MODULES SATISFYING CONDITIONS THAT ARE OPPOSITES OF ABSOLUTE PURITY AND FLATNESS

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ABSTRACT

MODULES SATISFYING CONDITIONS THAT ARE OPPOSITES OF ABSOLUTE PURITY AND FLATNESS

The main purpose of this thesis is to study the properties which are opposites of absolute pure and flat modules. A right module M is said to be *test for flatness by subpurity* (for short, t.f.b.s.) if its subpurity domain is as small as possible, namely, consisting of exactly the flat left modules. A left module M is said to be *rugged* if its flatness domain is the class of all regular right R-modules. Every ring has a t.f.b.s. module. For a right Noetherian ring R every simple right R-module is t.f.b.s. or absolutely pure if and only if R is a right V-ring or $R \cong A \times B$, where A is right Artinian with a unique non-injective simple right R-module and $\mathrm{Soc}(A_A)$ is homogeneous and B is semisimple. A characterization of t.f.b.s. modules over commutative hereditary Noetherian rings is given. Rings all (cyclic) modules of whose are rugged are shown to be von Neumann regular rings. Over a right Noetherian ring every left module is rugged or flat if and only if every right module is poor or injective if and only if $R = S \times T$, where S is semisimple Artinian and T is either Morita equivalent to a right PCI-domain, or T is right Artinian whose Jacobson radical contains no properly nonzero ideals. Connections between rugged and poor modules are shown. Rugged Abelian groups are fully characterized.

ÖZET

MUTLAK SAFLIK VE DÜZLÜK İLE ZIT OLAN KOŞULLARI SAĞLAYAN MODÜLLER

Bu tezde modüllerin mutlak saflık ve düzlük ile zıt olan özelliklerinin çalışılması amaçlanmaktadır. Bir sağ modülün alt saflık bölgesi mümkün olduğu kadar küçükse, yani sadece düz sol modüllerden oluşuyorsa bu sağ M modülüne *alt saflık bölgesi yoluyla düzlük için test modülü* (kısaca, t.f.b.s.) denir. Bir sol M modülünün düzlük bölgesi tüm düzenli sağ modüllerin sınıfı ise bu M modülüne *pürüzlü modül* denir. Her halka t.f.b.s. modüle sahiptir. Sağ Noether halkası için her basit modülün t.f.b.s. ya da mutlak saf olması ancak ve ancak halkanın V-halkası ya da A tek injektif olmayan basit sağ modüle sahip Artin halka, Soc(A_A) homojen ve B yarı basit olmak üzere $R \cong A \times B$ şeklinde olmasıdır. Değişmeli kalıtsal Noether halka üzerinde t.f.b.s. modüllerin karakterizasyonları verildi. Tüm (devirli) modüllerin pürüzlü olduğu halkaların von Neumann düzenli halkalar olduğu gösterildi. Bir sağ Noether halkası üzerinde her sol modül pürüzlüdür ya da düzdür ancak ve ancak her sağ modül fakirdir ya da injektiftir ancak ve ancak S yarı basit halka ve T sağ PCI-bölgesine Morita denk ya da T radikali sıfırdan farklı ideal içermeyen bir Artin halka olmak üzere $R = S \times T$ şeklindedir. Pürüzlü ve fakir modüller arasındaki bağlantılar gösterildi. Pürüzlü Abelian gruplar tam olarak karakterize edildi.

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LIST OF ABBREVIATIONS

R	an associative ring with unit unless otherwise stated
$\mathbb{Z}$, $\mathbb{Z}^+$	the ring of integers, the set of all positive integers
$\mathbb{Q}$	the field of rational numbers
$\mathbb{Z}_{p^\infty}$	the Prüfer (divisible) group for the prime p (the p-primary part of the torsion group $\mathbb{Q}/\mathbb{Z}$)
R-$\mathcal{M}od$	the category of *left* R-modules
$\mathcal{M}od$-R	the category of *right* R-modules
$\mathrm{Hom}_R(M, N)$	all R-module homomorphisms from M to N
$M \otimes_R N$	the tensor product of the *right* R-module M and the *left* R-module N
$\mathrm{Ker}\, f$	the kernel of the map f
$\mathrm{Im}\, f$	the image of the map f
M^+	the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$
$\mathrm{Soc}\, M$	the socle of the R-module M
$\mathrm{Rad}\, M$	the radical of the R-module M
$\mathrm{E}(M)$	the injective envelope (hull) of a module M
$\mathrm{T}(M)$	the torsion submodule of a module M
$Z(M)$	the singular submodule of a module M
$\mathrm{J}(R)$	the Jacobson radical of the ring R
$\mathcal{I}n^{-1}(M)$	the injectivity domain of a module M
$\mathcal{F}^{-1}(M)$	the flat domain of a module M
$\mathcal{S}(M)$	the subpurity domain of a module M
$i\mathcal{P}(R_R)$	the right injective profile of R
$f\mathcal{P}(_RR)$	the left flat profile of R
$\mathrm{Ann}^l_R(X)$	$= \{r \in R \mid rX = 0\}$ = the *left* annihilator of a subset X of a *left* R-module M
$\mathrm{Ann}^r_R(X)$	$= \{r \in R \mid Xr = 0\}$ = the *right* annihilator of a subset X of a *right* R-module M
$\mathrm{Ext}_R(C, A) = \mathrm{Ext}^1_R(C, A)$	set of all equivalence classes of short exact sequences starting with the R-module A and ending with the R-module C
$\cong$	isomorphic
$\leq$	submodule

CHAPTER 1

INTRODUCTION

In module and ring theory, some recent work has moved away from focusing on classical injectivity in order to consider the extent of injectivity of modules. While traditionally the study of non-injective modules emphasized those modules that are as injective as possible, the recent trend has been to make injectivity domain as small as possible.

Given right modules M and N, M is said to be injective relative to N (or M is N-injective) if, for any submodule K of N, any R-homomorphism $f : K \to M$ extends to an R-homomorphism $g : N \to M$. A right module M is injective relative to every right R-module is called an injective right R-module. The classes of modules N such that M is N-injective is called the injectivity domain of M and denoted by $\mathcal{I}n^{-1}(M)$. Clearly $\mathcal{I}n^{-1}(M)$ contains the class of semisimple right modules. As an opposite notion of injectivity, the authors of (Alahmadi, Alkan and López-Permouth, 2010), defined a right module M to be poor if its injectivity domain is exactly the class of semisimple right modules. In ((Er, Lòpez-Permouth and Sökmez, 2011), Proposition 1), it is proven that every ring has poor modules. In recent years, there is an appreciably interest to poor modules and to the rings defined via these modules (see, (Alizade and Büyükaşık, 2017), (Aydoğdu and Saraç, 2013), (Er, Lòpez-Permouth and Sökmez, 2011), (Er, Lòpez-Permouth and Tung, 2016)). A ring R is said to have no right middle class if every right module is poor or injective.

Given right modules M and N, M is said to be projective relative to N (or M is N-projective) if for each epimorphism $g : N \to K$ and each homomorphism $f : M \to K$ there is an R-homomorphism $\overline{f} : M \to N$ such that $g\overline{f} = f$. Following (Anderson and Fuller, 1992), the classes of modules N such that M is N-projective is called the projectivity domain of M and denoted by $\mathcal{P}r^{-1}(M)$. M is projective if and only if $\mathcal{P}r^{-1}(M) = \mathcal{M}od\text{-}R$. As an opposite notion of projectivity p-poor modules are studied in ((Holston, López-Permouth and Ertaş, 2012) and (López-Permouth and Simental, 2012)). In that paper, it is proven that every ring has a (semisimple) p-poor module.

Another kind of injectivity (namely, subinjectivity) is introduced in a similar vein. The opposite of injectivity induced by subinjectivity offer a new perspective on modules and rings and unearth a lot of interesting questions.

Given right modules M and N, M is said to be A-subinjective if for every extension B of A any homomorphism $\varphi : A \rightarrow M$ can be extended to a homomorphism $\phi : B \rightarrow M$ (see, (Aydoğdu and López-Permouth, 2011)). It is easy to see that M is injective if and only if M is A-subinjective for each module A. A module M is called *indigent* if M is subinjective relative to only injective modules.

In (Alizade, Büyükaşık and Er, 2014), a module A is said to be a *test for injectivity by subinjectivity (or t.i.b.s)* if whenever a module M is A-subinjective implies M is injective. It is known that every ring has a t.i.b.s. The indigent and t.i.b.s modules do not imply each other. On the other hand, each module is indigent or injective if and only if each module is t.i.b.s. or injective ((Alizade, Büyükaşık and Er, 2014), Proposition 2).

The idea and notion of subinjectivity can be used in order to study opposites of some other homological objects such as, absolutely pure and flat modules.

For a left module N, *the absolutely pure domain of N* is defined to be the collection of all right modules M such that M is absolutely N-pure. N is flat if and only if its absolutely pure domain consists of the entire class $\mathcal{M}od$-R. As an opposite to flatness, a module M is called *f-indigent* if its absolutely pure domain is as small as possible, namely, consisting of exactly the absolutely pure right modules (see (Durğun, 2016)).

In Chapter 2 some basic definitions, results and preliminary notions are given.

In Chapter 3 the notion of t.f.b.s. which is a sort of dual to the notion of f-indigence is given.

For a right module M, *the subpurity domain of M* is defined to be the collection of all left modules N such that M is absolutely N-pure (see (Durğun, 2016)). M is absolutely pure if and only if its subpurity domain consists of the entire class R-$\mathcal{M}od$. As an opposite to absolute purity, a module M is called *test module for flatness by subpurity* (t.f.b.s.) if its subpurity domain is as small as possible, namely, consisting of exactly the flat left modules.

Every ring has a t.f.b.s. module. The ring R is right t.f.b.s. and right S-ring (i.e. each finitely generated flat right ideal is projective) if and only if R is right semihereditary. A commutative domain R is t.f.b.s. if and only if R is Prüfer.

The right t.i.b.s. modules are t.f.b.s. The converse is not true in general. Over a right Noetherian ring any right t.f.b.s. module with finitely generated injective hull is a t.i.b.s. module (Proposition 3.10). We prove that every simple right module is t.i.b.s. or injective if and only if R is a right V-ring or R is right Noetherian and every simple right module is t.f.b.s. or injective if and only if R is a V-ring or $R \cong A \times B$, where A is right Artinian with a unique non-injective simple right module and $\mathrm{Soc}(A_A)$ is homogeneous and B is semisimple. We also prove necessary conditions for a right Noetherian ring whose right modules are t.f.b.s. or absolutely pure. Namely, we prove that, if every right module is t.f.b.s. or absolutely pure, then $R \cong A \times B$, where B is semisimple, and (i) A is right hereditary right Artinian serial ring with homogeneous socle, $J(A)^2 = 0$ and A has a unique noninjective simple right A-module, or, (ii) A is a QF-ring that is isomorphic to a matrix ring over a local ring, or, (iii) A is right SI with $\mathrm{Soc}(A_A) = 0$ (Theorem 3.4).

Prüfer domains are characterized as those domains all finitely generated ideals of whose are t.f.b.s. (Proposition 3.14). Finally, we give a characterization of t.f.b.s. modules over commutative hereditary Noetherian rings. Namely, it is shown that a module N is t.f.b.s. if and only if $\mathrm{Hom}(N/Z(N), S) \neq 0$ for each singular simple module S, where $Z(N)$ is the singular submodule of N (Theorem 3.6).

In Chapter 4 the properties of flat domain and rugged modules are studied.

Recall that a right module M is called *regular* if every submodule is pure in the sense of Cohn (see, (Cheatham and Smith, 1976)). That is, for every submodule K of M and left module N, the map $K \otimes N \to M \otimes N$ is exact. Given a left module M and a right module N, M is N-flat if for every submodule K of N the map $1_M \otimes i : K \otimes M \to N \otimes M$ is a monomorphism, where $i : K \to N$ is the inclusion map and 1_M is the identity map on M. The *flat domain* $\mathfrak{F}^{-1}(M)$ of a left module M, is defined to be the collection of right modules N such that M is N-flat (see, (Anderson and Fuller, 1992), page 232, Question 15). It is evident from the definitions that regular right modules are contained in $\mathfrak{F}^{-1}(M)$ for each left module M.

We study the modules whose flat domain is as small as possible. We call a left module M rugged if $\mathfrak{F}^{-1}(M)$ is exactly the class of regular right modules. Every ring has a rugged module. The ring is von Neumann regular if and only if every left module is rugged. Any left module that contains a pure and rugged submodule is itself rugged. If R is a ring such that regular right modules are semisimple, then a left module M is rugged if and only if its character module M^+ is poor.

Let R be a right Noetherian ring. It is proven that a left module M is rugged if and only if the character right module M^+ is poor. Every right module is poor or injective if and only if every left module is rugged or flat if and only if $R = S \times T$, where S is semisimple Artinian and T is Morita equivalent to a right PCI-domain, or T is right Artinian whose Jacobson radical properly contains no nonzero ideals.

A ring R is said to be right simple-destitute if every simple right module is poor. Simple-destitute rings are studied in (Alahmadi, Alkan and López-Permouth, 2010) and (Aydoğdu and Saraç, 2013), where several examples of simple-destitute rings are given.

The structure of simple-destitute general rings is not known. For a commutative ring R, we prove that R is simple-destitute if and only if R is local or semisimple Artinian if and only if every simple module is rugged and regular modules are semisimple. For a right semiartinian ring R, it is proven that if every simple right module is poor or injective, then R is a right V-ring, or $R = S \times T$, where S is semisimple Artinian and T is a ring with a unique simple right module. In addition, if R is commutative, then every simple module is poor or injective if and only if $R = S \times T$, where S is semisimple Artinian and T is a local ring.

An abelian group G is rugged if and only if the torsion part of G contains a direct summand isomorphic to $\oplus_p \mathbb{Z}_p$, where p ranges over all primes and $\mathbb{Z}_p$ is the simple abelian group of order p. The notions of poor, rugged and p-poor are coincide over the ring of integers.

CHAPTER 2

PRELIMINARIES

Throughout this thesis, R will denote an associative ring with identity. If not state otherwise, the symbol R, stands for a general ring and modules will be unital right R-modules. An *Integral domain,* or shortly a *domain*, will mean a nonzero ring without zero divisors, *not* necessarily commutative.

Essentially, we assume the fundamentals of module and ring theory and homological algebra are known. All definitions which are not given here can be found in (Anderson and Fuller, 1992), (Rotman, 1979), (Goodearl, 1976), (Lam, 1999), (Fuchs, 1970) and (Enochs and Jenda, 2000).

In this chapter we introduce our basic terminology for rings and modules, as well as the fundamental results to be used in this thesis.

2.1. Chain Conditions

A family of subsets $\{A_i : i \in I\}$ in a set $\mathfrak{A}$ is said to satisfy the *Ascending Chain Condition (ACC)* if, for any ascending chain $A_{i_1} \subseteq A_{i_2} \subseteq A_{i_3} \subseteq \cdots$ in the family, there exists an integer n such that $A_{i_n} = A_{i_{n+k}}$ for each $k \in \mathbb{N}$. A family of subsets $\{A_i : i \in I\}$ in a set $\mathfrak{A}$ is said to satisfy the *Descending Chain Condition (DCC)* if, for any descending chain $A_{i_1} \supseteq A_{i_2} \supseteq A_{i_3} \supseteq \cdots$ in the family, there exists an integer n such that $A_{i_n} = A_{i_{n+k}}$ for each $k \in \mathbb{N}$.

A module M is called *Artinian (Noetherian)* if the family of all submodules of M satisfies *DCC (ACC)*. A ring R is called right *Artinian (Noetherian)* if R_R is *Artinian (Noetherian)*. A similar definition can be made on the left. R is Artinian (Noetherian) if it is both right and left Artinian (Noetherian). M is Noetherian if and only if every submodule of M is finitely generated. The Artinian and Noetherian properties are inherited by submodules and factor modules. Finitely generated modules over a right Artinian (Noetherian) ring are Artinian (Noetherian). If R is right Noetherian, then every finitely generated module is finitely presented.

For any module M, the Jacobson radical of M is defined as the intersection of all maximal submodules of M and is denoted by $Rad(M)$. For a ring R, $J(R_R)$, $J({}_RR)$ are equal, and we denote both of them by $J(R)$.

Theorem 2.1 *(Hopkins-Levitzki) A ring R is right Artinian if and only if it is right Noetherian, $J(R)$ is nilpotenet and $R/J(R)$ is semisimple.*

A *composition series* for a module M is a chain of submodules

$$0 = M_0 \subset M_1 \subset M_2 \subset ... \subset M_{n-1} \subset M_n = M$$

such that each of the factors M_i/M_{i-1} is a simple module. n is called the length of the composition series. If a module M has a composition series of length n, then M is said to have length n, and we denoted $l(M) = n$. A module M has finite length if and only if M is both Noetherian and Artinian.

