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Affine inflation

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Affine gravity, a gravity theory based on affine connection with no notion of metric, supports scalar field dynamics only if scalar fields have nonvanishing potential. The nonvanishing vacuum energy ensures that the cosmological constant is nonvanishing. It also ensures that the energy-momentum tensor of vacuum gives the dynamically generated metric tensor. We construct this affine setup and study primordial inflation in it. We study inflationary dynamics in affine gravity and general relativity, comparatively. We show that nonminimally coupled inflaton dynamics can be transformed into minimally coupled ones with a modified potential. We also show that there is one unique frame in affine gravity, as opposed to the Einstein and Jordan frames in general relativity. Future observations with higher accuracy may be able to test affine gravity.

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I. INTRODUCTION

Inflation, the exponential expansion of the early universe to facilitate its flatness and homogeneity properties, rests on negative-pressure sources like vacuum energy or slow-moving scalar fields [1–4]. This conceptional idea also gives us the origin of the nearly scale-invariant spectrum of cosmological perturbations. These predictions are tested at some level by the anisotropy of cosmic microwave background radiation as well as large scale structure galaxy surveys [5].

In the modern view, the basic idea of inflation is to postulate the existence of a scalar field, named the "inflaton," which fills a region which existed in the early stage of the universe. This field is supposed to start with values slightly larger than Planck mass and lead to inflated domains. The inflationary dynamics have been studied mainly in metrical gravity [general relativity (GR)].

In GR, which is the purely metrical theory of gravity, scalar fields can be coupled minimally and nonminimally to gravity. In the first, the inflaton is coupled directly to the metric tensor and the inflationary regime is attained for the standard slow roll conditions applied to the scalar field. In this framework, inflationary models differ from each other in the potential of the scalar field [1–4]. Observations of density perturbations have severely constrained these models. In view of this, generalizations to nonminimal coupling have been proposed in the literature [6–11], including the standard model Higgs boson as an inflaton [12]. The nonminimal coupling ξ enters into the theory as $\xi\phi^2\mathcal{R}$, where ϕ is the inflaton and $\mathcal{R}$ the scalar curvature.

The minimal and nonminimal couplings are both studied in the GR, where the metric tensor is the fundamental variable. This is precisely the structure we observe at large distances. However, the spacetime structure may have been different to start with in the early universe. In other words, the metrical description of GR might have arisen dynamically as the universe evolves. To this end, affine gravity (AG) [13–19], based solely on connection with no notion of metric, stands out a viable framework to study. The AG framework necessitates scalar fields to have nonvanishing potentials, and thus, studying inflation in AG is important by itself. We find that the nonzero vacuum energy dynamically leads to a metric tensor as its energy-momentum tensor. This metric tensor is the consequence of the structure of the affine actions where the kinetic and the potential energies of scalars come out not in addition but in division. We will study salient consequences of this novel structure, and apply our findings to the inflationary epoch as a concrete testbed. We will show how a nonminimally coupled scalar can be turned into a minimally coupled one in AG by a field redefinition. We will study cosmological inflationary parameters in affine inflation (AfI) as functions of the nonminimal coupling parameter ξ, and compare them with the predictions of GR.

The paper is organized as follows. In Sec. II, we discuss minimally coupled scalar field in GR and AG, and reveal the differences and similarities between the two. We show therein how a metric tensor arises dynamically in the AG and how it relates to the energy-momentum tensor of the vacuum. In Sec. III, we extend our analysis to nonminimally coupled scalar fields and again study the GR and AG comparatively. Therein we point out an interesting property in that in the AG a nonminimally coupled scalar field can be transformed into a minimally coupled one by a field redefinition. (This is achieved in the GR by a conformal transformation of the metric plus field redefinition.) In Sec. IV, we apply our findings on scalar field dynamics to primordial inflation. We study in detail basic inflationary parameters in AG and GR, and we depict our results in tables and plots. In Sec. V we conclude.

