J. Plasma Physics (2007), *vol.* 73, *part* 2, *pp.* 257–272.
doi:10.1017/S0022377806004648 First published online 3 July 2006 Printed in the United Kingdom

The resonant nonlinear Schrödinger equation in cold plasma physics. Application of Bäcklund–Darboux transformations and superposition principles

J.-H. LEE,[1] O.K. PASHAEV,[2]
C. ROGERS[3,4] and W.K. SCHIEF[3,4]

[1]Institute of Mathematics, Academia Sinica, Taiwan
(leejh@math.sinica.edu.tw)

[2]Department of Mathematics, Izmir Institute of Technology, Turkey
(pashaev@math.sinica.edu.tw)

[3]School of Mathematics, University of New South Wales, Sydney, Australia
(colinr@maths.unsw.edu.au)

[4]Australian Research Council Centre of Excellence for Mathematics and Statistics of Complex Systems
(schief@maths.unsw.edu.au)

(Received 16 March 2006)

Abstract. A system of nonlinear equations governing the transmission of uni-axial waves in a cold collisionless plasma subject to a transverse magnetic field is reduced to the recently proposed resonant nonlinear Schrödinger (RNLS) equation. This integrable variant of the standard nonlinear Schrödinger equation admits novel nonlinear superposition principles associated with Bäcklund–Darboux transformations. These are used here, in particular, to construct analytic descriptions of the interaction of solitonic magnetoacoustic waves propagating through the plasma.

1. Introduction

The nonlinear Schrödinger (NLS) equation arises in the description of a wide range of physical phenomena. It is an integrable model which admits envelope soliton solutions of application notably in nonlinear optics and plasma physics. Dependent on the sign of dispersion (positive or negative), two types of NLS equation are known corresponding, in turn, to defocusing and focusing. These admit dark and bright solitons respectively. The envelope wave function is a complex quantity and the quadratic dispersion, in general, consists of two parts: the phase dispersion and the modulus dispersion. The first corresponds to geometrical optics effects, while the second is associated with diffraction. In both the focusing and defocusing cases of the standard NLS equation, the contribution to dispersion from the phase and the modulus have the same sign (positive and negative, respectively). However, in the phenomenological description of certain hypothetical nonlinear media, the sign may change as a result of competition between the phase and modulus dispersion. Then, the response of the medium to the action of a quasi-monochromatic wave with complex amplitude $\psi(x, t)$, which is a slowly varying function of the coordinate x and the time t is described by a novel integrable version of the NLS equation namely [1],

$$i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+\frac{\Lambda}{4}|\psi|^2\psi=s\frac{1}{|\psi|}\frac{\partial^2|\psi|}{\partial x^2}\psi. \tag{1.1}$$

This has been termed the *resonant nonlinear Schrödinger* (RNLS) equation. It can be regarded as a third version of the NLS equation, intermediate between the defocusing and focusing cases. The additional nonlinear term in the NLS equation can be viewed as due to an additional electrostriction pressure or diffraction term [2]. Even though the RNLS equation is integrable for arbitrary values of the coefficient s, the critical value $s = 1$ separates two distinct regions of behaviour. Thus, for $s < 1$ the model is reducible to the conventional NLS equation (focusing for $\Lambda > 0$ and defocusing for $\Lambda < 0$). However, for $s > 1$ it is not reducible to the usual NLS equation, but rather to a reaction–diffusion (RD) system. In this case, the model exhibits novel solitonic phenomena [1].

The RNLS equation can be interpreted as an NLS-type equation with an additional 'quantum potential' $U_{\rm Q} = |\psi|_{xx}/|\psi|$. This latter potential was introduced by de Broglie [3] and was subsequently used by Bohm [4] to develop a hidden-variable theory in quantum mechanics [5]. It also appears in stochastic mechanics [6]. Connections between such non-classical motions with the *internal* spin motion and the *zitterbewegung* have been considered in a series of papers (see [7]). Quantum potentials also appear in proposed nonlinear extensions of quantum mechanics with regard both to stochastic quantization [8,9] and to corrections from quantum gravity [10]. It is noted that the RNLS equation, like the conventional NLS equation, may also be derived in the context of capillarity models [11, 12].

In the present paper, our interest is in the RNLS equation as it appears in plasma physics. Thus, it is here shown to describe the propagation of one-dimensional long magnetoacoustic waves in a cold collisionless plasma subject to a transverse magnetic field. A bilinear representation of the RNLS equation is given. Bäcklund–Darboux transformations along with novel associated nonlinear superposition principles are presented and used to generate, in particular, solutions descriptive of the interaction of solitonic magnetoacoustic waves.

2. The dynamics of cold collisionless plasma

The dynamics of two-component cold collisionless plasma in the presence of an external magnetic field is described by the system of equations [13, 14]

$$m_{\rm i}\left[\frac{\partial}{\partial t}+\mathbf{v}_{\rm i}\cdot\nabla\right]\mathbf{v}_{\rm i}=e[\mathbf{E}+(\mathbf{v}_{\rm i}\times\mathbf{B})], \tag{2.1}$$

$$m_{\rm e}\left[\frac{\partial}{\partial t}+\mathbf{v}_{\rm e}\cdot\nabla\right]\mathbf{v}_{\rm e}=-e[\mathbf{E}+(\mathbf{v}_{\rm e}\times\mathbf{B})], \tag{2.2}$$

$$\frac{\partial n_{\rm i}}{\partial t}+\nabla\cdot(n_{\rm i}\mathbf{v}_{\rm i})=0, \tag{2.3}$$

$$\frac{\partial n_{\rm e}}{\partial t}+\nabla\cdot(n_{\rm e}\mathbf{v}_{\rm e})=0, \tag{2.4}$$

$$\operatorname{curl}\mathbf{B}=e\mu_0(n_{\rm i}\mathbf{v}_{\rm i}-n_{\rm e}\mathbf{v}_{\rm e}), \tag{2.5}$$

$$\frac{\partial\mathbf{B}}{\partial t}=-\operatorname{curl}\mathbf{E}, \tag{2.6}$$

$$\nabla\cdot\mathbf{B}=0, \tag{2.7}$$

where $m_\mathrm{i}, m_\mathrm{e}$, $\mathbf{v}_\mathrm{i}, \mathbf{v}_\mathrm{e}$, $n_\mathrm{i}, n_\mathrm{e}$ denote, in turn, masses, velocities and concentrations of ions and electrons respectively. $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, e is the electric charge and μ_0 is the magnetic permeability. If the frequency of oscillations is much smaller than the ion Langmuir frequency then plasma quasi-neutrality is implied, i.e. $n_\mathrm{i} \approx n_\mathrm{e} = n$. The mass density ρ and velocity $\mathbf{v}$ of the plasma may then be introduced via

$$\rho = (m_\mathrm{i} + m_\mathrm{e})n, \quad \mathbf{u} = \frac{m_\mathrm{i}\mathbf{v}_\mathrm{i} + m_\mathrm{e}\mathbf{v}_\mathrm{e}}{m_\mathrm{i} + m_\mathrm{e}}, \tag{2.8}$$

whence (2.3), (2.4), imply the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{2.9}$$

