t)$ is a solution of (2.1) with boundary feedback controller

$$U(t) = \int_0^L k(0,y)u(t,y)dy, \tag{2.5}$$

$w(t,x)$ is a solution of the following stable (target) system,

$$\begin{cases} w_t(t,x) + w_x(t,x) + w_{xxx}(t,x) + \lambda w(t,x) = 0, & \text{in } [0,L], \quad t > 0, \\ w(t,0) = 0, \quad w(t,L) = 0, \quad w_x(t,L) = 0, & \text{in } \mathbb{R}_+, \\ w(0,x) = w_0(x), & \text{in } [0,L]. \end{cases} \tag{2.6}$$

For any $t \geq 0$, the target system is exponentially stable with rate λ.
We can easily prove this fact by using the Lyapunov function defined by

$$V(t) = \frac{1}{2}\int_0^L |w(t,x)|^2 dx.$$

Differentiating the Lyapunov function and using the given boundary conditions, we have

$$\begin{aligned}
\frac{d}{dt}V(t) &= \int_0^L w(t,x)w_t(t,x)dx \\
&= -\int_0^L w(t,x)w_x(t,x)dx - \int_0^L w(t,x)w_{xxx}(t,x)dx \\
&\quad -\lambda\int_0^L w^2(t,x)dx \\
&= w(t,x)w_{xx}(t,x)|_0^L + \int_0^L w_x(t,x)w_{xx}(t,x)dx \\
&\quad -\lambda\int_0^L w^2(t,x)dx \\
&= -w_x^2(t,0) - \lambda\int_0^L w^2(t,x)dx \\
&\leq -\lambda\int_0^L w^2(t,x)dx.
\end{aligned}$$

The above inequality implies that

$$\|w(t,.)\|_{L^2(0,L)} \leq e^{-\lambda t}\|w(0,.)\|_{L^2(0,L)}, \quad \forall t \geq 0. \tag{2.7}$$

Remark 2.1 *Taking $L^2(0,L)$-norms of both sides of (2.4), one can easily show that*

$$\|w(t)\|_{L^2(0,L)} \lesssim \|u(t)\|_{L^2(0,L)}, \tag{2.8}$$

where the constant of the inequality depends on the function k.

2.2. Gain Kernel PDE and Method of Successive Approximation

We want to find out what conditions $k(x,y)$ has to satisfy. In order to do this, we substitute the given transformation into the target system. Let us first introduce the following notations:

$$\begin{cases}
k_x(x,x) = \frac{\delta}{\delta x}k(x,y)|_{y=x}, \\
k_y(x,x) = \frac{\delta}{\delta y}k(x,y)|_{y=x}, \\
\frac{d}{dx}k(x,x) = k_x(x,x) + k_y(x,x).
\end{cases} \tag{2.9}$$

At first, we compute the derivative of (2.4) with respect to t

$$
\begin{aligned}
w_t(t,x) &= u_t(t,x) - \int_x^L u_t(t,y)k(x,y)dy \\
&= u_t(t,x) + \int_x^L u_y(t,y)k(x,y)dy + \int_x^L u_{yyy}(t,y)k(x,y)dy \\
&= u_t(t,x) + k(x,y)u(t,y)|_x^L - \int_x^L u(t,y)k_y(x,y)dy \\
&\quad +k(x,y)u_{xx}(t,y)|_x^L - \int_x^L u_{yy}(t,y)k_y(x,y)dy \\
&= u_t(t,x) - k(x,x)u(t,x) - \int_x^L u(t,y)k_y(x,y)dy \\
&\quad +k(x,L)u_{xx}(t,L) - k(x,x)u_{xx}(t,x) - u_x(t,y)k_y(x,y)|_x^L \\
&\quad + \int_x^L u_y(t,y)k_{yy}(x,y)dy \\
&= u_t(t,x) - k(x,x)u(t,x) - \int_x^L u(t,y)k_y(x,y)dy \\
&\quad +k(x,L)u_{xx}(t,L) - k(x,x)u_{xx}(t,x) + u_x(t,x)k_y(x,x) \\
&\quad +u(t,y)k_{yy}(x,y)|_x^L - \int_x^L u(t,y)k_{yyy}(x,y)dy \\
&= u_t(t,x) - k(x,x)u(t,x) - \int_x^L u(t,y)k_y(x,y)dy \\
&\quad +k(x,L)u_{xx}(t,L) - k(x,x)u_{xx}(t,x) - u_x(t,x)k_y(x,x) \\
&\quad -u(t,x)k_{yy}(x,x) - \int_x^L u(t,y)k_{yyy}(x,y)dy,
\end{aligned}
$$

and then with respect to x

$$
\begin{aligned}
w_x(t,x) &= u_x(t,x) + k(x,x)u(t,x) - \int_x^L k_x(x,y)u(t,y)dy, \\
w_{xx}(t,x) &= u_{xx}(t,x) + k(x,x)u_x(t,x) + u(t,x)\frac{d}{dx}k(x,x) \\
&\quad k_x(x,x)u(t,x) - \int_x^L k_{xx}(x,y)u(t,y)dy, \\
w_{xxx}(t,x) &= u_{xxx}(t,x) + u(t,x)\frac{d^2}{dx^2}k(x,x) + 2u_x(t,x)\frac{d}{dx}k(x,x) \\
&\quad +u_{xx}(t,x)k(x,x) + u(t,x)\frac{d}{dx}k_x(x,x) + u_x(t,x)k_x(x,x) \\
&\quad +u(t,x)k_{xx}(x,x) - \int_x^L k_{xxx}(x,y)u(t,y)dy.
\end{aligned}
$$

Plugging the above equations into the target system, we obtain the following

$$
\begin{aligned}
& w_t(t,x) + w_x(t,x) + w_{xxx}(t,x) + \lambda w(t,x) = \\
& -\int_x^L u(t,y)\left\{k_{xxx}(x,y) + k_x(x,y) + k_{yyy}(x,y) + k_y(x,y) + \lambda k(x,y)\right\} dy \\
& +k(x,L)u_{xx}(t,L) + u_x(t,x)\left\{k_y(x,x) + k_x(x,x) + 2\frac{d}{dx}k(x,x)\right\} \\
& +u(t,x)\left\{\lambda + k_{xx}(x,x) - k_{yy}(x,x) + \frac{d}{dx}k_x(x,x) + \frac{d^2}{dx^2}k(x,x)\right\},
\end{aligned} \tag{2.10}
$$

If the kernel $k(x,y)$ satisfies the third order PDE below (gain kernel PDE), the right-hand side of (2.10) is zero:

$$
\begin{cases}
k_{xxx}(x,y) + k_{yyy}(x,y) + k_x(x,y) + k_y(x,y) = -\lambda k(x,y), & \text{in } T, \\
k(x,L) = 0, & \text{in } [0,L], \\
k(x,x) = 0, & \text{in } [0,L], \\
k_x(x,x) = \frac{\lambda}{3}(L-x), & \text{in } [0,L],
\end{cases} \tag{2.11}
$$

where $(x,y) \in T := \{(x,y) \,|\, x \in [0,L], y \in [x,L]\}$.

In order to solve the above equation, we transform it into an integral equation. In order to do that let us make the following change of variables

$$
t = y - x, \qquad s = x + y, \tag{2.12}
$$

and define the function $G(s,t) := k(x,y)$. We compute

$$
\begin{aligned}
k_x &= \frac{\partial G}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial G}{\partial t}\frac{\partial t}{\partial x} = G_s - G_t, \\
k_y &= \frac{\partial G}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial G}{\partial t}\frac{\partial t}{\partial y} = G_s + G_t, \\
k_{xx} &= \frac{\partial G_s}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial G_s}{\partial t}\frac{\partial t}{\partial x} - \frac{\partial G_t}{\partial s}\frac{\partial s}{\partial x} - \frac{\partial G_t}{\partial t}\frac{\partial t}{\partial x} = G_{ss} - 2G_{st} + G_{tt}, \\
k_{yy} &= G_{ss} + 2G_{st} + G_{tt},
\end{aligned}
$$

$$
\begin{aligned}
k_{xxx} &= \frac{\partial G_{ss}}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial G_{ss}}{\partial t}\frac{\partial t}{\partial x} - 2\left(\frac{\partial G_{st}}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial G_{st}}{\partial t}\frac{\partial t}{\partial x}\right) \\
&\quad + \frac{\partial G_{tt}}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial G_{tt}}{\partial t}\frac{\partial t}{\partial x}, \\
&= G_{sss} - 3G_{sst} + 3G_{tts} - G_{ttt}, \\
k_{yyy} &= G_{sss} + 3G_{sst} + 3G_{tts} + G_{ttt}.
\end{aligned}
$$

Hence, we have $k(x, y) = G(x + y, y - x)$ such that

$$
\begin{cases}
k_x = G_s - G_t, \quad k_y = G_s + G_t, \\
k_{xx} = G_{ss} - 2G_{st} + G_{tt}, \\
k_{yy} = G_{ss} + 2G_{st} + G_{tt}, \\
k_{xxx} = G_{sss} - 3G_{sst} + 3G_{tts} - G_{ttt}, \\
k_{yyy} = G_{sss} + 3G_{sst} + 3G_{tts} + G_{ttt}.
\end{cases}
$$

Therefore, the function $G(s, t)$ must solve the boundary value problem given by

$$
\begin{cases}
6G_{tts}(s, t) + 2G_{sss}(s, t) + 2G_s(s, t) = -\lambda G(s, t), & \text{in } T_0, \\
G(s, 2L - s) = 0, & \text{in } [L, 2L], \\
G(s, 0) = 0, & \text{in } [0, 2L], \\
G_t(s, 0) = \frac{\lambda}{6}(s - 2L), & \text{in } [0, 2L],
\end{cases}
\tag{2.13}
$$

defined in the triangle $T_0 := \{(s, t) \,|\, t \in [0, L], s \in [t, 2L - t]\}$.

In order to solve (2.13), we will integrate the first equation of (2.13) by using the given boundary conditions:

$$
\begin{aligned}
&\int_0^\tau 6G_{tts}(\eta, \xi) d\xi = -\int_0^\tau \{2G_{sss}(\eta, \xi) + 2G_s(\eta, \xi) + \lambda G(\eta, \xi)\}\, d\xi, \\
&6G_{ts}(\eta, \xi) - 6G_{ts}(\eta, 0) = -\int_0^\tau \{2G_{sss}(\eta, \xi) + 2G_s(\eta, \xi) + \lambda G(\eta, \xi)\}\, d\xi, \\
&\int_0^\tau 6G_{ts}(\eta, \tau) d\tau = -\int_0^t \lambda d\tau - \int_0^t \int_0^\tau \{2G_{sss}(\eta, \xi) + 2G_s(\eta, \xi) + \lambda G(\eta, \xi)\}\, d\xi d\tau, \\
&6G_s(\eta, t) - 6G_s(\eta, 0) = \lambda t - \int_0^t \int_0^\tau \{2G_{sss}(\eta, \xi) + 2G_s(\eta, \xi) + \lambda G(\eta, \xi)\}\, d\xi d\tau, \\
&\int_s^{2L-t} 6G_s(\eta, t) d\eta = \int_s^{2L-t} \lambda t d\eta - \int_s^{2L-t} \int_0^t \int_0^\tau \{2G_{sss}(\eta, \xi) + 2G_s(\eta, \xi) + \lambda G(\eta, \xi)\}\, d\xi d\tau d\eta, \\
&6G(s, t) = -\lambda t(2L - t - s) + \int_s^{2L-t} \int_0^t \int_0^\tau \{2G_{sss}(\eta, \xi) + 2G_s(\eta, \xi) + \lambda G(\eta, \xi)\}\, d\xi d\tau d\eta.
\end{aligned}
$$

$G(s,t)$ must satisfy the integral equation

$$G(s,t) = -\frac{\lambda t}{6}(2L-t-s) + \frac{1}{6}\int_{s}^{2L-t}\int_{0}^{t}\int_{0}^{\tau}\{2G_{sss}(\eta,\xi)+2G_{s}(\eta,\xi)+\lambda G(\eta,\xi)\}\,d\xi d\tau d\eta. \quad (2.14)$$

Since the solution of (2.13) is also a solution of the gain kernel PDE (2.11), if $G(s,t)$ exists, then so is $k(x,y)$. To prove the existence of $G(s,t)$, one uses the method of successive approximations.

