Eur. Phys. J. C (2017) 77:834
https://doi.org/10.1140/epjc/s10052-017-5418-0

THE EUROPEAN PHYSICAL JOURNAL C

CrossMark

Regular Article - Theoretical Physics

S- and T-self-dualities in dilatonic $f(R)$ theories

Tonguç Rador[1,2,a]

[1] Department of Physics, Boğaziçi University, 34342 Istanbul, Turkey
[2] Department of Physics, İzmir Institute of Technology, 35430 Izmir, Turkey

Received: 25 August 2017 / Accepted: 27 November 2017 / Published online: 6 December 2017

Abstract We search for theories, in general spacetime dimensions, that would incorporate a dilaton and higher powers of the scalar Ricci curvature such that they have exact S- or T-self-dualities. The theories we find are free of Ostrogradsky instabilities. We also show that within the framework we are confining ourselves, a theory of the form mentioned above cannot have both T- and S-dualities except for the case where the action is linear in the scalar curvature.

1 Introduction

Theories that contain higher powers of the curvature scalar have attracted much attention over the past years, after their introduction some time ago [1,2], especially in view of the accelerated expansion of the universe and the possible avenues related to extra dimensions inspired by string theory. The literature has become too voluminous to cite even partially, so we refer to the reviews on the subject [3–9] and the references therein.

As is well known the low energy string theory action is that of Einstein gravity with a non-minimally coupled dilaton. This lowest order action has two important symmetries: T- and S-dualities. In this work we investigate the conditions on how to have S- or T-symmetries in $f(R)$ type theories. The reason to confine the study to actions that contain only powers of the Ricci curvature scalar is the fact that pure $f(R)$ theories are free of Ostrogradsky instabilities [11]; as a consequence related to this the models we find are also free of the mentioned problem.

2 An observation

Let us consider the following action in d dimensions:

$$\int d^d x\sqrt{-\tilde{G}}\; e^{-(\frac{d-2}{4})\tilde{\phi}} \left[\tilde{R} + A(\tilde{\nabla}\tilde{\phi})^2\right] , \tag{1}$$

where A is a real number. It can be shown that the action is form invariant under the following transformations:

$$\tilde{G}_{\mu\nu} = e^{-\phi} G_{\mu\nu} \;, \tag{2a}$$
$$\tilde{\phi} = -\phi \;, \tag{2b}$$

which are called S-duality transformations. If the action is form invariant one may call the model S-self-dual.

However, to reach this conclusion one has to perform an integration by parts and hence convert some of the terms to boundary integrals which in turn, as is well known, will not affect the Euler–Lagrange equations of motion since to find them one assumes that the variations vanish at the boundary.

Furthermore as is also well known, one can represent the theory in the Einstein frame as opposed to the original Jordan frame by the following conformal transformation of the metric:

$$\tilde{G}_{\mu\nu} = e^{\tilde{\phi}/2} G_{\mu\nu}. \tag{3}$$

This will transform the original action as follows:

$$\int d^d x\sqrt{-G}\left[R + \left(A - \frac{(d-1)(d-2)}{16}\right)\left(\nabla\tilde{\phi}\right)^2 - \frac{(d-1)}{2}\nabla^2\tilde{\phi}\right]. \tag{4}$$

Here the non-tilde derivatives are those related to G. Here we also realize the last term as a surface term and it can thus be ignored from the perspective of getting the equations of motion. Furthermore if one also requires the scalar field kinetic term to be normalized to the canonical value of $-1/2$ one requires $A = 4$ for $d = 10$; the canonical numbers of low energy string theory. Also, the original S-self-duality presents itself here as the $\tilde{\phi} \to -\tilde{\phi}$ invariance.

We see that in both instances a manipulation of the lagrangian via integration by parts to get rid of the surface terms that are not invariant is necessary. However, this will not necessarily be possible if one has a theory that involves higher powers of R and this may obstruct S-self-duality of the

[a] e-mail: tonguc.rador@boun.edu.tr

theory. Our aim is to find an algebraic realization of the mentioned symmetry which will circumvent this impediment. That is, we seek ways to find lagrangian densities which are exactly symmetric under Eq. (2).

3 S-self-dual theories

First of all let us note that one has

$$\left[\tilde{R} + \frac{(d-1)}{2}\tilde{\nabla}^2\tilde{\phi}\right] = e^{\phi}\left[R + \frac{(d-1)}{2}\nabla^2\phi\right], \tag{5a}$$

$$A(\tilde{\nabla}\tilde{\phi})^2 = e^{\phi}A(\nabla\phi)^2, \tag{5b}$$

under the S-duality transformations. These algebraic properties can be exploited to form theories that will involve higher powers of R in such a way that the theory can be made S-self-dual.