Let M be a right R-module. Then the *Loewy series (or socle series)* $\{S_\alpha\}$ of M is defined as: $S_1 = \mathrm{Soc}(M)$, $S_\alpha/S_{\alpha-1} = \mathrm{Soc}(M/S_{\alpha-1})$, and for a limit ordinal α, $S_\alpha = \cup_{\beta<\alpha} S_\beta$. Put $S = \cup S_\alpha$. Then, by construction, M/S has zero socle. M is semiartinian (i.e. every proper factor of M has a simple module) if and only if $S = M$ (see, for example, (Dung, Huynh, Smith and Wisbauer, 1994)). Submodules and factor modules of semiartinian modules are semiartinian. A ring R is called a right *semiartinian ring* if R_R is semiartinian, equivalently, every right module has essential socle. A right Artinian ring is semiartinian. On the other hand , a right semiartinian right Noetherian ring is right Artinian.

A module M is said to be *uniserial* if the lattice of submodules of M is totally ordered by inclusion. A ring R is called a right (left) *uniserial ring* if R_R (${}_RR$) is a uniserial module. Any direct sum of uniserial modules called a *serial module*. A ring R is said to be right (left) *serial ring* if the module R_R (${}_RR$) is serial. A ring R is called a serial ring if R is both left as well as right serial. If R is a right or left serial ring, then $R/J(R)$ is a semisimple Artinian ring.

The ring R is called a *QF* (quasi-Frobenius) ring if R is right and left self-injective and Artinian. Equivalently, R is a right self injective ring which is right or left Noetherian.

2.2. Local, Semilocal, Perfect and Semiperfect Rings

In commutative algebra, a local ring is defined to be a nonzero ring which has a unique maximal ideal. In noncommutative algebra, as a natural generalization of a local ring, one calls a nonzero ring R *local* if R has a unique maximal left ideal, or, equivalently R has unique maximal right ideal.

Proposition 2.1 *((Lam, 2001), Theorem 19.1) The following statements are equivalent for an arbitrary ring R.*

(1) R has a unique maximal left ideal;

(2) R has a unique maximal right ideal;

(3) $R/J(R)$ is a division ring;

(4) $J(R)$ is the set of all non-invertible elements of R;

(5) the sum of two non-invertible elements of R is non-invertible.

If R is a ring satisfying equivalent conditions of Proposition 2.1, then R is called a local ring.

Definition 2.1 *A ring R is said to be semilocal if $R/J(R)$ is semisimple Artinian.*

Let R be a semilocal ring. Then, for every right module M, $RadM = MJ(R)$. Thus $M/RadM$ is a semisimple $R/J(R)$- module. Hence, $M/RadM$ is a semisimple R-module ((Anderson and Fuller, 1992)).

Lemma 2.1 *((Lam, 2001), Lemma 19.27) Let R be a ring and $\overline{R} = R/I$, where I is an ideal of R contained in $J(R)$. Let P, Q be finitely generated projective right R-modules. Then $P \cong Q$ as R-modules if and only if $P/PI \cong Q/QI$ as $\overline{R}$-modules.*

Two idempotents $e_1, e_2 \in R$ are said to be *orthogonal* if $e_1e_2 = e_2e_1 = 0$.

Proposition 2.2 *((Lam, 2001), Proposition 21.8) The following statements are equivalent for any nonzero idempotent $e \in R$.*

(1) eR is indecomposable as a right R-module;

(2) Re is indecomposable as a left R-module;

(3) eRe has no nontrivial idempotents;

(4) e has no decomposition such that $e_1 + e_2$ where e_1, e_2 are nonzero orthogonal idempotents in R.

If the idempotent $e \neq 0$ satisfies any of these conditions, e is said to be a primitive idempotent of R.

Definition 2.2 *An idempotent e is said to be a local idempotent if eRe is a local ring.*

Clearly, a local idempotent is a primitive idempotent.

Proposition 2.3 *((Lam, 2001), Proposition 21.18) Let e be an idempotent in R, and let* $\overline{R} = R/J(R)$*. Then e is a local idempotent in R if and only if* $eR/eJ(R)$ *is a simple right* $\overline{R}$*-module.*

Now, we give the definitions of perfect and semiperfect rings.

Definition 2.3 *A ring R is called semiperfect if R is semilocal and idempotents of* $R/J(R)$ *can be lifted to R.*

In a semiperfect ring, any primitive idempotent is local.

Theorem 2.2 *((Lam, 2001), Theorem 23.6) A ring R is semiperfect if and only if the identity element* 1 *can be decomposed into* $e_1 + e_2 + ... + e_n$*, where the* e_i'*s are mutually orthogonal local idempotents.*

Our next goal is to introduce the notion of left and right perfect rings. For this, we need a new notion of nilpotency called T-nilpotency.

Definition 2.4 *A subset A of a ring R is called left (right) T-nilpotent if, for any sequence of elements* $\{a_1, a_2, \cdots\} \subseteq A$*, there exists an integer* $n \geq 1$ *such that* $a_1a_2...a_n = 0$ *(*$a_n...a_2a_1 = 0$*).*

Definition 2.5 *A ring R is called right (left) perfect if* $R/J(R)$ *is semisimple and* $J(R)$ *is right (left) T-nilpotent. If R is right and left perfect, we call R is a perfect ring.*

Semiperfect rings are left-right symmetric, while left (right) perfect rings are always semiperfect. Both of these notions are generalizations of one-sided Artinian rings.

The following result offers various other characterizations for right perfect rings. This Theorem says that one sided Artinian rings are right and left perfect rings.

Theorem 2.3 *((Lam, 2001), Theorem 23.20) The following are equivalent for any ring R.*

(1) R is right perfect;

(2) R satisfies DCC on principal left ideals;

(3) Any left module M satisfies DCC on cyclic submodules;

(4) R does not contain an infinite orthogonal set of nonzero idempotents, and any nonzero left module M contains a simple module.

From the definitions of above, it is clear that a ring R is left perfect if and only if R is semiperfect and right semiartinian.

2.3. Relatively Injective Modules

In this section we give some properties of injective and relative injective modules which can be found in (Anderson and Fuller, 1992) and (Mohamed and Müller, 1990).

Definition 2.6 *Given right modules M and N, M is said to be injective relative to N (or M is N-injective) if, for any submodule K of N, any R-homomorphism $f : K \to M$ extends to an R-homomorphism $g : N \to M$. A right module M is injective relative to every right R-module is called an injective right R-module.*

Lemma 2.2 *Let p be a prime integer and $m, n \in \mathbb{Z}^+$. If $m \leq n$, then $\mathbb{Z}_{p^n}$ is $\mathbb{Z}_{p^m}$-injective.*

Proposition 2.4 *(**Baer's Criterion**) A right module M is injective if and only if for any right ideal I of R, any R-homomorphism $f : I \to M$ can be extended to $g : R \to M$.*

The following Proposition can be viewed as a generalization of Baer's Criterion.

Proposition 2.5 *((Mohamed and Müller, 1990), Proposition 1.4) A module M is N-injective if and only if M is nR-injective for every $n \in N$.*

Direct summands and direct product of injective modules are injective. On the other hand, it is not true that direct sum of injective modules are injective.

Theorem 2.4 *((Mohamed and Müller, 1990), Theorem 1.11) The direct sum of any family of N-injective modules is N-injective if and only if every cyclic (or finitely generated) submodule of N is Noetherian.*

Theorem 2.5 *((Rotman, 1979), Theorem 4.10) The following are equivalent for a ring R.*

(1) R is right Noetherian;

(2) Every direct limit (directed index set) of injective right modules is injective;

(3) Every direct sum of injective right modules is injective.

Remark 2.1 *Let R be a right Noetherian ring and M a right R-module. Let Γ be an ascending chain of injective submodules of M. Then $\cup_{E\in\Gamma}E$ is injective by ((Rotman, 1979), Exercise 2.31) and Theorem 2.5. Hence, by Zorn's Lemma, M contains a largest injective submodule, and so M can be written as $M = K \oplus N$, where K is injective and N has no nonzero injective submodule.*

Proposition 2.6 *((Mohamed and Müller, 1990), Proposition 1.6) $\prod_{i\in I} M_\alpha$ is N-injective if and only if M_α is N-injective for every $\alpha \in I$.*

The class of modules N such that M is N-injective is called the injectivity domain of M and denoted by $\mathcal{I}n^{-1}(M)$. Injectivity domain is closed under submodules, epimorphic images and also arbitrary direct sums. Clearly, $\mathcal{I}n^{-1}(M)$ contains the class of semisimple right modules. Also, it is immediate that M is injective if and only if $\mathcal{I}n^{-1}(M) = \mathcal{M}od\text{-}R$ (see (Anderson and Fuller, 1992)).

2.4. Relatively Projective Modules

Definition 2.7 *Given right R-modules M and N, M is said to be projective relative to N (or M is N-projective) for each epimorphism $g : N \to K$ and each homomorphism $f : M \to K$ there is an R-homomorphism $\overline{f} : M \to N$ such that $g\overline{f} = f$. A right module M is projective relative to every right R-module is called a projective right R-module.*

Direct sums and direct summands of projective modules are projective. A ring is a projective module over itself. Every free module is a projective module.

Proposition 2.7 *((Anderson and Fuller, 1992), Proposition 17.2) The following properties hold for a right R-module P.*

(1) P is projective;

(2) Every epimorphism $M \to P \to 0$ splits;

(3) P is isomorphic to a direct summand of a free R-module.

The class of modules N such that M is N-projective is called the projectivity domain of M and denoted $\mathcal{P}r^{-1}(M)$. Projectivity domain is closed under submodules, epimorphic images and finite direct sums. Clearly, $\mathcal{P}r^{-1}(M)$ contains the class of semisimple right modules. Also, it is immediate that M is projective if and only if $\mathcal{P}r^{-1}(M) = \mathcal{M}od\text{-}R$ (see (Anderson and Fuller, 1992)).

Proposition 2.8 *((Anderson and Fuller, 1992), Proposition 17.9) Let P be a projective right R-module. Then the following statements are equivalent.*

(1) P is a generator;

(2) $\mathrm{Hom}_R(P, T) \neq 0$ *for all simple right R-modules T;*

(3) P generates every simple right R-module.

2.5. Relatively Flat Modules

The purpose of this section is to give some properties of flat modules which can also be found in (Lam, 1999) and (Rotman, 1979).

Definition 2.8 *Given a left module M and a right module N, M is N-flat if for every submodule K of N the map* $1_M \otimes i : K \otimes M \to N \otimes M$ *is a monomorphism, where* $i : K \to N$ *is the inclusion map and* 1_M *is the identity map on M. A left R-module M that is flat relative to every right R-module is called a flat left R-module.*

Proposition 2.9 *((Rotman, 1979), Proposition 3.53) The following statements are equivalent for a right R-module M.*

(1) M is flat;

(2) The sequence $0 \to M \otimes I \to M \otimes R$ *is exact for every left ideal I of R;*

(3) The sequence $0 \to M \otimes I \to M \otimes R$ *is exact for every finitely generated left ideal I of R.*

Lemma 2.3 *((Lam, 1999), Lemma 4.66) For any ring R, the following are equivalent.*

(1) All right ideals of R are flat;

(2) All left ideals of R are flat;

(3) Submodules of flat right modules are flat;

(4) Submodules of flat left modules are flat.

Proposition 2.10 *((Vasconcelos, 1969), Corollary 3.1) Let R be a commutative ring and I a finitely generated ideal of R. Then I is projective if and only if it is flat and its annihilator is finitely generated.*

Theorem 2.6 *((Lam, 1999), Theorem 4.30) Let M be a finitely presented module over any ring R. Then M is flat if and only if M is projective.*

The *flat domain* $\mathcal{F}^{-1}(M)$ of a left module M is defined to be the collection of right modules N such that M is N-flat (see, ((Anderson and Fuller, 1992), page 232, Question 15). It is evident from the definitions M is flat if and only if $\mathcal{F}^{-1}(M)$ contains all left R modules.

2.6. Hereditary, Semihereditary and Regular Rings

In this section we give definitions and properties of hereditary, semihereditary and regular rings.

Definition 2.9 *A ring R is called right hereditary if each right ideal of R is projective. A ring R is called right semihereditary if each finitely generated right ideal of R is projective.*

Dedekind domains and Prüfer domains are hereditary and semihereditary rings, respectively. The following Theorem shows that if R is hereditary, projective modules closed under submodules and injective modules are closed under quotient modules.

Theorem 2.7 *((Rotman, 1979), Theorem 4.23) The following are equivalent for a ring R.*

(1) R is right hereditary (semihereditary);

(2) Every submodule (finitely generated submodule) of a projective module is projective;

(3) Every factor module of an injective module is injective (f.g. injective).

Definition 2.10 *A ring R is called a von Neumann regular ring if for each $x \in R$, there exists $y \in R$ such that $xyx = x$.*

Theorem 2.8 *((Goodearl, 1979), Theorem 1.1) The following are equivalent for a ring R.*

(1) R is von Neumann regular;

(2) Every principal right (left) ideal of R is generated by an idempotent;

(3) Every finitely generated right (left) ideal of R is generated by an idempotent.

The following Theorem gives relation between flat modules and von Neumann regular rings.

Theorem 2.9 *((Goodearl, 1979), Corollary 1.13) For any ring R, the following are equivalent.*

(1) R is von Neumann regular;

(2) Every right module is flat;

(3) Every cyclic right module is flat.

Theorem 2.10 *((Goodearl, 1979), Theorem 1.6) Let R be a commutative ring. Then R is von Neumann regular if and only if all simple modules are injective.*

2.7. Basic Subgroups

In this section we recall the definition of basic subgroups and results which will be useful in the sequel. For more details, we refer to (Fuchs, 1970).

Let p be a prime integer. A group G is called *p-group* if every nonzero element of G has order p^n for some $n \in \mathbb{Z}^+$. For a group G, $T(G)$ denote the torsion submodule of G. The set $T_p(G) = \{a \in G : \ p^k a = 0 \ for \ some \ k \in \mathbb{Z}^+\}$ is a subgroup of G, which is called the *p-primary component* of G. For every torsion group G, we have $G = \oplus_p T_p(G)$. A subgroup A of a group B is *pure* in B if $nA = A \cap nB$ for each integer n. A monomorphism $\alpha : A \to B$ of abelian groups is called *pure* if $\alpha(A)$ is pure in B. For any group G, the subgroups $T(G)$ and $T_p(G)$ are pure in G. A group G is said to be *bounded* if $nG = 0$, for some integer n. Bounded groups are direct sum of cyclic groups (see (Fuchs, 1970), Theorem 17.2). A group G is called a *divisible group* if $nG = G$ for each positive integer n. A group G is called a *reduced group* if G has no proper divisible subgroup.

Definition 2.11 *Let B be a subgroup of A which is satisfying the following three conditions for a fixed prime p.*

(1) B is a direct sum of cyclic p-groups and infinite cyclic groups;

(2) B is p-pure in A i.e. $pA = A \cap pB$;

(3) A/B is p-divisible i.e. $p(A/B) = A/B$.

In this case B is said to be p-basic subgroup of A.

Theorem 2.11 *(1) (Fuchs, 1959) Every group G contains a p-basic subgroup for each prime p.*

(2) (Kulikov, 1941) If H is a pure and bounded subgroup of a group G, then H is a direct summand of G.

For any prime p and q, $p \neq q$, q-basic subgroups of p-groups are 0, the only p-basic subgroups of p-groups may be nontrivial. Therefore, they are usually called simply basic subgroups. Clearly, basic subgroups of p-groups are pure.

2.8. Pure Submodules

This section is devoted to pure submodules of a module, pure exact sequences and regular submodules. For details, we refer to (Cheatham and Smith, 1976) and (Facchini, 1998).

Definition 2.12 *Let R be a ring. A short exact sequence $0 \to A_R \to B_R \to C_R \to 0$ of right R-modules is pure if the induced sequence of abelian groups*

$$0 \to Hom_R(E, A) \to Hom_R(E, B) \to Hom_R(E, C) \to 0$$

is exact for every finitely presented right R-module E.

A submodule A is a *pure submodule* of B if the exact sequence $0 \to A \to B \to B/A \to 0$ is pure. A pure monomorphism is a monomorphism $A \to B$ whose image is a pure submodule of B. Any split short exact sequence is pure. For any family of right R-modules $\{B_i\}$, $\oplus_{i \in I} B_i$ is a pure submodule of $\prod_{i \in I} B_i$ for any index set I.

The following Theorem gives us some important characterizations of pure exact sequences.