First, we start with an initial guess

$$G^{1}(s,t) = -\left(\frac{\lambda}{6}\right) t\,[((2L-t)-s)], \quad (2.15)$$

and set up the recursive formula for (2.14) as follows

$$G^{n+1}(s,t) = G^{1}(s,t)+\frac{1}{6}\int_{s}^{2L-t}\int_{0}^{t}\int_{0}^{\tau}\{2G^{n}_{sss}(\eta,\xi)+2G^{n}_{s}(\eta,\xi)+\lambda G^{n}(\eta,\xi)\}\,d\xi d\tau d\eta, \text{ for } n\geq 1. \quad (2.16)$$

After some computations, we have the following

$$\begin{aligned} G^{2}(s,t) &= [(2L-t)-s]\left[\left(\frac{\lambda}{6}\right)^{2}\frac{t^{4}}{4.3}-\left(\frac{\lambda}{6}\right)^{2}(2L)\frac{t^{3}}{3.2}+\left(\frac{\lambda}{6}\right)\frac{t^{3}}{3.2}-\left(\frac{\lambda}{6}\right)t\right] \\ &\quad +\frac{1}{2}\left[(2L-t)^{2}-s^{2}\right]\left[\left(\frac{\lambda}{6}\right)^{2}\frac{t^{3}}{3.2}\right], \end{aligned}$$

$$\begin{aligned} G^{3}(s,t) &= [(2L-t)-s]\left[-\left(\frac{\lambda}{6}\right)^{3}\frac{t^{7}}{7.6.4.3}+\left(\frac{1}{2}\right)\left(\frac{\lambda}{6}\right)^{3}\frac{t^{7}}{7.6.3.2}\right. \\ &\quad +\left(\frac{\lambda}{6}\right)^{3}(2L)\frac{t^{6}}{6.5.4.3}+\left(\frac{\lambda}{6}\right)^{3}(2L)\frac{t^{6}}{6.5.3.2}-\left(\frac{\lambda}{6}\right)^{2}\frac{t^{6}}{6.5.3.2}-\left(\frac{1}{2}\right)(4L)\left(\frac{\lambda}{6}\right)^{3}\frac{t^{6}}{6.5.3.2} \\ &\quad -\left(\frac{\lambda}{6}\right)^{2}\left(\frac{1}{3}\right)\frac{t^{6}}{6.5.4.3}-\left(\frac{\lambda}{6}\right)^{3}(2L)^{2}\frac{t^{5}}{5.4.3.2}+\left(\frac{\lambda}{6}\right)^{2}(2L)\frac{t^{5}}{5.4.3.2} \\ &\quad +\left(\frac{1}{2}\right)(2L)^{2}\left(\frac{\lambda}{6}\right)^{3}\frac{t^{5}}{5.4.3.2}+\left(\frac{\lambda}{6}\right)^{2}\left(\frac{1}{3}\right)(2L)\frac{t^{5}}{5.4.3.2}+\left(\frac{\lambda}{6}\right)\left(\frac{1}{3}\right)\frac{t^{5}}{5.4.3.2} \end{aligned}$$

$$
+\left(\frac{\lambda}{6}\right)^2 \frac{t^4}{4.3} - \left(\frac{\lambda}{6}\right)^2 (2L) \frac{t^3}{3.2} + \left(\frac{\lambda}{6}\right) \frac{t^3}{3.2} - \left(\frac{\lambda}{6}\right) t \Bigg]
$$

$$
+\frac{1}{2}\left[(2L-t)^2 - s^2\right]\Bigg[-\left(\frac{\lambda}{6}\right)^3 \frac{t^6}{6.5.4.3} + \left(\frac{\lambda}{6}\right)^3 (2L) \frac{t^5}{5.4.3.2} + \left(\frac{\lambda}{6}\right)^2 \frac{t^5}{5.4.3.2}
$$

$$
-\left(\frac{\lambda}{6}\right)^2 \left(\frac{1}{3}\right) \frac{t^5}{5.4.3.2} + \left(\frac{\lambda}{6}\right)^2 \frac{t^3}{3.2}\Bigg]
$$

$$
+\frac{1}{3}\left[(2L-t)^3 - s^3\right]\left[\left(\frac{1}{2}\right)\left(\frac{\lambda}{6}\right)^3 \frac{t^5}{5.4.3.2}\right].
$$

The series $\sum_{n=1}^{\infty} G^n(s,t)$ is uniformly convergent in T_0 since for any $k \geq 1$ and any $(s,t) \in T_0$

$$
|G^k(s,t)| \leq M \frac{B^k}{(2k)!} (t^{2k-1} + t^{2k}), \tag{2.17}
$$

where M, B are positive constants. Hence the series defines a continuous function $G : T_0 \to \mathbb{R}$

$$
G(s,t) = \sum_{n=1}^{\infty} G^n(s,t). \tag{2.18}
$$

This leads us to the solution of our integral equation as follows

$$
\begin{aligned}
G &= G^1 + \sum_{n=1}^{\infty} G^{n+1} \\
&= G^1 + \frac{1}{6} \sum_{n=1}^{\infty} \int_s^{2L-t} \int_0^t \int_0^\tau \{2G^n_{sss}(\eta,\xi) + 2G^n_s(\eta,\xi) + \lambda G^n(\eta,\xi)\}\, d\xi d\tau d\eta \\
&= G^1 + \frac{1}{6} \int_s^{2L-t} \int_0^t \int_0^\tau \left\{2 \sum_{n=1}^{\infty} G^n_{sss}(\eta,\xi) + 2 \sum_{n=1}^{\infty} G^n_s(\eta,\xi) + \lambda \sum_{n=1}^{\infty} G^n(\eta,\xi)\right\} d\xi d\tau d\eta \\
&= G^1 + \frac{1}{6} \int_s^{2L-t} \int_0^t \int_0^\tau \{2G^n_{sss}(\eta,\xi) + 2G^n_s(\eta,\xi) + \lambda G^n(\eta,\xi)\}\, d\xi d\tau d\eta,
\end{aligned}
$$

where $\sum_{n=1}^{\infty} G^n_s(s,t)$ and $\sum_{n=1}^{\infty} G^n_{sss}(s,t)$ are also uniformly convergent.
Therefore, $G(s,t)$ is a solution of (2.14) as well as the boundary problem in (2.13) so that we obtain the existence of the kernel $k(x,y)$.

2.3. Stability of the Linear System

In this section we show that the back-stepping transformation is invertible. In other words, we will conclude that the stability of the target system implies the stability of the linear system . To complete the design, let us define the inverse transformation, Π^{-1}

$$u(t,x) \quad = \quad \Pi^{-1}(w(x)) := w(t,x) + \int_x^L \ell(x,y)w(t,y)dy, \tag{2.19}$$

where ℓ is a continuous kernel function.
Similar to the computations in the previous section we obtain the following third order PDE

$$\begin{cases} \ell_{xxx}(x,y) + \ell_{yyy}(x,y) + \ell_x(x,y) + \ell_y(x,y) = \lambda\ell(x,y), & \text{in } T, \\ \ell(x,L) = 0, & \text{in } [0,L], \\ \ell(x,x) = 0, & \text{in } [0,L], \\ \ell_x(x,x) = \frac{\lambda}{3}(L-x), & \text{in}[0,L]. \end{cases} \tag{2.20}$$

Moreover, the existence of such the kernel $\ell(x,y)$ can be proved similar to the kernel $k(x,y)$. The important thing is to realise that by taking L^2-norms both sides of the inverse transformation (2.19), we obtain

$$||u(t)||_{L^2(0,L)} \lesssim ||w(t)||_{L^2(0,L)} \tag{2.21}$$

with the constant depens on the kernel funtion ℓ.
(2.21) means that the exponential stability of w implies the exponential decay of the linearized system which proves the following proposition

Proposition 2.1 *Let λ. Then, there exists a kernel function $k = k(x,y)$ such that the solution of the linearized KdV equation in (2.3) with the boundary feedback controller $U(t) = \int_0^L k(0,y)u(t,y)dy$ satisfies*

$$||u(t)||_{L^2(\Omega)} \lesssim e^{-\lambda t}||u_0||_{L^2(\Omega)} \tag{2.22}$$

for $t \geq 0$.

2.4. Stability of the Nonlinear System

Assume that $u(t,x)$ is a solution of the nonlinear system

$$\begin{cases} u_t(t,x) + u_x(t,x) + u_{xxx}(t,x) + u(t,x)u_x(t,x) = 0, & \text{in } T, \\ u(t,0) = U(t), \quad u(t,L) = 0, \quad u_x(t,L) = 0, & \text{in } \mathbb{R}_+, \\ u(0,x) = u_0(x), & \text{in } [0,L], \end{cases} \tag{2.23}$$

where the control $U(t)$ is given by (2.5). Then, $w = \Pi(u(t,x))$ satisfies the following

$$\begin{aligned} w_t(t,x) &+ w_x(t,x) + w_{xxx}(t,x) + \lambda w(t,x) \\ &= -\left(w(t,x) + \int_x^L \ell(x,y)w(t,y)dy\right)\left(w_x(t,x) + \int_x^L \ell_x(x,y)w(t,y)dy\right) \end{aligned} \tag{2.24}$$

with homogeneous boundary conditions

$$w(t,0) = 0, \quad w(t,L) = 0, \quad \text{and} \quad w_x(t,L) = 0. \tag{2.25}$$

Multiplying the previous equation by w and integrating in $(0,L)$, we have

$$\begin{aligned} \int_0^L w(t,x)w_t(t,x)dx \quad = \quad & -\int_0^L w(t,x)w_x(t,x)dx - \int_0^L w(t,x)w_{xxx}(t,x)dx \\ & -\lambda \int_0^L w^2(t,x)dx - \int_0^L w^2(t,x)\left[\int_x^L \ell_x(x,y)w(t,y)dy\right]dx \\ & -\int_0^L w(t,x)w_x(t,x)\left[\int_x^L \ell(x,y)w(t,y)dy\right]dx \\ & -\int_0^L w(t,x)\left[\int_x^L \ell_x(x,y)w(t,y)dy\right]\left[\int_x^L \ell(x,y)w(t,y)dy\right]dx \end{aligned}$$

$$
\begin{aligned}
&\int_0^L w(t,x)w_t(t,x)dx = -w(t,x)w_{xx}(t,x)|_0^L + \int_0^L w_x(t,x)w_{xx}(t,x)dx \\
&-\lambda \int_0^L w^2(t,x)dx - \int_0^L w^2(t,x)\left[\int_x^L \ell_x(x,y)w(t,y)dy\right]dx \\
&+ \int_0^L \tfrac{1}{2}w^2(t,x)\ell(x,x)w(t,x)dx - \int_0^L \tfrac{1}{2}w^2(t,x)\left[\int_x^L \ell_x(x,y)w(t,y)dy\right]dx \\
&- \int_0^L w(t,x)\left[\int_x^L \ell_x(x,y)w(t,y)dy\right]\left[\int_x^L \ell(x,y)w(t,y)dy\right]dx \\
&= -\tfrac{1}{2}|w_x(t,0)|^2 - \lambda\int_0^L w^2(t,x)dx - \tfrac{3}{2}\int_0^L w^2(t,x)\left[\int_x^L \ell_x(x,y)w(t,y)dy\right]dx \\
&- \int_0^L w(t,x)\left[\int_x^L \ell_x(x,y)w(t,y)dy\right]\left[\int_x^L \ell(x,y)w(t,y)dy\right]dx.
\end{aligned}
$$

Using integration by parts and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(0,L)} + \lambda\|w(t)\|^2_{L^2(0,L)} \\
&\qquad\qquad \le -\frac{1}{2}|w_x(t,0)|^2 + \left(\frac{3}{2}\|\ell\|_{C^1(T)} + \|\ell\|^2_{C^1(T)}\right)\|w(t)\|^3_{L^2(0,L)}. \quad (2.26)
\end{aligned}
$$

Hence, one has the following inequality

$$
y' + 2\lambda y - Cy^{\frac{3}{2}} \quad \le \quad 0, \tag{2.27}
$$

where $y(t) \equiv \|w(t)\|^2_{L^2(0,L)}$ and $C = 2\left(\frac{3}{2}\|\ell\|_{C^1(T)} + \|\ell\|^2_{C^1(T)}\right)$. If we assume that $\|w_0\|_{L^2(0,L)} < \frac{\lambda}{C}$ and solve the inequality (2.27), then we get

$$
\|w(t)\|^2_{L^2(0,L)} = y(t) \le \frac{1}{\left[\left(\frac{1}{\|w_0\|_{L^2(0,L)}} - \frac{C}{2\lambda}\right)e^{\lambda t} + \frac{C}{2\lambda}\right]^2} < \frac{1}{\left[\frac{e^{\lambda t}}{2\|w_0\|_{L^2(0,L)}}\right]^2}. \tag{2.28}
$$

From Remark 2.1, we have $\|w_0\|_{L^2(0,L} \lesssim \|u_0\|_{L^2(0,L}$. Combining this with (2.21) and (2.28), we conclude

$$
\|u(t)\|_{L^2(0,L} \quad \lesssim \quad e^{-\lambda t}\|w(t)\|_{L^2(0,L)}, \qquad \text{for } t \ge 0. \tag{2.29}
$$

Therefore, we proved Theorem 2.1.

CHAPTER 3

STABILIZATION FOR THE KDV-BURGERS EQUATION FROM THE LEFT DIRICHLET BOUNDARY CONDITION

In this chapter we study the stabilization problem for Korteweg-de Vries Burgers (KdVB) equation

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + u_{xxx}(x,t) + u(x,t)u_x(x,t) = 0, & x \in [0,1], \quad t > 0, \\ u(0,t) = U(t), \quad u(1,t) = 0, \quad u_x(1,t) = 0, & \text{in } \mathbb{R}_+, \\ u(x,0) = u_0(x), & \text{in } [0,1]. \end{cases} \tag{3.1}$$

The aim is to construct a feedback $U(t)$ by using back-stepping technique such that the solutions of (3.1) goes to zero as $t \to \infty$ with a predetermined exponential decay rate.
Let us state our main theorem

Theorem 3.1 *Let $\lambda > 0$. There exists $\delta > 0$, $D > 0$ and a kernel function $k = k(x,y)$ such that the solution of (3.1) with the control law $U(t) = u(t,.) = \int_0^1 k(0,y)u(t,y)dy$ satisfies*

$$\|u(t,.)\|_{L^2(0,1)} \leq De^{-\lambda t}\|u_0\|_{L^2(0,1)}, \quad \forall t \geq 0 \tag{3.2}$$

for any solution of (3.1) satisfying $\|u(0,.)\|_{L^2(0,1)} \leq \delta$.