Elimination of the electric field $\mathbf{E}$ if $m_\mathrm{e}/m_\mathrm{i} \ll 1$ and introduction of appropriate dimensionless variables leads to the system

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}(\mathbf{B} \times \operatorname{curl} \mathbf{B}) - \left(\frac{\operatorname{curl} \mathbf{B}}{\rho} \cdot \nabla\right)\left(\frac{\operatorname{curl} \mathbf{B}}{\rho}\right) = \mathbf{0}, \tag{2.10}$$

$$\begin{aligned} \frac{\partial}{\partial t}\left[\mathbf{B} + \frac{m_\mathrm{i}}{m_\mathrm{e}} \operatorname{curl} \mathbf{u} + \operatorname{curl}\left(\frac{\operatorname{curl} \mathbf{B}}{\rho}\right)\right] &= \operatorname{curl}(\mathbf{u} \times \mathbf{B}) - \frac{m_\mathrm{i}}{m_\mathrm{e}} \operatorname{curl}(\mathbf{u} \cdot \nabla)\mathbf{u} \\ &\quad - \operatorname{curl}\left[(\mathbf{u} \cdot \nabla)\frac{\operatorname{curl} \mathbf{B}}{\rho} + \left(\frac{\operatorname{curl} \mathbf{B}}{\rho} \cdot \nabla\right)\mathbf{u}\right], \end{aligned} \tag{2.11}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{2.12}$$

3. Magnetoacoustic waves: Uni-axial propagation

For uni-axial plasma propagation with

$$\mathbf{u} = u(x,t)\,\mathbf{e}_x, \quad \mathbf{B} = B(x,t)\,\mathbf{e}_z \tag{3.1}$$

the system (2.9)–(2.12) reduces to the form [15]

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \tag{3.2}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{B}{\rho}\frac{\partial B}{\partial x} = 0, \tag{3.3}$$

$$\frac{\partial}{\partial t}\left[B - \frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial B}{\partial x}\right)\right] + \frac{\partial}{\partial x}\left[u\left(B - \frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial B}{\partial x}\right)\right)\right] = 0. \tag{3.4}$$

Equations (3.2) and (3.4) together imply that

$$B - \frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial B}{\partial x}\right) = C\rho \tag{3.5}$$

where $C = C(m)$ with $m_x = -\rho, m_t = \rho u$. If we set $B = 1$ and $\rho = 1$ at infinity then $C = 1$ and the system (3.2)–(3.4) reduces to

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \tag{3.6}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{B}{\rho}\frac{\partial B}{\partial x} = 0, \tag{3.7}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial B}{\partial x}\right) = B - \rho. \tag{3.8}$$

This system is equivalent to that of Whitham [16] and has also been derived by Gurevich and Meshcherkin [17]. It describes the propagation of nonlinear magnetoacoustic waves in a cold plasma with a transverse magnetic field.

It has been shown recently by El, Khodorovskii and Tyurina [18] that a system of the type (3.6)–(3.8) also occurs in the context of hypersonic flow past slender bodies.

4. A shallow water approximation

Here, we consider a shallow water approximation to the magnetoacoustic system (3.6)–(3.8). Thus, rescaling the space and time variables via $x' = \beta x$ and $t' = \beta t$, we have

$$\frac{\partial \rho}{\partial t'} + \frac{\partial}{\partial x'}(\rho u) = 0, \tag{4.1}$$

$$\frac{\partial u}{\partial t'} + u\frac{\partial u}{\partial x'} + \frac{B}{\rho}\frac{\partial B}{\partial x'} = 0, \tag{4.2}$$

$$\beta^2\frac{\partial}{\partial x'}\left(\frac{1}{\rho}\frac{\partial B}{\partial x'}\right) = B - \rho. \tag{4.3}$$

On expansion of B as a power series in the parameter β^2 according to

$$B = \rho + \beta^2 b_2(\rho, \rho_{x'}, \rho_{x'x'}, \dots) + O(\beta^4), \tag{4.4}$$

insertion into (4.3) yields

$$b_2 = \frac{\partial}{\partial x'}\left(\frac{1}{\rho}\frac{\partial \rho}{\partial x'}\right). \tag{4.5}$$

Substitution of (4.4) into (4.2) yields

$$\frac{\partial u}{\partial t'} + u\frac{\partial u}{\partial x'} + \frac{\partial \rho}{\partial x'} + \beta^2\left[\frac{1}{\rho}\frac{\partial^3 \rho}{\partial x'^3} - \frac{2}{\rho^2}\frac{\partial \rho}{\partial x'}\frac{\partial^2 \rho}{\partial x'^2} + \left(\frac{1}{\rho}\frac{\partial \rho}{\partial x'}\right)^3\right] = 0 \tag{4.6}$$

to $O(\beta^2)$. Accordingly, the following system results:

$$\frac{\partial \rho}{\partial t'} + \frac{\partial}{\partial x'}(\rho u) = 0, \tag{4.7}$$

$$\frac{\partial u}{\partial t'} + u\frac{\partial u}{\partial x'} + \frac{\partial \rho}{\partial x'} + \beta^2\frac{\partial}{\partial x'}\left[\frac{1}{\rho}\frac{\partial^2 \rho}{\partial x'^2} - \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial \rho}{\partial x'}\right)^2\right] = 0. \tag{4.8}$$

This describes the propagation of long magnetoacoustic waves in a cold plasma of density ρ with velocity and magnetic field as given by (3.1), (4.4).

5. The resonant NLS equation

Introduction of the velocity potential S via $u = -2\partial S/\partial x'$ into the system (4.7), (4.8) leads to

$$\frac{\partial \rho}{\partial t'} - 2\frac{\partial}{\partial x'}\left(\rho\frac{\partial S}{\partial x'}\right) = 0, \tag{5.1}$$

together with the Bernoulli-type integral

$$-\frac{\partial S}{\partial t'} + \left(\frac{\partial S}{\partial x'}\right)^2 + \frac{1}{2}\rho + \frac{\beta^2}{2}\left[\frac{1}{\rho}\frac{\partial^2 \rho}{\partial x'^2} - \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial \rho}{\partial x'}\right)^2\right] = \mathbb{B}(t). \tag{5.2}$$

Here, the arbitrary function $\mathbb{B}(t)$ may be absorbed into the potential S and so may be set to zero without loss of generality.

