Renormalization of Dirac Delta Potentials Through Minimal Extension of Heisenberg Algebra

Fatih Erman*

Department of Mathematics, İzmir Institute of Technology, Urla 35430, İzmir, Turkey

Abstract *We renormalize the model of multiple Dirac delta potentials in two and three dimensions by regularizing it through the minimal extension of Heisenberg algebra. We show that the results are consistent with the other regularization schemes given in the literature.*

1 Introduction

In quantum field theory, we encounter ultraviolet divergences and a well-known procedure, called renormalization is used to remove these infinities.[1] This is essentially based on two-step procedure, and the first step is known as regularization. There are many different regularization schemes, such as cut-off regularization,[2] Pauli–Villars regularization,[3] Schwinger's proper time regularization,[4] and dimensional regularization.[5−7] After regularization, the bare parameters in the given model are redefined in such a way that they cancel all the divergences.

The divergences also occur in quantum mechanics for some singular potentials, namely, Dirac delta potentials in two and three dimensions. These infinities can be removed by following the idea of renormalization in quantum field theory[8−13] with different regularization schemes. One regularization scheme in real coordinate space is to modify the point-like Dirac delta potential $\delta(\boldsymbol{r})$ by a ring-type Dirac delta potential so that the singularity at the origin is removed.[12−13] Moreover, Dirac delta potentials have been studied from the self-adjoint extension point of view[14] and have many applications in various areas of physics (see Ref. [15] the references in Ref. [14]).

Quantum mechanics based on a generalized uncertainty relation has recently attracted a great deal of attention and it implies the existence of a minimal observable length.[16−26] The idea of deforming the usual canonical commutation relations between position and momentum operators is first introduced in the context of string theory, which restricts that the uncertainties in the position can not be smaller than some minimal length.[27−28] This is necessary since strings cannot probe distances below the Planck scale. Moreover, it has been shown that this minimal length may regularize the divergences in quantum field theory.[29−30]

The point-like Dirac delta potentials in the context of the minimal extension of Heisenberg algebra has been very recently discussed in Ref. [31]. The deformation parameter introduced through the extended Heisenberg algebra provides a natural cut-off and the bound state energies of the problem are finite without introducing any condition (renormalization condition). The main aim of this article is to renormalize finitely many Dirac delta potentials in D-dimensions by regularizing it through the modified minimal Heisenberg algebra. We show that this new regularization method is actually similar to the one where the pointlike Dirac delta interaction is modified with a finite range interaction. After the regularization of the Hamiltonian, we compute the resolvent (Green's functions) associated with it and then apply renormalization condition by appropriately choosing the coupling constants. We show that the result at the end is consistent with the one obtained by different regularization schemes.

The paper is organized as follows. In Sec. 2, we shortly review the minimal extension of Heisenberg algebra. Then we discuss the renormalization of the model by first regularizing it through the minimal extended Heisenberg algebra in Sec. 3.

2 Minimal Extension of Heisenberg Algebra

The modified Heisenberg algebra is

$$[X, P] = \mathrm{i}\hbar(1 + \beta P^2) \,, \tag{1}$$

where β is a positive parameter. This leads to the following uncertainty relations for position and momentum operators[16]

$$\Delta X \Delta P \geq \frac{\hbar}{2}(1 + \beta(\Delta P)^2 + \beta\langle P\rangle^2) \,. \tag{2}$$

This implies that

$$\Delta X_{\min} \geq \hbar\sqrt{\beta} \,, \tag{3}$$

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*E-mail: fatih.erman@gmail.com

and there is no minimal momentum uncertainties (so that momentum space representation exists). In D-dimensions, we have[18]

$$[X_i, P_j] = i\hbar(\delta_{ij} + \beta\delta_{ij}P^2) + \beta' P_i P_j \,, \tag{4}$$

where $\beta' > 0$. If P_i's assume to commute with each other, then the Jacobi's identity implies that the commutation relations for X_i are uniquely determined (up to possible extensions):

$$[X_i, X_j] = i\hbar\Big(\frac{2\beta - \beta' + \beta(2\beta + \beta')P^2}{(1 + \beta P^2)}\Big)(P_i X_j - P_j X_i). \tag{5}$$

For simplicity, we assume that β and β' are small and $\beta' = 2\beta$. Then, $[X_i, X_j] = 0$. This is known as the minimal extension of the Heisenberg algebra.[16,18,29] In the first order in β, a representation of the operators X_i and P_i satisfying the commutation relations (4) is given by

$$X_i = x_i \,, \quad P_i = (1 + \beta p^2)p_i \,, \tag{6}$$

where x_i and p_i are the usual position and momentum operators satisfying standard commutation relations in quantum mechanics. In this new representation,[17] the Schrödinger equation becomes

$$\Big(\frac{\boldsymbol{p}^2 + 2\beta\boldsymbol{p}^4}{2m}\Big)\psi + V(\boldsymbol{x})\psi = E\psi \,, \tag{7}$$

where the terms of the order β^2 are neglected.