In fact one can further generalize the possible terms in (5) by multiplying them with functions which are even in $\tilde{\phi}$. That is, under (2), one still has

$$\alpha(\tilde{\phi})\left[\tilde{R} + \frac{(d-1)}{2}\tilde{\nabla}^2\tilde{\phi}\right] = e^{\phi}\alpha(\phi)\left[R + \frac{(d-1)}{2}\nabla^2\phi\right], \tag{6a}$$

$$A(\tilde{\phi})(\tilde{\nabla}\tilde{\phi})^2 = e^{\phi}A(\phi)(\nabla\phi)^2, \tag{6b}$$

provided that $\alpha(-\phi) = \alpha(\phi)$ and $A(-\phi) = A(\phi)$. Furthermore one can also add a pure potential term for the scalar field which has the same algebraic transformation rule

$$e^{-\tilde{\phi}/2}V(\tilde{\phi}) = e^{\phi}\ e^{-\phi/2}V(\phi) \tag{7}$$

provided again that one has $V(-\phi) = V(\phi)$.

Thus the most general term that has the same algebraic transformation rule is the following:

$$\mathcal{U}_i(\tilde{G}_{\mu\nu}, \tilde{\phi}) \equiv \alpha_i(\tilde{\phi})\left[\tilde{R} + \frac{(d-1)}{2}\tilde{\nabla}^2\tilde{\phi}\right] + A_i(\tilde{\phi})(\tilde{\nabla}\tilde{\phi})^2 + e^{-\tilde{\phi}/2}V_i(\tilde{\phi}) \tag{8}$$

and it becomes

$$\mathcal{U}_i(\tilde{G}_{\mu\nu}, \tilde{\phi}) = e^{\phi}\mathcal{U}_i(G_{\mu\nu}, \phi) \tag{9}$$

under the S-duality transformations, provided α_i, A_i and V_i are all even functions.

Now let us define the following measure object:

$$M(\tilde{G}, \tilde{\phi}) \equiv \sqrt{-\tilde{G}}e^{-(d-2)\tilde{\phi}/4}, \tag{10}$$

which, under S-duality, transforms as

$$M(\tilde{G}, \tilde{\phi}) = e^{-\phi}M(G, \phi). \tag{11}$$

With these ingredients one can form a density which is S-self-dual and that contains higher powers of the curvature scalar. All we have to do is to multiply terms of the type in Eq. (8) that have different functions α_i, A_i and V_i. To this product we multiply the measure form in Eq. (10). To compensate for the resulting powers of the scalar field we shall need another factor. One readily arrives at the following object:

$$L^{(n)}(\tilde{G}_{\mu\nu}, \tilde{\phi}) \equiv M(\tilde{G}, \tilde{\phi})e^{(n-1)\tilde{\phi}/2}\prod_{i=1}^{n}\mathcal{U}_i(\tilde{G}_{\mu\nu}, \tilde{\phi}), \tag{12}$$

which is S-self-dual in the algebraic sense mentioned above. That is, one has

$$L^{(n)}(\tilde{G}_{\mu\nu}, \tilde{\phi}) = L^{(n)}(G_{\mu\nu}, \phi). \tag{13}$$

The role of the factor $e^{(n-1)\tilde{\phi}/2}$ in Eq. (12) is to compensate for factors of e^{ϕ} that come from the product of $\mathcal{U}_i$ terms and the factor $e^{-\phi}$ that comes from the measure object in Eq. (10).

So the most general S-self-dual action that contains higher powers of the scalar curvature can be formed via summing various forms of the type in Eq. (12);

$$S = \int d^dx \sum_n L^{(n)}(\tilde{G}_{\mu\nu}, \tilde{\phi}). \tag{14}$$

We note again that this action is S-self-dual in an algebraic sense; no integration by parts is necessary to make non-invariant terms surface terms.

3.1 Ostrogradski instabilities

Whenever one has a theory with a lagrangian that contains higher derivative terms one is facing the troublesome Ostrogradski instabilities. Since the S-self-dual theories we have introduced has a generic lagrangian which depends on the second derivatives of the dilaton field we have to assess if one truly has this problem.