If we now let

$$\psi = \sqrt{\rho}e^{-iS} \tag{5.3}$$

then the system (5.1), (5.2) reduces to the RNLS equation

$$i\frac{\partial \psi}{\partial t'} + \frac{\partial^2 \psi}{\partial x'^2} - \frac{1}{2}|\psi|^2\psi = (1+\beta^2)\frac{1}{|\psi|}\frac{\partial^2|\psi|}{\partial x'^2}\psi \tag{5.4}$$

corresponding to $\Lambda = -2$, $s = 1+\beta^2$ in (1.1). Since $s > 1$, the RNLS equation (5.4) cannot be transformed into the conventional NLS equation. However, on introduction of $e^{(+)}$ and $e^{(-)}$ according to

$$e^{(+)} = \sqrt{\rho}e^{(1/\beta)S}, \quad e^{(-)} = -\sqrt{\rho}e^{-(1/\beta)S} \tag{5.5}$$

so that $e^{(+)} > 0$, $e^{(-)} < 0$, we obtain the RD system

$$\mp\frac{\partial e^{(\pm)}}{\partial \tau} + \frac{\partial^2 e^{(\pm)}}{\partial x'^2} - \frac{1}{2\beta^2}e^{(+)}e^{(-)}e^{(\pm)} = 0, \tag{5.6}$$

where $\tau = \beta t'$. Without loss of generality, we can choose $\beta > 0$. On the other hand, for $e^{(+)} > 0$, $e^{(-)} > 0$ we set

$$e^{(+)} = \sqrt{\rho}e^{(1/\beta)S}, \quad e^{(-)} = \sqrt{\rho}e^{-(1/\beta)S}, \tag{5.7}$$

whence for ρ and S we retrieve a system of the type (5.1), (5.2) but with ρ replaced by $-\rho$. The RNLS equation

$$i\frac{\partial \psi}{\partial t'} + \frac{\partial^2 \psi}{\partial x'^2} + \frac{1}{2}|\psi|^2\psi = (1+\beta^2)\frac{1}{|\psi|}\frac{\partial^2|\psi|}{\partial x'^2}\psi \tag{5.8}$$

results. The original RNLS equation (5.4) and its variant (5.8) can be viewed as defocusing and focusing NLS-type equations, respectively, but perturbed by the quantum potential. However, the behaviour of the soliton solutions for the RNLS equation is novel. Thus, in general, the defocusing RNLS equation admits bright soliton solutions, while the focusing RNLS equation admits dark soliton solutions [1].

6. Bilinear form

Gurevich and Krylov [19] constructed the travelling wave solutions $u = u(x - u_0 t)$, $\rho = \rho(x - u_0 t)$ of a magnetoacoustic wave system of the type (4.7), (4.8). Indeed, it is readily shown that this system admits solutions with

$$u = u_0 - \frac{(\alpha_1\alpha_2\alpha_3)^{1/2}}{\rho}, \tag{6.1}$$

$$\rho = \alpha_1 + (\alpha_3 - \alpha_1)\mathrm{dn}^2[\tfrac{1}{2}(\alpha_3 - \alpha_1)^{1/2}(x - u_0 t), \kappa], \tag{6.2}$$

where dn is the Jacobian elliptic function with modulus κ given by $\kappa^2 = (\alpha_3 - \alpha_2)/(\alpha_3 - \alpha_1)$ and α_i are real constants. The associated solution of the RNLS equation (5.4) for $\alpha_1 = 0$ adopts the form

$$\psi = 2\beta c\,\mathrm{dn}[c(x' - u_0 t'), \kappa] \exp\left(-i\left\{\phi_0 - \frac{u_0}{2}x' + \left[\frac{u_0^2}{4} + \beta^2 c^2(2 - \kappa^2)\right]t'\right\}\right), \tag{6.3}$$

where c, ϕ_0 are arbitrary constants and $x' = \rho x, t' = \beta t$. The particular case $\alpha_1 = \alpha_2 = 0$ leading to $\kappa = 1$ produces the envelope soliton

$$\psi = 2\beta c \exp\left(-i\left[\phi_0 - \frac{u_0}{2}x' + \left(\frac{u_0^2}{4} + \beta^2 c\right)t'\right]\right)\mathrm{sech}\, c(x' - u_0 t'). \tag{6.4}$$

Correspondingly, for the RD system (5.6) we obtain, in turn, a 'dissipative-periodic' solution

$$e^{(\pm)} = \pm 2\beta c\,\mathrm{dn}[c(x' - v\tau), \kappa] \exp\left(\pm\left\{\phi_0 - \frac{vx'}{2} + \left[\frac{v^2}{4} + c^2(2 - \kappa^2)\right]\tau\right\}\right), \tag{6.5}$$

and a dissipative analogue of the envelope soliton, namely the so-called 'dissipaton' [1]

$$e^{(\pm)} = \pm 2\beta c \exp\left(\pm\left[\phi_0 - \frac{v}{2}x' + \left(\frac{v^2}{4} + c^2\right)\tau\right]\right)\mathrm{sech}\, c(x' - v\tau), \tag{6.6}$$

where v is a constant velocity of propagation. In a similar manner, corresponding to the envelope soliton

$$\psi = 2\beta c \exp\left(-i\left[\phi_0 - \frac{u_0}{2}x' + \left(\frac{u_0^2}{4} - 2\beta^2 c^2\right)t'\right]\right)\tanh c(x' - u_0 t') \tag{6.7}$$

of the RNLS equation (5.8), the associated RD system admits the dissipative, 'dark' soliton given by

$$e^{(\pm)} = 2\beta c \exp\left(\pm\left[\phi_0 - \frac{v}{2}x' + \left(\frac{v^2}{4} - 2c^2\right)\tau\right]\right)\tanh c(x' - v\tau). \tag{6.8}$$

It turns out that the RD version of the RNLS equation is well-suited to the generation of multi-soliton solutions via a bilinear representation. Thus, if we introduce

$$e^{(\pm)} = 2\beta G^{(\pm)}/F, \tag{6.9}$$

where $G^{(\pm)}$ and F are real functions, then the bilinear representation

$$(\pm D_\tau - D_{x'}^2)(G^{(\pm)} \cdot F) = 0, \quad D_{x'}^2(F \cdot F) = -2G^{(+)}G^{(-)} \tag{6.10}$$

results. Here, Hirota's bilinear operators [20] are defined by

$$D_y^n(f \cdot g) = (\partial_y - \partial_{y'})^n f(y)\, g(y')|_{y'=y}. \tag{6.11}$$

The corresponding solution of the RNLS equation (5.4) has

$$|\psi(x', t')|^2 = \rho = -e^{(+)}e^{(-)} = 2\beta^2\frac{D_{x'}^2(F \cdot F)}{F^2} = 4\beta^2\frac{\partial^2 \ln F}{\partial x'^2}. \tag{6.12}$$

The one-dissipaton is given by the solution of system (6.10) with

$$G^{\pm} = \pm e^{\eta_1^{\pm}}, \quad F = 1 + e^{\eta_1^+ + \eta_1^- + \phi_{1,1}}, \quad e^{\phi_{1,1}} = (k_1^+ + k_1^-)^{-2}, \tag{6.13}$$

where $\eta_1^{\pm} = k_1^{\pm} x' \pm (k_1^{\pm})^2 \tau + \eta_1^{\pm(0)}$ and $k_1^{\pm}$, $\eta_1^{\pm(0)}$ are constants. If we set $c = (k_1^+ + k_1^-)/2$, $v = -(k_1^+ - k_1^-)$ then it acquires the form (6.6).

Multi-dissipaton and additional exotic solutions of the RNLS equation may now, in principle, be generated via its bilinear representation [1]. Here, however, an alternative approach via Bäcklund transformations and associated nonlinear superposition principles is adopted. This allows the construction of solutions descriptive of the interaction of solitonic magnetoacoustic waves which exhibit the required asymptotic behaviour.