3.1. Control System

We consider the linear part of main system with the same boundary conditions

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + u_{xxx}(x,t) = 0, & x \in [0,1], \quad t > 0, \\ u(0,t) = U(t), \quad u(1,t) = 0, \quad u_x(1,t) = 0, & \text{in } \mathbb{R}_+, \\ u(x,0) = u_0(x), & \text{in } [0,1]. \end{cases} \tag{3.3}$$

We use the Volterra transformation $\Pi : L^2(0,1) \to L^2(0,1)$ defined by

$$w(x,t) = \Pi(u(x)) := u(x,t) - \int_x^1 k(x,y)u(y,t)dy. \tag{3.4}$$

The solution of (3.3) with the boundary feedback controller

$$U(t) = \int_0^1 k(0,y)u(y,t)dy \tag{3.5}$$

is mapped to the solution of the following target system

$$\begin{cases} w_t(x,t) - w_{xx}(x,t) + w_{xxx}(x,t) + \lambda w(t,x) = 0, & x \in [0,1], \quad t > 0, \\ w(0,t) = 0, \quad w(1,t) = 0, \quad w_x(1,t) = 0, & \text{in } \mathbb{R}_+, \\ w(x,0) = w_0(x), & \text{in } [0,1]. \end{cases} \tag{3.6}$$

Our aim is to convert (3.3) into the target system since it is exponentially stable with the decay rate λ. In order to show this, we use the Lyapunov function

$$V(t) = \frac{1}{2}\int_0^1 w(x,t)^2 dx. \tag{3.7}$$

Differentiating the Lyapunov function we get

$$\begin{aligned} \frac{d}{dt}V(t) &= \int_0^1 w(x,t)w_{xx}(x,t)dx - \int_0^1 w(x,t)w_{xxx}(x,t)dx \\ &\quad -\lambda \int_0^1 w^2(x,t)dx, \\ &= w(x,t)w_x(x,t)|_0^1 - \int_0^1 w_x^2(x,t)dx - w(x,t)w_{xx}(x,t)|_0^1 \\ &\quad + \int_0^1 w_x(x,t)w_{xx}(x,t)dx - \lambda \int_0^1 w^2(x,t)dx, \\ &= -\int_0^1 w_x^2(x,t)dx + \frac{1}{2}w_x^2(x,t)|_0^1 - \lambda \int_0^1 w^2(x,t)dx, \\ &= -\int_0^1 w_x^2(x,t)dx - \frac{1}{2}w_x^2(0,t) - \lambda \int_0^1 w^2(x,t)dx, \\ &\leq -\lambda \int_0^1 w^2(x,t)dx. \end{aligned}$$

We deduce that $w(x,t)$ decays to zero exponentially fast with the rate λ in the L^2-sense:

$$\|w(.,t)\|_{L^2(0,1)} \leq e^{-\lambda t}\|w(.,0)\|_{L^2(0,1)}, \quad \forall t \geq 0. \tag{3.8}$$

3.2. Gain Kernel PDE and Method of Successive Approximation

If u is a solution of the main system (3.1) with the boundary feedback controller $U(t)$ introduced in (3.5), then w is a solution of the target system with the initial data

$$w_0 = u_0 - \int_x^1 k(x,y)u_0(y)dy. \tag{3.9}$$

Now, we assume that u is solution of (3.3). Then, if we substitute the integral transformation (3.9) into the target system, we can find the conditions $k(x,y)$ has to satisfy. In order to do this let us take derivative of (3.4) with respect to t and x:

$$\begin{aligned}
w_t(x,t) &= u_t(x,t) - \int_x^1 k(x,y)u_t(y,t)dy, \\
&= u_t(x,t) - \int_x^1 k(x,y)\left\{u_{yy}(y,t) - u_{yyy}(y,t)\right\}dy, \\
&= u_t(x,t) - \int_x^1 u_{yy}(y,t)k(x,y)dy + \int_x^1 u_{yyy}(y,t)k(x,y)dy, \\
&= u_t(x,t) - u_x(y,t)k(x,y)|_x^1 + \int_x^1 u_y(y,t)k_y(x,y)dy + u_{xx}(y,t)k(x,y)|_x^1 \\
&\quad - \int_x^1 u_{yy}(y,t)k_y(x,y)dy, \\
&= u_t(x,t) + u_x(x,t)k(x,x) + u(y,t)k_y(x,y)|_x^1 - \int_x^1 u(y,t)k_{yy}(x,y)dy \\
&\quad + u_{xx}(1,t)k(x,1) - u_{xx}(x,t)k(x,x) - u_x(y,t)k_y(x,y)|_x^1 + \int_x^1 u_y(y,t)k_{yy}(x,y)dy,
\end{aligned}$$

$$
\begin{aligned}
&= u_t(x,t) + u_x(x,t)k(x,x) - u(x,t)k_y(x,x) - \int_x^1 u(y,t)k_{yy}(x,y)dy + u_{xx}(1,t)k(x,1) \\
&\quad -u_{xx}(x,t)k(x,x) + u_x(x,t)k_y(x,x) + u(y,t)k_{yy}(x,y)|_x^1 - \int_x^1 u(y,t)k_{yyy}(x,y)dy, \\
&= u_t(x,t) + u_x(x,t)k(x,x) - u(x,t)k_y(x,x) - \int_x^1 u(y,t)k_{yy}(x,y)dy + u_{xx}(1,t)k(x,1) \\
&\quad -u_{xx}(x,t)k(x,x) + u_x(x,t)k_y(x,x) - u(x,t)k_{yy}(x,x) - \int_x^1 u(y,t)k_{yyy}(x,y)dy.
\end{aligned}
$$

$$
w_x(x,t) = u_x(x,t) + u(x,t)k(x,x) - \int_x^1 k_x(x,y)u(y,t)dy.
$$

$$
\begin{aligned}
w_{xx}(x,t) &= u_{xx}(x,t) + u_x(x,t)k(x,x) + u(x,t)\frac{d}{dx}k(x,x) \\
&\quad +u(x,t)k_x(x,x) - \int_x^1 k_{xx}(x,y)u(y,t)dy.
\end{aligned}
$$

$$
\begin{aligned}
w_{xxx}(x,t) &= u_{xxx}(x,t) + u(x,t)\frac{d^2}{dx^2}k(x,x) + 2u_x(x,t)\frac{d}{dx}k(x,x) \\
&\quad +u_{xx}(x,t)k(x,x) + u(x,t)\frac{d}{dx}k_x(x,x) + u_x(x,t)k_x(x,x) \\
&\quad +u(x,t)k_{xx}(x,x) - \int_x^1 k_{xxx}(x,y)u(y,t)dy.
\end{aligned}
$$

By using the target system, we obtain

$$
\begin{gathered}
w_t(x,t) - w_{xx}(x,t) + w_{xxx}(x,t) + \lambda w(x,t) = \\
u(x,t)\left\{k_{xx}(x,x) + 2k_{xy}(x,x) + k_{xx}(x,x) + k_{xy}(x,x) + k_{xx}(x,x) + \lambda\right\} \\
+u_x(x,t)\left\{k_y(x,x) + k_x(x,x) + k_y(x,x) + k_x(x,x) + k_y(x,x) + k_x(x,x)\right\} \\
+u_{xx}(1,t)k(x,1) + \int_x^1 u(y,t)\left\{-k_{yy}(x,y) - k_{yyy}(x,y) + k_{xx}(x,y) - k_{xxx}(x,y) - \lambda k(x,y)\right\}dy.
\end{gathered}
$$

Note that we use the same notations (2.9) in Chapter 2.

In order for the right hand side of (3.1) to be zero, the kernel $k(x,y)$ defined in the triangle $T := \{(x, y) \,|\, x \in [0, 1], y \in [x, 1]\}$ (see Figure 3.1) must satisfy

$$\begin{cases} k_{yy}(x, y) + k_{yyy}(x, y) - k_{xx}(x, y) + k_{xxx}(x, y) = -\lambda k(x, y), & \text{in } T, \\ k(x, 1) = 0, & \text{in } [0, 1], \\ k(x, x) = 0, & \text{in } [0, 1], \\ k_x(x, x) = \frac{\lambda}{3}(1 - x), & \text{in } [0, 1]. \end{cases} \tag{3.10}$$

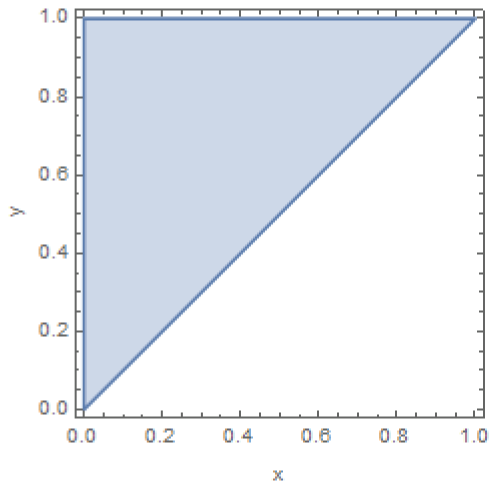

Figure 3.1. Triangular domain T

In order to find appropriate $k(x, y)$ and prove the existence of the solution, we convert the gain kernel PDE into an integral equation. Let us define $G(s, t) := k(x, y)$ and make the following change of variables

$$t = y - x, \qquad s = x + y.$$

Then, we obtain

$$k_x = G_s - G_t, \quad k_y = G_s + G_t,$$
$$k_{xx} = G_{ss} - 2G_{st} + G_{tt}, \quad k_{yy} = G_{ss} + 2G_{st} + G_{tt},$$
$$k_{xxx} = G_{sss} - 3G_{sst} + 3G_{tts} - G_{ttt},$$
$$k_{yyy} = G_{sss} + 3G_{sst} + 3G_{tts} + G_{ttt}.$$

Therefore, the function $G(s,t)$ must be a solution of the following third order PDE

$$\begin{cases} 2G_{sss}(s,t) + 6G_{stt}(s,t) + 4G_{st}(s,t) = -\lambda G(s,t), & \text{in } T_0, \\ G(s, 2-s) = 0, & \text{in } [1,2], \\ G(s,0) = 0, & \text{in } [0,2], \\ G_t(s,0) = -\frac{\lambda}{6}(2-s), & \text{in } [0,2], \end{cases} \tag{3.11}$$

on the triangle $T_0 := \{(s,t)\,|\,t \in [0,1], s \in [t, 2-t]\}$ (see Figure 3.2).

To transform the previous system into an integral equation, we integrate (3.11) by using the given boundary conditions

$$\begin{aligned}
&\int_0^\tau G_{stt}(\eta,\xi)d\xi = -\tfrac{1}{6}\int_0^\tau \{2G_{sss}(\eta,\xi) + 4G_{st}(\eta,\xi) + \lambda G(\eta,\xi)\}\,d\xi,\\
&G_{st}(\eta,\tau) - G_{st}(\eta,0) = -\tfrac{1}{6}\int_0^\tau \{2G_{sss}(\eta,\xi) + 4G_{st}(\eta,\xi) + \lambda G(\eta,\xi)\}\,d\xi,\\
&\int_0^t G_{st}(\eta,\tau)d\tau = \int_0^t \tfrac{\lambda}{6}d\tau - \tfrac{1}{6}\int_0^t\int_0^\tau \{2G_{sss}(\eta,\xi) + 4G_{st}(\eta,\xi) + \lambda G(\eta,\xi)\}\,d\xi d\tau,\\
&G_s(\eta,t) - G_s(\eta,0) = \tfrac{\lambda}{6}t - \tfrac{1}{6}\int_0^t\int_0^\tau \{2G_{sss}(\eta,\xi) + 4G_{st}(\eta,\xi) + \lambda G(\eta,\xi)\}\,d\xi d\tau,\\
&\int_s^{2-t} G_s(\eta,t)d\eta = \int_s^{2-t} \tfrac{\lambda}{6}t - \tfrac{1}{6}\int_s^{2-t}\int_0^t\int_0^\tau \{2G_{sss}(\eta,\xi) + 4G_{st}(\eta,\xi) + \lambda G(\eta,\xi)\}\,d\xi d\tau d\eta,\\
&G(2-t,t) - G(s,t) = \tfrac{\lambda}{6}t(2-t-s) - \tfrac{1}{6}\int_s^{2-t}\int_0^t\int_0^\tau \{2G_{sss}(\eta,\xi) + 4G_{st}(\eta,\xi) + \lambda G(\eta,\xi)\}\,d\xi d\tau d\eta,\\
&G(s,t) = -\tfrac{\lambda}{6}t(2-t-s) + \tfrac{1}{6}\int_s^{2-t}\int_0^t\int_0^\tau \{2G_{sss}(\eta,\xi) + 4G_{st}(\eta,\xi) + \lambda G(\eta,\xi)\}\,d\xi d\tau d\eta.
\end{aligned}$$

Hence, we have the following integral equation

$$G(s,t) = -\frac{\lambda}{6}t(2-t-s) + \frac{1}{6}\int_s^{2-t}\int_0^t\int_0^\tau \{2G_{sss}(\eta,\xi) + 4G_{st}(\eta,\xi) + \lambda G(\eta,\xi)\}\,d\xi d\tau d\eta. \tag{3.12}$$

To find out $G(s,t)$, we use the method of successive approximations. Let us define the recursive formula,

$$G^{n+1}(s,t) = G^1(s,t) + \frac{1}{6}\int_s^{2-t}\int_0^t\int_0^\tau \{2G^n_{sss}(\eta,\xi) + 4G^n_{st}(\eta,\xi) + \lambda G^n(\eta,\xi)\}\,d\xi d\tau d\eta, \text{ for } n \geq 1, \tag{3.13}$$

and set below $G^1(s,t)$ as an initial guess

$$G^1(s,t) = \frac{\lambda}{6}t(s+t-2). \tag{3.14}$$

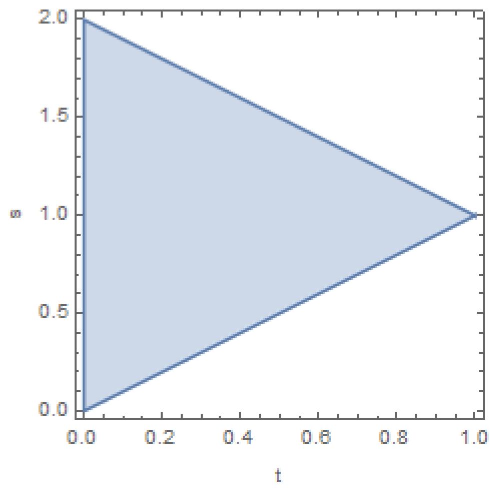

Figure 3.2. Triangular domain T_0

We have the following lemma.