3 Renormalization of Delta Potentials

We now consider the problem where the point particle interacts with the finitely many Dirac delta potentials in two or three dimensions. The formal Hamiltonian in standard quantum mechanical formulation is given by

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \sum_{i=1}^{N} \lambda_i \delta(\boldsymbol{x} - \boldsymbol{a}_i) \,. \tag{8}$$

Let us consider the Schrödinger equation for the above Hamiltonian (8) in the following form

$$\begin{aligned} \langle x|H|\psi\rangle &= \langle x|H_0|\psi\rangle - \sum_{i=1}^{N} \lambda_i \delta(\boldsymbol{x} - \boldsymbol{a}_i)\psi(\boldsymbol{x}) \\ &= \langle x|\Big(H_0 - \sum_{i=1}^{N} \lambda_i |\boldsymbol{a}_i\rangle\langle\boldsymbol{a}_i|\Big)|\psi\rangle = E\,\psi(\boldsymbol{x}) \,, \end{aligned} \tag{9}$$

where H_0 is the free part of the Hamiltonian and the kets $|\boldsymbol{a}_i\rangle$ are the eigenkets of the position operator with eigenvalue $\boldsymbol{a}_i$.

We first find the regularized resolvent for the regularized version of the above Hamiltonian. Instead of regularizing the Hamiltonian by modifying the point interaction with the finite range one, we regularize the Hamiltonian by modifying the Heisenberg algebra introduced above. Hence, we have

$$H_\beta = H_0^\beta - \sum_{i=1}^{N} \lambda_i(\beta)|\boldsymbol{a}_i\rangle\langle\boldsymbol{a}_i| \,, \tag{10}$$

where

$$H_0^\beta \hat\psi(\boldsymbol{p}) = \Big(\frac{\boldsymbol{p}^2 + 2\beta\boldsymbol{p}^4}{2m}\Big)\hat\psi(\boldsymbol{p}) \,. \tag{11}$$

Note that we recover the original Hamiltonian when the parameter β goes to zero. The parameter β is interpreted as the cut-off parameter.

In order to find the regularized resolvent $R_\beta(E) = (H_\beta - E)^{-1}$, we will solve the following inhomogenous equation

$$\Big(H_0^\beta - \sum_{j=1}^{N} \lambda_i(\beta)|\boldsymbol{a}_j\rangle\langle\boldsymbol{a}_j| - E\Big)|\psi\rangle = |\rho\rangle \,. \tag{12}$$

Let $|\boldsymbol{f}_i^\beta\rangle = \sqrt{\lambda_i(\beta)}|\boldsymbol{a}_i\rangle$ or $\langle\boldsymbol{x}|\boldsymbol{f}_i^\beta\rangle = \sqrt{\lambda_i(\beta)}\delta(\boldsymbol{x} - \boldsymbol{a}_i)$. Then, acting the operator $(H_0^\beta - E)^{-1}$ on both sides of Eq. (12) from left, we get

$$|\psi\rangle = \sum_{j=1}^{N} (H_0^\beta - E)^{-1}|\boldsymbol{f}_j^\beta\rangle\langle\boldsymbol{f}_j^\beta|\psi\rangle + (H_0^\beta - E)^{-1}|\rho\rangle \,. \tag{13}$$

If we take the inner product of this with $\langle\boldsymbol{f}_i^\beta|$, we obtain

$$\sum_{j=1}^{N} T_{ij}(\beta, k)\langle\boldsymbol{f}_j^\beta|\psi\rangle = \langle\boldsymbol{f}_i^\beta|(H_0^\beta - E)^{-1}|\rho\rangle \,, \tag{14}$$

where

$$T_{ij}(\beta, E) = \begin{cases} 1 - \langle\boldsymbol{f}_i^\beta|(H_0^\beta - E)^{-1}|\boldsymbol{f}_i^\beta\rangle & \text{if } i = j \,, \\ -\langle\boldsymbol{f}_i^\beta|(H_0^\beta - E)^{-1}|\boldsymbol{f}_j^\beta\rangle, & \text{if } i \neq j \,. \end{cases} \tag{15}$$