7. Bäcklund–Darboux transformations. Nonlinear superposition principles

The RD system (5.6) remarkably represents the simplest two-component integrable system contained in the AKNS hierarchy of integrable systems [21]. Here, for convenience, we consider the normalization

$$p_t = p_{xx} - \tfrac{1}{2} p^2 q, \quad -q_t = q_{xx} - \tfrac{1}{2} q^2 p \tag{7.1}$$

which is equivalent to (5.6). The RD system is retrieved on application of an appropriate scaling of the independent variables x and t which intrudes the parameter β. For $pq < 0$, the variant

$$p = \kappa e^{\int \tau\, dx}, \quad q = -\kappa e^{-\int \tau\, dx} \tag{7.2}$$

of the Hasimoto transformation [22] for the standard nonlinear Schrödinger equation then produces the system

$$\begin{aligned} \kappa_t &= 2\kappa_x \tau + \kappa \tau_x \\ \tau_t &= \left(\frac{\kappa_{xx}}{\kappa} + \tau^2 + \frac{\kappa^2}{2} \right)_x , \end{aligned} \tag{7.3}$$

which, in turn, constitutes the analogue of the classical Da Rios system [23] descriptive of the self-induced motion of a thin isolated vortex filament travelling without stretching in an incompressible fluid. The variables ρ, u of the cold plasma system (4.7), (4.8) are related to those of the above Da Rios-type system by

$$\rho = \kappa^2, \quad u = -2\tau \tag{7.4}$$

so that

$$\rho = -pq, \quad u = \frac{q_x}{q} - \frac{p_x}{p}. \tag{7.5}$$

7.1. Elementary Darboux transformations

It may be verified that the linear system

$$\begin{aligned} \Phi_x &= \frac{1}{2} \left[\begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \Phi \\ \Phi_t &= \frac{1}{2} \left[\begin{pmatrix} -\frac{1}{2}pq & p_x \\ -q_x & \frac{1}{2}pq \end{pmatrix} + \lambda \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \Phi \end{aligned} \tag{7.6}$$

for a non-singular matrix-valued function Φ is compatible if and only if p and q obey the evolution equations (7.1). Here, λ is an arbitrary constant parameter. Indeed, $(7.6)_1$ constitutes the AKNS 'scattering problem' corresponding to the system (7.1) with λ being the 'spectral' parameter. This fact may be exploited to generate large classes of solutions of the *nonlinear* system (7.1) via the inverse scattering transform (IST) method applied to the *linear* representation (Lax pair) [24]. Alternatively, standard transformations of Bäcklund–Darboux type may be used to construct iteratively sequences of solutions from known seed solutions (see, e.g., [25–27] and references therein). However, in contrast to the complex nonlinear Schrödinger equation which is formally obtained from (7.1) by letting $t \to it$ and $p = \bar{q}$, the simplest 'regular' matrix Darboux transformation associated with the 'real' coupled system (7.1) may be decomposed into two 'singular' elementary transformations. The decomposition of Bäcklund transformations into the product of commuting elementary Bäcklund transformations was introduced by Konopelchenko [28] and subsequently developed in a series of papers [29–31]. A review of the procedure is given in [32]. Interestingly, this observation turns out to be significant in connection with the generation of solutions which represent solitonic magnetoacoustic waves displaying the required asymptotic behaviour $\rho \to 1$ at infinity.

Thus, the simplest matrix Darboux transformation $\tilde{\mathbb{B}} : (\Phi, p, q) \to (\tilde{\Phi}, \tilde{p}, \tilde{q})$ which leaves form-invariant the linear representation (7.6) is given by (cf. [32])

$$\tilde{\mathbb{B}} : \begin{cases} \tilde{\Phi} = \left[\begin{pmatrix} \lambda - \mu & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}p \\ 1 \end{pmatrix} (-\phi^{-1} 1) \right] \Phi \\ \tilde{p} = p_x - \mu p - \frac{1}{2} p^2 \phi^{-1}, \quad \tilde{q} = -2\phi^{-1}, \end{cases} \tag{7.7}$$

where $\phi = \phi_1/\phi_2$ and

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{7.8}$$

constitutes a vector-valued solution of (7.6) associated with the parameter μ. The validity of this transformation may be verified directly. It induces a Bäcklund transformation which maps any seed solution (p, q) to another solution $(\tilde{p}, \tilde{q})$ of the coupled system (7.1). It is observed that if the 'Darboux matrix' $\tilde{\Phi}\Phi^{-1}$ is regarded as a matrix-valued polynomial in the parameter λ then the coefficient multiplying λ is singular. Moreover, the Darboux matrix is parametrized in terms of the quantity ϕ which obeys the compatible Riccati system

$$\begin{aligned} \phi_x &= \tfrac{1}{2}p + \mu\phi - \tfrac{1}{2}q\phi^2 \\ \phi_t &= \tfrac{1}{2}p_x + \tfrac{1}{2}\mu p + (-\tfrac{1}{2}pq + \mu^2)\phi + (\tfrac{1}{2}q_x - \tfrac{1}{2}\mu q)\phi^2. \end{aligned} \tag{7.9}$$

By symmetry, another elementary matrix Darboux transformation $\hat{\mathbb{B}} : (\Phi, p, q) \to (\hat{\Phi}, \hat{p}, \hat{q})$ may be introduced according to

$$\hat{\mathbb{B}} : \begin{cases} \hat{\Phi} = \left[\begin{pmatrix} 0 & 0 \\ 0 & \lambda - \sigma \end{pmatrix} + \begin{pmatrix} 1 \\ -\frac{1}{2}q \end{pmatrix} (1 - \psi) \right] \Phi \\ \hat{p} = 2\psi, \quad \hat{q} = -q_x - \sigma q + \frac{1}{2} q^2 \psi, \end{cases} \tag{7.10}$$

where $\psi = \psi_1/\psi_2$ and

$$\boldsymbol{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{7.11}$$

constitutes another vector-valued solution of (7.6) associated with the 'Bäcklund parameter' σ. The corresponding Riccati system for ψ reads

$$\begin{aligned}\psi_x &= \tfrac{1}{2}p + \sigma\psi - \tfrac{1}{2}q\psi^2 \\ \psi_t &= \tfrac{1}{2}p_x + \tfrac{1}{2}\sigma p + (-\tfrac{1}{2}pq + \sigma^2)\psi + (\tfrac{1}{2}q_x - \tfrac{1}{2}\sigma q)\psi^2\end{aligned} \tag{7.12}$$

and any seed solution (p, q) is mapped to another solution $(\hat{p}, \hat{q})$ of the coupled system (7.1).

If we choose the seed solution $p = q = 0$ then ϕ and ψ are given by

$$\phi = -e^{\mu x + \mu^2 + \tilde{c}}, \quad \psi = e^{\sigma x + \sigma^2 t + \hat{c}} \tag{7.13}$$

and

$$\tilde{q} = 2e^{-\mu x - \mu^2 t - \tilde{c}}, \quad \hat{p} = 2e^{\sigma x + \sigma^2 t + \hat{c}}. \tag{7.14}$$

It is noted that $\tilde{q}$ and $\hat{p}$ are particular solutions of the backward and forward heat equations

$$-\tilde{q}_t = \tilde{q}_{xx}, \quad \hat{p}_t = \hat{p}_{xx}, \tag{7.15}$$

respectively, which constitute special reductions of the coupled system (7.1) corresponding to $\tilde{p} = 0$, and $\hat{q} = 0$, respectively.