Lemma 3.1 *Let G^1 and G^{n+1} be defined by* (3.14) *and* (3.13)*. Then there exists a C^3-function G such that $\lim_{n\to\infty} G^n = G$ (uniformly), and moreover, G solves the integral equation* (3.12) *as well as the boundary value problem given in* (3.11).

Proof 3.1 *In order to prove the desired result we will show that G^n is Cauchy in $C(T_0)$. Let us first introduce the following notations:*
Let P be the linear differential operator given by

$$P\varphi = \frac{1}{3}\varphi_{sss} + \frac{2}{3}\varphi_{st} + \frac{\lambda}{6}\varphi$$

for $\varphi = \varphi(s,t)$, and define the linear integration operator below

$$I[\varphi](s,t) \equiv \int_s^{2-t} \int_0^t \int_0^\tau \varphi(\eta,\xi)d\xi d\tau d\eta.$$

Let us set $H^1 \equiv G^1$ and $H^{n+1} \equiv I[PH^n]$ for $n \geq 1$. Then, we get

$$G^2 \equiv G^1 + I[PH^1] = G^1 + H^2, \quad and$$

$$
\begin{aligned}
G^3 = G^1 + I[PG^2] = G^1 + I[PG^1 + PH^2] &= G^1 + I[PG^1] + I[PH^2] \\
&= G^1 + I[PH^1] + H^3 \\
&= G^2 + H^3.
\end{aligned}
$$

More generally, $G^{n+1} = G^n + H^{n+1}$*. Then, for* $m > n$

$$
\max_{T_0} |G^m - G^n| = \max_{T_0} \left| \sum_{k=n+1}^{m} (G^k - G^{k-1}) \right| \leq \sum_{k=n+1}^{m} \max_{T_0} |G^k - G^{k-1}| = \sum_{k=n+1}^{m} \max_{T_0} |H^k|.
$$

In order to prove the Cauchy criteria for G^n*, we need to prove that the series* $\sum_{k=1}^{\infty}$ *is absolutely convergent. This absolute convergence can be proved by obtaining a good estimate on* $|H^k|$*. To do this we start with observing the first few* H^k*'s. For* $k = 1$*, we get*

$$
H^1 = G^1 = -\left(\frac{\lambda}{6}\right) t \left[(2-t) - s\right]. \tag{3.15}
$$

Then,

$$
PH^1 = -\left(\frac{\lambda}{6}\right)^2 t \left[(2-t) - s\right] + \left(\frac{\lambda}{6}\right)\left(\frac{2}{3}\right). \tag{3.16}
$$

For $k = 2$*, we have*

$$
\begin{aligned}
H^2 = I[PH^1] = -\left(\frac{\lambda}{6}\right)^2 \Bigg[& \frac{2t^3}{3 \cdot 2} \left[(2-t) - s\right] \\
& - \frac{t^4}{4 \cdot 3} \left[(2-t) - s\right] - \frac{t^3}{3 \cdot 2} \frac{\left[(2-t)^2 - s^2\right]}{2} \Bigg] + \left(\frac{\lambda}{6}\right)\left(\frac{2}{3}\right) \frac{t^2}{2 \cdot 1}\left[(2-t) - s\right] \\
= \left[(2-t) - s\right] & \left[\left(\frac{\lambda}{6}\right)^2 \frac{t^4}{4 \cdot 3} - \left(\frac{\lambda}{6}\right)^2 \frac{2t^3}{3 \cdot 2} + \left(\frac{\lambda}{6}\right)\left(\frac{2}{3}\right) \frac{t^2}{2 \cdot 1} \right] \\
& + \frac{1}{2}\left[(2-t)^2 - s^2\right] \left[\left(\frac{\lambda}{6}\right)^2 \frac{t^3}{3 \cdot 2} \right].
\end{aligned} \tag{3.17}
$$

Then,

$$PH^2 = [(2-t)-s]\left[\left(\frac{\lambda}{6}\right)^3 \frac{t^4}{4\cdot 3} - \left(\frac{\lambda}{6}\right)^3 \frac{2t^3}{3\cdot 2} + \left(\frac{\lambda}{6}\right)^2\left(\frac{2}{3}\right)\frac{t^2}{2\cdot 1}\right]$$
$$+ \frac{1}{2}\left[(2-t)^2 - s^2\right]\left[\left(\frac{\lambda}{6}\right)^3 \frac{t^3}{3\cdot 2}\right]$$
$$- \left[\left(\frac{\lambda}{6}\right)^2\left(\frac{2}{3}\right)\frac{4\cdot t^3}{4\cdot 3} - \left(\frac{\lambda}{6}\right)^2\left(\frac{2}{3}\right)\frac{2\cdot 3t^2}{3\cdot 2} + \left(\frac{\lambda}{6}\right)\left(\frac{2}{3}\right)^2 \frac{2t}{2\cdot 1}\right] - \frac{1}{2}2s\left[\left(\frac{\lambda}{6}\right)^2\left(\frac{2}{3}\right)\frac{3t^2}{3\cdot 2}\right]. \quad (3.18)$$

For $k = 3$, we have

$$H^3 = I[PH^2]$$
$$= [(2-t)-s]\left[\left(\frac{\lambda}{6}\right)^3 \frac{2t^6}{6\cdot 5\cdot 4\cdot 3} - \left(\frac{\lambda}{6}\right)^3 \frac{2^2t^5}{5\cdot 4\cdot 3\cdot 2} + \left(\frac{\lambda}{6}\right)^2\left(\frac{2}{3}\right)\frac{2t^4}{4\cdot 3\cdot 2\cdot 1}\right]$$
$$+ [(2-t)-s]\left[-\left(\frac{\lambda}{6}\right)^3 \frac{t^7}{7\cdot 6\cdot 4\cdot 3} + \left(\frac{\lambda}{6}\right)^3 \frac{2t^6}{6\cdot 5\cdot 3\cdot 2} - \left(\frac{\lambda}{6}\right)^2\left(\frac{2}{3}\right)\frac{t^5}{5\cdot 4\cdot 2\cdot 1}\right]$$

$$+ \frac{1}{2}\left[(2-t)^2 - s^2\right]\left[-\left(\frac{\lambda}{6}\right)^3 \frac{t^6}{6\cdot 5\cdot 4\cdot 3} + \left(\frac{\lambda}{6}\right)^3 \frac{2t^5}{5\cdot 4\cdot 3\cdot 2} - \left(\frac{\lambda}{6}\right)^2\left(\frac{2}{3}\right)\frac{2t^4}{4\cdot 3\cdot 2\cdot 1}\right]$$
$$+ [(2-t)-s]\,\frac{1}{2}\left[\left(\frac{\lambda}{6}\right)^3 \frac{2^2t^5}{5\cdot 4\cdot 3\cdot 2}\right]$$

$$+ [(2-t)-s]\,\frac{1}{2}\left[-\left(\frac{\lambda}{6}\right)^3 \frac{4t^6}{6\cdot 5\cdot 3\cdot 2}\right] + [(2-t)-s]\,\frac{1}{2}\left[\left(\frac{\lambda}{6}\right)^3 \frac{t^7}{7\cdot 6\cdot 3\cdot 2}\right]$$
$$+ \frac{1}{2\cdot 3}\left[(2-t)^3 - s^3\right]\left[\left(\frac{\lambda}{6}\right)^3 \frac{t^5}{5\cdot 4\cdot 3\cdot 2}\right]$$
$$- [(2-t)-s]\left[\left(\frac{\lambda}{6}\right)^2\left(\frac{2}{3}\right)\frac{4\cdot t^5}{4\cdot 3\cdot 4\cdot 5} - \left(\frac{\lambda}{6}\right)^2\left(\frac{2}{3}\right)\frac{2\cdot 3t^4}{3\cdot 2\cdot 3\cdot 4} + \left(\frac{\lambda}{6}\right)\left(\frac{2}{3}\right)^2 \frac{2t^3}{2\cdot 1\cdot 2\cdot 3}\right]$$
$$- \frac{1}{2}\left[(2-t)^2 - s^2\right]\left[\left(\frac{\lambda}{6}\right)^2\left(\frac{2}{3}\right)\frac{3t^4}{3\cdot 4\cdot 3\cdot 2}\right]$$
$$= [(2-t)-s]\left[-\left(\frac{\lambda}{6}\right)^3 \frac{t^7}{7\cdot 6\cdot 4\cdot 3} + \frac{1}{2}\left(\frac{\lambda}{6}\right)^3 \frac{t^7}{7\cdot 6\cdot 3\cdot 2}\right.$$

$$
\begin{aligned}
&+\left(\frac{\lambda}{6}\right)^3 \frac{2t^6}{6\cdot 5\cdot 4\cdot 3}+\left(\frac{\lambda}{6}\right)^3 \frac{2t^6}{6\cdot 5\cdot 3\cdot 2}-\frac{1}{2}\left(\frac{\lambda}{6}\right)^3 \frac{4t^6}{6\cdot 5\cdot 3\cdot 2}\\
&-\left(\frac{\lambda}{6}\right)^3 \frac{2^2t^5}{5\cdot 4\cdot 3\cdot 2}-\left(\frac{\lambda}{6}\right)^2\left(\frac{2}{3}\right)\frac{t^5}{5\cdot 4\cdot 2\cdot 1}+\frac{1}{2}\left(\frac{\lambda}{6}\right)^3 \frac{2^2t^5}{5\cdot 4\cdot 3\cdot 2}-\left(\frac{\lambda}{6}\right)^2\left(\frac{2}{3}\right)\frac{4\cdot t^5}{4\cdot 3\cdot 4\cdot 5}\\
&\left(\frac{\lambda}{6}\right)^2\left(\frac{2}{3}\right)\frac{2t^4}{4\cdot 3\cdot 2\cdot 1}+\left(\frac{\lambda}{6}\right)^2\left(\frac{2}{3}\right)\frac{2\cdot 3t^4}{3\cdot 2\cdot 3\cdot 4}\\
&+\left(\frac{\lambda}{6}\right)\left(\frac{2}{3}\right)^2\frac{2t^3}{2\cdot 1\cdot 2\cdot 3}\Bigg]\\
&+\frac{1}{2}\left[(2-t)^2-s^2\right]\Bigg[-\left(\frac{\lambda}{6}\right)^3\frac{t^6}{6\cdot 5\cdot 4\cdot 3}+\left(\frac{\lambda}{6}\right)^3\frac{2t^5}{5\cdot 4\cdot 3\cdot 2}\\
&-\left(\frac{\lambda}{6}\right)^2\left(\frac{2}{3}\right)\frac{2t^4}{4\cdot 3\cdot 2\cdot 1}-\left(\frac{\lambda}{6}\right)^2\left(\frac{2}{3}\right)\frac{3t^4}{3\cdot 4\cdot 3\cdot 2}\Bigg]\\
&+\frac{1}{2\cdot 3}\left[(2-t)^3-s^3\right]\left[\left(\frac{\lambda}{6}\right)^3\frac{t^5}{5\cdot 4\cdot 3\cdot 2}\right]. \qquad (3.19)
\end{aligned}
$$

We catch a pattern from above calculations which yields us to the following structure for H^k:

$$
H^k=\sum_{i=1}^{k}\frac{1}{i!}\left[(2-t)^i-s^i\right]\cdot\left[c^k_{3k-1-i,i}t^{3k-1-i}+c^k_{3k-2-i,i}t^{3k-2-i}+...+c^k_{k-1+i,i}t^{k-1+i}\right]. \qquad (3.20)
$$

When we calculate $I[PH^{k-1}]$, two important observations appear for H^k: the maximum number of terms of type $[(2-t)^i-s^i]t^j$ is $3k-5(k\geq 2)$ and the lowest denominator of the terms of $c^k_{j,i}$ is $k!$ (when $i=1$). Hence, the coefficient $3k-5$ coming from the previous step of succession when we cancel it with k in the $k!$ and observe that $\frac{3k-5}{k}<3$. We would obtain a bound in the form 3^k, in k steps. Considering the other terms, we estimate

$$
\left|c^k_{j,i}\right|\leq\frac{3^k2^{k-i}\alpha^k}{(k-2+i)!}
$$

for $j\in\{k-1+i,k+i,...,3k-1-i\}$ with $\alpha\equiv\max\left\{\frac{\lambda}{6},\frac{2}{3}\right\}$.