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II. MINIMALLY COUPLED SCALAR FIELD

A. GR perspective

The spacetime of GR is equipped with a metric tensor $g_{\mu\nu}$ which makes the notions of distances and angles possible and also forms the invariant volume via the $\sqrt{-g}$ factor. In this theory, the gravity-scalar field coupling is described by the following action:

$$S^{(1)}_{\rm GR} = \int d^4x\sqrt{-g}\left[\frac{M^2_{\rm Pl}}{2}\mathcal{R}(g) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \quad (1)$$

where $\mathcal{R}(g)$ is the Ricci scalar curvature and $V(\phi)$ is the potential associated with the scalar field ϕ. The reduced Planck mass relates to Newton's constant G_N as $M^2_{\rm Pl} = (8\pi G_N)^{-1}$.

The theory of Eq. (1), including the celebrated Einstein-Hilbert action, is based on the metric tensor $g_{\mu\nu}$ as a fundamental quantity. GR then is a *purely metric* theory of gravity. The gravitational equations are then given by

$$M^2_{\rm Pl}G_{\mu\nu}(g) = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2 - g_{\mu\nu}V(\phi), \quad (2)$$

where $G_{\mu\nu}(g)$ is the Einstein's tensor constructed from $g_{\mu\nu}$ and the right-hand side is the energy-momentum tensor of the scalar field

$$T^{\phi}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2 - g_{\mu\nu}V(\phi). \quad (3)$$

The dynamics of the scalar field ϕ is described by the following equation derived from Eq. (1) by varying with respect to ϕ:

$$\Box\phi - V'(\phi) = 0, \quad (4)$$

where prime stands for differentiation with respect to ϕ.

The scalar-tensor action in Eq. (1) sets the minimally coupled scalar field dynamics. It possesses two important properties:

(1) As in the case of the flat spacetime action, kinetic terms (derivatives) of the scalar field and potentials appear in the action in the same line as a sum of two terms.
(2) As a result of the first property, all potentials $V(\phi)$ (zero or nonzero) are admissible, and in the vacuum where $\phi = \phi_{\rm min}$ one can optionally set $V(\phi) = 0$ or leave it nonzero depending on the model.

Next, we will consider the purely affine theory where the metric tensor is absent and see that these two properties no longer hold.

B. AG perspective

This geometry possesses only affine connection. This is all we need to define curvature. There is no metric tensor to start with; gravity is purely affine.

A real scalar field ϕ with scalar potential $V(\phi)$ in affine spacetime of Ricci curvature $\mathcal{R}_{\mu\nu}(\Gamma)$ is governed by the action

$$S^{(1)}_{\rm AG} = \int d^4x\frac{\sqrt{{\rm Det}[M^2_{\rm Pl}\mathcal{R}_{\mu\nu}(\Gamma) - \partial_\mu\phi\partial_\nu\phi]}}{V(\phi)}, \quad (5)$$

wherein the connection $\Gamma^\lambda_{\mu\nu}$ is taken to be symmetric.

The AG model in Eq. (5) is the simplest form of a pure affine theory of gravity coupled to a scalar field. This theory was shown to be equivalent to general relativity for $V(\phi) = m^2\phi^2/2$, where the metric tensor arises as the momentum canonically conjugates to the connection [14]. This proof can be straightforwardly extended to a general potential $V(\phi)$ [15,18].

Unlike the action Eq. (1) of GR, the AG action assumes the two properties below:

(1) The derivatives of the scalar field (kinetic part) enter the dynamics along with the curvature tensor. They both appear in the determinant needed for the invariant volume element. The nonderivative parts of the scalar field (potential part) appear in the denominator not to add to but to divide the kinetic part.
(2) The action Eq. (5) is then singular at $V(\phi) = 0$. This means that the scalar field must always have a nonzero potential energy. If $\phi = \phi_{\rm min}$ is the value of the scalar field for which $V(\phi)$ attains its minimum and if $V(\phi_{\rm min}) \neq 0$ then the theory makes sense, physically. In general, $\phi_{\rm min}$ is constant (it may be zero), and hence $V(\phi_{\rm min})$ is the vacuum energy.