By acting the inverse of the matrix T_{ij} on Eq. (14), we can find $\langle\boldsymbol{f}_j^\beta|\psi\rangle$ and then substituting back this into Eq. (13), we find the regularized resolvent

$$R_\beta(E) = (H_0^\beta - E)^{-1} + (H_0^\beta - E)^{-1}\Big(\sum_{i,j=1}^{N} |\boldsymbol{f}_i^\beta\rangle[T^{-1}(\beta, E)]_{ij}\langle\boldsymbol{f}_j^\beta|\Big)(H_0^\beta - E)^{-1} \,. \tag{16}$$

If we now return to the original variables and define

$$\Phi_{ij}(\beta, E) = \begin{cases} \dfrac{1}{\lambda_i(\beta)} - \langle\boldsymbol{a}_i|(H_0^\beta - E)^{-1}|\boldsymbol{a}_i\rangle, & \text{if } i = j \,, \\ -\langle\boldsymbol{a}_i|(H_0^\beta - E)^{-1}|\boldsymbol{a}_j\rangle, & \text{if } i \neq j \,, \end{cases} \tag{17}$$

we get

$$R_\beta(E) = (H_0^\beta - E)^{-1} + (H_0^\beta - E)^{-1}\Big(\sum_{i,j=1}^{N} |\boldsymbol{a}_i\rangle[\Phi^{-1}(\beta, E)]_{ij}\langle\boldsymbol{a}_j|\Big)(H_0^\beta - E)^{-1} \,. \tag{18}$$

By inserting the completeness for the momentum eigenstates, it is easy to see that the diagonal term of the matrix Φ_{ij}

$$\langle \boldsymbol{a}_i | (H_0^\beta - E)^{-1} | \boldsymbol{a}_i \rangle = \int \frac{\mathrm{d}^D p}{(2\pi\hbar)^D} \frac{(2m)}{p^2 + 2\beta p^4 - 2mE} \quad (19)$$

is divergent as $\beta \to 0$. In order to remove the divergence from the model, let us first consider the one-center case ($N = 1$) for simplicity. Suppose that i-th center is isolated from all other centers. Then $\Phi_{ij}(\beta, E)$ is just a single function for the i-th center and reads

$$\Phi_{ii}(\beta, E) = \frac{1}{\lambda_i(\beta)} - \int_0^\infty \frac{\mathrm{d}p}{(2\pi\hbar)^D} \frac{2mp^{D-1}}{p^2 + 2\beta p^4 - 2mE} \quad (20)$$

for any $i = 1, \ldots, N$. Note that this integral is divergent for large values of momentum if $\beta = 0$. As long as $\beta \neq 0$, it is finite for $D < 4$, as emphasized in Ref. [31]. If we choose the bare running coupling constants

$$\frac{1}{\lambda_i(\beta)} = \frac{1}{\lambda_i^R(M_i)} + \int_0^\infty \frac{\mathrm{d}p}{(2\pi\hbar)^D} \frac{2mp^{D-1}}{p^2 + 2\beta p^4 - 2mM_i} , (21)$$

where M_i is the renormalization scale and takes the limit as $\beta \to 0^+$, we obtain a non-trivial finite expression for the resolvent of a single delta potential problem:

$$R(E) = (H_0 - E)^{-1} + (H_0 - E)^{-1} | \boldsymbol{a}_i \rangle$$

$$\times [\Phi_{ii}^{-1}(E)] \langle \boldsymbol{a}_i | (H_0 - E)^{-1} , \quad (22)$$

where the function Φ_{ii} is

$$\Phi_{ii}(k) = \frac{1}{\lambda_i^R(M_i)} - \int_0^\infty \frac{\mathrm{d}p}{(2\pi\hbar)^D}$$
$$\times \frac{(2m)^2 p^{D-1}(E - M_i)}{(p^2 + 2m\mu_i^2)(p^2 - 2mE)} , \quad (23)$$

for all i. Since the poles of the resolvent correspond to the bound state energies, and the resolvent formula (23) includes the reciprocal of $\Phi_{ii}(E)$, its zeroes determine the bound state spectrum. The renormalization scale M_i could possibly be eliminated in favour of a physical parameter by imposing the renormalization condition. For instance, the renormalization scale can be chosen to be equal to the bound state energy of the particle to the i-th center, say $-\mu_i^2$, so that:

$$\Phi_{ii}(-\mu_i^2) = 0 . \quad (24)$$

Therefore, for bound state problems, it is very convenient to choose the renormalization scale to be the bound state energy by setting $1/\lambda_i^R = 0$ so that we eliminate the unphysical scale M_i.