Not that the maximum number of terms of H^k is $3k-1-1-(k-1+1)+1=2k-1$ (when $i=1$). It follows that

$$
\max_{T_0}|H^k|\leq\frac{2^{k-1}\alpha^k3^k(2k-1)}{(k-1)!}.
$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{2^{k-1}\alpha^k 3^k (2k-1)}{(k-1)!} \leq 6\alpha \sum_{k=1}^{\infty} \frac{(12\alpha)^{k-1}}{(k-1)!} < 6\alpha e^{12\alpha} < \infty. \tag{3.21}$$

3.3. Stability of the Linear System

As we know from Cerpa and Coron (2013), we were able to say that the linearized KdV system is stable based on the stability of the target system since the back-stepping transformation is invertible. We can use the same method to get the conclusion. Let us try to find an inverse transformation in the form

$$u(x,t) \quad = \quad \Pi^{-1}(w(x)) := w(x,t) + \int_x^1 \ell(x,y)w(y,t)dy. \tag{3.22}$$

Differentiating the inverse transformation with respect to t

$$\begin{aligned}
u_t(x,t) \quad &= \quad w_t(x,t) + \int_x^1 \ell(x,y)w_t(y,t)dy, \\
&= \quad w_t(x,t) + \int_x^1 \ell(x,y)\left\{w_{yy}(y,t) - w_{yyy}(y,t) - \lambda w(y,t)\right\}dy, \\
&= \quad w_t(x,t) + \int_x^1 \ell(x,y)w_{yy}(y,t)dy - \int_x^1 \ell(x,y)w_{yyy}(y,t)dy - \int_x^1 \lambda\ell(x,y)w(y,t)dy, \\
&= \quad w_t(x,t) + w_x(y,t)\ell(x,y)|_x^1 - \int_x^1 \ell_y(x,y)w_y(y,t)dy \\
&\quad\; -w_{xx}(y,t)\ell(x,y)|_x^1 + \int_x^1 \ell_y(x,y)w_{yy}(y,t)dy - \int_x^1 \lambda\ell(x,y)w(y,t)dy, \\
&= \quad w_t(x,t) - w_x(x,t)\ell(x,x) - w(y,t)\ell_y(x,y)|_x^1 + \int_x^1 \ell_{yy}(x,y)w(y,t)dy \\
&\quad\; -w_{xx}(1,t)\ell(x,1) + w_{xx}(x,t)\ell(x,x) + w_x(y,t)\ell_y(x,y)|_x^1 \\
&\quad\; - \int_x^1 \ell_{yy}(x,y)w_y(y,t)dy - \int_x^1 \lambda\ell(x,y)w(y,t)dy,
\end{aligned}$$

$$
\begin{aligned}
&= w_t(x,t) - w_x(x,t)\ell(x,x) + w(x,t)\ell_y(x,x) + \int_x^1 \ell_{yy}(x,y)w(y,t)dy \\
&\quad - w_{xx}(1,t)\ell(x,1) + w_{xx}(x,t)\ell(x,x) - w_x(x,t)\ell_y(x,x) - w(y,t)\ell_{yy}(x,y)|_x^1 \\
&\quad + \int_x^1 \ell_{yyy}(x,y)w(y,t)dy - \int_x^1 \lambda\ell(x,y)w(y,t)dy, \\
&= w_t(x,t) - w_x(x,t)\ell(x,x) + w(x,t)\ell_y(x,x) + \int_x^1 \ell_{yy}(x,y)w(y,t)dy \\
&\quad - w_{xx}(1,t)\ell(x,1) + w_{xx}(x,t)\ell(x,x) - w_x(x,t)\ell_y(x,x) + w(x,t)\ell_{yy}(x,x) \\
&\quad + \int_x^1 \ell_{yyy}(x,y)w(y,t)dy - \int_x^1 \lambda\ell(x,y)w(y,t)dy.
\end{aligned}
$$

Differentiating the inverse transformation with respect to x, we obtain

$$
\begin{aligned}
u_{xx}(x,t) &= w_{xx}(x,t) - w_x(x,t)\ell(x,x) - w(x,t)\frac{d}{dx}\ell(x,x) - w(x,t)\ell_x(x,x) \\
&\quad + \int_x^1 \ell_{xx}(x,y)w(y,t)dy,
\end{aligned}
$$

and

$$
\begin{aligned}
u_{xxx}(x,t) &= w_{xxx}(x,t) - w_{xx}(x,t)\ell(x,x) - w_x(x,t)\frac{d}{dx}\ell(x,x) \\
&\quad - w_x(x,t)\frac{d}{dx}\ell(x,x) - w(x,t)\frac{d^2}{dx^2}\ell(x,x) - w_x(x,t)\ell_x(x,x) \\
&\quad - w(x,t)\frac{d}{dx}\ell_x(x,x) - w(x,t)\ell_{xx}(x,x) + \int_x^1 \ell_{xxx}(x,y)w(y,t)dy.
\end{aligned}
$$

Substituting these equations into the linear equation, we have the following kernel PDE system

$$
\begin{cases}
\ell_{yyy}(x,y) + \ell_{xxx}(x,y) + \ell_{yy}(x,y) - \ell_{xx}(x,y) = \lambda\ell(x,y), & \text{in } T, \\
\ell(x,1) = 0, & \text{in } [0,1], \\
\ell(x,x) = 0, & \text{in } [0,1], \\
\ell_x(x,x) = \frac{\lambda}{3}(1-x), & \text{in } [0,1],
\end{cases}
\tag{3.23}
$$

defined in the triangle T.

Existence of the kernel $\ell(x,y)$ can be proved as the existence of $k(x,y)$. More specifically, the above PDE can be transformed into another third order PDE which can be written as an integral equation. Moreover, the solution of the integral equation can be obtained by the method of successive approximations. Here, the crucial point is to realise that, from (3.22), we have the following estimation

$$\|u(t)\|_{L^2(0,1)} \lesssim \|w(t)\|_{L^2(0,1)} \tag{3.24}$$

which means that u has an exponential decay rate since w satisfies (3.8).

3.4. Stability of the Nonlinear System

Suppose that $u(x,t)$ solves the nonlinear system

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + u_{xxx}(x,t) + u(x,t)u_x(x,t) = 0, & x \in [0,1], \quad t > 0, \\ u(0,t) = U(t), \quad u(1,t) = 0, \quad u_x(1,t) = 0, & \text{in } \mathbb{R}_+, \\ u(x,0) = u_0(x), & \text{in } [0,1]. \end{cases} \tag{3.25}$$

with the control input $U(t)$ defined in (3.5). Then, the following PDE can be obtained by using the transformation introduced in (3.4)

$$\begin{aligned} & w_t(x,t) - w_{xx}(x,t) + w_{xxx}(x,t) + \lambda w(x,t) \\ = & -\left(w(x,t) + \int_x^1 \ell(x,y)w(y,t)dy\right)\left(w_x(x,t) + \int_x^1 \ell_x(x,y)w(y,t)dy\right) \end{aligned} \tag{3.26}$$

with homogeneous boundary conditions

$$w(0,t) = 0, \quad w(1,t) = 0, \quad \text{and} \quad w_x(1,t) = 0. \tag{3.27}$$

Multiplying (3.26) by $w(x,t)$ and integrating in $(0,1)$, we obtain

$$\begin{aligned}
\int_0^1 w(x,t)w_t(x,t)dx &= \int_0^1 w(x,t)w_{xx}(x,t)dx - \int_0^1 w(x,t)w_{xxx}(x,t)dx \\
&\quad -\lambda \int_0^1 w^2(x,t)dx - \int_0^1 w^2(x,t)w_x(x,t)dx \\
&\quad - \int_0^1 w^2(x,t)\left[\int_x^1 \ell_x(x,y)w(y,t)dy\right]dx \\
&\quad - \int_0^1 w(x,t)w_x(x,t)\left[\int_x^1 \ell(x,y)w(y,t)dy\right]dx \\
&\quad - \int_0^1 w(x,t)\left[\int_x^1 \ell(x,y)w(y,t)dy\right]\left[\int_x^1 \ell_x(x,y)w(y,t)dy\right]dx.
\end{aligned}$$

$$\begin{aligned}
\int_0^1 w(x,t)w_t(x,t)dx &= w(x,t)w_x(x,t)|_0^1 - \int_0^1 w_x^2(x,t) - w(x,t)w_{xx}(x,t)|_0^1 \\
&\quad + \int_0^1 w_x(x,t)w_{xx}(x,t)dx - \lambda \int_0^1 w^2(x,t)dx \\
&\quad + \int_0^1 \frac{1}{2}w^2(x,t)\ell(x,x)w(x,t)dx \\
&\quad - \int_0^1 \frac{1}{2}w^2(x,t)\left[\int_x^1 \ell_x(x,y)w(y,t)dy\right]dx \\
&\quad - \int_0^1 w^2(x,t)\left[\int_x^1 \ell_x(x,y)w(y,t)dy\right]dx \\
&\quad - \int_0^1 w(x,t)\left[\int_x^1 \ell(x,y)w(y,t)dy\right]\left[\int_x^1 \ell_x(x,y)w(y,t)dy\right]dx \\
&= -\int_0^1 w_x^2(t,x)dx - \frac{1}{2}|w_x(0,t)|^2 - \lambda \int_0^1 w^2(x,t)dx \\
&\quad -\frac{3}{2}\int_0^1 w^2(x,t)\left[\int_x^1 \ell_x(x,y)w(y,t)dy\right]dx \\
&\quad - \int_0^1 w(x,t)\left[\int_x^1 \ell(x,y)w(y,t)dy\right]\left[\int_x^1 \ell_x(x,y)w(y,t)dy\right]dx.
\end{aligned}$$

Using integration by parts and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(0,1)} + \lambda\|w(t)\|^2_{L^2(0,1)} \\
&\qquad \leq -\|w_x(t)\|^2_{L^2(0,1)} - \frac{1}{2}|w_x(0,t)|^2 + \left(\frac{3}{2}\|\ell\|_{C^1(T)} + \|\ell\|^2_{C^1(T)}\right)\|w(t)\|^3_{L^2(0,1)}. \quad (3.28)
\end{aligned}$$

Then, we have the following inequality:

$$y' + 2\lambda y - cy^{\frac{3}{2}} \le 0, \tag{3.29}$$

where $y(t) \equiv \|w(t)\|^2_{L^2(\Omega)}$ and $c = 2\left(\frac{3}{2}\|\ell\|_{C^1(T)} + \|\ell\|^2_{C^1(T)}\right)$.

Let us now assume that $\|w_0\|_{L^2(0,1)} < \frac{\lambda}{c}$ and solve the inequality (3.29). Then, we obtain

$$\|w(t)\|^2_{L^2(0,1)} = y(t) \le \frac{1}{\left[\left(\frac{1}{\|w_0\|_{L^2(0,1)}} - \frac{c}{2\lambda}\right)e^{\lambda t} + \frac{c}{2\lambda}\right]^2} < \frac{1}{\left[\frac{e^{\lambda t}}{2\|w_0\|_{L^2(0,1)}}\right]^2}, \tag{3.30}$$

from which it follows that

$$\|w(t)\|_{L^2(0,1)} \lesssim \|w_0\|_{L^2(0,1)}. \tag{3.31}$$

Recalling that $\|w_0\|_{L^2(\Omega)} \lesssim \|u_0\|_{L^2(\Omega)}$ and combining this with (3.24) and (3.31), we conclude that

$$\|u(t)\|_{L^2(\Omega)} \lesssim e^{-\lambda t}\|u_0\|_{L^2(\Omega)}e^{-\lambda t}, \text{ for } t \ge 0. \tag{3.32}$$

As a conclusion, the nonlinear system is exponentially stable with decay rate λ under the assumption that the initial datum is small. Hence, the proof of Theorem 3.1 is complete.

CHAPTER 4

OUTPUT FEEDBACK STABILIZATION OF THE LINEAR KORTEWEG-DE VRIES EQUATION

In chapter 2, we designed a control system for the following KdV system

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) + u(x,t)u_x(x,t) = 0, & \text{in } T, \\ u(0,t) = U(t), \quad u(L,t) = 0, \quad u_x(L,t) = 0, & \text{in } \mathbb{R}_+, \\ u(x,0) = u_0(x), & \text{in } [0,L], \end{cases} \tag{4.1}$$

where $U(t)$ is the boundary controller and u_0 is the initial condition. We proved the exponential stability of both the linearized system and nonlinear system with decay rate λ. We also observe that this rate can be chosen as large as we desire.

In this chapter, based on Krstic and Smyshlyaev (2008b), Marx and Cerpa (2016), and Hasan (2016), review the construction of an observer for the linearized model so that the exponential stabilization can be achieved where the system is not fully observable. The main result is the following theorem:

Theorem 4.1 *For any positive parameter λ, there exist a control input $U(t) := U(\hat{u}(x,t))$, a function $p_1 = p_1(x)$, and a positive constant C such that the coupled system* (4.3)-(4.4) *is globally exponential stable with a decay rate equals to λ. That is,*

$$\|u(.,t)\|_{H^3_{(0,L)}} + \|\hat{u}(.,t)\|_{L^2_{(0,L)}} \leq Ce^{-\lambda t}\|u_0\|_{H^3_{(0,L)}}. \tag{4.2}$$

4.1. Observer Design

In this section, we will construct an observer for the linearized KdV equation

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) = 0, \\ u(0,t) = U(t), \quad u(L,t) = 0, \quad \text{and} \quad u_x(L,t) = 0, \\ y(t) = u_{xx}(L,t). \end{cases} \tag{4.3}$$

We consider the observer

$$\begin{cases} \hat{u}_t(x,t) + \hat{u}_x(x,t) + \hat{u}_{xxx}(x,t) + p_1(x)[y(t) - \hat{u}_{xx}(L,t)] = 0, \\ \hat{u}(0,t) = U(t),\ \hat{u}(L,t) = \hat{u}_x(L,t) = 0. \end{cases} \tag{4.4}$$

We define the error $\tilde{u} := u - \hat{u}$ and it satisfies

$$\begin{cases} \tilde{u}_t(x,t) + \tilde{u}_x(x,t) + \tilde{u}_{xxx}(x,t) - p_1(x)\tilde{u}_{xx}(L,t) = 0, \\ \tilde{u}(0,t) = \tilde{u}(L,t) = \tilde{u}_x(L,t) = 0. \end{cases} \tag{4.5}$$

For $\lambda > 0$, let us introduce the back-stepping transformation Π_0 as follows

$$\tilde{u}(x,t) \;=\; \Pi_0(\tilde{w}(x)) = \tilde{w}(x,t) - \int_x^L p(x,y)\tilde{w}(y,t)dy. \tag{4.6}$$

The above map transforms (4.5) into the following linear system

$$\begin{cases} \tilde{w}_t(x,t) + \tilde{w}_x(x,t) + \tilde{w}_{xxx}(x,t) + \lambda\tilde{w}(x,t) = 0, \\ \tilde{w}(0,t) = \tilde{w}(L,t) = \tilde{w}_x(L,t) = 0, \end{cases} \tag{4.7}$$

whose solution is exponentially stable with rate λ.