In the following, we will generate a metric and its dynamical equations (Einstein field equations) through the action Eq. (5) by utilizing its above-mentioned properties. The important point here is that the potential energy, which must have a nonvanishing part always, is nothing but the energy-momentum tensor of vacuum, and it creates by itself a notion of metric. (In fact, even in GR, metric can well be interpreted as the energy-momentum tensor of vacuum [20].) In this sense, affine spacetime filled with vacuum energy $V(\phi_{\rm min})$ provides a very simple background which turns out to be the maximally symmetric spacetime (see the discussion at the end of this section and at the beginning of Sec IV.)

This nonvanishing vacuum energy, speaking covariantly, implies the existence of a vacuum energy-momentum tensor, $T_{\mu\nu}$. It is a nonvanishing, invertible rank-two tensor giving a covariant description of the vacuum energy. It is implicitly contained in affine spacetime, and acts as a "dimensionful" metric tensor by the nature of vacuum.

With the nonsingular inverse $(T^{-1})^{\lambda\rho}$, it defines the Levi-Civita connection

$$^{T}\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}(T^{-1})^{\lambda\rho}(\partial_\mu T_{\nu\rho} + \partial_\nu T_{\rho\mu} - \partial_\rho T_{\mu\nu}), \tag{6}$$

with respect to which

$$\nabla^{T}_{\mu} T_{\alpha\beta} = 0 \tag{7}$$

naturally holds.

To reveal more the structure of this energy-momentum tensor, one can also note that the identity tensor δ^μ_ν is inherently contained in affine spacetime, and thus $T_{\mu\nu}$ can be incorporated in its mixed form as

$$T^{\mu}_{\nu} \equiv V(\phi_{\rm min})\delta^{\mu}_{\nu} = V(\phi_{\rm min})T_{\nu\alpha}(T^{-1})^{\alpha\mu}. \tag{8}$$

This manifests itself as part of the affine spacetime. It does not arise from raising the indices of the tensor $T_{\mu\nu}$ though it will do so when the metric tensor is defined through $T_{\mu\nu}$ [see Eq. (11) below].

Accordingly, $T_{\mu\nu}$ is essentially a "dimensionful" metric tensor. In a sense, the vacuum sets a metrical geometry. However, one thing remains in that it is necessary to generate $T_{\mu\nu}$ dynamically from the equations of motion. (See the discussions in [20].)

The equation of motion arising from the affine action Eq. (5) takes the form

$$\nabla_\mu \left\{ \frac{\sqrt{{\rm Det}[M^2_{\rm Pl}\mathcal{R}_{\mu\nu} - \partial_\mu\phi\partial_\nu\phi]}}{V(\phi)} ((M^2_{\rm Pl}\mathcal{R} - \partial\phi\partial\phi)^{-1})^{\alpha\beta} \right\} = 0, \tag{9}$$

where the covariant derivative operator ∇_μ is with respect to the arbitrary affine connection Γ that defines the Ricci tensor. Now, by taking into account the property Eq. (7), the last equation of motion is solved as

$$M^2_{\rm Pl}\mathcal{R}_{\mu\nu} - \partial_\mu\phi\partial_\nu\phi = \left(\frac{V(\phi)}{V(\phi_{\rm min})}\right) T_{\mu\nu}, \tag{10}$$

so the affine connection coincides with $^{T}\Gamma^{\lambda}_{\mu\nu}$ given in Eq. (6). This is, in a sense, the vacuum connection, the connection that is generated by the vacuum stress energy tensor. Here, the metric tensor of GR is nothing but a tensor $g_{\mu\nu}$ where its existence is guaranteed by the nonzero vacuum energy via

$$g_{\mu\nu} = T_{\mu\nu}/V(\phi_{\rm min}). \tag{11}$$

Clearly, this tensor is defined only for $V(\phi_{\rm min}) \neq 0$, the condition that makes the theory derived from the action Eq. (5) factual. To that end, indices raising, lowering, and contracting tensors take their standard form by this metric tensor. As a result, the gravitational equations (10) can be easily brought into the form of Einstein's equations (2).