7.2. *Compound elementary Darboux transformations. A superposition principle*

Solutions of arbitrary complexity may be obtained by means of iterative application of elementary matrix Darboux transformations. For instance, the action of the compound transformation

$$\mathbb{B}' = \hat{\mathbb{B}} \circ \tilde{\mathbb{B}} \tag{7.16}$$

may be obtained in the following manner. Successive application of the transformation laws $(7.7)_1$ and $(7.10)_1$ produces the 'eigenfunction'

$$\Phi' = \left[\begin{pmatrix} 0 & 0 \\ 0 & \lambda - \sigma \end{pmatrix} + \begin{pmatrix} 1 \\ -\frac{1}{2}\tilde{q} \end{pmatrix} (1 - \tilde{\psi})\right] \left[\begin{pmatrix} \lambda - \mu & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}p \\ 1 \end{pmatrix} (-\phi^{-1} 1)\right] \Phi, \tag{7.17}$$

where $\tilde{\psi}$ is a solution of the Riccati system (7.12) corresponding to the potentials $\tilde{p}$ and $\tilde{q}$. If we make the canonical choice $\tilde{\boldsymbol{\psi}} = \tilde{\mathbb{B}}(\boldsymbol{\psi})$ then $\tilde{\psi}$ is given by

$$\tilde{\psi} = \frac{1}{2}p - (\sigma - \mu)\frac{\psi\phi}{\psi - \phi}. \tag{7.18}$$

Hence, the action of $\mathbb{B}'$ is readily shown to be

$$\mathbb{B}' : \begin{cases} \Phi' = \left[\left(\lambda - \dfrac{\sigma + \mu}{2}\right)\mathbb{1} + \dfrac{\sigma - \mu}{2(\psi - \phi)}\begin{pmatrix} -(\psi + \phi) & 2\psi\phi \\ -2 & \psi + \phi \end{pmatrix}\right]\Phi \\ p' = p - 2(\sigma - \mu)\dfrac{\psi\phi}{\psi - \phi}, \quad q' = q - 2(\sigma - \mu)\dfrac{1}{\psi - \phi}. \end{cases} \tag{7.19}$$

The transformation laws $(7.7)_1$ and $(7.10)_1$ imply that $\tilde{\mathbb{B}}(\boldsymbol{\phi}) = \mathbf{0}$ and $\hat{\mathbb{B}}(\boldsymbol{\psi}) = \mathbf{0}$. Accordingly, the matrix Darboux transform Φ' obeys the algebraic conditions

$$\Phi' = (\lambda\mathbb{1} + Q)\Phi, \quad \Phi'[\Phi = \boldsymbol{\phi}] = \mathbf{0}, \quad \Phi'[\Phi = \boldsymbol{\psi}] = \mathbf{0}. \tag{7.20}$$

Since Φ' is uniquely determined by these conditions, it is evident that Φ' also constitutes an eigenfunction associated with the compound transformation $\tilde{\mathbb{B}} \circ \hat{\mathbb{B}}$

if we make use of the eigenfunction $\hat{\boldsymbol{\phi}} = \hat{\mathbb{B}}(\boldsymbol{\phi})$. We are therefore led to the *permutability theorem*

$$\mathbb{B}' = \hat{\mathbb{B}} \circ \tilde{\mathbb{B}} = \tilde{\mathbb{B}} \circ \hat{\mathbb{B}} \tag{7.21}$$

which expresses the fact that the elementary matrix Darboux transformations $\tilde{\mathbb{B}}$ and $\hat{\mathbb{B}}$ commute provided that the eigenfunctions $\tilde{\boldsymbol{\psi}}$ and $\hat{\boldsymbol{\phi}}$ are chosen in the above-mentioned manner. In this connection, it is noted that $\mathbb{B}'$ coincides with the standard simplest 'regular' matrix Darboux transformation for the AKNS hierarchy [33] since the latter is characterized by the conditions (7.20).

Elimination of ϕ and ψ in the transformation laws $(7.19)_{2,3}$ by means of $(7.7)_3$ and $(7.10)_2$ leads to the nonlinear superposition principle

$$p' = p + 4(\sigma - \mu)\frac{\hat{p}}{\tilde{q}\hat{p} + 4}, \quad q' = q - 4(\sigma - \mu)\frac{\tilde{q}}{\tilde{q}\hat{p} + 4}. \tag{7.22}$$

Thus, if (p, q) is a solution of the nonlinear system (7.1) and $\tilde{q}$ and $\hat{p}$ are elementary Bäcklund transforms of q and p associated with the Bäcklund parameters μ and σ respectively then another solution (p', q') of (7.1) is given by (7.22). For instance, nonlinear superposition of the solutions (7.14) of the heat equations (7.15) generates the dissipaton (cf. (6.6))

$$\begin{aligned} p' &= (\sigma - \mu)\frac{e^{\xi}}{\cosh \eta}, \quad \xi = \frac{\sigma + \mu}{2}x + \frac{\sigma^2 + \mu^2}{2}t + \frac{\hat{c} + \tilde{c}}{2} \\ q' &= (\mu - \sigma)\frac{e^{-\xi}}{\cosh \eta}, \quad \eta = \frac{\sigma - \mu}{2}x + \frac{\sigma^2 - \mu^2}{2}t + \frac{\hat{c} - \tilde{c}}{2} \end{aligned} \tag{7.23}$$

so that

$$\rho' = \frac{(\sigma - \mu)^2}{\cosh^2 \eta}, \quad u' = -(\sigma + \mu). \tag{7.24}$$

Hence, the quantity ρ' exhibits the typical sech^2 profile of a soliton.

7.3. A novel superposition principle for 'regular' matrix Darboux transformations

In principle, multi-dissipatons may now be generated by means of a well-known permutability theorem associated with the matrix Darboux transformation $\mathbb{B}'$ (see [25, 26] and references therein). However, the implementation of this solution generation technique is somewhat arduous due to the complex nature of the corresponding nonlinear superposition principle. This impediment is circumvented here by adopting the analogue of a novel 'universal' superposition principle which has been established only recently in [34].

7.3.1. A conservation law. The additional ingredient in the above-mentioned compact superposition principle is the potential r defined by

$$r_x = pq, \quad r_t = p_x q - p q_x \tag{7.25}$$

corresponding to the simplest conservation law

$$(pq)_t = (p_x q - p q_x)_x \tag{7.26}$$

associated with the RD system (7.1). Up to arbitrary additive constants, the action of the elementary Darboux transformations $\tilde{\mathbb{B}}$ and $\hat{\mathbb{B}}$ on r is readily shown to be

$$\tilde{r} = r - 2p\phi^{-1}, \quad \hat{r} = r - 2q\psi. \tag{7.27}$$

We may therefore choose the constant of integration in the potential r' such that

$$r' = r - 2(\sigma - \mu)\frac{\psi + \phi}{\psi - \phi}. \tag{7.28}$$

This convention is adopted in the remainder of this paper. If, for instance, $p = q = r = 0$ then the potential related to the dissipaton (7.23) reads

$$r' = 2(\mu - \sigma)\tanh\eta. \tag{7.29}$$

In general, the potential r may be used to reformulate the transformation law $(7.19)_1$ as

$$\Phi' = \left[\left(\lambda - \frac{\sigma + \mu}{2}\right)\mathbb{1} + \begin{pmatrix} \frac{1}{4}(r' - r) & -\frac{1}{2}(p' - p) \\ \frac{1}{2}(q' - q) & -\frac{1}{4}(r' - r) \end{pmatrix}\right]\Phi. \tag{7.30}$$

This is the key observation which will be exploited below.