It is a well-known fact that pure $f(R)$ theories are free of Ostrogradski instabilities even though they contain higher derivatives of the metric. Now let us switch to the Einstein frame via (3). This will result in the following:

$$\mathcal{U}_i(\tilde{G}_{\mu\nu}, \tilde{\phi}) = e^{-\tilde{\phi}/2}\mathcal{V}_i(G_{\mu\nu}, \tilde{\phi}), \tag{15}$$

where

$$\mathcal{V}_i(G_{\mu\nu}, \tilde{\phi}) = \alpha_i(\tilde{\phi})R + \left[A_i(\tilde{\phi}) + \alpha_i(\tilde{\phi})\frac{(d-1)(d-2)}{8}\right]\left(\nabla\tilde{\phi}\right)^2 + V_i(\tilde{\phi}). \tag{16}$$

Consequently the original lagrangians of the form (14) do not have higher derivatives of the scalar field in the Einstein frame. We therefore conclude that the theories we have introduced will be free of Ostrogradskian instabilities.

One can understand this absence also in the original Jordan frame. In that case, one has double derivatives of the scalar field in the action but they always come accompanied by the curvature term in the form $\tilde{R}+(d-1)\tilde{\nabla}^2\tilde{\phi}/2$ and thus always get mixed with the degrees of the geometry and hence do not create a truly independent canonical momentum.

The fact that Ostrogradskian instabilities are absent for the dilaton field does not necessarily mean that none of the equations of motion involves higher derivatives of it. It simply means that there is at least an equation of motion that depends only on its derivatives of second or first degree. To expose this let us evaluate the full equations of motion in the Einstein frame. Without loss of generality we use a simplified form for the action

$$\int d^d x \sqrt{-g}\, f(\mathcal{V}) \tag{17}$$

with

$$\mathcal{V} = \alpha(\chi) R + \beta(\chi)\,(\nabla\chi)^2 + V(\chi) \tag{18}$$

with $\beta(\chi) = [A(\chi) + \alpha(\chi)(d-1)(d-2)/8]$. Note that we have set $\tilde{\phi} = \chi$ and got rid of the tildes.

One can show that the variation of the action with respect to the metric gives the following equation:

$$\alpha f'\, R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f + (g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu)(\alpha f') + \beta f'\,(\nabla_\mu\chi)(\nabla_\nu\chi) = 0 \tag{19}$$

where primes indicate derivatives with respect to the full argument of the corresponding function. This equation obviously incorporates derivative of the scalar field to third order. But the variation of the action with respect to the scalar field yields

$$\left[\alpha' R + \beta'(\nabla\chi)^2 + V'\right] f' = \nabla_\mu \left[2\beta f' \nabla^\mu \chi\right], \tag{20}$$

and thus it is manifestly clear from the above that the equations of motion for the scalar field involve only up to and including the second derivative. Therefor third order derivative is fixed by this equation.

3.2 Pure $f(R)$ theories in the Einstein frame

One by-product of our discussion is that a pure $f(R)$ theory can be thought of as an Einstein frame representation of an S-self-dual dilatonic theory, provided one has all α_i, A_i and V_i as constants and satisfying the following condition:

$$A_i + \alpha_i \frac{(d-1)(d-2)}{8} = 0. \tag{21}$$

This condition is quite non-trivial in the sense that it binds the coefficients of the scalar curvature to those of the coefficient of the kinetic term for the scalar field in the Jordan frame. The resulting theory in the Jordan frame, though by construction S-self-dual, may have instabilities due to a wrong signature.

In fact, one can envisage a more general framework. If we require an originally S-self-dual theory to end up in a pure $f(R)$ theory in the Einstein frame one does not necessarily require the coefficients α_i, A_i and V_i to be constants. We may still require them to be functions of the field $\tilde{\phi}$. In this case the condition above becomes

$$A_i(\tilde{\phi}) + \alpha_i(\tilde{\phi}) \frac{(d-1)(d-2)}{8} = 0. \tag{22}$$

These conditions of the field $\tilde{\phi}$ should be simultaneously satisfied. The simplest way for this to happen is to assume that the theory in the original Jordan frame takes, modulo the measure term, the form of $f(\mathcal{U})$ where

$$\mathcal{U} = \alpha(\tilde{\phi}) \left(\tilde{R} + \frac{d-1}{2}\tilde{\nabla}^2\tilde{\phi}\right) + A(\tilde{\phi})(\tilde{\nabla}\tilde{\phi})^2 + e^{-\tilde{\phi}/2} V(\tilde{\phi}) \tag{23}$$

and (22) becomes a single equation and may more easily be accommodated. Thus in this generalized case the dilaton in the Jordan frame must have already been stabilized in a subtle way: A quite different and somewhat richer condition than (21). If this is realized somehow the stable point of the dilaton field must obey this condition. In view of the equations of motion presented in the previous subsection this will also mandate a constant Ricci scalar curvature.