Unlike the metric tensor (and its Levi-Civita connection) which is usually supposed to be resulted from the dynamical equation (9) as a constant of integration, the vacuum stress energy tensor (and hence the vacuum connection) is given *a priori* in the affine spacetime, which translates a nonzero minimum potential energy of matter into a metrical geometry [20].

It must be emphasized here that the structure of the vacuum given by the stress-energy connection Eq. (6) and the energy-momentum tensor Eq. (8) is not restricted to local minima of the potential. All one needs is a nonzero primordial piece in $V(\phi)$, which can be defined as a minimum value of $V(\phi)$ corresponding to $\phi_{\rm min}$. This constant value saves the action Eq. (5) from going singular. In general, the potential $V(\phi)$ is model dependent and its minimum can be reached even asymptotically. This does not affect the definition of the Levi-Civita connection Eq. (6).

With all these at hand, variation of the action with respect to the scalar field ϕ leads to the dynamical equation of motion of ϕ,

$$\Box\phi - V'(\phi) = 0, \tag{12}$$

where we have used the solution Eq. (10) to get the operator $\Box$.

As we have seen, the equations of motion (10) and (12) derived from the affine action Eq. (5) are already familiar from the field equations of GR derived from Einstein-Hilbert action, where the scalar field is coupled minimally. This shows that coupling matter in AG through action Eq. (5) is equivalent to minimal coupling in GR. A summary of this comparison is given in Table I.

Generation of the metric tensor can be understood through the fact that Eq. (9), with $\phi = \phi_{\rm min}$, has a solution of the form $\mathcal{R}_{\mu\nu} = \Lambda g_{\mu\nu}$, which defines the metric tensor as in Eq. (11). For $\Lambda = 0$, curvature vanishes, metric becomes irrelevant, and metric description fails. Thus, the Eddington solution with $\Lambda \propto V(\phi_{\rm min})$ defines the metric tensor [13].

The formalism presented here goes beyond the original Eddington approach, where matter fields are not included. It produces dynamically the metric and the cosmological

TABLE I. AG vs GR for minimal coupling cases.

	GR	AG	AG vs GR
Fundamental quantity	Metric	Connection	
Action	Eq. (1)	Eq. (5)	Equivalent
Field equations	Eq. (2)	Eq. (2)	

constant as constants of integration. In our case, however, the affine spacetime is filled with scalar matter from the start. In this sense, the theory described by the action Eq. (5) improves on Eddington's approach. The vacuum $V(\phi_{\rm min})$ manifests itself as the nonzero energy required for elucidating the metric tensor. In other words, the metric tensor, though an integration constant as in Eddington's approach, can be structured in our case as the energy-momentum tensor of vacuum.

III. NONMINIMALLY COUPLED SCALAR FIELD

A. GR case

Nonminimal coupling in GR corresponds to a direct coupling between the scalar field and the curvature scalar. In this case, the action Eq. (1) is extended by adding an explicit interaction term between ϕ and $\mathcal{R}(g)$ as follows:

$$S_{\rm GR}^{(2)} = S_{\rm GR}^{(1)} + \int d^4x\sqrt{-g}\left(\frac{\xi}{2}\phi^2\mathcal{R}(g)\right), \tag{13}$$

where ξ is a dimensionless parameter. It is then straightforward to obtain the gravitational field equations

$$M_{\rm Pl}^2 G_{\mu\nu}(g) = T_{\mu\nu}^{\phi} + \xi\nabla_\mu\nabla_\nu\phi^2 - \xi\Box\phi^2 g_{\mu\nu} - \xi\phi^2 G_{\mu\nu}(g), \tag{14}$$

where $T_{\mu\nu}^{\phi}$ is the energy momentum of the scalar field given in Eq. (3). Similarly, the equation of motion for the scalar field takes the form

$$\Box\phi - V'(\phi) + \xi\phi\mathcal{R}(g) = 0. \tag{15}$$

In consequence, the following properties concerning the form of the action and the equations of motion are worth noting:

(1) As we see from the total action Eq. (13), the nonminimal coupling term $\xi\phi^2\mathcal{R}(g)$ appears in the theory as an additional term. This is a property of coupling to gravity in the pure metrical picture.