7.3.2. A novel superposition principle. We now consider the action of two matrix Darboux transformations $\mathbb{B}_1$ and $\mathbb{B}_2$ of the type $\mathbb{B}'$ generated by the pairs of eigenfunctions $(\boldsymbol{\phi}^1, \boldsymbol{\psi}^1)$ and $(\boldsymbol{\phi}^2, \boldsymbol{\psi}^2)$ with parameters (μ_1, σ_1) and (μ_2, σ_2), respectively. The corresponding eigenfunctions Φ_1 and Φ_2 are of the form

$$\begin{aligned} \Phi_1 &= [(\lambda - \alpha)\mathbb{1} + Q]\Phi, \quad \alpha = \frac{\sigma_1 + \mu_1}{2}, \quad Q = U_1 - U \\ \Phi_2 &= [(\lambda - \beta)\mathbb{1} + R]\Phi, \quad \beta = \frac{\sigma_2 + \mu_2}{2}, \quad R = U_2 - U \end{aligned} \tag{7.31}$$

with

$$U = \begin{pmatrix} \frac{1}{4}r & -\frac{1}{2}p \\ \frac{1}{2}q & -\frac{1}{4}r \end{pmatrix} \tag{7.32}$$

and the Bäcklund transforms $U_1 = \mathbb{B}_1(U)$ and $U_2 = \mathbb{B}_2(U)$. The eigenfunctions

$$\boldsymbol{\phi}_1^2 = \mathbb{B}_1(\boldsymbol{\phi}^2), \quad \boldsymbol{\psi}_1^2 = \mathbb{B}_1(\boldsymbol{\psi}^2), \quad \boldsymbol{\phi}_2^1 = \mathbb{B}_2(\boldsymbol{\phi}^1), \quad \boldsymbol{\psi}_2^1 = \mathbb{B}_2(\boldsymbol{\psi}^1) \tag{7.33}$$

may then be used to construct eigenfunctions associated with the compound matrix Darboux transformations $\mathbb{B}_2 \circ \mathbb{B}_1$ and $\mathbb{B}_1 \circ \mathbb{B}_2$, respectively, denoted by

$$\Phi_{12} = \mathbb{B}_2(\mathbb{B}_1(\Phi)), \quad \Phi_{21} = \mathbb{B}_1(\mathbb{B}_2(\Phi)). \tag{7.34}$$

Specifically, the expressions

$$\begin{aligned} \Phi_{12} &= [(\lambda - \beta)\mathbb{1} + R^1]\Phi_1, \quad R^1 = U_{12} - U_1 \\ \Phi_{21} &= [(\lambda - \alpha)\mathbb{1} + Q^2]\Phi_2, \quad Q^2 = U_{21} - U_2 \end{aligned} \tag{7.35}$$

are obtained, where $U_{12} = \mathbb{B}_2(\mathbb{B}_1(U))$ and $U_{21} = \mathbb{B}_1(\mathbb{B}_2(U))$.

It is well established that the regular matrix Darboux transformations $\mathbb{B}_1$ and $\mathbb{B}_2$ commute provided that the choice of eigenfunctions (7.33) is made [25, 26]. Thus, the *permutability theorem*

$$\Phi_{12} = \Phi_{21} \tag{7.36}$$

produces the identity

$$[(\lambda - \beta)\mathbb{1} + R^1][(\lambda - \alpha)\mathbb{1} + Q] = [(\lambda - \alpha)\mathbb{1} + Q^2][(\lambda - \beta)\mathbb{1} + R]. \tag{7.37}$$

Since the latter must be valid for any value of λ, insertion of the parametrizations $(7.31)_{3,6}$ and $(7.35)_{2,4}$ produces the two relations

$$U_{12} = U_{21} \tag{7.38}$$

and

$$(\beta-\alpha)(U_{12}-U_1-U_2+U)=(U_{12}-U_2)(U_2-U)-(U_{12}-U_1)(U_1-U). \quad (7.39)$$

The first relation not only confirms the permutability theorem at the nonlinear level, that is $(p_{12},q_{12})=(p_{21},q_{21})$, but also extends its validity to the potential r. The second relation provides explicit expressions for the Bäcklund transform (p_{12},q_{12},r_{12}) in terms of the seed (p,q,r) and its intermediate Bäcklund transforms (p_1,q_1,r_1) and (p_2,q_2,r_2). This is summarized in the following theorem.

THEOREM 7.1. *Let (p,q) be a solution of the RD equations (7.1), r be a corresponding potential defined by (7.25) and U be the matrix*

$$U=\begin{pmatrix}\frac{1}{4}r & -\frac{1}{2}p\\ \frac{1}{2}q & -\frac{1}{4}r\end{pmatrix}. \quad (7.40)$$

Let U_1 and U_2 be Bäcklund transforms of U generated by $\mathbb{B}_1$ and $\mathbb{B}_2$ associated with the pairs of Bäcklund parameters (μ_1,σ_1) and (μ_2,σ_2), respectively. If the unique solution U_{12} of the linear equation

$$(\sigma_2+\mu_2-\sigma_1-\mu_1)(U_{12}-U_1-U_2+U)=[U_{12}-U,U_2-U_1] \quad (7.41)$$

is parametrized according to

$$U_{12}=\begin{pmatrix}\frac{1}{4}r_{12} & -\frac{1}{2}p_{12}\\ \frac{1}{2}q_{12} & -\frac{1}{4}r_{12}\end{pmatrix} \quad (7.42)$$

then (p_{12},q_{12}) constitutes another solution of (7.1) with r_{12} being a corresponding potential. (p_{12},q_{12},r_{12}) is the image of both (p_1,q_1,r_1) and (p_2,q_2,r_2) under the Darboux transformations $\mathbb{B}_2$ and $\mathbb{B}_1$ respectively.