4 T-duality

Let us go back to the original action (1) and study the effect of T-duality on it. T-duality is a symmetry that acts on the internal dimensions; so a separation of what is compactified and what is not is mandatory. This is accomplished via the following separation of the metric:

$$ds^2 = \tilde{G}_{\mu\nu} dX^\mu dX^\nu = g_{mn}(x) dx^m dx^n + e^{2\tilde{C}(x)} \gamma_{ij}(y) dy^i dy^j, \tag{24}$$

which can be called the compactification or warped ansatz: The metric is assumed to be block diagonal with respect to the co-ordinates y of the compactified extra dimensions and the co-ordinates x of the observed dimensions, which include time. Here $\tilde{C}(x)$ is called the radion field and $\gamma(y)_{ij}$ is a metric for the manifold of extra dimensions which does not depend on the co-ordinates of the observed dimensions. We set the number of the extra dimensions to be p. In view of this ansatz we also assume that the dilaton field $\tilde{\phi}$ does not depend on the co-ordinates of the internal dimensions.

As is well known, this ansatz allows us to evaluate the Ricci scalar of the full spacetime as follows [13,14]:

$$\begin{aligned}\tilde{R} = R[\tilde{G}] &= R[g] + e^{-2\tilde{C}} R[\gamma] \\ &- 2pe^{-\tilde{C}} g^{mn} \nabla_m \nabla_n e^{\tilde{C}} \\ &- p(p-1)e^{-2\tilde{C}} g^{mn} \left(\nabla_m e^{\tilde{C}}\right)\left(\nabla_n e^{\tilde{C}}\right), \end{aligned} \tag{25}$$

where the covariant derivatives ∇ are now those of the metric g_{mn}. We also have

$$\sqrt{-\tilde{G}}\; e^{-(d-2)\tilde{\phi}/4} = \sqrt{-g}\sqrt{\gamma}\; e^{p\tilde{C}-(d-2)\tilde{\phi}/4} \tag{26}$$

and

$$A(\tilde{\nabla}\tilde{\phi})^2 = A\tilde{G}^{\mu\nu}(\tilde{\nabla}_\mu\tilde{\phi})(\tilde{\nabla}_\nu\tilde{\phi}) = Ag^{mn}(\nabla_m\tilde{\phi})(\nabla_n\tilde{\phi}), \tag{27}$$

following directly from the ansatz.

Now let us define the following transformation rules:

$$\tilde{C} = -C, \tag{28a}$$

$$\tilde{\phi} = \phi - \frac{8p}{d-2}C, \tag{28b}$$

which automatically leave the measure in (26) invariant. These are called T-duality transformations and one can show that the action (1) is form invariant under them provided one fixes

$$A = \left(\frac{d-2}{4}\right)^2 \longrightarrow A = 4 \text{ for } d = 10, \tag{29a}$$

$$R[\gamma] = 0. \tag{29b}$$

However, one must realize, in view of the form in (25), which involves double derivatives of $\tilde{C}$, that again a manipulation involving integration by parts is required. This works in a similar way to that of S-duality: some terms are non-invariant but they are surface terms. So in conclusion the action in (1) is both S- and T-self-dual provided the coefficient of A is fixed to be predetermined constant in (29). These are of course well-known properties of the low energy action of string theory. The general point we make is that manipulations of it via integration by parts is necessary for both symmetries. We were able to generalize S-self-duality to actions that are required to involve higher powers of the scalar curvature that circumvent this impediment. We now look for generalizing T-self-duality. That is, we shall now seek to find lagrangian densities which involve higher powers of the curvature scalar that are exactly symmetric under T-duality.

4.1 T-self-dual theories

Let us remember that the measure prefactor is invariant under the T-duality transformations. Thus, it is clear that if we are to get an algebraic invariance similar to what happened in the generalization of S-duality transformations, we need to add a term to $R[\tilde{G}]$ that would compensate for the appearance of the double derivative of $\tilde{C}$.