(2) Correction to the energy-momentum tensor of the scalar field due to nonminimal coupling has the following form:

$$T_{\mu\nu}^{\rm GR} = \xi\nabla_\mu\nabla_\nu\phi^2 - \xi\Box\phi^2 g_{\mu\nu} - \xi\phi^2 G_{\mu\nu}(g). \tag{16}$$

The first two terms of this tensor arise here due to the nonlinearity of the action; they contain second derivatives of the metric tensor. This creates derivatives of the scalar field, and then the improved energy-momentum tensor emerges as kinetic terms of matter. For a constant field $\phi = \phi_0$, these terms disappear leaving behind no contribution to the cosmological constant.

Next we will study the corresponding nonminimal coupling in AG and see the differences.

B. AG case

Equivalence between the gravitational field equations that are derived from AG and GR actions in the minimal coupling case leads to the following questions:

(1) Is the gravity-scalar field coupling given in the action Eq. (5) minimal?

(2) If yes, what is the generalization of this action to a nonminimal case? Are the field equations derived from this new theory equivalent to the associated nonminimal case of GR?

Firstly, as we have seen so far, in GR, the invariant volume element, which is required for integration on spacetime, is independent of matter fields. It is determined by the scalar density $\sqrt{-g}$ of the metric tensor. Invariant quantities are then formed by matter fields contracted with this metric. However, in AG, the invariant integration measure explicitly involves the matter fields, as is clear from the action Eq. (5). Thus, the comparison with the nonminimal case of GR may not be straightforward. Here we propose a possible and simple generalization of action Eq. (5) as follows:

$$S_{\rm AG}^{(2)} = \int d^4x \frac{\sqrt{{\rm Det}[(M_{\rm Pl}^2 + \xi\phi^2)\mathcal{R}_{\mu\nu}(\Gamma) - \partial_\mu\phi\partial_\nu\phi]}}{V(\phi)}. \tag{17}$$

Construction of this affine action is performed by using kinetic terms of the matter fields and their coupling terms to the Ricci tensor. This automatically coincides with the action Eq. (5) for $\xi = 0$.

Our aim in this paper is to study the gravitational dynamics and the dynamics of the scalar field which is nonminimally coupled to gravity in affine spacetime through action Eq. (17). To that purpose, it is important to shed light again on some points concerning the structure of this action:

(1) Unlike GR where the nonminimal coupling term in action Eq. (13) arises as an additional term, the $\xi\phi^2\mathcal{R}_{\mu\nu}(\Gamma)$ interaction in Eq. (17) is part of the invariant integration measure and does not come in an additive action piece.

(2) The theory becomes singular if at some values of ϕ, the potential vanishes. This means that there must be a primordial piece in $V(\phi)$. There is, however, an alternative view. It may be said that a constant ϕ defines complete absence of the scalar field [see Eq. (37) in Sec IV]. The interesting point is that the requisite primordial piece V_0 in the potential can be interpreted as $V(\phi_{\rm min})$.

(3) Needless to say, kinetic terms of matter fields vanish for a constant potential $V(\phi_{\rm min}) \neq 0$. It is this structure of affine spacetime that accommodates

the vacuum energy as an essential ingredient needed to forbid the singular behavior of the theory.

Action Eq. (17) is the simplest possible generalization of Eq. (5). First of all, this choice is structured as the one that gives the minimal form in Eq. (5) in the limit $\xi = 0$. The action Eq. (17) maintains the same fundamental structure, in which the kinetic term (not modified here) takes part in defining the volume element (square root of the determinant), and the potential divides the volume element. The step taken here is to couple explicitly the field ϕ with the Ricci tensor, which is the only geometric quantity in the action. It is this form that goes beyond minimal coupling as it provides direct coupling between ϕ and the connection Γ. This can, of course, be generalized to more general forms like $\mathcal{F}(\phi)$ rather than $\xi\phi^2$. However, higher powers of ϕ are expected to be suppressed by the Planck mass.