If we apply the same argument to N-center case, we obtain the following resolvent formula

$$R(E) = (H_0 - E)^{-1} + (H_0 - E)^{-1} \left(\sum_{i,j=1}^N | \boldsymbol{a}_i \rangle [\Phi^{-1}(E)]_{ij} \langle \boldsymbol{a}_j | \right) (H_0 - E)^{-1} , \quad (25)$$

where

$$\Phi_{ij}(E) = \begin{cases} -\int_0^\infty \dfrac{\mathrm{d}p}{(2\pi\hbar)^D} \dfrac{(2m)^2 p^{D-1}(E + \mu_i^2)}{(p^2 + 2m\mu_i^2)(p^2 - 2mE)} , & \text{if } i = j , \\ -\int \dfrac{\mathrm{d}^D p}{(2\pi\hbar)^D} \dfrac{(2m)\, \mathrm{e}^{(\mathrm{i}/\hbar)\boldsymbol{p}\cdot(\boldsymbol{a}_i - \boldsymbol{a}_j)}}{p^2 - 2mE} , & \text{if } i \neq j , \end{cases} \quad (26)$$

defined on the complex E plane. We call the matrix $\Phi_{ij}(E)$ the principal matrix and it is simply the inverse of $-T$-matrix in the standard formulation of scattering theory ($R = R_0 - R_0 T R_0$).[32] The above formula can be extended onto the largest possible subset of the complex plane by analytic continuation. Here it is important to note that the principal matrix satisfies $\Phi^\dagger(E) = \Phi(E^*)$ and the resolvent formula (25) that we have obtained is a kind of Krein's formula.[33] Let $E = k^2 \in \mathbb{C} \setminus \mathbb{R}$ and $\Im k > 0$. For $D = 2$, the principal matrix is then

$$\Phi_{ij}(k) = \begin{cases} \dfrac{m}{\pi\hbar^2} \log\left(-\dfrac{k}{\mu_i}\right) & \text{if } i = j , \\ -\dfrac{\mathrm{i}m}{4\hbar^2} H_0^{(1)}\left(\dfrac{\sqrt{2mk}|\boldsymbol{a}_i - \boldsymbol{a}_j|}{\hbar}\right) , & \text{if } i \neq j , \end{cases} \quad (27)$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero. For $D = 3$

$$\Phi_{ij}(k) = \begin{cases} -\dfrac{\sqrt{m}}{2\sqrt{2}\pi\hbar^3}(\mathrm{i}k + \mu_i) , & \text{if } i = j , \\ -\dfrac{m}{2\pi\hbar^2|\boldsymbol{a}_i - \boldsymbol{a}_j|} \exp\left(\dfrac{\mathrm{i}k\sqrt{2m}|\boldsymbol{a}_i - \boldsymbol{a}_j|}{\hbar}\right) , & \text{if } i \neq j . \end{cases} \quad (28)$$

Here we have used the integral representation of $H_0^{(1)}$[34]

$$H_0^{(1)}(z) = \frac{2}{\mathrm{i}\pi} \int_1^\infty \frac{\mathrm{e}^{\mathrm{i}zt}}{\sqrt{t^2 - 1}} \mathrm{d}t . \quad (29)$$

These results are completely consistent with the literature obtained with the other regularization schemes[8−10,13−14] for $N = 1$.

4 Conclusion

We have shown that the renormalization of the finitely many Dirac delta potentials through the regularization with minimal extension of the Heisenberg algebra is consistent with the one obtained by different regularization schemes. The advantage of using the Green's function or

resolvent for the renormalization procedure is that it includes all the information about the spectrum, e.g., bound states and scattering states and it allows us to extend the model to Dirac delta potentials supported by curves in $\mathbb{R}^3$ and the Salpeter type Hamiltonians with Dirac delta potentials.

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