Theorem 2.12 *((Facchini, 1998), Theorem 1.27) Let R be a ring and $0 \to A_R \to B_R \to C_R \to 0$ an exact sequence of right R-modules. The following conditions are equivalent.*

(1) The sequence $0 \to A_R \to B_R \to C_R \to 0$ is pure;

(2) For every finitely presented left module F, the induced sequence of abelian groups

$$0 \to A \otimes F \to B \otimes F \to C \otimes F \to 0$$

is exact;

(3) *For every left module F, the induced sequence of abelian groups*

$$0 \to A \otimes F \to B \otimes F \to C \otimes F \to 0$$

is exact.

The relationship between flat modules and pure exact sequences is given in the following theorem.

Theorem 2.13 *((Lam, 1999), Proposition 4.14) Let $0 \to K \to P \to M \to 0$ be a short exact sequence, where P is a projective module. Then M is flat if and only if K is a pure submodule of P.*

A right module M is called *regular* if every submodule is pure in the sense of Cohn (see, (Fieldhouse, 1967)). That is, for every submodule K of M and left module N, the map $0 \to K \otimes N \to M \otimes N$ is exact.

Semisimple modules are regular. There are regular modules which are not semisimple. We note that if the ring is right Noetherian or semilocal then every regular right module is semisimple (Cheatham and Smith, 1976).

Proposition 2.11 *((Cheatham and Smith, 1976), Proposition 2) If R is a ring such that all maximal left ideals of R are finitely generated, then each regular left R-module is semisimple.*

2.9. Absolutely Pure Modules

Maddox (Maddox, 1967) has called a module *absolutely pure* if it is pure in every module containing it as a submodule. The following Proposition gives us the characterization of absolutely pure modules.

Proposition 2.12 *((Enochs and Jenda, 2000), 6.2.3) The following statements are equivalent for a right module N.*

(1) *N is absolutely pure;*

(2) *$N \otimes M \to E(N) \otimes M$ is a monomorphism for each finitely presented left module M;*

(3) *$N \otimes M \to E(N) \otimes M$ is a monomorphism for each left module M;*

(4) *$Ext^1_R(F, N) = 0$ for each finitely presented right module F.*

An R-module M is finitely presented if and only if there is an exact sequence $0 \to A \to R^n \to M \to 0$ with A finitely generated. Hence $\mathrm{Ext}^1(M, N) = 0$ for all finitely presented R-modules M if and only if $\mathrm{Ext}^1(R^n/A, N) = 0$ for every $n > 0$ and finitely generated $A \subseteq R^n$. So an R-module N is absolutely pure if and only if $Hom(R^n, N) \to Hom(A, N) \to 0$ is exact for every $n > 0$ and finitely generated $A \subseteq R^n$.

It is known that injective modules are absolutely pure, but the converse is not true in general.

Example 2.1 *Let F_i be a field for each $i \in I$, where I is an infinite set and $M = \oplus_{i\in I} F_i$ and $R = \prod_{i\in I} F_i$. Then M_R is absolutely pure but not injective.*

Theorem 2.14 *((Megibben, 1970), Theorem 3) R is right Noetherian if and only if each absolutely pure right module is injective.*

Theorem 2.15 *((Megibben, 1970), Theorem 5) A ring R is von Neumann regular if and only if every R-module is absolutely pure.*

2.10. Singular Submodule

In this section we recall the definition of a singular module and state some results about singular and nonsingular modules.

Given any right module M, the *singular submodule of M* is the set

$$Z(M) = \{m \in M : mI = 0 \; for\ some\ essential\ right\ ideal\ \mathrm{I}\ of\ \mathrm{R}\}.$$

Equivalently, $Z(M)$ is the set of those $m \in M$ for which the right ideal $ann_R(m)$ is essential in R. An R-module M is called singular if $Z(M) = M$, and it is called a nonsingular module if $Z(M) = 0$. A ring R is called a right nonsingular ring if R is nonsingular as a right R-module. $Z_r(R)$ will be used for $Z(R_R)$. Similarly, we say that R is *left nonsingular ring* if $Z_l(R) = 0$. Right and left nonsingular rings are not equivalent ((Goodearl, 1976), Exercise 1).

Proposition 2.13 *(Goodearl, 1976) The following hold for any ring R.*

(1) A module N is nonsingular if and only if $Hom(M, N) = 0$ for all singular modules M.

(2) If R is a right semihereditary ring, then $Z_r(R) = 0$.

(3) *If* $Z_r(R) = 0$*, then* $Z(M/Z(M)) = 0$ *for all right R-modules M.*

(4) *If* $N \leq M$*, then* $Z(N) = N \cap Z(M)$*.*

(5) *Suppose that* $Z_r(R) = 0$*. A right module M is singular if and only if* $Hom(M, N) = 0$ *for all nonsingular right modules N.*

Let M be an R-module and $N \leq M$. If N is an essential submodule of M, then M/N is singular. Converse is not true in general. For example, let $M = \mathbb{Z}/2\mathbb{Z}$ and $N = 0$. M/N is singular but N is not an essential submodule of M. The following Proposition shows that when the converse true.

Proposition 2.14 *((Goodearl, 1976), Proposition 1.21) Let M be a nonsingular module and* $N \leq M$*. Then* M/N *is singular if and only if N is an essential submodule of M.*

Proposition 2.15 *((McConnell and Robson, 2001), Proposition 4.5) Let R be a hereditary Noetherian ring and I any essential right ideal. Then the singular right module* R/I *has finite length.*

The class of all singular right modules is closed under submodules, factor modules and direct sums. On the other hand, the class of all nonsingular right modules is closed under submodules, direct products, essential extensions, and module extensions.

Proposition 2.16 *((Goodearl, 1976), Proposition 1.24) If M is any simple right R-module, then M is either singular or projective, but not both.*

The relation between flat and nonsingular modules is given in the following two Propositions.

Proposition 2.17 *((Goodearl, 1972), Proposition 2.3) If R is a nonsingular commutative ring, then all nonsingular modules are flat if and only if R is semihereditary.*

Proposition 2.18 *((Goodearl, 1976), Exercise 12) If* $Z_r(R) = 0$ *and* R_R *is finite-dimensional, all flat right modules are nonsingular.*

A ring R is called a right *SI-ring* if every singular right R-module is injective. A ring R is called a right *PCI-ring* if each proper cyclic right R-module is injective. Right PCI-rings are right Noetherian and right hereditary. The right SI-ring and right PCI-ring conditions are equivalent for domains.

Theorem 2.16 *((Goodearl, 1972), Theorem 3.11) A ring R is right SI if and only if R is right non-singular and $R = K \oplus R_1 \oplus R_2 \oplus \cdots R_n$, where $K/\operatorname{Soc}(K_K)$ is semisimple and each R_i is Morita equivalent to a right PCI-domain.*

2.11. Covers and Envelopes

Definition 2.13 *Let R be a ring and $\mathcal{F}$ a class of R-modules. Then for an R-module M, a morphism $\varphi : M \to F$, where $F \in \mathcal{F}$, is called $\mathcal{F}$-envelope of M if*

(1) any diagram with $F' \in \mathcal{F}$

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "f"] & F \arrow[dl, "g"] \\ F' \end{tikzcd}$$

can be completed such that $g\varphi = f$ and

(2)

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\varphi"] & F \arrow[dl, "g"] \\ F \end{tikzcd}$$

can be completed only by automorphisms of F such that $g\varphi = \varphi$.

If $\varphi : M \to F$ satisfies (1) *but may be not* (2)*, then it is called an $\mathcal{F}$-preenvelope of M.*

If envelopes exist, they are unique up to isomorphism.

Definition 2.14 *Let R be a ring and $\mathcal{F}$ a class of R-modules. Then, for an R-module M, a morphism $\varphi : C \to M$, where $C \in \mathcal{F}$ is called an $\mathcal{F}$-cover of M if*

(1) any diagram with $C' \in \mathcal{F}$

$$\begin{tikzcd} C' \arrow[d, "g"] \arrow[dr, "f"] & \\ C \arrow[r, "\varphi"] & M \end{tikzcd}$$

can be completed to a commutative diagram such that $\varphi g = f$ and

(2) the diagram

$$\begin{tikzcd} C \arrow[d, "g"] \arrow[dr, "\varphi"] & \\ C \arrow[r, "\varphi"] & M \end{tikzcd}$$

can be completed only by automorphisms of Csuch that $\varphi g = \varphi$.

If $\varphi : C \to M$ satisfies (1) *but may be not* (2)*, then it is called an $\mathcal{F}$-precover of M.*

If an $\mathcal{F}$-cover exists, then it is unique up to isomorphism.

2.12. Some Useful Results

In this section we give some known results which we will use or cite throughout the dissertation. The following is a key Theorem.

Theorem 2.17 *((Enochs and Jenda, 2000), Theorem 3.2.11) Let M and N be right R-modules. If M is finitely presented, then $M \otimes_R N^+ \cong \mathrm{Hom}_R(M_R, N_R)^+$.*

Proof Consider the exact sequence $F_1 \to F_0 \to M \to 0$ with F_0 and F_1 finitely generated and free. Then we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_1 \otimes N^+ \arrow[r] \arrow[d, "\cong"] & F_0 \otimes N^+ \arrow[r] \arrow[d, "\cong"] & M \otimes N^+ \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & Hom(F_1, N)^+ \arrow[r] & Hom(F_0, N)^+ \arrow[r] & Hom(M, N)^+ \arrow[r] & 0
\end{tikzcd}$$

with exact rows. But the first two vertical maps are isomorphisms, so φ is an isomorphism.

□

While flat modules are related to projective modules, there is also an interesting relationship between flat modules and injective modules, discovered by J. Lambek (Lambek, 1964).

Theorem 2.18 *((Anderson and Fuller, 1992), Lemma 19.14) A right module M is N-flat if and only if M^+ is N-injective. In particular, M is flat if and only if M^+ is injective.*

Theorem 2.19 *((Cheatham and Stone, 1981), Theorem 3.2.16) If R is right Noetherian, then a right module M is injective if and only if M^+ is flat.*

Proposition 2.19 *((Fuchs and Salce, 2001)) Let R be a commutative ring and I be any ideal of R. Then*

$$Hom_R(R/I, M) \cong M[I],$$

where $M[I] = \{m \in M : Im = 0\}$.

For convenience and for the sake of self-containment, we include here the following known result.

Lemma 2.4 *Let R be a commutative ring and S a simple module. Then $S \cong S^+$.*

Proof Let S be a simple module and P the maximal ideal of R such that $S \cong R/P$. Let $\{S_i\}_{i\in I}$ be the complete set of non isomorphic simple modules. Then $\prod_{i\in I} E(S_i)$ is an injective cogenerator of R. $S^+ = \mathrm{Hom}(S, \prod_{i\in I} E(S_i)) \cong \prod_{i\in I} \mathrm{Hom}(S, E(S_i)) \cong \mathrm{Hom}(S, E(S))$. By Proposition 2.19, $\mathrm{Hom}(S, E(S)) \cong S$. The proof is completed. □

Lemma 2.5 *((Ware, 1971), Lemma 2.6) Suppose R is a commutative ring and S is a simple R-module. Then S is flat if and only if S is injective.*

Lemma 2.6 *((Vasconcelos, 1969), Proposition 1.1) Let R be a commutative ring and I a finitely generated ideal of R. If $I^2 = I$, then I is a direct summand of R.*

Proposition 2.20 *((Lam, 1999), Corollary 3.86) Let R be a commutative Noetherian ring. Then R is Artinian if and only if every injective indecomposable module over R is finitely generated.*

Proposition 2.21 *((Dung, Huynh, Smith and Wisbauer, 1994), 13.5) The following conditions are equivalent for any ring R.*

(1) R is (left and right) Artinian serial and $J(R)^2 = 0$;

(2) R is a direct sum of minimal left ideals and injective left ideals of length 2.

CHAPTER 3

THE OPPOSITE OF ABSOLUTE PURITY

The purpose of this chapter is to mention the study of an alternative perspective on the analysis of the absolute purity of a module.

Given a right R-module M and a left R-module N, the module M is said to be absolutely N-pure if the natural homomorphism $M \otimes N \rightarrow K \otimes N$ is monic for each extension K of M (see (Durğun, 2016)). For a right module M, *the subpurity domain of* M is defined to be the collection of all left modules N such that M is absolutely N-pure. M is absolutely pure if and only if its subpurity domain consists of the entire class R-$\mathcal{M}od$. As an opposite to absolute purity, a module M is called *test for flatness by subpurity* (t.f.b.s.) if its subpurity domain is as small as possible, namely, consisting of exactly the flat left modules. Every ring has a t.f.b.s. Rings all of whose modules are t.f.b.s. are shown to be precisely the von Neumann regular rings.

For a right Noetherian ring R, we prove that every simple right R-module is t.f.b.s. or absolutely pure if and only if R is a right V-ring or $R \cong A \times B$, where A is right Artinian with a unique non-injective simple right R-module and $Soc(A_A)$ is homogeneous and B is semisimple (Theorem 3.3). We also prove necessary conditions for a right Noetherian ring whose (cyclic, finitely generated) right modules are t.f.b.s. or absolutely pure.

A domain R is Prüfer if and only if each finitely generated ideal is t.f.b.s. (Proposition 3.14). Finally, we give a characterization of t.f.b.s. modules over commutative hereditary Noetherian rings. It is proved that an R-module N is t.f.b.s. if and only if $Hom(N/Z(N), S) \neq 0$ for each singular simple R-module S, where $Z(N)$ is the singular submodule of N.

3.1. The Notion of Subpurity Domain of a Module

In this section, we study the properties of subpurity domain and we also give relations between subinjectivity and subpurity domains.

The next lemma shows that for an R-module M to be absolutely N-pure, one only needs to extend maps to $E(M)$.

Lemma 3.1 *The following statements are equivalent for a right module M and a left module N.*

(1) M is absolutely N-pure.

(2) For every right R-module K and essential submodule M of K, the sequence $0 \to M \otimes N \to K \otimes N$ is exact.

(3) The sequence $0 \to M \otimes N \to E(M) \otimes N$ is exact.

(4) The sequence $0 \to M \otimes N \to E \otimes N$ is exact for some injective extension E of M.

Proof The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear. To show (4) $\Rightarrow$ (1), let T be an extension of M. Consider the map $f = i \otimes 1_N : M \otimes N \to T \otimes N$, where $i : M \to T$ is the inclusion. By (4), there is an injective extension E of M such that $g : M \otimes N \to E \otimes N$ is monic. On the other hand, there is an injective extension E' of T such that $E \leq E'$. Then we have the following commutative diagram.

$$\begin{tikzcd} M \otimes N \arrow[r, "f"] \arrow[d, "g"] & T \otimes N \arrow[d, "h"] \\ E \otimes N \arrow[r, "t"] & E' \otimes N \end{tikzcd}$$

Since E is a direct summand of E', t is a monomorphism. Then $hf = tg$ is a monomorphism, and so f is a monomorphism. This implies that $N \in \mathcal{S}(M)$. □

It is known that, $M \otimes (\oplus_{i\in I} N_i) \cong \oplus_{i\in I}(M \otimes N_i)$ for any index set I. From this fact, we get the following Proposition.

Proposition 3.1 *The following properties hold for a right module M.*

(1) M is absolutely $\oplus_{j\in I} N_j$-pure if and only if M is absolutely N_j-pure for each $j \in I$.

(2) If M is absolutely N-pure, then M is absolutely K-pure for every summand K of N.

(3) If K is a pure submodule of the left module N, then M is absolutely N-pure if and only if M is absolutely K-pure and absolutely N/K-pure.

Proof (1) We have $M \otimes (\oplus_{j\in I} N_j) \cong \oplus_{j\in I}(M \otimes N_j)$. Therefore $i \otimes 1_{\oplus_{j\in I} N_j} : M \otimes (\oplus_{j\in I} N_j) \to E(M) \otimes (\oplus_{j\in I} N_j)$ is a monomorphism if and only if $i \otimes 1_{N_j}$ is a monomorphism for each $j \in I$. This proves (1).

(2) is clear by (1).

(3) For completeness we give the proof here which can be also found in ((Durğun, 2016), Proposition 2.5). Consider the exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$. Since K is a pure submodule of N, we have the following commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & M \otimes K \arrow[r, "\beta"] \arrow[d, "\alpha"] & M \otimes N \arrow[r, "\delta"] \arrow[d, "\gamma"] & M \otimes \frac{N}{K} \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & E(M) \otimes K \arrow[r, "\eta"] \arrow[d] & E(M) \otimes N \arrow[r, "\vartheta"] \arrow[d] & E(M) \otimes \frac{N}{K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{E(M)}{M} \otimes K \arrow[r] \arrow[d] & \frac{E(M)}{M} \otimes N \arrow[r] \arrow[d] & \frac{E(M)}{M} \otimes \frac{N}{K} \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Assume that M is absolutely N-pure. Then γ is a monomorphism. Since K is a pure submodule of N, the rows are exact. By diagram chasing we get $\eta\alpha = \gamma\beta$. Since γ and β are monic, then α is monic. By 3×3-Lemma θ is monic. For the converse, α and θ are monomorphism. By Five Lemma, γ is a monomorphism.