Proof. The matrix equation (7.39) may be brought into the form

$$\begin{aligned}&2(\beta-\alpha)(U_{12}-U_1-U_2+U)\\&\quad=[U_{12}-U,U_2-U_1]-(U_{12}-U_2)^2+(U_1-U)^2+(U_{12}-U_1)^2-(U_2-U)^2.\end{aligned} \quad (7.43)$$

Since the square of any trace-free 2×2 matrix is proportional to the unit matrix, decomposition of (7.43) into its trace and trace-free parts leads to (7.41) and

$$(U_{12}-U_2)^2-(U_1-U)^2-(U_{12}-U_1)^2+(U_2-U)^2=0. \quad (7.44)$$

Furthermore, the trace terms of (7.41) multiplied by $U_{12}-U_1+U_2-U$ give rise to

$$\operatorname{tr}[(U_{12}-U_1)^2-(U_2-U)^2]=0 \quad (7.45)$$

so that

$$(U_{12}-U_1)^2=(U_2-U)^2. \quad (7.46)$$

Similarly, we obtain the additional relation

$$(U_{12}-U_2)^2=(U_1-U)^2. \quad (7.47)$$

Accordingly, the condition (7.44) is redundant. □

It is evident that the singularity structure of (p_{12},q_{12},r_{12}) depends crucially on the regularity of the matrix

$$(\beta-\alpha)\mathbb{1}+U_1-U_2 \quad (7.48)$$

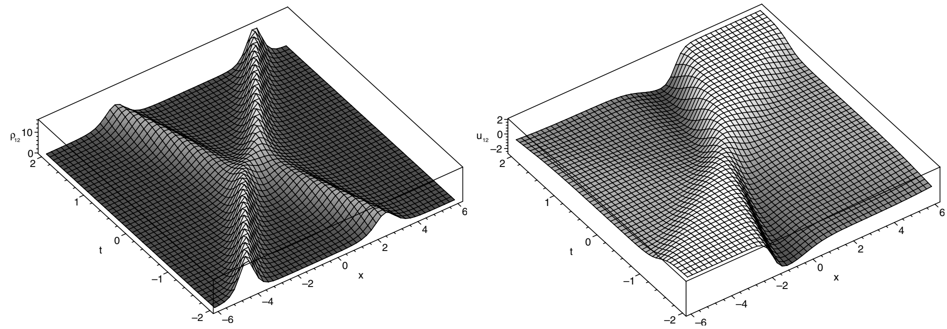

Figure 1. ρ_{12} and u_{12} associated with a two-dissipaton solution for $\mu_2 = -3$, $\sigma_2 = -1$, $\mu_1 = 1$, $\sigma_1 = 2$.

which multiplies U_{12} in the nonlinear superposition principle (7.39). The corresponding determinant is proportional to

$$4(\sigma_2 + \mu_2 - \sigma_1 - \mu_1)^2 + 4(p_2 - p_1)(q_2 - q_1) - (r_2 - r_1)^2. \tag{7.49}$$

For instance, in the case of the nonlinear superposition of two dissipatons of the form (7.23), (7.29), regularity of the two-dissipaton solution requires that

$$c \cosh(\eta_2 + \eta_1) + c' \cosh(\eta_2 - \eta_1) + c'' \cosh(\xi_2 - \xi_1) \neq 0, \tag{7.50}$$

where

$$c = (\sigma_2 - \sigma_1)(\mu_2 - \mu_1), \quad c' = (\sigma_2 - \mu_1)(\mu_2 - \sigma_1), \quad c'' = (\sigma_2 - \mu_2)(\sigma_1 - \mu_1). \tag{7.51}$$

This is achieved by choosing the Bäcklund parameters such that

$$\mu_2 < \sigma_2 < 0 < \mu_1 < \sigma_1 \quad \Rightarrow \quad c, c', c'' > 0. \tag{7.52}$$

The quantities ρ_{12} and u_{12} associated with a proto-typical two-dissipaton solution are shown in Fig. 1.

7.4. Generation of solitonic magnetoacoustic waves

In order to generate solutions of the RD system (7.1) corresponding to magnetoacoustic waves in a cold plasma which exhibit the correct behaviour at infinity, it is required to choose the non-zero seed

$$p = e^{t/2}, \quad q = -e^{-t/2} \tag{7.53}$$

so that the 'background' plasma density and velocity are

$$\rho = 1, \quad u = 0. \tag{7.54}$$

This choice constitutes the natural analogue of the seed solution of the nonlinear Schrödinger equation on which the generation of 'smoke rings' is based [35, 36].

Integration of the Riccati systems (7.9) and (7.12) yields

$$\begin{aligned} \phi &= e^{t/2}\left[-\mu - \sqrt{\mu^2 - 1}\,\tanh\left(\frac{\sqrt{\mu^2-1}}{2}(x + \mu t + \tilde{c})\right)\right] \\ \psi &= e^{t/2}\left[-\sigma - \sqrt{\sigma^2 - 1}\,\tanh\left(\frac{\sqrt{\sigma^2-1}}{2}(x + \sigma t + \hat{c})\right)\right]. \end{aligned} \tag{7.55}$$

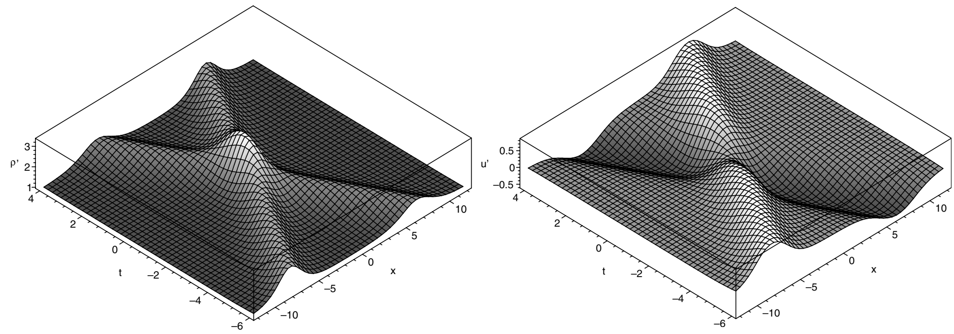

Figure 2. Interaction of two solitonic magnetoacoustic waves for $\mu = 4/3$ and $\sigma = -3/2$.

For $|\mu|, |\sigma| > 1$, it is then readily verified that the elementary Bäcklund transforms

$$\begin{gathered} \tilde{p} = p_x - \mu p - \frac{1}{2}p^2\phi^{-1}, \quad \tilde{q} = -2\phi^{-1} \\ \hat{p} = 2\psi, \quad \hat{q} = -q_x - \sigma q + \frac{1}{2}q^2\psi \end{gathered} \tag{7.56}$$

give rise to solutions $(\tilde{\rho}, \tilde{u})$ and $(\hat{\rho}, \hat{u})$ of the cold plasma system which represent single magnetoacoustic solitons and exhibit the required asymptotic behaviour $(\tilde{\rho}, \tilde{u}) \to (1, 0)$ and $(\hat{\rho}, \hat{u}) \to (1, 0)$. Specifically, relations (7.5) deliver

$$\begin{aligned} \hat{\rho} &= 1 + (\sigma^2 - 1)\operatorname{sech}^2\left(\frac{\sqrt{\sigma^2 - 1}}{2}(x + \sigma t + \hat{c})\right) \\ \hat{u} &= \sigma(1 - \sigma^2)\Big/\left[\sigma^2 + \sinh^2\left(\frac{\sqrt{\sigma^2 - 1}}{2}(x + \sigma t + \hat{c})\right)\right] \end{aligned} \tag{7.57}$$

and similar expressions for $\tilde{\rho}$ and $\tilde{u}$. It is noted that the above solution may be retrieved from (6.1), (6.2) by making the choice $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = \sigma^2, u_0 = -\sigma$ so that, in particular, $\kappa = 1$.