Now, the dynamical field equations derived from action Eq. (17) take the form

$$\nabla_\mu \left\{ \sqrt{\mathrm{Det}[(M_{\mathrm{Pl}}^2 + \xi\phi^2)\mathcal{R} - \partial\phi\partial\phi]}\,\frac{(M_{\mathrm{Pl}}^2 + \xi\phi^2)}{V(\phi)} \times (((M_{\mathrm{Pl}}^2 + \xi\phi^2)\mathcal{R} - \partial\phi\partial\phi)^{-1})^{\alpha\beta} \right\} = 0, \tag{18}$$

which can be integrated as

$$(M_{\mathrm{Pl}}^2 + \xi\phi^2)\mathcal{R}_{\mu\nu} - \partial_\mu\phi\partial_\nu\phi = \left(\frac{V(\phi)}{V(\phi_{\min})}\right)\left(\frac{M_{\mathrm{Pl}}^2}{M_{\mathrm{Pl}}^2 + \xi\phi^2}\right) T_{\mu\nu}. \tag{19}$$

Again, in terms of the metric tensor Eq. (11), Eq. (19) is written as

$$M_{\mathrm{Pl}}^2 G_{\mu\nu}(g) = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2} g_{\mu\nu}(\partial\phi)^2 - g_{\mu\nu}\frac{V(\phi)}{\mathcal{F}(\phi)} - \xi\phi^2 G_{\mu\nu}(g), \tag{20}$$

where we have defined for brevity the function $\mathcal{F}(\phi)$ as

$$\mathcal{F}(\phi) = 1 + \frac{\xi\phi^2}{M_{\mathrm{Pl}}^2}. \tag{21}$$

These are the gravitational field equations resulting from the nonminimal coupling of the scalar field to gravity in affine spacetime. The right-hand side term of Eq. (20) is the generalized energy-momentum tensor of the scalar field, which can be written as

$$T_{\mu\nu}(\phi) = T_{\mu\nu}^{\phi} + T_{\mu\nu}^{\mathrm{AG}}(\phi), \tag{22}$$

where $T_{\mu\nu}^{\phi}$ is the standard energy-momentum tensor, Eq. (3), derived from the minimal coupling case. The term $T_{\mu\nu}^{\mathrm{AG}}$ is an improved energy-momentum tensor,

$$T_{\mu\nu}^{\mathrm{AG}} = \frac{\xi\phi^2}{M_{\mathrm{Pl}}^2 + \xi\phi^2} V(\phi) g_{\mu\nu} - \xi\phi^2 G_{\mu\nu}(g). \tag{23}$$

Obviously, this quantity vanishes for $\xi = 0$—the minimal coupling case.

Now variation of the action Eq. (17) with respect to the scalar field ϕ leads to the following equation of motion:

$$\Box\phi - V'(\phi) + \xi\phi\mathcal{R}(g) + \Psi(\phi) = 0, \tag{24}$$

where we have used the identity Eq. (18) and then Eq. (11). Here the function $\Psi(\phi)$ is given by

$$\Psi(\phi) = \frac{\xi\phi^2}{M_{\mathrm{Pl}}^2 + \xi\phi^2} V'(\phi) - \left(\frac{2\xi\phi}{M_{\mathrm{Pl}}^2 + \xi\phi^2}\right) g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi. \tag{25}$$

Equation (24) implies the covariant conservation of the energy-momentum tensor Eq. (22),

$$\nabla^\mu T_{\mu\nu}(\phi) = 0. \tag{26}$$

This is a consequence of the general covariance of the affine action Eq. (17).