Now, our goal is to find the kernel $p(x,y)$ such that $\tilde{u}(x,t)$ solves (4.5). For that purpose, we differentiate the transformation with respect to t and x, respectively.

$$\begin{aligned} \tilde{u}_t(x,t) \;=\; & \tilde{w}_t(x,t) - \tilde{w}(x,t)p(x,x) + \tilde{w}_{xx}(L,t)p(x,L) \\ & -\tilde{w}_{xx}(x,t)p(x,x) + \tilde{w}_x(x,t)p_y(x,x) - \tilde{w}(x,t)p_{yy}(x,x) \\ & - \int_x^L \tilde{w}(y,t)\left\{p_{yyy}(x,y) + p_y(x,y) - \lambda p(x,y)\right\} dy. \end{aligned}$$

$$\tilde{u}_x(x,t) \;=\; \tilde{w}_x(x,t) + \tilde{w}(x,t)p(x,x) - \int_x^L p_x(x,y)\tilde{w}(y,t)dy.$$

$$
\begin{aligned}
\tilde{u}_{xxx}(x,t) &= \tilde{w}_{xxx}(x,t) + \tilde{w}_{xx}(x,t)p(x,x) + 2\tilde{w}_x(x,t)\frac{d}{dx}p(x,x) \\
&\quad + \tilde{w}(x,t)\frac{d^2}{dx^2}p(x,x) + \tilde{w}_x(x,t)p_x(x,x) + \tilde{w}(x,t) \\
&\quad + \tilde{w}(x,t)p_{xx}(x,x) - \int_x^L p_{xxx}(x,y)\tilde{w}(y,t)dy.
\end{aligned}
$$

Inserting above equations into the error system, we obtain that

$$
\begin{aligned}
\tilde{u}_t(x,t) + \tilde{u}_x(x,t) + \tilde{u}_{xxx}(x,t) - p_1(x)\tilde{u}_{xx}(L,t) &= \\
\tilde{w}_t(x,t) + \tilde{w}_x(x,t) + \tilde{w}_{xxx}(x,t) + \lambda\tilde{w}(x,t) & \\
+\tilde{w}(x,t)\left(p_{xx}(x,x) + \frac{d^2}{dx^2}p(x,x) + \frac{d}{dx}p_x(x,x) - p_{yy}(x,x) - \lambda\right) & \\
+\tilde{w}_x(x,t)\left(2\frac{d}{dx}p(x,x) + p_x(x,x) + p_y(x,x)\right) + \tilde{w}_{xx}(L,t)\left(p(x,L) - p_1(x)\right) & \\
-\int_x^L \tilde{w}(y,t)\left\{p_{yyy}(x,y) + p_y(x,y) + p_{xxx}(x,y) + p_x(x,y) - \lambda p(x,y)\right\}dy. &
\end{aligned}
$$

we have four condition on $T := \{(x,y)\,|\,x \in [0,L], y \in [x,L]\}$ in order to make right-hand side equal to zero

$$
\begin{cases}
p_{yyy}(x,y) + p_{xxx}(x,y) + p_y(x,y) + p_x(x,y) = \lambda p(x,y), & (x,y) \in T, \\
3p_x(x,x) + 3p_y(x,y) = 0, & x \in [0,L], \\
3p_{xx}(x,x) + 3p_{xy}(x,x) - \lambda = 0, & x \in [0,L], \\
p(x,L) = p_1(x), & x \in [0,L].
\end{cases}
\tag{4.8}
$$

In addition, if we set $x = 0$ in the transformation Π_0, we see the following equality

$$
p(0,y) = 0, \quad \forall\, y \in [0,L]. \tag{4.9}
$$

Hence, the gain kernel PDE can be written as

$$
\begin{cases}
p_{xxx}(x,y) + p_{yyy}(x,y) + p_y(x,y) + p_x(x,y) = \lambda p(x,y), & (x,y) \in T, \\
p(x,x) = 0, & x \in [0,L], \\
p_x(x,x) = \frac{\lambda}{3}(x), & x \in [0,L], \\
p(0,y) = 0, & x \in [0,L].
\end{cases}
\tag{4.10}
$$

By using the method of successive approximation, we will find the kernel $p(x,y)$ which solves (4.10). In order to do this, let us introduce new variables

$$\bar{x} = L - y, \qquad \bar{y} = L - x, \tag{4.11}$$

and define $F(\bar{x},\bar{y}) := p(x,y)$. So, the expectation is that $F(\bar{x},\bar{y})$ solves the following PDE

$$\begin{cases} F_{\bar{x}\bar{x}\bar{x}}(\bar{x},\bar{y}) + F_{\bar{y}\bar{y}\bar{y}}(\bar{x},\bar{y}) + F_{\bar{y}}(\bar{x},\bar{y}) + F_{\bar{x}}(\bar{x},\bar{y}) = -\lambda F(\bar{x},\bar{y}), & (\bar{x},\bar{y}) \in T, \\ F(\bar{x},\bar{x}) = 0, & \bar{x} \in [0,L], \\ F_{\bar{x}}(\bar{x},\bar{x}) = \frac{\lambda}{3}\left(L - \bar{x}\right), & \bar{x} \in [0,L], \\ F(\bar{x},L) = 0, & \bar{y} \in [0,L]. \end{cases} \tag{4.12}$$

One can realise that we have dealt with the same system as (4.12) in Chapter 2 (2.11). Therefore, we can say that there exists $F(\bar{x},\bar{y})$ solving (4.12). It follows that there also exists $p(x,y)$ solving (4.10). It follows that the function Π_0 is linear and continuous.

4.2. Stability Analysis of the Closed Loop System

In order to prove that the error $\tilde{u} = u - \hat{u}$ tends to 0 as t tends to infinity, let us define the back-stepping transformation and its inverse for $\hat{u}$ as follows

$$\hat{w}(x,t) = \hat{u}(x,t) - \int_x^L k(x,y)\hat{u}(y,t)dy \tag{4.13}$$

and

$$\hat{u}(x,t) = \hat{w}(x,t) + \int_x^L \ell(x,y)\hat{w}(y,t)dy, \tag{4.14}$$

where k and ℓ satisfy (2.11) and (2.20), respectively. From (4.13) we obtain the following equation for $\hat{w}$

$$\begin{cases} \hat{w}_t(x,t) + \hat{w}_x(x,t) + \hat{w}_{xxx}(x,t) + \lambda\hat{w}(x,t) = \\ -\left\{p_1(x) - \int_x^L k(x,y)p_1(y)dy\right\}\tilde{w}_{xx}(L,t), \\ \hat{w}(0,t) = \hat{w}(L,t) = \hat{w}_x(L,t) = 0, \end{cases} \tag{4.15}$$

Here, the important thing is that the exponential decay of (4.15) would imply that the closed loop system decays exponentially since the maps Π and Π_0 are continuous, invertible and their inverse maps are continuous as well. In order to prove the stability of the target system, we use the Lyapunov argument by setting the following function

$$V(t) = V_1(t) + V_2(t) + V_3(t), \tag{4.16}$$

and for appropriate coefficients A, B

$$V_1(t) = \frac{A}{2}\int_0^L |\hat{w}(x,t)|^2 dx, \tag{4.17}$$

$$V_2(t) = \frac{B}{2}\int_0^L |\tilde{w}(x,t)|^2 dx, \tag{4.18}$$

$$V_3(t) = \frac{B}{2}\int_0^L |\tilde{w}_t(x,t)|^2 dx, \tag{4.19}$$

Let us take derivative of the function $V(t)$:

$$\begin{aligned}\dot{V}_1(t) &= A\int_0^L \hat{w}_t(x,t)\hat{w}(x,t)dx \\ &\leq (-A\lambda + D^2)\int_0^L |\hat{w}(x,t)|^2 dx + A^2|\tilde{w}_{xx}(L,t)|^2 \\ &= 2\left(-\lambda + \frac{D^2}{A}\right)V_1(t) + A^2|\tilde{w}_{xx}(L,t)|^2\end{aligned}$$

where $D := max_{x\in[0,L]}\left\{p_1(x) - \int_x^L k(x,y)p_1(y)dy\right\}$.
One can notice the fact that the target system in chapter 2 and the target system satisfied by $\tilde{w}(x,t)$ in this chapter are actually the same type of systems. Taking derivative of $V_2(t)$, we obtain

$$\dot{V}_2(t) \leq -2\lambda V_2.$$

Differentiating $V_3(t)$, we get

$$\begin{aligned}\dot{V}_3(t) &= \int_0^L z_t(x,t)z(x,t)dx \\ &= -\frac{B}{2}|z_x(0,t)|^2 - B\lambda\int_0^L |z(x,t)|^2 dx \\ &\leq -2\lambda V_3(t).\end{aligned}$$

Combining the above inequalities, we have

$$\begin{aligned}\dot{V}(t) &\leq 2\left(-\lambda + \frac{D^2}{A}\right)V_1(t) + A^2|\tilde{w}_{xx}(L,t)|^2 \\ &\quad -2\lambda V_2 - 2\lambda V_3(t).\end{aligned} \tag{4.20}$$

At this point, we need more information about the term $|\tilde{w}_{xx}(L,t)|^2$. We will obtain an estimate on the term $|\tilde{w}_{xx}(L,t)|^2$. For this purpose, we multiply the target system

$$\begin{cases}\tilde{w}_t(x,t) + \tilde{w}_x(x,t) + \tilde{w}_{xxx}(x,t) + \lambda\tilde{w}(x,t) = 0, \\ \tilde{w}(0,t) = \tilde{w}(L,t) = \tilde{w}_x(L,t) = 0,\end{cases} \tag{4.21}$$

by $x\tilde{w}_{xx}$, then we obtain

$$\begin{aligned}|\tilde{w}_{xx}(L,t)|^2 &\leq \left(\frac{1}{L} + L\right)||\tilde{w}_{xx}||^2_{L^2(0,L)} + \left(2\lambda + \frac{1}{L}\right)||\tilde{w}_x||^2_{L^2(0,L)} \\ &\quad +\frac{1}{L}||\tilde{w}_t||^2_{L^2(0,L)}.\end{aligned}$$

Indeed, one can even say that there exists $a, b > 0$ such that

$$|\tilde{w}_{xx}(L,t)|^2 \leq a||\tilde{w}||^2_{L^2(0,L)} + b||\tilde{w}_t||^2_{L^2_{(0,L)}}. \tag{4.22}$$

Before completing the proof, we need to observe the following two remarks

Remark 4.1 *Let $V_3(t)$ be defined by*

$$\tilde{V}_3(t) = \frac{B}{2}\int_0^L |\tilde{w}_{xxx}(x,t)|^2 dx,$$

Then, there exist positive constants d_1, d_2 such that

$$d_1(V_2(t) + \tilde{V}_3(t)) \leq V_2(t) + V_3(t) \leq d_2(V_2(t) + \tilde{V}_3(t)).$$

Remark 4.2 *Based on Remark 4.1 , we have used the fact that the norm $\|f\|_{H^3(0,L)}$ and the norm $\|f\|_{L^2(0,L)} + \|f_{xxx}\|_{L^2(0,L)}$ are equivalent.*

Plugging the estimate on $|\tilde{w}_{xx}(L,t)|^2$ into (4.20) and using the given remarks, we obtain

$$\begin{aligned} \dot{V}(t) \;&\leq\; 2\left(-\lambda + \frac{D^2}{A}\right)V_1(t) + 2a\frac{A^2}{B}V_2(t) \\ &\quad +2b\frac{A^2}{B}V_3(t) - 2\lambda V_2(t) - 2\lambda V_3(t). \end{aligned}$$

Therefore,

$$\begin{aligned} \dot{V}(t) \;&\leq\; 2\left(-\lambda + \frac{D^2}{A}\right)V_1(t) + 2\left(-\lambda + \frac{aA^2}{B}\right)V_2(t) \\ &\quad +2\left(-\lambda + \frac{bA^2}{B}\right)V_3(t). \end{aligned} \tag{4.23}$$

Finally, the above inequality implies that by choosing $A, B > 0$ sufficiently large, one has

$$\dot{V}(t) \;\leq\; 2\left(-\lambda + \varepsilon\right)V(t). \tag{4.24}$$

This gives the rapid stabilization since we can choose the parameter λ as large as we want.

This proves Theorem 4.1, from which it follows that the closed loop system with the output feedback control law depending on a boundary measurement of the state is exponentially stable with decay rate as close to λ as desired.

CHAPTER 5

OUTPUT FEEDBACK STABILIZATION OF THE LINEAR KDV-BURGER'S EQUATION

In Chapter 3, the stabilization of KdVB equation was obtained by using a feedback controller acting on the left Dirichlet boundary condition. The related model was given by

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + u_{xxx}(x,t) + u(x,t)u_x(x,t) = 0, & x \in [0,1], \quad t > 0, \\ u(0,t) = U(t), \quad u(1,t) = 0, \quad u_x(1,t) = 0, & \text{in } \mathbb{R}_+, \\ u(x,0) = u_0(x), & \text{in } [0,1]. \end{cases} \tag{5.1}$$

where $U(t)$ is the control input and u_0 is the initial condition.
It was proved that both the linearized system and the nonlinear system are exponentially stable with a positive decay rate λ.

In this chapter, our purpose is to design an output feedback controller for the KdV-Burger's equation and prove the stability of closed-loop system by constructing an appropriate observer. An output feedback controller is used when a type of boundary measurement is available while there is no full access to the medium.
Our main result in this section is the following:

Theorem 5.1 *For any positive parameter λ, there exist a control input $U(t) := U(\hat{u}(x,t))$, and a kernel function $k(x,y)$ obtained in Theorem* (3.1) *and a function $p_1 = p_1(x)$ such that the solution* (5.3)-(5.4) *satisfies*

$$\|u - \hat{u}\|_{H^3(0,1)} + \|\hat{u}\|_{L^2(0,1)} \lesssim e^{-\lambda t}\left(\|u_0 - \hat{u}_0\|_{H^3(0,1)} + \|\hat{u}_0\|_{L^2(0,1)}\right), \tag{5.2}$$

for $t \geq 0$.