The only term that can be added in the original variables is the following:

$$\tilde{G}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{\phi}.$$

Using the compactification ansatz (24) we can simply infer that the inverse metric $\tilde{G}^{\mu\nu}$ is also block diagonal. This, along with the assumption that the dilaton field is blind to co-ordinates y^i of the extra dimensional metric, means that one has

$$\tilde{G}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{\phi} = g^{mn}\left(\nabla_m\nabla_n\tilde{\phi} + p\nabla_m\tilde{C}\nabla_n\tilde{\phi}\right) \tag{30}$$

where the covariant derivatives on the right hand side are those related to the metric g_{mn}. With these considerations, after a slightly tedious calculation, one can show that the object

$$R[\tilde{G}] + A(\tilde{\nabla}\tilde{\phi})^2 + \sigma\tilde{\nabla}^2\tilde{\phi} \tag{31}$$

after the metric compactification ansatz is T-self-dual, provided the following conditions are met:

$$A = -\left(\frac{d-2}{4}\right)^2, \tag{32a}$$

$$R[\gamma] = 0, \tag{32b}$$

$$\sigma = \frac{d-2}{2}. \tag{32c}$$

The difference in the value of A in the above as opposed to the one in (29) is to be attributed to the fact that in (29) an integration by parts is incorporated to show the T-self-duality of the action linear in R. Thus they are not actually different conditions if one considers only the action (1). We also see that in view of its somewhat contrived form, T-duality forbids a potential term for the dilaton.[1]

It is now possible to take powers of the object in Eq. (31) to form lagrangians that involve higher powers of the scalar curvature such that the theory is exactly T-self-dual in the sense we have previously described.

[1] This is generically true, however, one can envisage a periodic potential for the dilaton field and this may end up being T-self-dual provided the dilaton and radion are both stabilized somehow.

5 Incompatibility of T- and S-self-dualities for $f(R, \phi)$ theories

Note that the coefficients of the term $\tilde{\nabla}^2\tilde{\phi}$ are found to be different for S-self-duality and T-self-duality conditions: $(d-1)/2$ in the former case and $(d-2)/2$ in the latter. To get a resolution, fixing the functions $\alpha_i(\tilde{\phi})$ and $A_i(\tilde{\phi})$ to be constants and $V_i(\tilde{\phi}) = 0$ in (12) is necessary of course but cannot save the situation. This discrepancy can be attributed to the fact that S-duality transformations act on the full spacetime defined by the general metric $\tilde{G}_{\mu\nu}$, whereas for T-duality transformations one must first assume the compactification ansatz. These two are very different symmetries and the dilaton simply becomes overworked especially in the absence of help from ”integrating by parts and ignoring the surface terms” strategies. One can show that using two dilaton fields, one responsible for T-duality and one for S-duality, does not work either; nor contemplating a workaround by the 3-form field of string theory will help, since it cannot produce a laplacian of the dilaton, to compensate for the mismatch.

Furthermore this is not the whole issue. Even we could have found an algebraic TS-self-dual combination of fields which is linear in the curvature we cannot use its powers as the form of S-duality we found for theories that involve higher powers of the curvature needs powers of the exponential of the dilaton to compensate for the conformal factors arising from the transformations and T-duality transformations will mess these up. Thus we can state the following.

There are no both S- and T-self-dual categories of theories, in the algebraic sense defined in this work, which are free of Ostrogradski instabilities containing a single dilaton field and incorporating higher powers of the scalar curvature.

However, we must remind the reader that for instance relaxing the condition on the non-existence of Ostrogradski instabilities may yield theories that involve higher powers of curvature objects, not necessarily the Ricci scalar, and still incorporate both T- and S-dualities.

Funded by SCOAP^3.

References

1. H.A. Buchdahl, Month. Not. R. Astron. Soc. **150**, 18 (1970)
2. A.A. Starobinsky, Phys. Lett. B. **91**, 99102 (1980)
3. S. Nojiri, S.D. Odintsov, Int. J. Geom. Meth. Mod. Phys. **4**, 115–146 (2007)
4. A. De Felice, S. Tsujikawa, Living Rev. Relat. **13**, 3 (2010)
5. T.P. Sotiriou, V. Faraoni, Rev. Mod. Phys. **82**, 451–497 (2010)
6. S. Capozziello, M. De Laurentis, Phys. Rep. **509**(4–5), 167–321 (2011)
7. S. Nojiri, S.D. Odintsov, Phys. Rept. **505**, 59–144 (2011)
8. S. Capozziello, V. Faraoni, *Beyond Einstein gravity: a survey of gravitational theories for cosmology and astrophysics* (Springer, New York, 2011). ISBN: 978-94-007-0165-6
9. V. Faraoni. Volume 38 of the series Astrophysics and Space Science Proceedings, 19–32 (2013)
10. M. Ostrogradsky, Mem. Ac. St. Petersbourg VI **4**, 385 (1850)
11. R.P. Woodard, Lect. Notes. Phys. **720**, 403 (2007)
12. R.P. Woodard. arxiv:1506.02210v2 (2015)
13. F. Dobarro, E. Lami Dozo, Trans. Am. Math. Soc **303**, 161–168 (1987)
14. H. Yamabe, Osaka Math. J. **12**, 21–37 (1960)