□

Definition 3.1 *The subpurity domain of M, $\mathcal{S}(M)$, is defined as the collection of all left modules N such that M is absolutely N-pure.*

Proposition 3.2 $\bigcap_{M \in \mathcal{M}od\text{-}R} \mathcal{S}(M) = \{N \in R\text{-}\mathcal{M}od | N \textit{ is flat}\}$.

Proof Let $N \in \bigcap_{M \in \mathcal{M}od\text{-}R} \mathcal{S}(M)$. Then $N \in \mathcal{S}(I)$ for each right ideal I of R, i.e. $I \otimes N \to R \otimes N$ is a monomorphism. Hence N is flat by Proposition 2.9. The reverse containment is obvious. □

Proposition 3.3 *The following properties hold for any right module M and left module N.*

(1) $\oplus_{i=1}^{n} M_i$ is absolutely N-pure if and only if M_i is absolutely N-pure for each $i = 1, 2, \cdots, n$.

(2) If R is right Noetherian and I is any index set, then $\oplus_{i \in I} M_i$ is absolutely N-pure if and only if M_i is absolutely N-pure for each $i \in I$.

(3) If R is a right hereditary ring and M is absolutely N-pure, then M/K is absolutely N-pure for any submodule K of M.

Proof

(1) Set $M = \oplus_{i=1}^{n} M_i$ and suppose that M is absolutely N-pure. We have $E(M) = \oplus_{i=1}^{n} E(M_i)$ and $(\oplus_{i=1}^{n} M_i) \otimes N \cong \oplus_{i=1}^{n} (M_i \otimes N)$. Then $(\oplus_{i=1}^{n} M_i) \otimes N \to (\oplus_{i=1}^{n} E(M_i)) \otimes N$ is a monomorphism if and only if $M_i \otimes N \to E(M_i) \otimes N$ is a monomorphism for each $i = 1, \cdots, n$. Therefore M is absolutely N-pure if and only if M_i is absolutely N-pure for each $i = 1, \cdots, n$.

(2) Since R is Noetherian, $E(M) = \oplus_{i \in I} E(M_i)$. The rest of the proof is similar to that of (1).

(3) Let K be a submodule of M and E an injective hull of M. Then we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[d, "g"] & M/K \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & K \arrow[r] & E \arrow[r] & E/K \arrow[r] & 0 \end{tikzcd}$$

with f is an isomorphism. Applying $- \otimes N$ to the diagram above gives the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \otimes N \arrow[r] \arrow[d, "f \otimes 1_N"] & M \otimes N \arrow[r] \arrow[d, "g \otimes 1_N"] & M/K \otimes N \arrow[r] \arrow[d, "h \otimes 1_N"] & 0 \\ 0 \arrow[r] & K \otimes N \arrow[r] & E \otimes N \arrow[r] & E/K \otimes N \arrow[r] & 0 \end{tikzcd}$$

Since $f \otimes 1_N$ and $g \otimes 1_N$ is a monomorphism, $h \otimes 1_N$ is a monomorphism by the Five Lemma. On the other hand, E/K is injective by the hereditary condition. Hence M/K is absolutely N-pure by Lemma 3.1.

□

The following is a consequence of Proposition 3.3(1).

Corollary 3.1 $\mathcal{S}(\oplus_{i=1}^{n} M_i) = \cap_{i=1}^{n} \mathcal{S}(M_i)$.

Proposition 3.4 *Let F be a flat right module. Suppose that F is absolutely M-pure for some left module M. Then F is absolutely K-pure for any submodule K of M.*

Proof Let K be a submodule of M. We have the commutative diagram

$$\begin{array}{ccc} F \otimes K & \xrightarrow{h} & E(F) \otimes K \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle t} \\ F \otimes M & \xrightarrow{g} & E(F) \otimes M \end{array}$$

induced by the inclusions $F \to E(F)$ and $K \to M$. Since F is flat and absolutely M-pure, the maps f and g are monomorphisms. Then, by the commutativity of the diagram, $gf = th$ is a monomorphism. Then h is a monomorphism, and so F is absolutely K-pure. □

A module M is said to be N-subinjective if every homomorphism $f : N \to M$ can be extended to a homomorphism $h : E(N) \to M$ (see (Aydoğdu and López-Permouth, 2011)).

Lemma 3.2 *((Durğun, 2016), Proposition 2.6) Given a right module M and a left module N, M is absolutely N-pure if and only if N^+ is M-subinjective.*

Proposition 3.5 *((Durğun, 2016), Proposition 2.8) Let $R = R_1 \oplus R_2$ be a ring decomposition. Then M_R is absolutely $_RN$-pure if and only if MR_i is absolutely R_iN-pure for $i = 1, 2$.*

Proof By assumption, we have $M = MR_1 \oplus MR_2$ for any right module M and $N = R_1N \oplus R_2N$ for any left module N. Now it is clear by Proposition 3.1(1) and 3.3(1). □

3.2. T.f.b.s. Modules

Clearly, a right R-module M is absolutely pure if and only if $\mathcal{S}(M) = R\text{-}\mathcal{M}od$. On the other hand, it makes sense to consider the opposite case: What are the modules whose subpurity domain is as small as possible? It is clear that $\mathcal{S}(M)$ consists of the class of left flat modules.

Definition 3.2 *A right R-module M is called test module for flatness by subpurity (for short, t.f.b.s.) if $\mathcal{S}(M_R)$ consists of only flat left R-modules.*

Proposition 3.6 *Every ring has a t.f.b.s.*

Proof Let R be a ring and $N = \oplus I$, where I ranges among all finitely generated right ideals of R. Assume that a right R-module N is absolutely A-pure. Let I be a finitely generated right ideal. Since I is a direct summand of N, I is absolutely A-pure. So the map $I \otimes A \to R \otimes A$ is a monomorphism. Therefore A is flat by Proposition 2.9. □

Proposition 3.7 *The following hold for a right R-module M.*

(1) If M has a pure submodule N which is t.f.b.s., then M is t.f.b.s.

(2) If M is t.f.b.s., then $M \oplus N$ is t.f.b.s. for any module N.

(3) Let A be an absolutely pure right module. Then $M \oplus A$ is t.f.b.s. if and only if M is t.f.b.s.

(4) M is t.f.b.s. if and only if M^n is t.f.b.s.

(5) If M is flat and t.f.b.s., then submodules of flat left modules are flat.

Proof (1) Let M be an absolutely A-pure module. Then $g : M \otimes A \to E(M) \otimes A$ is a monomorphism. As N is pure in M the map $f : N \otimes A \to M \otimes A$ is also a monomorphism. Now the map $gf : N \otimes A \to E(M) \otimes A$ is a monomorphism. Then N is absolutely A-pure, and so A is flat, because N is t.f.b.s. Hence M is t.f.b.s.

(2) is clear by Corollary 3.1.

(3) follows from the equality $\mathcal{S}(M \oplus A) = \mathcal{S}(M) \cap \mathcal{S}(A) = \mathcal{S}(M)$.

(4) follows from $\mathcal{S}(M^n) = \cap \mathcal{S}(M) = \mathcal{S}(M)$.

(5) is clear by Proposition 3.4.

□

Proposition 3.8 *The following statements are equivalent for a ring R.*

(1) R is von Neumann regular;

(2) Every right R-module is t.f.b.s.;

(3) There exists a right absolutely pure t.f.b.s. R-module.

Proof (1) $\Rightarrow$ (2) Let M be a right R-module. Then $E(M)/M$ is flat by (1), so that M is absolutely pure. Therefore $\mathcal{S}(M) = R\text{-}\mathcal{M}od$, i.e. M is t.f.b.s.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Let M be an absolutely pure sp-poor right R-module. Since M is absolutely pure, $\mathcal{S}(M) = R\text{-}\mathcal{M}od$. But M is t.f.b.s., hence every left R-module is flat. This implies R is von Neumann regular.

□

In order to investigate when the ring is t.f.b.s. as a right module over itself, we need the following definition.

Definition 3.3 *A ring R is called a right S-ring if every finitely generated flat right ideal is projective.*

Right coherent rings, right semihereditary rings, local rings and semiperfect rings are examples of right S-rings.

Theorem 3.1 *A ring R is right t.f.b.s. and a right S-ring if and only if R is right semihereditary.*

Proof Suppose R_R is t.f.b.s. Then every left ideal of R is flat by Proposition 3.7(5). Hence every right ideal of R is flat by Lemma 2.3. Now the right S-ring condition implies that every finitely generated right ideal of R is projective. Therefore, R is right semihereditary.

Conversely, assume that R is right semihereditary. Then R is a right S-ring. To prove that R_R is t.f.b.s., suppose R is absolutely A-pure. Let I be a finitely generated right ideal of R. Then $R^m = I \oplus K$ because R is right semihereditary. So, I is absolutely A-pure by Proposition 3.3(1). Hence, $0 \to I \otimes A \to R \otimes A$ is a monomorphism, and so A is flat. This gives that R_R is t.f.b.s. □

Corollary 3.2 *A commutative domain is Prüfer if and only if it is t.f.b.s.*

Proof A commutative ring is an S-ring if and only if $\mathrm{Ann}(I)$ is finitely generated for each finitely generated flat ideal I by Proposition 2.10. Now it is clear by Theorem 3.1.

□

Corollary 3.3 *Let R be a semiperfect ring. Then the following are equivalent.*

(1) R_R is t.f.b.s.;

(2) ${}_RR$ is t.f.b.s.;

(3) R is semihereditary.

Recall that a module M is said to be a test for injectivity by subinjectivity (t.i.b.s.) if the only M-subinjective modules are injective modules (see (Alizade, Büyükaşık and Er, 2014)).

Proposition 3.9 *If N is right t.i.b.s., then N is right t.f.b.s.*

Proof Let M be an arbitrary left R-module, and suppose that the exact sequence $0 \to N \otimes M \to E(N) \otimes M$ is a monomorphism. Then $(E(N) \otimes M)^+ \to (N \otimes M)^+$ is epic. By the First Adjoint Isomorphism Theorem, we get the following diagram.

$$\begin{tikzcd}
(E(N) \otimes M)^+ \arrow[r] \arrow[d, "\cong"] & (N \otimes M)^+ \arrow[r] \arrow[d, "\cong"] & 0 \\
\mathrm{Hom}(E(N), M^+) \arrow[r] & \mathrm{Hom}(N, M^+) \arrow[r] & 0
\end{tikzcd}$$

Hence M^+ is N-subinjective, and since N is t.i.b.s., M^+ is injective. Therefore, M is flat by Theorem 2.18, and so N is t.f.b.s.. □

Theorem 3.2 *((Alizade, Büyükaşık and Er, 2014), Theorem 19) The following are equivalent for a ring R.*

(1) R_R is t.i.b.s.;

(2) R is right hereditary right Noetherian.

There are t.f.b.s. modules which are not t.i.b.s., for example, every semihereditary ring is t.f.b.s. as a right module over itself. On the other hand, right t.i.b.s. rings are charcterized as follows. In searching the converse of Proposition 3.9, we have the following.

Proposition 3.10 *Let R be a right Noetherian ring. If M is right t.f.b.s. and $E(M)$ is finitely generated, then M is right t.i.b.s.*

Proof Suppose a right module N is M-subinjective i.e. the sequence $\mathrm{Hom}_R(E(M), N) \to \mathrm{Hom}_R(M, N) \to 0$ is epic. Then we get the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(M, N)^+ \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(E(M), N)^+ \arrow[d, "\cong"] \\
0 \arrow[r] & M \otimes_R N^+ \arrow[r] & E(M) \otimes_R N^+
\end{tikzcd}$$

whose columns are isomorphisms by Theorem 2.17. Since M is t.f.b.s., N^+ is flat, and so N is injective by the Noetherianity of R. □

Proposition 3.11 *The following are equivalent for a ring R.*

(1) R_R is t.f.b.s. and Noetherian;

(2) R_R is a t.i.b.s.

Proof (1) $\Rightarrow$ (2) Suppose that R_R is t.f.b.s. Since R is right Noetherian, it is a right S-ring. So, R_R is right semihereditary by Theorem 3.1. Then R is right hereditary since R is Noetherian. Hence R_R is t.i.b.s. by Theorem 3.2

(2) $\Rightarrow$ (1) By Proposition 3.9, R_R is t.f.b.s., and R_R is Noetherian by Theorem 3.2. □

3.3. Rings Whose Simple Modules are Absolutely Pure or T.f.b.s.

In this subsection, we characterize the right Noetherian rings over which each simple right module is t.f.b.s. or injective.

Theorem 3.3 *The following are equivalent for a ring R.*

(1) Every simple right module is t.i.b.s. or injective;

(2) R is a right V-ring or R is right Noetherian and every simple right module is t.f.b.s. or absolutely pure;

(3) R is a right V-ring or $R \cong A \times B$, where A is right Artinian with a unique non-injective simple right R-module and $\mathrm{Soc}(A_A)$ *is homogeneous and B is semisimple.*

Proof (1) $\Rightarrow$ (2) Suppose that every simple right module is t.i.b.s. or injective and suppose that R is not a V-ring. Then there is a noninjective simple right module T. Then T is right t.i.b.s. by the hypothesis. Since T is finitely generated, arbitrary direct sum of injective modules is T-subinjective. So R is right Noetherian. Then every simple right module is t.f.b.s. or absolutely pure by Proposition 3.9.

(2) $\Rightarrow$ (3) Suppose that every simple right module is t.f.b.s. or injective. Assume that R is not a right V-ring. Then there exits a non-injective simple right R-module T, which is t.f.b.s. by the hypothesis. Let U be a simple right R-module which is not isomorphic to T. Then $\mathrm{Hom}(T, U) = 0$. Hence, by Theorem 2.19, $T \otimes U^+ \cong \mathrm{Hom}(T, U)^+ = 0$. This means that T is absolutely U^+-pure. Since T is t.f.b.s., U^+ is flat. Thus, U is injective by Theorem 2.19. This implies that T is the unique non-injective simple right R-module up to isomorphism. We shall prove that R is right semiartinian. Suppose there

is a non-zero right R-module M such that $\mathrm{Soc}(M) = 0$. Let N be a submodule of M. Then $\mathrm{Hom}(T, N) = 0$, and so $0 = \mathrm{Hom}(T, N)^+ \cong T \otimes N^+$ by Theorem 2.17. That is, T is absolutely N^+-pure. Since T is t.f.b.s., N^+ is flat. Hence, N is injective again by Theorem 2.19. Therefore N is a direct summand of M, and so M is semisimple. This is a contradiction. Hence R is right semiartinian, and R is right Artinian by the right Noetherian assumption. Let $R_R = e_1R \oplus \cdots \oplus e_tR \oplus e_{t+1}R \oplus \cdots \oplus e_nR$, where $\{e_1, \cdots, e_n\}$ is a complete set of primitive orthogonal idempotents. Without loss of generality we can assume that $e_{t+1}R, \cdots, e_nR$ are the injective minimal right ideals of R. Set $A = e_1R \oplus \cdots \oplus e_tR$ and $B = e_{t+1}R \oplus \cdots \oplus e_nR$. Then B is a two sided ideal of R, and $\mathrm{Hom}(A, B) \cong \oplus_{i=1}^{t} \oplus_{j=t+1}^{n} \mathrm{Hom}(e_iR, e_jR) = 0$. Otherwise, we have $\mathrm{Hom}(e_iR, e_jR) \neq 0$ for some $1 \leq i \leq t$ and $t + 1 \leq j \leq n$, which implies $e_iR/e_iJ \cong e_jR$, and so $e_iR \cong e_jR$, a contradiction because e_iR is not injective. Thus A is a two sided ideal, and $R = A \oplus B$ is a ring direct sum.

(3) $\Rightarrow$ (1) Being a V-ring is sufficient for the stated condition. Assume $R \cong A \times B$, where A is right Artinian with a unique non-injective simple right R-module and $\mathrm{Soc}(A_A)$ is homogeneous and B is semisimple. Let T be the unique noninjective simple right R-module and M is T-subinjective. We shall prove that M is injective. Since R is right Artinian, $M = E \oplus N$, for some injective submodule E and a submodule N which does not contain non-zero injective submodule by Remark 2.1. We need to show that M is injective, or equivalently, $N = 0$. Suppose the contrary that $N \neq 0$. Note that $\mathrm{Soc}(N) \neq 0$. Let S be a simple submodule of N. Since S is not injective, $S \cong T$. Let $f : T \to N$ be a non-zero homomorphism. Since M is T-subinjective and N is a direct summand of M, N is T-subinjective. Therefore, f extends to a homomorphism $g : E(S) \to N$. As f is one to one and S essential in $E(S)$, g is one to one. Therefore $g(E(S))$ is a nonzero injective submodule of N, a contradiction. Hence M is injective, and so T_R is t.i.b.s. □

By ((Alizade, Büyükaşık and Er, 2014), Proposition 25), every nonzero cyclic module is t.i.b.s. if and only if R is semisimple Artinian. It is natural ask what happens if all simple modules are t.i.b.s. Theorem 3.3 in hand, we have the following.