5.1. Observer Design

We propose the following observer system

$$\begin{cases} \hat{u}_t(x,t) - \hat{u}_{xx}(x,t) + \hat{u}_{xxx}(x,t) + p_1(x)[y(t) - \hat{u}_{xx}(1,t)] = 0, \\ \hat{u}(0,t) = U(t), \ \hat{u}(1,t) = \hat{u}_x(1,t) = 0, \end{cases} \tag{5.3}$$

for the plant

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + u_{xxx}(x,t) = 0, \\ u(0,t) = U(t), \quad u(1,t) = 0, \quad \text{and} \quad u_x(1,t) = 0, \\ y(t) = u_{xx}(1,t). \end{cases} \tag{5.4}$$

We set $\tilde{u} := u - \hat{u}$. This function is called the error and it satisfies

$$\begin{cases} \tilde{u}_t(x,t) - \tilde{u}_{xx}(x,t) + \tilde{u}_{xxx}(x,t) - p_1(x)\tilde{u}_{xx}(1,t) = 0, \\ \tilde{u}(0,t) = \tilde{u}(1,t) = \tilde{u}_x(1,t) = 0. \end{cases} \tag{5.5}$$

We define the back-stepping transformation Π_0

$$\tilde{u}(x,t) \quad = \quad \Pi_0(\tilde{w}(x)) = \tilde{w}(x,t) - \int_x^1 p(x,y)\tilde{w}(y,t)dy \tag{5.6}$$

so that the solution $\tilde{u}$ is mapped into the solution of the linear system

$$\begin{cases} \tilde{w}_t(x,t) - \tilde{w}_{xx}(x,t) + \tilde{w}_{xxx}(x,t) + \lambda\tilde{w}(x,t) = 0, \\ \tilde{w}(0,t) = \tilde{w}(1,t) = \tilde{w}_x(1,t) = 0, \end{cases} \tag{5.7}$$

which is exponentially stable with decay rate λ as it was shown in Chapter 3.

Finding $p(x, y)$ is the important step here. In order to obtain the kernel $p(x, y)$, we differentiate the transformation (5.6) with respect to t and x, respectively:

$$
\begin{aligned}
\tilde{u}_t(x,t) \quad = \quad & \tilde{w}_t(x,t) + \tilde{w}_x(x,t)p(x,x) - \tilde{w}(x,t)p_y(x,x) \\
& + \tilde{w}_{xx}(1,t)p(x,1) - \tilde{w}_{xx}(x,t)p(x,x) \\
& + \tilde{w}_x(x,t)p_y(x,x) - \tilde{w}(x,t)p_{yy}(x,x) \\
& - \int_x^1 \tilde{w}(y,t)\left\{p_{yyy}(x,y) + p_{yy}(x,y) - \lambda p(x,y)\right\} dy.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{u}_{xx}(x,t) \quad = \quad & \tilde{w}_{xx}(x,t) + \tilde{w}_x(x,t)p(x,x) + \tilde{w}(x,t)\frac{d}{dx}p(x,x) \\
& \tilde{w}(x,t)p_x(x,x) - \int_x^1 \tilde{w}(y,t)p_{xx}(x,y)dy.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{u}_{xxx}(x,t) \quad = \quad & \tilde{w}_{xxx}(x,t) + \tilde{w}_{xx}(x,t)p(x,x) + 2\tilde{w}_x(x,t)\frac{d}{dx}p(x,x) \\
& + \tilde{w}(x,t)\frac{d^2}{dx^2}p(x,x) + \tilde{w}_x(x,t)p_x(x,x) \\
& + \tilde{w}(x,t)\frac{d}{dx}p_x(x,x) + \tilde{w}(x,t)p_{xx}(x,x) \\
& - \int_x^1 \tilde{w}(y,t)p_{xxx}(x,y)dy.
\end{aligned}
$$

Substituting the above equations into the error system, we obtain

$$
\begin{aligned}
\tilde{u}_t(x,t) - \tilde{u}_{xx}(x,t) + \tilde{u}_{xxx}(x,t) - p_1(x)\tilde{u}_{xx}(1,t) \quad = \\
\tilde{w}_t(x,t) - \tilde{w}_{xx}(x,t) + \tilde{w}_{xxx}(x,t) + \lambda\tilde{w}(x,t) + \tilde{w}_{xx}(1,t)\left(p(x,1) - p_1(x)\right) \\
+\tilde{w}(x,t)\Big(3p_{xy}(x,y) + 3p_{xx}(x,y) - \lambda\Big) + \tilde{w}_x(x,t)\Big(3p_y(x,x) + 3p_x(x,x)\Big) \\
+ \int_x^1 \tilde{w}(y,t)\left\{\lambda p(x,y) - p_{yy}(x,y) - p_{yyy}(x,y) + p_{xx}(x,y) - p_{xxx}(x,y)\right\} dy.
\end{aligned}
$$

This equation gives four conditions on $T := \{(x, y) \,|\, x \in [0, 1],\ y \in [x, 1]\}$

$$\begin{cases} p_{yy}(x, y) + p_{yyy}(x, y) - p_{xx}(x, y) + p_{xxx}(x, y) = \lambda p(x, y), & (x, y) \in T, \\ 3p_x(x, x) + 3p_y(x, y) = 0, & x \in [0, 1], \\ 3p_{xx}(x, x) + 3p_{xy}(x, x) - \lambda = 0, & x \in [0, 1], \\ p(x, 1) = p_1(x), \end{cases} \tag{5.8}$$

with another one obtained by setting $x = 0$ in the transformation (5.6)

$$p(0, y) \quad = \quad 0, \quad \forall y \in [0, 1]. \tag{5.9}$$

These four conditions yield that p satisfies the following gain kernel PDE:

$$\begin{cases} p_{yy}(x, y) + p_{yyy}(x, y) - p_{xx}(x, y) + p_{xxx}(x, y) = \lambda p(x, y), & (x, y) \in T, \\ p(x, x) = 0, & x \in [0, 1], \\ p_x(x, x) = \frac{\lambda}{3} x, & x \in [0, 1], \\ p(0, y) = 0, & x \in [0, 1]. \end{cases} \tag{5.10}$$

We will use the method of successive approximation to obtain $p(x, y)$. Now, let us introduce the following change of variables

$$\bar{x} = 1 - y, \qquad \bar{y} = 1 - x, \tag{5.11}$$

with $F(\bar{x}, \bar{y}) := p(x, y)$. We expect that $F(\bar{x}, \bar{y})$ solves the following third order PDE system

$$\begin{cases} F_{\bar{x}\bar{x}}(\bar{x}, \bar{y}) - F_{\bar{x}\bar{x}\bar{x}}(\bar{x}, \bar{y}) - F_{\bar{y}\bar{y}}(\bar{x}, \bar{y}) - F_{\bar{y}\bar{y}\bar{y}}(\bar{x}, \bar{y}) = \lambda F(\bar{x}, \bar{y}), & (\bar{x}, \bar{y}) \in T, \\ F(\bar{x}, \bar{x}) = 0, & \bar{x} \in [0, 1], \\ F_{\bar{x}}(\bar{x}, \bar{x}) = \frac{\lambda}{3}(1 - \bar{x}), & \bar{x} \in [0, 1], \\ F(\bar{x}, 1) = 0, & \bar{y} \in [0, 1]. \end{cases} \tag{5.12}$$

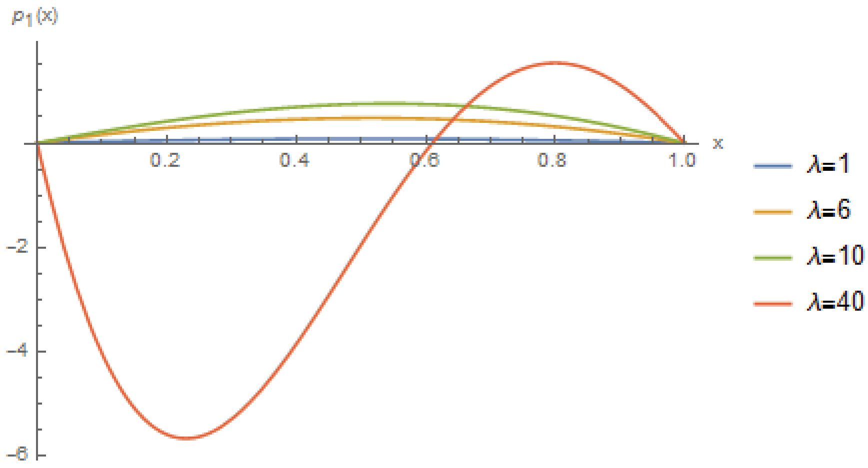

Figure 5.1. Control gain $p_1(x)$ for different values of λ

The solution of (5.12) can be found as the solution of the gain kernel PDE introduced in Chapter 3.

5.2. Stability Analysis of the Closed Loop System

In order to prove the stabilization of the closed-loop system, let us consider the backstepping transformation and its inverse for $\hat{u}$. So, recall that

$$\hat{w}(x,t) = \hat{u}(x,t) - \int_x^1 k(x,y)\hat{u}(y,t)dy \tag{5.13}$$

and

$$\hat{u}(x,t) = \hat{w}(x,t) + \int_x^1 \ell(x,y)\hat{w}(y,t)dy, \tag{5.14}$$

where k and ℓ satisfy (3.10) and (3.23), respectively. Then, $\hat{w}$ satisfies the following system:

$$\begin{cases} \hat{w}_t - \hat{w}_{xx} + \hat{w}_{xxx} + \lambda\hat{w} = -\left\{p_1(x) - \int_x^1 k(x,y)p_1(y)dy\right\}\tilde{w}_{xx}(1,t), \\ \hat{w}(0,t) = \hat{w}(1,t) = \hat{w}_x(1,t) = 0. \end{cases} \tag{5.15}$$

The following lemma estimates the term $\tilde{w}_{xx}^2(1,t)$ at the right hand side of (5.15).

Lemma 5.1 *Let $\tilde{w}$ be a solution of* (5.7). *Then,*

$$|\tilde{w}_{xx}(1,t)| \lesssim \left(\|\tilde{w}(t)\|_{L^2(0,1)} + \|\tilde{w}_t(t)\|_{L^2(0,1)}\right).$$

Proof Let us first multiply (5.7) with $x\tilde{w}_{xx}$ and integrate over (0, 1), we have

$$\begin{aligned} \int_0^1 x\tilde{w}_t\tilde{w}_{xx}dx - \int_0^1 x\tilde{w}_{xx}^2 dx + \int_0^1 x\tilde{w}_{xx}\tilde{w}_{xxx}dx + \lambda\int_0^1 x\tilde{w}_{xx}\tilde{w}dx &= 0, \\ \int_0^1 x\tilde{w}_t\tilde{w}_{xx}dx - \int_0^1 x\tilde{w}_{xx}^2 dx + \frac{1}{2}|\tilde{w}_{xx}(1,t)|^2 - \frac{1}{2}\int_0^1 \tilde{w}_{xx}^2 dx - \lambda\int_0^1 x\tilde{w}_{xx}\tilde{w}dx &= 0. \end{aligned}$$

By using Cauchy-Schwarz inequality, we estimate

$$|\tilde{w}_{xx}(1,t)|^2 \leq \|\tilde{w}_t\|^2_{L^2(0,1)} + 4\|\tilde{w}_{xx}\|^2_{L^2(0,1)} + \lambda\|\tilde{w}_x\|^2_{L^2(0,1)}. \quad (5.16)$$

Then, we obtain the following inequality from the main equation and the triangle inequality

$$\|\tilde{w}_{xxx}\|^2_{L^2(0,1)} \leq 3\left(\lambda^2\|\tilde{w}\|^2_{L^2(0,1)} + \|\tilde{w}_{xx}\|^2_{L^2(0,1)} + \|\tilde{w}_t\|^2_{L^2(0,1)}\right). \quad (5.17)$$

Let us recall the Gagliardo-Nirenberg inequality:

Let $(0,1) \subset \mathbb{R}^2$. Let $m \in \mathbb{N}$, $r, q \in [1,\infty]$, and $w \in L^q(0,1) \cap L^r(0,1)$. Assume that $\partial_x^m w \in L^q(0,1)$. Define p by

$$\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-\alpha}{q}, \quad \frac{j}{m} \leq \alpha \leq 1.$$

Then,

$$\|D^j w\|_{L^p} \leq C\|D^m w\|^{\alpha}_{L^r}\|w\|^{1-\alpha}_{L^q}.$$

Now, for this case take $p = 2, j = 2, m = 3, r = 2, \text{and} q = 2$, that is $\alpha = \frac{2}{3}$, one has

$$\|\tilde{w}_{xx}\|_{L^2(0,1)} \leq \|\tilde{w}_{xxx}\|^{\frac{2}{3}}_{L^2(0,1)}\|\tilde{w}\|^{\frac{1}{3}}_{L^2(0,1)}. \quad (5.18)$$

Secondly, take $p = 2, j = 1, m = 3, r = 2,$ and$q = 2$, that is $\alpha = \frac{1}{3}$, one has

$$||\tilde{w}_x||_{L^2(0,1)} \leq ||\tilde{w}_{xxx}||^{\frac{1}{3}}_{L^2(0,1)} ||\tilde{w}||^{\frac{2}{3}}_{L^2(0,1)}. \tag{5.19}$$

Let us take squares both sides of (5.18)-(5.19) and apply them $\epsilon-$Young's inequality, then we have

$$||\tilde{w}_{xx}||^2_{L^2(0,1)} \lesssim ||\tilde{w}_{xxx}||^2_{L^2(0,1)} + ||\tilde{w}||^2_{L^2(0,1)}, \tag{5.20}$$

for $\epsilon > 0$ small enough and fixed. We get the following inequality by using (5.20) in (5.17)

$$||\tilde{w}_{xxx}||^2_{L^2(0,1)} \lesssim ||\tilde{w}||^2_{L^2(0,1)} + ||\tilde{w}_t||^2_{L^2(0,1)}. \tag{5.21}$$

From (5.21), we can write the following

$$||\tilde{w}_{xx}||^2_{L^2(0,1)} \lesssim ||\tilde{w}||^2_{L^2(0,1)} + ||\tilde{w}_t||^2_{L^2(0,1)}. \tag{5.22}$$

Combining all of these we can rewrite (5.16) in the form:

$$|\tilde{w}_{xx}(1,t)|^2 \lesssim ||\tilde{w}||^2_{L^2(0,1)} + ||\tilde{w}_t||^2_{L^2(0,1)}. \tag{5.23}$$

□

Now, let us an energy functional for the above system by

$$E(t) \equiv ||\hat{w}(t)||^2_{L^2(0,1)} + ||\tilde{w}(t)||^2_{L^2(0,1)} + ||\tilde{w}_t(t)||^2_{L^2(0,1)}. \tag{5.24}$$

Multiplying (5.15) by $\hat{w}$ and integrating over $(0, 1)$, we have

$$\frac{1}{2}\frac{d}{dt}||\hat{w}(t)||^2_{L^2(0,1)} + ||\hat{w}_x(t)||^2_{L^2(0,1)} + \frac{1}{2}|\hat{w}_x(0,t)| + \lambda||\hat{w}(t)||^2_{L^2(0,1)} = \tilde{w}_{xx}(1,t)\int_0^1 \psi\hat{w}dx, \tag{5.25}$$

where $\psi(x) = \left\{p_1(x) - \int_x^1 k(x,y)p_1(y)dy\right\}$. Let us combine this with Lemma 5.1 and use $\epsilon-$Young's inequality to obtain

$$\frac{d}{dt}\|\hat{w}(t)\|^2_{L^2(0,1)} \leq (-2\lambda - \epsilon)\|\hat{w}(t)\|^2_{L^2(0,1)} + C_\epsilon\left(\|\tilde{w}\|^2_{L^2(0,1)} + \|\tilde{w}_t\|^2_{L^2(0,1)}\right) \tag{5.26}$$

where C_ϵ depends on ϵ, λ, and $\|\psi\|_\infty$.