Solutions that correspond to the interaction of two magnetoacoustic solitons may be obtained by means of the compound Bäcklund transforms

$$p' = p - 2(\sigma - \mu)\frac{\psi\phi}{\psi - \phi}, \quad q' = q - 2(\sigma - \mu)\frac{1}{\psi - \phi} \tag{7.58}$$

provided that $\sigma\mu < 0$. In Fig. 2, the pair (ρ', u') is displayed for $\mu = 4/3$ and $\sigma = -3/2$. The matrix Darboux transformations and nonlinear superposition principles discussed in the preceding may now be used to generate solutions representing the interaction of an arbitrary number of magnetoacoustic solitons.

Acknowledgements

J.H.L. and O.K.P. would like to thank M. Pavlov for valuable discussions and for providing them with corresponding papers. The research of J.H.L. was supported by the Institute of Mathematics, Academia Sinica, Taipei, Taiwan. The research of O.K.P. was supported partially by the Izmir Institute of Technology grant 2005-IYTE-13, Izmir, Turkey.

References

[1] Pashaev, O. K. and Lee, J.-H. 2002 Resonance solitons as black holes in Madelung fluid. *Mod. Phys. Lett.* A **17**, 1601–1619.

[2] Hasegawa, A. and Kodama, Y. 1995 *Solitons in Optical Communications*. Oxford: Clarendon Press.

[3] de Broglie, L. 1926 Sur le possibilité de relier les phénomènes d'interfère et de diffraction à la théorie des quanta de lumière. *C. R. Acad. Sci. (Paris)* **183**, 447–448.

[4] Bohm, D. 1952 A suggested interpretation of the quantum theory in terms of 'hidden variables' I. *Phys. Rev.* **85**, 166–179.

[5] Akhmanov, A., Sukhorukov, A. P. and Khokhlov, R. V. 1968 Self-focusing and diffraction of light in a nonlinear medium. *Sov. Phys. Uspékhi* **93**, 609–636.

[6] Nelson, E. 1966 Derivation of the Schrödinger equation from Newtonian mechanics. *Phys. Rev.* **150**, 1079–1085.

[7] Salesi, G. 1996 Spin and Madelung fluid. *Mod. Phys. Lett.* A **11**, 1815–1823.

[8] Guerra, F. and Pusterla, M. 1982 A nonlinear Schrödinger equation and its relativistic generalization from basic principles. *Lett. Nuovo Cimento* **34**, 351–356.

[9] Smolin, L. 1986 Quantum fluctuations and inertia. *Phys. Lett.* A **113**, 408–412.

[10] Bertolami, O. 1991 Nonlinear connections to quantum mechanics from quantum gravity. *Phys. Lett.* A **154**, 225–229.

[11] Antonovskii, L. K., Rogers, C. and Schief, W. K. 1997 Note on a capillary model and the nonlinear Schrödinger equation. *J. Phys. A: Math. Gen.* **30**, L555–L557.

[12] Rogers, C. and Schief, W. K. 1999 The resonant nonlinear Schrödinger equation via an integrable capillarity model. *Nuovo Cimento* **114**, 1409–1412.

[13] Karpman, V. I. 1975 *Nonlinear Waves in Dispersive Media*. Oxford: Pergamon.

[14] Akhiezer, A. I. 1975 *Plasma Electrodynamics*. Oxford: Pergamon.

[15] Adlam, J. H. and Allen, J. E. 1958 The structure of strong collision-free hydromagnetic waves. *Philos. Mag.* **3**, 448–455.

[16] Whitham, G. B. 1965 Nonlinear dispersive waves. *Proc. R. Soc. London, Ser.* A **283**, 238–261.

[17] Gurevich, A. V. and Meshcherkin, A. P. 1984 Expanding self–similar discontinuities and shock waves in dispersive hydrodynamics. *Sov. Phys.–JETP* **60**, 732–740.

[18] El, G. A., Khodorovskii, V. V. and Tyurina, A. V. 2004 Hypersonic flow past slender bodies in dispersive hydrodynamics. *Phys. Lett.* A **333**, 334–340.

[19] Gurevich, A. V. and Krylov, A. L. 1988 A shock wave in dispersive hydrodynamics. *Sov. Phys. Dokl.* **32**, 73–74.

[20] Hirota, R. 1971 Exact solution of the Korteweg–de Vries equation for multiple collisions of solitons. *Phys. Rev. Lett.* **27**, 1192–1194.

[21] Ablowitz, M. J., Kaup, D. J., Newell, A. C. and Segur, H. 1973 Nonlinear evolution equations of physical significance. *Phys. Rev. Lett.* **31**, 125–127.

[22] Hasimoto, H. 1972 A soliton on a vortex filament. *J. Fluid. Mech.* **51**, 477–485.

[23] Da Rios, L. S. 1906 Sul moto d'un liquido indefinito con un filetto vorticoso. *Rend. Circ. Mat. Palermo* **22**, 117–135.

[24] Ablowitz, M. J. and Segur, H. 1981 *Solitons and the Inverse Scattering Transform*. Philadelphia, PA: SIAM.

[25] Rogers, C. and Schief, W. K. 2002 *Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory*. Cambridge: Cambridge University Press.

[26] Gu, C., Hu, H. and Zhou, Z. 2005 *Darboux Transformations in Integrable Systems: Theory and their Applications to Geometry*. Dordrecht: Springer.

[27] Rogers, C. and Shadwick, W. F. 1989 *Bäcklund Transformations and their Applications*. New York: Academic Press.

[28] Konopelchenko, B. G. 1979 The group structure of Bäcklund transformations. *Phys. Lett.* A **74**, 189–192.

[29] Konopelchenko, B. G. 1981 Transformation properties of the integrable evolution equations. *Phys. Lett.* B **100**, 254–260.

[30] Konopelchenko, B. G. 1982 Elementary Bäcklund transformations, nonlinear superposition principles and solutions of integrable equations. *Phys. Lett.* A **87**, 445–448.

[31] Calogero, F. and Degasperis, A. 1984 Elementary Bäcklund transformations, nonlinear superposition formulae and algebraic construction of solutions for the nonlinear evolution equations solvable by the Zakharov–Shabat spectral transform. *Physica.* D **14**, 103–116.

[32] Konopelchenko, B. G. and Rogers, C. 1992 Bäcklund and reciprocal transformations: gauge connections. In: *Nonlinear Equations in the Applied Sciences* (ed. W. F. Ames and C. Rogers). New York: Academic Press, pp. 317–362.

[33] Neugebauer, G. and Meinel, R. 1984 General N-soliton solution of the AKNS class on arbitrary background. *Phys. Lett.* A **100**, 467–470.

[34] Schief, W. K. Discrete Chebyshev nets and a universal permutability theorem. (In preparation.)

[35] Cieśliński, J., Gragert, P. K. H. and Sym, A. 1986 Exact solutions to localised induction-approximation equation modelling smoke-ring motion. *Phys. Rev. Lett.* **57**, 1507–151.

[36] Rogers, C. and Schief, W. K. 1998 Intrinsic geometry of the NLS equation and its auto-Bäcklund transformation. *Stud. Appl. Math.* **101**, 267–287.