Corollary 3.4 *Every simple module is a t.i.b.s. if and only if R is semisimple Artinian or right Artinian with a unique simple module.*

Over a von Neumann regular ring, every (simple) right module is t.f.b.s. by Proposition 3.8. Thus the rings whose simple right modules are $t.f.b.s.$ or injective need not be right Noetherian.

3.4. Rings Whose Modules are Absolutely Pure or T.f.b.s.

In this section, we consider rings over which each module is t.f.b.s. or absolutely pure and refer such rings as having no flat middle class. Over a right Noetherian ring we characterize the structure of a ring.

In (Durğun, 2016) the author investigate the absolutely pure domain of right module A as the collection of all left modules M such that M is absolutely A-pure. Absolutely pure domain of any module consists of the class of absolutely pure modules. A right module A is said to be *f-indigent* if its absolutely pure domain is exactly the class of absolutely pure left modules.

Lemma 3.3 *The following conditions are equivalent for a ring R:*

(1) Every right R-module is t.f.b.s. or absolutely pure;

(2) If A_R is absolutely ${}_RB$-pure then A_R is absolutely pure or ${}_RB$ is flat;

(3) Every left R-module is flat or f-indigent.

Proof (1) $\Rightarrow$ (2) Suppose that A_R is absolutely ${}_RB$-pure and A_R is not absolutely pure. By hypothesis, A_R is t.f.b.s., so ${}_RB$ is flat.
(2) $\Rightarrow$ (1) Let M_R be a right module which is not absolutely pure and M_R be absolutely ${}_RN$-pure for any left module ${}_RN$. By (2), ${}_RN$ is flat so M_R is t.f.b.s..
(2) $\Rightarrow$ (3) Suppose ${}_RM$ be a left module which is not flat and N_R is absolutely ${}_RM$-pure for any left module ${}_RM$. By hypothesis, N_R is absolutely pure so ${}_RM$ is f-indigent.
(3) $\Rightarrow$ (2) Let A_R be an absolutely ${}_RB$-pure and ${}_RB$ is not flat. By (3), ${}_RB$ is f-indigent, so A_R is absolutely pure.

$\square$

Lemma 3.4 *Let R be a right Noetherian right V-ring. Suppose every (cyclic) right module is t.f.b.s. or absolutely poor. Then $R \cong A \times B$, where B is semisimple and A is right SI with* $\mathrm{Soc}(A_A) = 0$.

Proof By the hypothesis, R_R is t.f.b.s. or injective. First suppose R_R is t.f.b.s.. Then R_R is hereditary by Theorem 3.1. We shall prove that every cyclic singular right module is injective. Let K_R be cyclic singular right R module. Since R_R is nonsingular, $\mathrm{Hom}(K, R) = 0$. Hence, by Theorem 2.17, $\mathrm{Hom}(K, R)^+ \cong K \otimes R^+ = 0$. This means that K is absolutely R^+-pure. Therefore, R^+ is flat or K is injective by Lemma 3.3. Since R is right Noetherian and non-injective, R^+ is not flat. So, K is injective. Hence R is a right SI-ring. Since

R is a right Noetherian right V-ring, all semisimple modules are injective, so $\mathrm{Soc}(R_R)$ is injective. Then $R = A \oplus \mathrm{Soc}(R_R)$. Set $B = \mathrm{Soc}(R_R)$. Then B is a two sided ideal of R and $\mathrm{Hom}(A, B) = 0$. Otherwise, we have $\mathrm{Hom}(A, B) \neq 0$ which implies $\frac{A}{K} \cong S$ for $K \leq A$ and simple ideal S. This gives $A \cong K \oplus S$, a contradiction because $\mathrm{Soc}(A_A) = 0$. Thus, A is a two sided ideal and $R \cong A \times B$ is a ring direct sum.

If R_R is not t.f.b.s., then R is right injective. So, R is right QF. Hence R is semisimple Artinian. This completes the proof. □

In ((Durğun, 2016), Theorem 4.2) the author prove some necessary conditions for a two sided Noetherian ring over which each right module is flat or f-indigent. In light of Lemma 3.3, the following corresponding result is a slight generalization of ((Durğun, 2016), Theorem 4.2) to right Noetherian rings.

Theorem 3.4 *Let R be a right Noetherian ring. Suppose that every right R-module is t.f.b.s. or absolutely pure. Then $R \cong A \times B$, where B is semisimple, and*

(1) A is right hereditary right Artinian serial with homogeneous socle, $J(A)^2 = 0$ and A has a unique noninjective simple right A-module, or;

(2) A is a QF-ring that is isomorphic to a matrix ring over a local ring, or;

(3) A is right SI with $\mathrm{Soc}(A_A) = 0$.

Proof Suppose that every simple right module is t.f.b.s. or absolutely pure. Then R is a right V-ring or $R \cong A \times B$, where A is right Artinian with a unique non-injective simple right module and $\mathrm{Soc}(A_A)$ is homogeneous and B is semisimple by Theorem 3.3.

Assume that R is not a V-ring, then A is right Artinian. So $A_A = e_1A \oplus e_2A \oplus \cdots \oplus e_nA$, where $e_1, \cdots, e_n$ are primitive orthogonal idempotents. By the hypothesis and the said property A has a unique noninjective minimal right ideal, say T, up to isomorphism. Also any simple right ideal which is not isomorphic to T is injective. Therefore, for each $1 \leq i \leq n$, e_iA/e_iJ is injective or isomorphic to T. If e_iA/e_iJ is injective, then $\mathrm{Hom}(e_iA, T) = 0$. Then $\mathrm{Hom}(e_iA, T)^+ \cong e_iA \otimes T^+ = 0$ by Theorem 2.17. Therefore, e_iA is absolutely T^+-pure and since T^+ is not flat, e_iA is injective by Lemma 3.3. In this case, we have $Soc(e_iA) \cong T$, by the injectivity of e_iA. As the ring is right Artinian, $Soc(e_iA)$ is essential in e_iA. So, there is a submodule $X \leq e_iA$ such that $X/Soc(e_iA)$ is singular. By singularity, $X/Soc(e_iA)$ is not isomorphic to T, hence it must be injective, and so it is a direct summand of $e_iA/Soc(e_iA)$. Since e_iA is local, $e_iA/Soc(e_iA)$ is indecomposable. Therefore, $X/Soc(e_iA) = e_iA/Soc(e_iA)$, and so the composition length of e_iA is 2.

Now, if $e_iA/e_iJ \cong T$, then e_iA/e_iJ is projective. So e_iJ, is a direct summand of e_iA. But e_iA is local, so e_iA must be simple.

As a consequence, A is a direct sum of right ideals which are simple or injective with composition length 2. Hence by Proposition 2.21 we obtain A is serial and $J(A)^2 = 0$. Now, by the hypothesis, R_R is hereditary or injective. If R is hereditary, we obtain (1). If R is injective, then R is right QF by the Noetherian assumption. Then $e_iA \cong e_jA$ for each i and j. That is, $A \cong (eA)^n$ for some local idempotent e in A. In conclusion we obtain (2).

If R is a right V-ring, then (3) follows by Lemma 3.4. This completes the proof.

□

For searching the converse of the above Theorem we need the following Proposition.

Proposition 3.12 *((Alizade, Büyükaşık and Er, 2014), Proposition 10) Let R be a hereditary Artinian serial ring which is indecomposable (or homogeneous right socle) with $J^2 = 0$. Then every right module is t.i.b.s. or injective.*

Note that, by Proposition 3.12 and Proposition 3.9, if R is a ring satisfying the condition (1) of Theorem 3.4, then any right module is t.f.b.s. or absolutely pure over such ring.

A right module M is called f-test module if for every left module N, $Tor_1(M, N) = 0$ implies N is flat, see (Alizade and Durğun, 2017). There exists f-indigent and f-test modules which is not t.f.b.s., on the other hand, there exist t.f.b.s. modules which are neither f-indigent nor f-test. The following two Corollaries help us find this example.

Corollary 3.5 *((Durğun, 2016), Corollary 5.1) An abelian group G is f-indigent if and only if, for each prime p, $T(G) \neq pT(G)$, where $T(G)$ is the torsion part of G.*

Corollary 3.6 *((Alizade and Durğun, 2017), Corollary 4.20) An abelian group is f-test if and only if it contains a submodule isomorphic to $\oplus_p \frac{\mathbb{Z}}{p\mathbb{Z}}$, where p ranges over all primes.*

Example 3.1 *Consider the semisimple $\mathbb{Z}$-module $\oplus_p \mathbb{Z}_p$, where p ranges over all primes and $\mathbb{Z}_p$ denotes the simple $\mathbb{Z}$-module of order p. Then $\oplus_p \mathbb{Z}_p$ is both f-indigent and f-test by Corollaries 3.5 and 3.6, respectively. The module $\oplus_p \mathbb{Z}_p$ is not t.f.b.s. by Corollary 3.8.*

On the other hand, the ring of integers $\mathbb{Z}$ is t.f.b.s. by Theorem 3.1. But $\mathbb{Z}$ is neither f-indigent nor f-test again by by Corollaries 3.5 and 3.6, respectively.

Proposition 3.13 *Let R be a right QF ring. Then every right module is t.f.b.s. or absolutely pure if and only if every right module is flat or f-test.*

Proof Let M be a right module which is not flat. Since R is a right QF ring, M is not absolutely pure. By hypothesis, M is t.f.b.s. Our aim is to show for any left module N, if $\mathrm{Tor}_1(M,N)=0$ then N is flat. Since R is QF-ring, $\mathrm{Tor}_1(E(M),N)=0$. Together with $\mathrm{Tor}_1(M,N)=0$ and $\mathrm{Tor}_1(E(M),N)=0$ implies that M_R is absolutely ${}_RN$-pure. Since M is t.f.b.s. then N is flat.

Conversely, let M be a non absolutely pure module, then by hypothesis M is not flat. Suppose that M is absolutely ${}_RN$-pure for left R-module N. Since $\mathrm{Tor}_1(M,N)=0$ and M is f-test then ${}_RN$ is flat. M is t.f.b.s.

□

3.5. T.f.b.s. Modules Over Commutative Rings

In this section we deal with t.f.b.s. modules over commutative rings. We also give a complete characterization of t.f.b.s. modules over commutative hereditary Noetherian rings.

Proposition 3.14 *The following are equivalent for a commutative domain R.*

(1) R is Prüfer;

(2) R is t.f.b.s.;

(3) Every nonzero finitely generated ideal is t.f.b.s.;

(4) A finitely generated R-module M is t.f.b.s. when $\mathrm{Hom}(M,R)\neq 0$.

Proof (1) $\Leftrightarrow$ (2) By Corollary 3.2.

(1) $\Rightarrow$ (3) Let $I\neq 0$ be a finitely generated ideal of R. Since R is Prüfer, I is projective. Let P be a maximal ideal of R. We claim that $I.P\neq I$. Suppose the contrary that $P.I=I$. Then the localization at P gives $I_P=(I.P)_P=I_P.P_P$. Since I_P is finitely generated and R_P is a local ring with maximal ideal P_P, we have $I_P=0$ by Nakayama's Lemma. This implies $I=0$. Contradiction. Therefore we have $I.P\neq I$ for each maximal ideal of R. Then $I/P.I$ is a semisimple R/P -module, and so semisimple as an R-module. Then $\mathrm{Hom}(I,R/P)\neq 0$, and so I is a projective generator by Proposition 2.8. Hence, there is an epimorphism $f: I^n\to R$, and so $I^n\cong R\oplus K$ for some $K\leq I^n$. Now (2) and Proposition 3.7(2) implies I^n is t.f.b.s. Then I is t.f.b.s. by Proposition 3.7(4).

(3) $\Rightarrow$ (2) is clear.

(3) $\Rightarrow$ (4) Let M be a finitely generated module. Let $0\neq f\in\mathrm{Hom}(M,R)$. Then $f(M)$ is a nonzero finitely generated ideal of R, and hence $f(M)$ is projective by the

equivalence (1) $\Leftrightarrow$ (3). Therefore, $M \cong f(M) \oplus K$ for some $K \leq M$. Since $f(M)$ is t.f.b.s. by (3), the module M is t.f.b.s. by Proposition 3.7(2).

(4) $\Rightarrow$ (2) is clear.

□

Over a Prüfer domain, each finitely generated module can be written as a direct sum of its torsion submodule and a projective submodule (see, (Fuchs and Salce, 2001)). Hence the following is clear by the Theorem above.

Corollary 3.7 *Let R be a Prüfer domain and M a finitely generated R-module. M is t.f.b.s. if and only if $T(M) \neq M$.*

Now we shall give a characterization of t.f.b.s. modules over commutative hereditary Noetherian rings. We begin with the following.

Theorem 3.5 *Let R be a commutative hereditary Noetherian ring and F a flat module. Then F is t.f.b.s. if and only if* $\mathrm{Hom}(F, S) \neq 0$ *for each singular simple R-module S.*

Proof Suppose F is t.f.b.s. and $S \cong R/I$ is a singular simple R-module. Since S is not flat (otherwise S is projective, so nonsingular), we have $0 \neq F \otimes S \cong F/FI$. Therefore F has a maximal submodule K such that $F/K \cong R/I$. This implies $\mathrm{Hom}(F, S) \neq 0$.

For the converse, assume that F is not t.f.b.s. Then there is a non-flat R-module M such that F is absolutely M-pure. Since M is not flat and the ring is hereditary, $Z(M) \neq 0$ by Proposition 2.17. So $Z(M)$ contains a (singular) simple R-module, say S, by Proposition 2.15. Set $E = E(F)$. As F is flat and absolutely M-pure, $F \otimes S \rightarrow E \otimes S$ is a monomorphism by Proposition 3.4. Since E is injective and R is Noetherian, E^+ is flat. Note that E^+ is nonsingular by Proposition 2.18. Then $(E \otimes S)^+ \cong \mathrm{Hom}(S, E^+) = 0$ because E^+ is nonsingular and S is singular. Therefore, $E \otimes S = 0$ and $F \otimes S = 0$. This implies $\mathrm{Hom}(F, S) = 0$. A contradiction. Hence M is nonsingular i.e. flat. This implies F is t.f.b.s. □

Lemma 3.5 *Let R be a commutative Noetherian ring and M an R-module. If $M \otimes R/P = 0$ for some maximal ideal P of R, then $M \otimes E(R/P) = 0$.*

Proof Let $A_i = \{x \in E(R/P) | P^i x = 0\}$. Then $E(R/P) = \cup_{i \geq 1} A_i$ by ((Matlis, 1958), Theorem 3.4). Moreover, A_i is a finitely generated module by ((Matlis, 1958), Theorem 3.4). Then A_i is a finitely generated module over the Artinian ring R/P^i. So, A_i has a finite composition length. Let

$$0 = T_0 \leq T_1 \leq \cdots \leq T_n = A_i$$

be a composition series of A_i. Then $T_{k+1}/T_k \cong R/P$ for each $k = 0, \cdots, i-1$. Consider the sequence $M \otimes T_1 \to M \otimes T_2 \to M \otimes T_2/T_1$. Now $M \otimes R/P = 0$ implies $M \otimes T_1 = M \otimes T_2/T_1$, and so $M \otimes T_2 = 0$. In the next step, from the sequence $M \otimes T_2 \to M \otimes T_3 \to M \otimes T_3/T_2$, we obtain $M \otimes T_3 = 0$. Continuing in this way, at the last step we shall get $M \otimes A_i = 0$. This fact together with $E(R/P) = \cup_{i \geq 1} A_i$ implies that $M \otimes E(R/P) = 0$. This completes the proof.

□

Theorem 3.6 *Let R be a commutative hereditary Noetherian ring and N an R-module. The following are equivalent.*

(1) N is t.f.b.s.;

(2) $N/Z(N)$ *is t.f.b.s.;*

(3) $\mathrm{Hom}(N/Z(N), S) \neq 0$ *for every singular simple R-module S;*

(4) $N/Z(N) \otimes S \neq 0$ *for every singular simple R-module S.*

Proof

(1) ⇒ (4) Suppose $N/Z(N) \otimes S = 0$ for some simple R-module S. Then, $N/Z(N) \otimes E(S) = 0$ by Lemma 3.5. On the other hand, $(Z(N) \otimes E(S))^+ \cong \mathrm{Hom}(Z(N), E(S)^+) = 0$ because $E(S)^+$ is flat i.e. nonsingular. Then, $Z(N) \otimes E(S) = 0$. Therefore, $N \otimes E(S) = 0$ i.e. N is absolutely $E(S)$-pure, then $E(S)$ is flat. This contradicts with the fact that $E(S)$ is singular. Therefore, we must have $N/Z(N) \otimes S \neq 0$.