One can observe that the target system in Chapter 3 and the system (5.7) are the same systems. Hence, we have right to state that

$$\frac{d}{dt}\|\tilde{w}(t)\|^2_{L^2(0,1)} \leq -2\lambda\|\tilde{w}(t)\|^2_{L^2(0,1)}, \quad t \geq 0, \tag{5.27}$$

which is equivalent to

$$\|\tilde{w}(t)\|^2_{L^2(0,1)} \leq e^{-2\epsilon t}\|\tilde{w}_0\|^2_{L^2(0,1)}. \tag{5.28}$$

Now, we take the derivative of (5.7) with respect to t, then multiply both sides by $\tilde{w}_t$ and integrate over $(0, 1)$. Hence, we obtain

$$\begin{aligned}\int_0^1 \tilde{w}_t\tilde{w}_{tt}dx &= \int_0^1 \tilde{w}_{xxt}\tilde{w}_t dx - \int_0^1 \tilde{w}_{xxxt}\tilde{w}_t dx - \lambda\int_0^1 \tilde{w}_t^2 dx,\\ \frac{1}{2}\frac{d}{dt}\|\tilde{w}_t(t)\|^2_{L^2(0,1)} &= -\|\tilde{w}_{tx}(t)\|^2_{L^2(0,1)} - \frac{1}{2}|\tilde{w}_{xt}|^2 - \lambda\|\tilde{w}_t(t)\|^2_{L^2(0,1)}.\end{aligned}$$

It follows that

$$\|\tilde{w}_t(t)\|_{L^2(0,1)} \leq e^{-\lambda t}\|\tilde{w}_t(0)\|_{L^2(0,1)} = e^{-\lambda t}\|\tilde{w}_0'' - \tilde{w}_0''' - \lambda\tilde{w}_0\|_{L^2(0,1)} \lesssim e^{-\lambda t}\|\tilde{w}_0\|_{H^3(0,1)}, \quad t \geq 0. \tag{5.29}$$

We deduce the following inequality by combining (5.26)-(5.29)

$$E'(t) \leq -(2\lambda - \epsilon)E(t) + C_{\epsilon,w_0}e^{-2\lambda t}, \tag{5.30}$$

where $C_{\epsilon,w_0} > 0$ is a constant that depends on ϵ, $\|\tilde{w}_0\|_{L^2(0,1)}$, $\|\tilde{w}_0'' - \tilde{w}_0''' - \lambda\tilde{w}_0\|_{L^2(0,1)}$, λ, and $\|\psi\|_\infty$. Let us multiply both sides of (5.30) by $e^{(2\lambda-\epsilon)t}$ and integrate, then we obtain

$$E(t) \leq C_{\epsilon,w_0}e^{-(2\lambda-\epsilon)t}, \quad t \geq 0. \tag{5.31}$$

Using the above analysis, now we can prove Theorem 5.1:

Proof Let us first pick some $\tilde{\lambda} > \lambda$, for example, $\tilde{\lambda} \equiv \lambda + \frac{\epsilon}{2}$. We can put $\tilde{\lambda}$ instead of λ in the entire analysis above and obtain $E(t) \leq C_{\epsilon,w_0} e^{-(2\tilde{\lambda}-\epsilon)t} \leq C_{\epsilon,w_0} e^{-2\lambda t}$ where the constant C_{ϵ,w_0} and all kernel functions depend on $\tilde{\lambda}$. It can be observed from the back-stepping transformation that $\|\tilde{u}(t)\|_{H^3(0,1)} \lesssim \|\tilde{w}(t)\|_{H^3(0,1)}$. Likewise, we obtain from (5.14) that $\|\hat{u}(t)\|_{L^2(0,1)} \lesssim \|\hat{w}(t)\|_{L^2(0,1)}$. Moreover, we can write a similar relation between initial data using the invertibility of the back-stepping transformations, that is $\|\tilde{w}_0\|_{H^3(0,1)} \lesssim \|u_0 - \tilde{u}_0\|_{H^3(0,1)}$ and $\|\hat{w}_0\|_{L^2(0,1)} \lesssim \|\hat{u}_0\|_{L^2(0,1)}$. Combining all of these arguments, we deduce that

$$\|u - \hat{u}\|_{H^3(0,1)} + \|\hat{u}\|_{L^2(0,1)} \lesssim e^{-\lambda t} \left(\|u_0 - \hat{u}_0\|_{H^3(0,1)} + \|\hat{u}_0\|_{L^2(0,1)} \right).$$

□

CHAPTER 6

CONCLUSION

This thesis is divided into two main parts. The first part is devoted to the study of boundary control designs for the Korteweg-de Vries (KdV) and the KdV-Burgers (KdVB) equations posed on a bounded interval. Exponential stabilization for the linearized versions of both equations is obtained by placing a feedback controller on the left Dirichlet boundary condition. This controller is constructed by using the back-stepping method. Exponential stabilization of the corresponding nonlinear systems can only be shown under a smallness assumption the initial datum. In the second part of this thesis, we studied the observer designs for the KdV and the KdVB equations posed on a finite interval. It is shown that stabilization can be still achieved when a type of boundary measurement is available while there is no full access to the medium. The results obtained in Chapter 3 and Chapter 5 in this thesis will also appear as a research article (see Özsarı and Arabacı (2017)).

REFERENCES

Aamo, O. M. (2013). Disturbance rejection in 2×2 linear hyperbolic systems. *IEEE Trans. Automat. Control 58*(5), 1095–1106.

Aamo, O. M., A. Smyshlyaev, and M. Krstic (2005). Boundary control of the linearized ginzburg–landau model of vortex shedding. *SIAM Journal on Control and Optimization 43*(6), 1953–1971.

Amick, C., J. Bona, and M. Schonbek (1989). Decay of solutions of some nonlinear wave equations. *Journal of Differential Equations 81*(1), 1 – 49.

Balogh, A., D. S. Gilliam, and V. I. Shubov (2001). Stationary solutions for a boundary controlled Burgers' equation. *Math. Comput. Modelling 33*(1-3), 21–37. Computation and control, VI (Bozeman, MT, 1998).

Balogh, A. and M. Krstic (2000). Boundary control of the Korteweg-de Vries-Burgers equation: further results on stabilization and well-posedness, with numerical demonstration. *IEEE Trans. Automat. Control 45*(9), 1739–1745.

Cavalcanti, M. M., V. N. Domingos Cavalcanti, V. Komornik, and J. H. Rodrigues (2014). Global well-posedness and exponential decay rates for a KdV-Burgers equation with indefinite damping. *Ann. Inst. H. Poincaré Anal. Non Linéaire 31*(5), 1079–1100.

Cerpa, E. (2007). Exact controllability of a nonlinear Korteweg-de Vries equation on a critical spatial domain. *SIAM J. Control Optim. 46*(3), 877–899.

Cerpa, E. (2012). Stabilization methods for the korteweg-de vries equation.

Cerpa, E. and J.-M. Coron (2013). Rapid stabilization for a Korteweg-de Vries equation from the left Dirichlet boundary condition. *IEEE Trans. Automat. Control 58*(7), 1688–1695.

Cerpa, E. and E. Crépeau (2009). Rapid exponential stabilization for a linear Korteweg-de Vries equation. *Discrete Contin. Dyn. Syst. Ser. B 11*(3), 655–668.

Coron, J.-M. and E. Crépeau (2004). Exact boundary controllability of a nonlinear KdV equation with critical lengths. *J. Eur. Math. Soc. (JEMS) 6*(3), 367–398.

Crépeau, E. and C. Prieur (2010). Discussion on: "Adaptive boundary control of the forced generalized Korteweg-de Vries-Burgers equation" [mr2656995]. *Eur. J. Control 16*(1), 85–87.

Di Meglio, F. (2011, July). *Dynamics and control of slugging in oil production*. Theses, École Nationale Supérieure des Mines de Paris.

Gao, W. and F. Deng (2007, 05). Stability of burgers-korteweg-de vries equation.

Glass, O. and S. Guerrero (2008). Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit. *Asymptot. Anal. 60*(1-2), 61–100.

Glass, O. and S. Guerrero (2010). Controllability of the Korteweg-de Vries equation from the right Dirichlet boundary condition. *Systems Control Lett. 59*(7), 390–395.

Hasan, A. (2015). Adaptive boundary observer for nonlinear hyperbolic systems: Design and field testing in managed pressure drilling. In *ACC*.

Hasan, A. (2016, June). Output-feedback stabilization of the korteweg-de vries equation. In *2016 24th Mediterranean Conference on Control and Automation (MED)*, pp. 871–876.

Hasan, A. and B. Foss (2011). Global stabilization of the generalized burgers-korteweg-de vries equation by boundary control. *IFAC Proceedings Volumes 44*(1), 6771 – 6776. 18th IFAC World Congress.

Jia, C. and B.-Y. Zhang (2012). Boundary stabilization of the Korteweg-de Vries equation and the Korteweg-de Vries-Burgers equation. *Acta Appl. Math. 118*, 25–47.

Korteweg, D. J. and G. de Vries (1895). On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Philos. Mag. (5) 39*(240), 422–443.

Krstic, M. (1999). On global stabilization of Burgers' equation by boundary control. *Systems Control Lett. 37*(3), 123–141.

Krstic, M. (2009). *Delay compensation for nonlinear, adaptive, and PDE systems*. Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA.

Krstic, M., B.-Z. Guo, and A. Smyshlyaev (2011). Boundary controllers and observers for the linearized Schrödinger equation. *SIAM J. Control Optim. 49*(4), 1479–1497.

Krstic, M. and A. Smyshlyaev (2008a). Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays. *Systems Control Lett. 57*(9), 750–758.

Krstic, M. and A. Smyshlyaev (2008b). *Boundary control of PDEs*, Volume 16 of *Advances in Design and Control*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA. A course on backstepping designs.

Liu, W.-J. and M. Krstić (2002). Global boundary stabilization of the Korteweg-de Vries-Burgers equation. *Comput. Appl. Math. 21*(1), 315–354. Special issue in memory of Jacques-Louis Lions.

Marx, S. and E. Cerpa (2016, September). Output feedback stabilization of the Korteweg-de Vries equation *. working paper or preprint.

Özsarı, T. and E. Arabacı (2017). Boosting the decay of solutions of the linearised korteweg–de vries–burgers equation to a predetermined rate from the boundary. *International Journal of Control*, 1–11.

Perla Menzala, G., C. F. Vasconcellos, and E. Zuazua (2002). Stabilization of the Korteweg-de Vries equation with localized damping. *Quart. Appl. Math. 60*(1), 111–129.

Qian, L., L. Tian, and D. Zhang (2014). Unique continuation property and decay for the Korteweg-de-Vries-Burgers equation with localized damping. *Int. J. Nonlinear Sci. 18*(2), 110–115.

Rosier, L. (1997). Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain. *ESAIM Control Optim. Calc. Var. 2*, 33–55.

Rosier, L. (2004). Control of the surface of a fluid by a wavemaker. *ESAIM Control Optim. Calc. Var. 10*(3), 346–380.

Rosier, L. and B.-Y. Zhang (2006). Global stabilization of the generalized Korteweg-de Vries equation posed on a finite domain. *SIAM J. Control Optim. 45*(3), 927–956.

Rosier, L. and B.-Y. Zhang (2009). Control and stabilization of the Korteweg-de Vries equation: recent progresses. *J. Syst. Sci. Complex. 22*(4), 647–682.

Russell, D. L. and B. Y. Zhang (1996). Exact controllability and stabilizability of the Korteweg-de Vries equation. *Trans. Amer. Math. Soc. 348*(9), 3643–3672.

Tang, S. and M. Krstic (2013). Stabilization of linearized korteweg-de vries systems with anti-diffusion. In *American Control Conference, ACC 2013, Washington, DC, USA, June 17-19, 2013*, pp. 3302–3307.

Vazquez, R. and M. Krstic (2007). A closed-form feedback controller for stabilization of the linearized 2-D Navier-Stokes Poiseuille system. *IEEE Trans. Automat. Control 52*(12), 2298–2312.