(2) ⇒ (1) Let N is an absolutely A-pure module. Then $N/Z(N)$ is absolutely A-pure by Proposition 3.3(3). Hence, A is flat by (2). This implies that N is t.f.b.s.

(2) ⇔ (3) The module $N/Z(N)$ is nonsingular i.e. flat. So, the proof is clear by Theorem 3.5.

(3) ⇔ (4) Clear. □

Corollary 3.8 *Let R be a Principal Ideal Domain. Then an R-module G is t.f.b.s. if and only if* $G/T(G) \neq p(G/T(G))$ *for every irreducible element p in R.*

By ((Alizade, Büyükaşık and Er, 2014), Theorem 26), an abelian group G is t.i.b.s. if and only if G contains a direct summand isomorphic to $\mathbb{Z}$. Now the following is clear.

Corollary 3.9 *Let G be a finitely generated abelian group. Then the following are equivalent.*

(1) G *is t.f.b.s.;*

(2) G *is t.i.b.s.;*

(3) $T(G) \neq G$.

CHAPTER 4

THE OPPOSITE OF FLATNESSS

The purpose of this chapter is to mention the modules which satisfy conditions that are opposite of flatness. We establish that such modules exist over arbitrary rings and we call them *Rugged Modules.*

Rings all of whose (cyclic) modules are rugged are shown to be precisely the von Neumann regular rings. We consider rings without a flatness middle class (i.e. rings for which modules must be either flat or rugged.) We obtain that, over a right Noetherian ring every left module is rugged or flat if and only if every right module is poor or injective if and only if $R = S \times T$, where S is semisimple Artinian and T is either Morita equivalent to a right PCI-domain, or T is right Artinian whose Jacobson radical properly contains no nonzero ideals.

Character modules serve to bridge results about flatness and injectivity profiles; in particular, connections between rugged and poor modules are explored. If R is a ring whose regular left modules are semisimple, then a right module M is rugged if and only if its character left module M^+ is poor.

Rugged Abelian groups are fully characterized and shown to coincide precisely with injectively poor and projectively poor Abelian groups.

4.1. Relative Flatness of Modules

This section is devoted to prove the basic properties about relative flatness of modules. We start by recalling what is understood by a relative flat module. For the results in this section we refer to (Büyükaşık et. al, 2017).

Definition 4.1 *Given a right R-module N, a left R-module M is said to be flat relative to N, relatively flat to N, or N-flat if the canonical morphism $K \otimes_R M \to N \otimes_R M$ is a monomorphism for every submodule K of N.*

Proposition 4.1 *Let M be a left R-module, N be any right R-module and $K \leq N$ be any submodule. If M is N-flat then M is K-flat and N/K-flat. If, in addition, K is pure in N, then if M is K-flat and N/K-flat then M is N-flat.*

Proof If M is N-flat then M is K-flat.

Now we only have to prove that M is N/K-flat. If $A \leq N/K$ is any submodule, tensoring with M the pullback diagram of the injection $A \hookrightarrow N/K$ and the projection $N \to N/K$ we get a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \otimes M \arrow[r, equal] \arrow[d] & K \otimes M \arrow[d] & & \\
0 \arrow[r] & P \otimes M \arrow[r] \arrow[d] & N \otimes M \arrow[r] \arrow[d] & C \otimes M \arrow[r] \arrow[d, equal] & 0 \\
 & A \otimes M \arrow[r] \arrow[d] & \frac{N}{K} \otimes M \arrow[r] \arrow[d] & C \otimes M \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

so $A \otimes M \to N/K \otimes M$ is a monomorphism.

Suppose now that M is K-flat and N/K-flat.

Given any submodule $A \leq N$, we know $A \cap K$ (together with the injection maps) is the pullback diagram of the injection maps of K and A into N, so the quotient module $A/(A \cap K)$ is, up to an isomorphism, a submodule of N/K. But M is N/K-flat so the map $\frac{A}{A \cap K} \otimes M \to \frac{N}{K} \otimes M$ is monic and we get the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & (A \cap K) \otimes M \arrow[r] \arrow[d] & A \otimes M \arrow[r] \arrow[d] & \frac{A}{A \cap K} \otimes M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \otimes M \arrow[r] \arrow[d] & N \otimes M \arrow[r] \arrow[d] & \frac{N}{K} \otimes M \arrow[r] & 0 \\
 & \frac{K}{A \cap K} \otimes M \arrow[r] \arrow[d] & \frac{N}{A} \otimes M \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

From the diagram, we immediately see that $A \otimes M \to N \otimes M$ is monic, and so that M is N-flat. □

Now we know how the relative flatness of a module behaves with respect to the modules of a short exact sequence, but, on the other hand, what can we say about the relative flatness of the modules of a short exact sequence respect to a given module?

Proposition 4.2 *Let N be any right R-module, M any left R-module and K any pure submodule of M. If K and M/K are both N-flat modules, then M is N-flat.*

Proof For any submodule $A \leq N$ we have a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & A \otimes K \arrow[r] \arrow[d] & A \otimes M \arrow[r] \arrow[d] & A \otimes \frac{M}{K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \otimes K \arrow[r] \arrow[d] & N \otimes M \arrow[r] \arrow[d] & N \otimes \frac{M}{K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{N}{A} \otimes K \arrow[r] \arrow[d] & \frac{N}{A} \otimes M \arrow[r] \arrow[d] & \frac{N}{A} \otimes \frac{M}{K} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

From by (3×3)- Lemma, we see that $A \otimes M \to N \otimes M$ is monic, and so that M is N-flat. □

From Proposition 4.1, we know that the relative flatness of a module preserves finite direct sums, that is, if M is N_i-flat then M is $\oplus_{i=1}^{n} N_i$-flat. Let us now prove that this property does not only holds for finite direct sums.

Proposition 4.3 *If $\{N_i;\ i \in I\}$ is any family of right R-modules and M is a left R-module, then M is $\oplus_i N_i$-flat if and only if M is N_i-flat for every i.*

Proof The necessary condition becomes clear from Proposition 4.1.

If we well order I, we can write our family of N_i's as $\{N_\alpha;\ \alpha < \omega\}$ for some ordinal number ω.

For any $\mu < \omega$, call $A_\mu = \sum_{\alpha<\mu} N_\alpha$. Then $\oplus_i N_i = A_\omega$ and so, using induction and Proposition 4.1, we get that if $\mu < \omega$ is any successor ordinal number and M is A_α-flat for every $\alpha < \mu$, then M is A_μ-flat.

Let μ be now any limit ordinal, $\mu < \omega$. If $K \leq A_\mu$ is any submodule, since A_μ is a direct union we see that $K = \sum_{\alpha<\mu}(K \cap A_\alpha)$ so we get that $K \otimes M \to A_\mu \otimes M$ is a monomorphism if and only if $\left(\sum_{\alpha<\mu}(K \cap A_\alpha)\right) \otimes M \to \left(\sum_{\alpha<\mu} A_\alpha\right) \otimes M$ is a monomorphism.

But the square

$$\begin{tikzcd} \left(\sum_{\alpha<\mu}(K \cap A_\alpha)\right) \otimes M \arrow[r] \arrow[d, "\cong"'] & \left(\sum_{\alpha<\mu} A_\alpha\right) \otimes M \arrow[d, "\cong"] \\ \sum_{\alpha<\mu}((K \cap A_\alpha) \otimes M) \arrow[r] & \sum_{\alpha<\mu}(A_\alpha \otimes M) \end{tikzcd}$$

is commutative so we get that $\left(\sum_{\alpha<\mu}(K \cap A_\alpha)\right) \otimes M \to \left(\sum_{\alpha<\mu} A_\alpha\right) \otimes M$ is a monomorphism if and only if $(K \cap A_\alpha) \otimes M \to A_\alpha \otimes M$ is a monomorphism for each $\alpha < \mu$.

Now, M is A_α-flat for every $\alpha < \mu$ by the induction hypothesis, so each $(K \cap A_\alpha) \otimes M \to A_\alpha \otimes M$ is a monomorphism. □

4.2. Rugged Modules

In this section we introduce and study the opposite notion of a flat module. We call these new modules rugged modules. For the results in this section we refer to (Büyükaşık et. al, 2017).

Definition 4.2 *The flatness extent of a left R-module M, $\mathcal{F}(M)$, is defined as the class of all M-flat right R-modules. The flatness domain of M, $\mathcal{F}^{-1}(M)$, is the class of all right R-modules relative to which M is flat.*

By Proposition 4.1 we know that $\mathcal{F}^{-1}(M)$ is always closed under submodules and quotient modules and that if $K \leq N$ is pure and $K \in \mathcal{F}^{-1}(M)$ then also $N \in \mathcal{F}^{-1}(M)$ (so this means in particular that $\mathcal{F}^{-1}(M)$ is always closed under direct sums).

We will call a module rugged if its flatness domain is as small as it can be, which begs the question of how small the flatness domain of a module can be.

It is clear that if a right R-module N is such that any of its submodules is pure in it then every left R-module is N-flat, that is, $\mathcal{F}(N) = R\text{-}\mathcal{M}od$, and vice versa. The type of modules for which every submodule is pure were defined as regular modules by D. J. Fieldhouse in his Ph. D. Thesis in 1967 ((Fieldhouse, 1967)) and have been studied by several authors (see for instance (Fieldhouse, 1972), (Cheatham and Smith, 1976)).

Thus, if N_R is regular then $\mathcal{F}(N) = R\text{-}\mathcal{M}od$ and then $N \in \mathcal{F}^{-1}(M)$ for every left R-module M. And conversely, if $N_R \in \bigcap\limits_{R\text{-}\mathcal{M}od} \mathcal{F}^{-1}(M)$ then $R\text{-}\mathcal{M}od = \mathcal{F}(N)$ and then N is regular. So, answering the question above, we see that $\bigcap\limits_{R\text{-}\mathcal{M}od} \mathcal{F}^{-1}(M)$ is the class of all regular right R-modules and so the answer to the question of what the domain of flatness of a rugged module should be is automatically answered.

Definition 4.3 *A left R-module M is said to be rugged if its flatness domain is the class of all regular right R-modules.*

A trivial example of the existence of rugged modules is that of a semisimple ring. Over these rings every right (left) module is regular and so every left (right) module is rugged.

Another example of rings over which every module is rugged is that of regular rings. Over these rings every module is flat and so every submodule of a given module is pure in it. This means that every module is regular and so again every module is rugged.

But of course, over an arbitrary ring not all modules are rugged in general. So for instance in the category of abelian groups $\mathcal{F}^{-1}(\mathbb{Z}) = \mathbb{Z}\text{-}\mathcal{M}od$ since $\mathbb{Z}$ is flat, but not every abelian group is regular since $\mathbb{Z}$ itself is not. Thus $\mathbb{Z}$ is not a rugged abelian group.

So the first question that comes to mind at this point is, what are the rings over which every module is rugged?

Proposition 4.4 *The following statements are equivalent for a ring R.*

(1) R is von Neumann regular;

(2) ${}_RR$ is rugged;

(3) Every cyclic left module is rugged;

(4) Every left module is rugged;

(5) Every right module is rugged.

Proof (1) $\Rightarrow$ (4) Since the ring is regular, every left (right) module is flat. Therefore every right (left) module is regular. This implies that every left module is rugged.

(2) $\Rightarrow$ (1) Since ${}_RR$ is flat, $\mathcal{F}^{-1}({}_RR) = \mathcal{M}od\text{-}R$. On the other hand, by (2), ${}_RR$ is rugged. Therefore every right module is regular, and so R is a von Neumann regular ring.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) is clear.

(1) $\Leftrightarrow$ (5) By left-right symmetry.

□

Theorem 4.1 *Rugged left (right) R-modules always exist for any ring R.*

Proof Let $\mathcal{S}$ be a set of representatives of all finitely presented left R-modules.

If a right R-module N is $\oplus_{M\in\mathcal{S}}M$-flat then, for any submodule $S \leq N$, the morphism $S \otimes (\oplus_{M\in\mathcal{S}}M) \to N \otimes (\oplus_{M\in\mathcal{S}}M)$ is a monomorphism and so $S \otimes M \to N \otimes M$ is a monomorphism for every $M \in \mathcal{S}$. But every left R-module is a direct limit of modules in $\mathcal{S}$, and $\otimes$ commutes with direct limits, so we have that $S \otimes U \to N \otimes U$ is a monomorphism for every left R-module U, and this means that S is pure in N. □

Our next step will be to see how rugged modules behave with respect to submodules.

Proposition 4.5 *If a module has a pure and rugged submodule then it is rugged itself. In particular, every module having a rugged direct summand is itself rugged and, as a consequence, direct summands of rugged modules need not be rugged and direct sums of rugged modules are rugged.*

Proof Suppose that M is a left R-module and that $P \leq M$ is a pure and rugged submodule. Then, for any right R-module N in the flatness domain of M and any submodule $S \leq N$, we have a commutative diagram

$$\begin{tikzcd} S \otimes P \arrow[r] \arrow[d, tail] & N \otimes P \arrow[d, tail] \\ S \otimes M \arrow[r, tail] & N \otimes M \end{tikzcd}$$

in which the bottom row and both columns are monomorphisms.

Then the upper row of the diagram is a monomorphism too and this means that $N \in \mathcal{F}^{-1}(P)$. But P is rugged so N is regular and then M is rugged. □

It is clear that if M is an N-flat module and $K \leq M$ is any pure submodule, then K is also N-flat, that is, $\mathcal{F}^{-1}(M) \subseteq \mathcal{F}^{-1}(K)$. This will be used in the following.

Proposition 4.6 *Let K be a submodule of a right module M. If M/K is flat for some $K \leq M$, then K is rugged if and only if M is rugged.*

Proof Since M/K is flat, K is pure in M and $\mathcal{F}^{-1}(M/K) = R\text{-}\mathcal{M}od$. Then $\mathcal{F}^{-1}(K) = \mathcal{F}^{-1}(K) \cap \mathcal{F}^{-1}(M/K) \subseteq \mathcal{F}^{-1}(M)$ by Proposition 4.2, so by the comment above we get that $\mathcal{F}^{-1}(K) = \mathcal{F}^{-1}(M)$. This implies that K is rugged if and only if M is rugged. □

Remark 4.1 *Let R be a right PCI domain. Then R_R is poor by ((Alahmadi, Alkan and López-Permouth, 2010), Theorem 3.2). But neither ${}_RR$ nor R_R are rugged by Proposition*

4.4. On the other hand, a ring R has an injective poor right module if and only if the ring is semisimple Artinian (see, (Alahmadi, Alkan and López-Permouth, 2010)). Thus over a nonsemisimple von Neumann regular ring every injective right module is rugged but not poor.

Semisimple left modules are left regular. There are left regular modules which are not semisimple. Coincidence of semisimple and regular modules leads to the following. Note that if the ring is right Noetherian or semilocal then every regular right module is semisimple (see (Cheatham and Smith, 1976)).

Proposition 4.7 *Let R be a ring such that every regular left module is semisimple. Then a right module M is rugged if and only if its character module M^+ is poor.*

Proof The equality $\mathcal{F}^{-1}(M) = \mathcal{I}n^{-1}(M^+)$ follows immediately by the adjoint isomorphism theorem without any further assumption on R. Now, with our hypothesis on R this implies that M is rugged if and only if M^+ is poor. □

Proposition 4.8 *Let R be a right Noetherian ring. Then $\mathcal{I}n^{-1}(M) = \mathcal{F}^{-1}(M^+)$ for each right module M. In particular, M is poor if and only if M^+ is rugged.*

Proof By Proposition 2.5 we know that a right R-module N holds in the class $\mathcal{I}n^{-1}(M)$ if and only if $nR \in \mathcal{I}n^{-1}(M)$ for every $n \in N$. Thus, if we prove that $C \in \mathcal{F}^{-1}({}_RM^+)$ if and only if $C \in \mathcal{I}n^{-1}(M)$ for each cyclic right R-module C we will have that $\mathcal{I}n^{-1}(M) = \mathcal{F}^{-1}({}_RM^+)$: if a right R-module N holds in $\mathcal{F}^{-1}(M^+)$ then $nR \in \mathcal{F}^{-1}(M^+)$ $\forall n \in N$. But this means that $nR \in \mathcal{I}n^{-1}(M)$ by Proposition 2.5 and so that $N \in \mathcal{I}n^{-1}(M)$.

On the other hand, if $N \in \mathcal{I}n^{-1}(M)$ then clearly $nR \in \mathcal{I}n^{-1}(M)$ for all $n \in N$. Again we have that $nR \in \mathcal{F}^{-1}(M^+)$ and then $\oplus nR \in \mathcal{F}^{-1}(M^+)$ (Proposition 4.3). But $\mathcal{F}^{-1}(M^+)$ is closed under quotients so $N = \sum_{n\in N} nR \in \mathcal{F}^{-1}(M^+)$ and we are done.

So let's now prove that $N \in \mathcal{F}^{-1}({}_RM^+)$ if and only if $N \in \mathcal{I}n^{-1}(M)$ for any cyclic N.

Let K be a submodule of N. Since K is finitely presented, we get the following commutative diagram whose columns are isomorphisms by Theorem 2.17

$$\begin{tikzcd}
K \otimes M^+ \arrow[r, "\alpha"] \arrow[d, "\cong"] & N \otimes M^+ \arrow[r] \arrow[d, "\cong"] & N/K \otimes M^+ \arrow[r] \arrow[d, "\cong"] & 0 \\
Hom(K, M)^+ \arrow[r, "\beta"] & Hom(N, M)^+ \arrow[r] & Hom(N/K, M)^+ \arrow[r] & 0
\end{tikzcd}$$

Clearly α is monic if and only if β is monic. This implies that $N \in \mathcal{F}^{-1}(M^+)$ if and only if $N \in \mathcal{I}n^{-1}(M)$.

Finally, regular right modules are semisimple over right Noetherian rings so in particular M is poor if and only if M^+ is rugged. □

Summing up Proposition 4.7 and Proposition 4.8 we get the following.

Corollary 4.1 *Let R be a right Noetherian ring. The following hold.*

(1) A left module N is rugged if and only if N^{++} is rugged.

(2) A right module M is poor if and only if M^{++} is poor.

4.3. The Flatness Profile of a Ring

Intuitively, the flatness domain of a module somehow tells us how far (or how close) such a module is from to being flat. We shall see now that for any ring we can construct, by means of the flatness domain of all its modules, a lattice that shows the “levels of flatness" that the category of modules over such a ring can have. For the results in this section we refer to (Büyükaşık et. al, 2017).

Definition 4.4 *The (left) flat profile of any ring R is defined as the class of flatness domains of all (left) R-modules,*

$$f\mathcal{P}({}_RR) = \left\{\mathcal{F}^{-1}(M);\ M \in R\text{-}\mathcal{M}od\right\}.$$

Recall (see (López-Permouth and Simental, 2012) for instance) that the injectivity domain of a right R-module M is defined as $\mathcal{I}n^{-1}(M) = \{N \in \mathcal{M}od\text{-}R;\ M \text{ is } N\text{-injective}\}$, that the right injective profile of R is $i\mathcal{P}(R_R) = \{\mathcal{I}n^{-1}(M) :\ M \in \mathcal{M}od\text{-}R\}$ (which is in bijective correspondence with a set), and that $\bigcap \mathcal{I}n^{-1}(M_i) = \mathcal{I}n^{-1}\left(\prod M_i\right)$ for any family of right R-modules $\{M_i :\ i \in I\}$.

Now, it immediately follows from the adjoint isomorphism theorem that if M is a left R-module then $N_R \in \mathcal{F}^{-1}({}_RM) \Leftrightarrow N_R \in \mathcal{I}n^{-1}(M_R^+)$.

But this means that there is a one to one map from the left flat profile of a ring to its right injective profile, and so that the flat profile of a ring can be considered as set.

Since $f\mathcal{P}(R)$ is ordered by the inclusion, to have a lattice structure we only need to find the minimum of every subset of $f\mathcal{P}(R)$. This becomes clear from the following.

Proposition 4.9 *$\bigcap_{i\in I}\mathcal{F}^{-1}(M_i)=\mathcal{F}^{-1}(\oplus_{i\in I}M_i)$ for any family $\{M_i;\ i\in I\}$ of left R-modules. Therefore $f\mathcal{P}({}_RR)$ is always a lattice.*

Proof $\oplus M_i$ is N_R-flat if and only if $0\to A\otimes(\oplus M_i)\to N\otimes(\oplus M_i)$ is exact for every submodule $A\le N$, that is, if and only if $0\to A\otimes M_i\to N\otimes M_i$ is exact for every $i\in I$ and every submodule $A\le N$. But this just means that every M_i is N-flat, so we are done. □

Therefore we now see that $f\mathcal{P}(R)$ is a lattice, and by the comments above, the map

$$\begin{aligned}\varphi: f\mathcal{P}({}_RR) &\rightarrow i\mathcal{P}(R_R)\\ \mathcal{F}^{-1}(M) &\mapsto \mathcal{I}n^{-1}(M^+)\end{aligned}$$

is one to one. Indeed φ is the canonical inclusion, and $\varphi\left(\bigcap\mathcal{F}^{-1}(M_i)\right)=\varphi\left(\mathcal{F}^{-1}(\oplus M_i)\right)=\mathcal{I}n^{-1}((\oplus M_i)^+)=\mathcal{I}n^{-1}\left(\prod M_i^+\right)=\bigcap\mathcal{I}n^{-1}(M_i)$ so this inclusion φ is a monomorphism of lattices and then we see that $f\mathcal{P}({}_RR)$ is a sublattice of $i\mathcal{P}(R_R)$.

Rings for which every module is either injective or poor, that is, its injective profile consists exactly of two elements (the whole category of modules and the class of all semisimple modules) which are introduced in (Alahmadi, Alkan and López-Permouth, 2010) and named as rings having no middle class. We shall call rings having a similar flat profile as rings having no flat middle class.

Definition 4.5 *A ring is said to have no flat middle class on the left if its left flat profile contains exactly two classes of modules, that is, every left R-module is either flat or rugged (and these two classes of modules are different).*

Now, can we determine, at least in some cases, the shape of the lattice $f\mathcal{P}(R)$? Here are some examples.

Examples.

(1) The flat profile of a ring has a unique element if and only if all modules are rugged, so we have that $|f\mathcal{P}(R)|=1$ if and only if R is regular.

(2) By ((López-Permouth and Simental, 2012), Proposition 2.13) we know that the right injective profile of any right Artinian uniserial ring has exactly $\ell(R)-1$ elements, where $\ell(R)$ denotes the length of its composition series. Since we the left flat profile is a sublattice of the right injective profile, it is easy to find rings with flat profile consisting of precisely 2 elements: non regular, right Artinian rings with

composition length 3 ($\mathbb{Z}/4\mathbb{Z}$ is one such ring). Then, the flat profile of these rings consists of the whole category Mod-R and the class of all regular right R-modules

(3) By the same argument of Example 2, we see that if we find a non-flat and non-rugged $\mathbb{Z}/8\mathbb{Z}$-module then we will have $|f\mathcal{P}(\mathbb{Z}/8\mathbb{Z})| = 3$.

$2\mathbb{Z}/8\mathbb{Z}$ is not a flat module so we only have to check that it is not rugged.

It is clear that $2\mathbb{Z}/8\mathbb{Z} \in \mathcal{F}^{-1}(2\mathbb{Z}/8\mathbb{Z})$ since the canonical map

$$4\frac{\mathbb{Z}}{8\mathbb{Z}} \otimes 2\frac{\mathbb{Z}}{8\mathbb{Z}} \longrightarrow 2\frac{\mathbb{Z}}{8\mathbb{Z}} \otimes 2\frac{\mathbb{Z}}{8\mathbb{Z}}$$

is a monomorphism, and on the other side the canonical map

$$4\frac{\mathbb{Z}}{8\mathbb{Z}} \otimes 4\frac{\mathbb{Z}}{8\mathbb{Z}} \longrightarrow 2\frac{\mathbb{Z}}{8\mathbb{Z}} \otimes 4\frac{\mathbb{Z}}{8\mathbb{Z}}$$

is not a monomorphism so $2\mathbb{Z}/8\mathbb{Z}$ is not a regular $\mathbb{Z}/8\mathbb{Z}$-module.

We see that the flat and the injective profiles of the rings in Examples 2 and 3 above coincide. We now prove that over general Noetherian rings we do always have an isomorphism between these two types of profiles.

Proposition 4.10 *If R is right Noetherian, then its left flat profile and its right injective profile coincide. In particular, R has no left flat middle class if and only if R has no right middle class.*

Proof We know that

$$\begin{aligned} \varphi : f\mathcal{P}({}_R R) &\rightarrow i\mathcal{P}(R_R) \\ \mathcal{F}^{-1}(M) &\mapsto \mathcal{I}n^{-1}(M^+) \end{aligned}$$

is one to one. On the other hand, by Proposition 4.8, $\mathcal{I}n^{-1}(N) = \mathcal{F}^{-1}(N^+)$ for every right R-module N. Then

$$\mathcal{I}n^{-1}(N) = \mathcal{F}^{-1}(N^+) = \mathcal{I}n^{-1}(N^{++})$$

for every N, by Theorem 2.18. This means that, the map φ is onto, and so φ is a bijection.

□

The rings with no right middle class are characterized in (Aydoğdu and Saraç, 2013) and (Er, Lòpez-Permouth and Sökmez, 2011). The question whether a ring with no right middle class is right Noetherian or not is not known. Now Proposition 4.10 in hand, we shall characterize the right Noetherian ring with no left flat middle class. First we recall the following result.

Theorem 4.2 *((Aydoğdu and Saraç, 2013), Theorem 3) Let R be any ring. Then R has no right middle class if and only if $R = S \times T$, where S is semisimple Artinian and T satisfies one of the following conditions:*

(i) *T is Morita equivalent to a right PCI-domain, or*

(ii) *T is a right SI right V-ring with the following properties:*

(a) *T has essential homogeneous right socle and*

(b) *for any submodule A of Q_T which does not contain the right socle of T properly, $QA = Q$, where Q is the maximal right quotient ring of T,or*

(iii) *T is a right Artinian ring whose Jacobson radical properly contains no nonzero ideals.*

Lemma 4.1 *((Aydoğdu and Saraç, 2013), Lemma 2.4) Suppose a ring $R = S \oplus T$ is a direct sum of two rings S and T where S is semisimple Artinian. Then R has no (simple) middle class if and only if T has no (simple) middle class.*

Corollary 4.2 *Let R be a right Noetherian, right SI and right V-ring. Then R has no right middle class if and only if $R = S \oplus T$, where S is semisimple Artinian and T is Morita equivalent to a right PCI-domain.*

Proof By Theorem 2.16, $R = K \oplus R_1 \oplus R_2 \oplus \cdots R_n$ where $K/\operatorname{Soc}(K_K)$ is semisimple and each R_i is Morita equivalent to a right SI-domain for each $i = 1, \cdots n$. By the right Noetherian and right V-ring assumptions $\operatorname{Soc}(K_K)$ is injective. This together with the fact that $K/\operatorname{Soc}(K_K)$ is semisimple implies that K is semisimple. On the other hand, by ((Er, Lòpez-Permouth and Sökmez, 2011), Lemma 2, Lemma 3) we must have $i = 1$. Setting $S = K$ and $T = R_1$, we get the desired decomposition. This proves the necessity. Sufficiency holds by Lema 4.1. □

Corollary 4.3 *Let R be a right Noetherian ring. The following are equivalent.*

(1) *R has no left flat middle class;*

(2) *R has no right middle class;*

(3) *$R = S \times T$, where S is semisimple Artinian and T satisfies one of the following conditions:*

Proof (1) $\Leftrightarrow$ (2) By Proposition 4.10.

(2) $\Leftrightarrow$ (3) By Theorem 4.2 and Corollary 4.2. □

Remark 4.2 *Any von Neumann regular ring is a ring with no flat middle class. There are von Neumann regular rings which are not with no right (left) middle class (see, (Er, Lòpez-Permouth and Sökmez, 2011), (Example 8)). We do not know whether any ring which is not regular and with no left flat middle class is right Noetherian. If this is the case, then we would have a complete characterization of the rings with no left flat middle class by Corollary 4.3.*

4.4. Rings Whose Simple Right Modules are Poor

In this section, we characterize the commutative simple-destitute rings. Also some results in (Aydoğdu and Saraç, 2013) are generalized.

The following is a generalization of ((Alahmadi, Alkan and López-Permouth, 2010), Theorem3.3) and it also shows that ((Aydoğdu and Saraç, 2013), Lemma 4.6) hold without the commutativity assumption on R.

Proposition 4.11 *Let R be a semilocal ring. The right R-module $R/J(R)$ is poor.*

Proof Set $S = R/J(R)$ and suppose S is B-injective for some cyclic right R-module B. We claim that $\mathrm{Rad}(B) = 0$. Suppose that there is a nonzero element $x \in \mathrm{Rad}(B)$. Let $f : xR \to R/J(R)$ be a nonzero homomorphism. Then f can be extended to a homomorphism $g : B \to R/J(R)$. This implies that $f(xR) = g(xR) \subseteq g(\mathrm{Rad}(B)) \subseteq \mathrm{Rad}(R/J(R)) = 0$, a contradiction. Therefore $\mathrm{Rad}(B) = 0$, i.e. $B.J(R) = 0$. Hence B is semisimple, because R is semilocal. □

Proposition 4.12 *Let R be a commutative ring. The following are equivalent.*

(1) Every simple R-module is either poor or injective;

(2) R satisfies one of the following two conditions:

(a) Simple R-modules are either rugged or flat and regular R-modules are semisimple, or

(b) R is a von Neumann regular ring.

Proof Let S be a simple module. Then $S \cong S^{+}$ by Lemma 2.4, and so $\mathcal{F}^{-1}(S) = \mathcal{I}n^{-1}(S)$ by Theorem 2.18. (1) $\Rightarrow$ (2) Suppose R is not von Neumann regular. Let S be a simple module. If S is injective, then $\mathcal{F}^{-1}(S) = R\text{-}\mathcal{M}od$. That is, S is flat. If S is poor, then $\mathcal{F}^{-1}(S)$ contains only the semisimple modules. Thus S is rugged. Since regular modules are always contained in $\mathcal{F}^{-1}(S)$ and S is poor, regular modules are semisimple. This proves (2).

(2) $\Rightarrow$ (1) If the ring is von Neumann regular, then every simple is injective. So (1) holds in this case. Let S be a noninjective simple module. Then S is rugged by (2). That is, $\mathcal{F}^{-1}(S)$ consist of exactly the regular modules. By (2) again, every regular module is semisimple. This fact together with $\mathcal{F}^{-1}(S) = \mathcal{I}n^{-1}(S)$ implies that S is poor. □

By ((Aydoğdu and Saraç, 2013), Lemma 4.6) commutative local rings are simple-destitute. The following theorem gives a characterization of commutative simple-destitute rings.

Theorem 4.3 *Let R be a commutative ring. The following are equivalent.*

(1) R is simple-destitute;

(2) Every simple module is rugged, and regular modules are semisimple;

(3) R is semisimple Artinian, or R is local.

Proof (1) $\Rightarrow$ (3) Suppose R is not local and let us prove that R is semisimple Artinian. If R has an injective simple module, then being poor and injective implies R is semisimple Artinian (see, (Alahmadi, Alkan and López-Permouth, 2010), Remark 2.3.)). Now assume that all simple modules are noninjective. Let I be an ideal which is not properly contained in $rad(R)$. We shall prove that R/I is semisimple. First suppose $I \neq rad(R)$. Then there is a maximal ideal P of R such that $R = P + I$. Set $S = R/P$. Then $S = S.R = S(P + I) = SP + SI = SI$. We shall prove that S is R/I-injective. Let $X/I \leq R/I$ and $f : R/I \to S$. Then $S = SI$ and $f(R/I) \subseteq S$ together implies that $f(R/I) \subseteq f(R/I).I = 0$. Thus S is R/I-injective, and so R/I is semisimple by the hypothesis.

Now for $I = rad(R)$, let us prove that R/I is semisimple. We shall prove this by showing that S is R/I-injective. Let $0 \neq X/I \leq R/I$ and $0 \neq f : X/I \to S$. If $\mathrm{Ker}(f) = 0$, then X/I is simple and so a direct summand of R/I, because $rad(R/I) = 0$. This, clearly, implies that f extends to R/I. Now suppose $0 \neq \mathrm{Ker}\, f = K/I$. Then K/I is a maximal submodule of X/I and I is properly contained in K. Thus R/K is semisimple by the previous paragraph, and so $R/K = X/K \oplus L/K$ for some $X/K \leq R/K$.

Let $\overline{f} : X/K \to S$ be the homomorphism induced by f, $\pi : R/I \to R/K$ the natural epimorphism, $p : R/K \to X/K$ projection homomorphism. Then $\overline{f}p\pi$ extends f. Thus S is R/I-injective, and so R/I is semisimple, by the hypothesis again.

Now let B be an ideal of R properly containing $J = rad(R)$. Since R/J is semisimple and cyclic, B/J is finitely generated and semisimple as a direct summand of R/J. Then there is a finitely generated ideal A of R such that $B = A + J$. Since A is not contained in J, R/A is semisimple again by the arguments above. This implies that $J \subseteq A$, and so $B = A + J = A$ is finitely generated. We conclude that every maximal ideal of R is finitely generated. Now let P, Q be distinct maximal ideals of R. Since $P + Q^2 = R$, R/P is (R/Q^2)-injective. Then R/Q^2 is semisimple by our hypothesis, and hence $Q = Q^2$. In the same way, we get that for every maximal ideal X of R, $X^2 = X$. Being idempotent and finitely generated implies every maximal ideal is a direct summand by Lemma 2.6. Therefore every simple module is projective, and so R is semisimple Artinian.

(3) $\Rightarrow$ (1) If R is semisimple Artinian, then every module is poor, in particular simple modules are poor. If the ring is local, then its unique simple is poor by Proposition 4.11.

(1) $\Rightarrow$ (2) By Proposition 4.12.

(2) $\Rightarrow$ (1) By Proposition 4.7 and Lemma 2.4.

□

In the following Theorem the authors of (Aydoğdu and Saraç, 2013) characterize the commutative Noetherian rings with no simple middle class.

Theorem 4.4 *((Aydoğdu and Saraç, 2013), Theorem 4.7) Let R be a commutative Noetherian ring. Then R has no simple middle class if and only if there is a ring decomposition* $R = S \oplus T$ *where S is semisimple Artinian and T is a local ring.*

The following result is a slight generalization of ((Aydoğdu and Saraç, 2013), Theorem 4.7).

Proposition 4.13 *Let R be a commutative ring with* $\mathrm{Soc}(R) = 0$*. Suppose every maximal ideal is finitely generated. The following are equivalent.*

(1) R has no simple middle class;

(2) R is simple-destitute;

(3) R is local.

Proof (2) ⇔ (3) By Theorem 4.3 and the fact that $\mathrm{Soc}(R) = 0$.

(2) ⇒ (1) is clear.

(1) ⇒ (2) Let S be a simple module. Suppose S is injective. Then S is flat by Lemma 2.5. Since maximal ideals are finitely generated, S is finitely presented. Then S is projective by Theorem 2.6. This implies that R has a direct summand isomorphic to S. This contradicts with the fact that $\mathrm{Soc}(R) = 0$. Therefore S is not injective, and so every simple module is poor by (1). This completes the proof.

□

In (Aydoğdu and Saraç, 2013), the right Artinian rings with no right simple middle class are characterized as follows.

Theorem 4.5 *((Aydoğdu and Saraç, 2013), Theorem 3.7) If R is a right Artinian non-semisimple ring with no simple middle class, then R is a ring decomposition $R = S \oplus T$ where S is semisimple Artinian and $\mathrm{Soc}(T_T)$ is poor homogenous.*

Lemma 4.2 *((Er, Lòpez-Permouth and Sökmez, 2011), Proof of Proposition 3) Let R be a right semiartinian ring and S_1 be a poor simple noninjective module and S_2 be a simple noninjective module. Then $S_1 \cong S_2$.*

Proof Let R be a right semiartinian ring. Then there exists some $S_2 \subseteq S'_2 \subseteq E(S_1)$ such that S_2 is maximal (and essential) in S'_2. Since S_1 is poor, S_1 must be a proper submodule of $\mathrm{Tr}(S'_2, E(S_1))$. Then there exist some homomorphism $f : S'_2 \to E(S_1)$ such that $f(S'_2)$ is not contained in S_1, whence $f(S'_2)$ properly contains S_1. Thus the composition length of $f(S'_2)$ greater than 1, forcing f to be a monomorphism. It follows immediately that $S_1 \cong S_2$. □

For right semiartinian rings we have the following.

Proposition 4.14 *Let R be a right semiartinian ring with no simple middle class. Then R is a right V-ring or, there is a ring direct sum $R = S \oplus T$, where S is semisimple Artinian and T has a unique noninjective simple right module up to isomorphism, and $\mathrm{Soc}(T)$ is homogeneous.*

Proof Suppose R is not a V-ring. Let U be a noninjective simple right module. Then U is poor by the hypothesis. By Lemma 4.2, R has a unique noninjective simple right module, up to isomorphism, under the stated hypothesis. Let S be the sum of the injective simple right ideals of R. We claim that, S is injective. Suppose the contrary and let E be the injective hull of S. By the right semiartinian condition the socle of E/S is nonzero. Let X/S be a simple submodule of E/S. We shall prove that U is X-injective. Let A

be a nonzero submodule of X and $f : A \to U$ be a homomorphism. If $A \leq S$, then $\mathrm{Hom}(A, U) = 0$ because U is noninjective and A is semisimple. So f extends to X, trivially. Suppose A is not contained in S. Then $A + S = X$, because S is a maximal submodule of X and A is nonzero. Since S is semisimple, $S = A \cap S \oplus S'$ for some $S' \leq S$. Then $X = A \oplus S'$, and clearly $f\pi : X \to U$ extends f, where $\pi : X \to A$ is the natural projection. This implies that U is X-injective. But U is poor, so X is semisimple, and then S is a direct summand of X. This contradicts with the fact that S is essential in X. Therefore S must be injective. Hence $R = S \oplus T$ for some right ideal T of R. By the choice of S, we have $\mathrm{Hom}(S, T) = 0$ and $\mathrm{Hom}(T, S) = 0$. Thus S and T are two-sided ideals, and so $R = S \oplus T$ is a ring direct sum. As R has a unique noninjective simple right module, T has the same property as well. This completes the proof. □

We do not know whether the converse of Proposition 4.14 is true or not. Regarding this, we have the following.

Proposition 4.15 *Let R be a right semiartinian ring with a unique noninjective simple right module U. Then $U^{(\mathbb{N})}$ is poor.*

Proof Suppose $U^{(\mathbb{N})}$ is B-injective for some (nonzero) cyclic right module B. The socle $\mathrm{Soc}(B)$ is essential in B by the semiartinian condition. Let $\mathrm{Soc}(B) = U^{(I)}$ for some index set I. Let us show that I must be finite. If I is infinite, then B has a (non finitely generated) semisimple submodule, say A, isomorphic to $U^{(\mathbb{N})}$. This implies that, A is B-injective, and so $B = A \oplus A'$ for some $A' \leq B$. This contradicts with the fact that B is cyclic. Therefore I must be finite.

If I is finite, then $\mathrm{Soc}(B) = U^{(I)}$ is B-injective. So $\mathrm{Soc}(B)$ is a direct summand of B. Thus B is semisimple, because $\mathrm{Soc}(B)$ is essential in B. Therefore $U^{(\mathbb{N})}$ is poor. □

The converse of Proposition 4.14 is true for commutative semiartinian rings. To see this, we need the following lemma.

Lemma 4.3 *Let R be a commutative ring and U be a simple R-module. If U is B-injective for some module B, then $U^{(I)}$ is B-injective for every index set I.*

Proof Let $P = ann_R(U)$ and I be an index set. U is B-injective by the hypothesis, so U^I is B-injective by Proposition 2.6. Since R is commutative and $U^I.P = 0$, the module U^I is a semisimple R/P-module. So U^I is also semisimple as an R-module. This implies $U^{(I)}$ is a direct summand of U^I, and so it is B-injective. □

The following is a consequence of Proposition 4.14 and Lemma 4.3.

Corollary 4.4 *Let R be a commutative semiartinian ring. Then R has no simple middle class if and only if $R = S \times T$, where S is semisimple Artinian and T is local.*

4.5. Rugged Abelian Groups

In this section, we characterize the rugged abelian groups. It turns out that, the notions of rugged, poor and p-poor coincide over the ring of integers.

Proposition 4.16 *Let R be a commutative hereditary Noetherian ring. Then a module N is rugged if and only if the singular submodule $Z(N)$ of N is rugged.*

Proof Since R is a commutative hereditary ring, $N/Z(N)$ is flat by Proposition 2.17. Thus $Z(N)$ is a pure submodule of N. Now the proof follows by Proposition 4.6. □

Example 4.1 *The socle $\oplus_p \mathbb{Z}_p$ of $\mathbb{Q}/\mathbb{Z}$, where p ranges over all primes and $\mathbb{Z}_p$ is the simple abelian group of order p, is a rugged $\mathbb{Z}$-module.*

Proof We have $(\oplus_p \mathbb{Z}_p)^+ = \mathrm{Hom}(\oplus_p \mathbb{Z}_p, \mathbb{Q}/\mathbb{Z}) \cong \prod_p \mathrm{Hom}(\mathbb{Z}_p, \mathbb{Q}/\mathbb{Z}) \cong \prod_p \mathbb{Z}_p$. The group $\prod_p \mathbb{Z}_p$ is poor by ((Alizade and Büyükaşık, 2017), Theorem 3.1). Since $\oplus_p \mathbb{Z}_p$ is the torsion part of $\prod_p \mathbb{Z}_p$, $\oplus_p \mathbb{Z}_p$ is pure in $\prod_p \mathbb{Z}_p$. Then $\oplus_p \mathbb{Z}_p$ is rugged by Proposition 4.7. □

Theorem 4.6 *An abelian group G is rugged if and only if its torsion part $T(G)$ has a direct summand isomorphic to $\oplus_p \mathbb{Z}_p$.*

Proof To prove the necessity, let G be a rugged group. Then $T(G)$ is rugged by Proposition 4.16. If $T_p(G) = 0$ for some prime p, then $pT(G) = T(G)$. Whence

$$p\mathbb{Z}_{p^2} \otimes T(G) \cong \mathbb{Z}/p\mathbb{Z} \otimes T(G) \cong T(G)/pT(G) = 0.$$

This implies that $T(G)$ is $\mathbb{Z}_{p^2}$-flat, a contradiction. Thus $T_p(G) \neq 0$ for each prime p. Fix a prime p, and let B be the p-basic subgroup of $T_p(G)$ by Theorem 2.11. B is a direct direct sum of cyclic p-groups i.e. groups isomorphic $\mathbb{Z}_{p^n}$. We claim that B has a direct summand isomorphic to $\mathbb{Z}_p$. Suppose the contrary, and let $B = \oplus_{i\in I} < a_i >$ where each $< a_i >$ is a cyclic group isomorphic to $\mathbb{Z}_{p^n}$ for some $n \geq 2$. Then, for each $i \in I$, $< a_i >$ is $\mathbb{Z}_p^2$-flat and so B is $\mathbb{Z}_p^2$-flat. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & B \otimes \mathbb{Z}_p \arrow[r, "\theta"] \arrow[d, "\alpha"] & T_p(G) \otimes \mathbb{Z}_p \arrow[r] \arrow[d, "\gamma"] & (T_p(G)/B) \otimes \mathbb{Z}_p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \otimes \mathbb{Z}_{p^2} \arrow[r, "\beta"] & T_p(G) \otimes \mathbb{Z}_{p^2} \arrow[r] & (T_p(G)/B) \otimes \mathbb{Z}_{p^2} \arrow[r] & 0
\end{tikzcd}$$

Since B is $\mathbb{Z}_{p^2}$-flat and pure in $T_p(G)$, α, β and θ are monomorphisms. Then $\beta\alpha = \gamma\theta$ is a monomorphism. $T_p(G)/B$ is divisible, so

$$T_p(G)/B \otimes \mathbb{Z}_p = (T_p(G)/B) \otimes \mathbb{Z}_{p^2} = 0.$$

Thus θ is an isomorphism. This clearly implies that γ is a monomorphism, and so $T_p(G)$ is $\mathbb{Z}_{p^2}$-flat. Since $T(G) = \oplus_p T_p(G)$ and $T_q(G) \otimes \mathbb{Z}_p = 0$ for all primes $q \neq p$, we have $T(G) \otimes \mathbb{Z}_p \cong T_p(G) \otimes \mathbb{Z}_p$ and $T(G) \otimes \mathbb{Z}_{p^2} \cong T_p(G) \otimes \mathbb{Z}_{p^2}$. Thus $T(G)$ is $\mathbb{Z}_{p^2}$-flat. This contradicts with the fact that, $T(G)$ is rugged. So B must have a direct summand, say A_p, isomorphic to $\mathbb{Z}_p$. Now A_p is bounded and pure in $T_p(G)$, so A_p is a direct summand of $T_p(G)$ by Theorem 2.11. Write $T_p(G) = A_p \oplus C_p$. Then

$$T(G) = \oplus_p T_p(G) = \oplus_p (A_p \oplus C_p) = (\oplus_p A_p) \oplus (\oplus_p C_p).$$

Note that $\oplus_p A_p \cong \oplus_p \mathbb{Z}_p$. This completes the proof of the necessity.

For the sufficiency, suppose that $T(G)$ contains a direct summand isomorphic to $\oplus_p \mathbb{Z}_p$. Then $\oplus_p \mathbb{Z}_p$ is a pure submodule of $T(G)$. On the other hand $T(G)$ is pure in G, whence $\oplus_p \mathbb{Z}_p$ is a pure submodule of G. Since $\oplus_p \mathbb{Z}_p$ is rugged, G is rugged by Proposition 4.5. $\square$

Recall that a module M is called p-poor if its domain of projectivity consists precisely of the semisimple modules (see (Holston, López-Permouth and Ertaş, 2012)).

Theorem 4.7 *((Alizade and Sipahi, 2017), Theorem 4.1) The following are equivalent for an abelian group G.*

(1) G is poor;

(2) G is p-poor;

(3) For every prime integer p, a p-basic subgroup $B_p(G)$ of G has a direct summand isomorphic to $\mathbb{Z}_p$;

(4) For every prime integer p, G has a direct summand isomorphic to $\mathbb{Z}_p$;

(5) The torsion part $T(G)$ of G has a direct summand isomorphic to $\oplus_{p\in P}\mathbb{Z}_p$.

By Theorem 4.7 and 4.6, we get the following.

Corollary 4.5 *For an abelian group G, the following are equivalent.*

(1) G is poor;

(2) G is rugged;

(3) G is p-poor.

4.6. Homological Properties

In this section we mention about rugged precovers and envelopes. For the results, we refer to (Büyükaşık et. al, 2017).

Proposition 4.17 *Every left (right) R-module has a surjective rugged precover for any ring R.*

Proof We know there exists at least a rugged left R-module P, so for any $M \in R$-Mod the module $M \oplus P$ is rugged. Thus the fist canonical projection $M \oplus P \to M$ is a surjective rugged precover of M. □

But what about rugged covers? Are they as easy to find as rugged precovers? The answer is no. Indeed, as we shall see in the next result, rugged covers become really rare in general. In fact every ring is very poor in its supply of rugged covers. They only occur in the trivial case when the module is itself rugged.

Theorem 4.8 *A left (right) module has a rugged cover if and only if it is rugged.*

Proof If M is any left R-module we know $p_1 : M \oplus P \to M$ (P is any rugged left module) is a rugged precover, so if M has a rugged cover, say $\varphi : F \to M$, we get a commutative diagram

$$\begin{tikzcd} & M \oplus P \arrow[d, "p_1"] \\ F \arrow[ur, dashed, "f"] \arrow[r, "\varphi"'] & M \end{tikzcd}$$

and we know that φ is surjective and that $f(F)$ is a direct summand of $M \oplus P$.

But since $f(F)$ is a direct summand of $M \oplus P$, $p_1 f(F) = M$ means that $f(F) = M \oplus T$ for some direct summand T of P.

Now, the diagram

$$\begin{tikzcd} & M \oplus T \arrow[d, "p_1"] \arrow[dl, dashed] \\ M \oplus T \arrow[r, "p_1"'] & M \end{tikzcd}$$

can be completed commutatively by $id_M \oplus 0$, so we see that $p_1 : M \oplus T \to M$ cannot be a rugged cover unless $T = 0$ and so M should be rugged and its rugged cover should be the trivial one, $M = M$. $\square$

The same applies to rugged preenvelopes and envelopes, so rugged preenvelopes always exist but nontrivial rugged envelopes never happen.

Proposition 4.18 *If P is any nontrivial rugged module, then for any module M the canonical injection $M \to M \oplus P$ is an injective rugged preenvelope. Furthermore, M has a rugged envelope if and only if M is rugged.*

CHAPTER 5

CONCLUSION

In this thesis, motivated by the poor and t.i.b.s. modules, we introduced the rugged and t.f.b.s. modules. The aim of this study is to investigate the rings and modules that are characterized via these modules. We characterized the right Noetherian rings over which every module is t.f.b.s. or absolutely pure and every module is rugged or flat. Connections between poor and rugged modules, also t.i.b.s. and t.f.b.s. modules are given. Rugged and t.f.b.s. abelian groups are fully characterized.

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