The last two terms of Eq. (24) measure the deviation from the dynamics of the scalar field in the minimal coupling case Eq. (12). The AG dynamics have the following properties:

(1) The dynamics of the scalar field is equivalent to the prescription of GR for $\xi = 0$. However, in the general case, the affine dynamics show no equivalence to GR due to the presence of $\Psi(\phi)$. Like the improved energy-momentum tensor of the scalar field Eq. (23) in the gravitational sector, the quantity $\Psi(\phi)$ might impose constraints on the propagation of matter fields in the curved background.

(2) The first term of the improved tensor Eq. (23) shows no dependence on the field derivatives; this is a consequence of the linearity of the affine action Eq. (17) where the fundamental quantity is an affine connection. Unlike the GR case, this term emerges in the theory as a potential term rather than a derivative of the field. For a general constant field $\phi = \phi_0$, the improved term does not vanish but rather creates a cosmological constant. Thus,

   (a) The first term of the improved energy-momentum tensor Eq. (23) is the measure of shifts between AG and GR in the nonminimal coupling case and new observable effects, if any, would arise through it.

   (b) The same term is essential in AG and it may enable us to shed light on some new features of the cosmological constant both classically and quantum mechanically [16,20–23].

TABLE II. AG vs GR for nonminimal coupling cases.

	GR	AG	AG vs GR
Fundamental quantity	Metric	Connection	
Action	Eq. (13)	Eq. (17)	Different
Field equations	Eq. (14)	Eq. (20)	

AG vs GR is summarized in Table II for the nonminimally coupled scalar fields.

We conclude this discussion by shedding light on an important point concerning the transformation between minimal and nonminimal coupling in GR and AG:

(1) In GR, the transition between the two actions, Eqs. (1) and (13), is made using the familiar conformal transformations where both actions are considered as the same theory written in two different frames. The Jordan and Einstein frames are described by *two metric tensors* $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ which are related by

$$\tilde{g}_{\mu\nu} = \mathcal{F}(\phi) g_{\mu\nu}. \tag{27}$$

The question then of which frame or which metric should be considered physical causes a serious ambiguity in GR.

(2) However, in AG, no such frames make sense in the theory. In fact, there is a unique metric tensor given by Eq. (11) which has originated from the nonzero vacuum energy. In this setup, the transition from the nonminimal affine action Eq. (17) to the minimal affine action Eq. (5) is obtained only through field redefinition,

$$d\varphi = \frac{d\phi}{\sqrt{\mathcal{F}(\phi)}}. \tag{28}$$

In terms of this new field, action Eq. (17) becomes

$$S_{\rm AG} = \int d^4x \frac{\sqrt{\text{Det}[M_{\rm Pl}^2 \mathcal{R}_{\mu\nu}(\Gamma) - \partial_\mu\varphi\partial_\nu\varphi]}}{U(\varphi)}, \tag{29}$$

which describes a minimally coupled scalar field φ in affine space with the potential

$$U[\varphi(\phi)] = \frac{V(\phi)}{\mathcal{F}^2(\phi)}. \tag{30}$$

This interesting feature of AG is not restricted to single fields but it holds true also in multiscalar theories [24]. The main impact of the passage from the nonminimal to minimal coupling cases is the new interactions induced. The multiscalar theories, for instance, can develop new interactions (even after diagonalizing their kinetic terms). It is therefore inferred that nonminimally coupled scalars can always be reduced to minimally coupled scalars with modified interactions with other matter fields.

Action Eq. (29) will be the basis for our discussion of the inflationary regime in the following section.

IV. AFFINE INFLATION

The gravitational field equations (20) take a simpler form when $\phi = \phi_{\rm min}$. This is the maximally symmetric vacuum case, and it leads to Einstein's equations with a cosmological constant (CC),

$$M_{\rm Pl}^2 G_{\mu\nu}(g) = -\frac{V(\phi_{\rm min})}{\mathcal{F}^2(\phi_{\rm min})} g_{\mu\nu}. \tag{31}$$

The solution to this equation is the maximally symmetric de Sitter (anti-de Sitter) spacetime. The nonzero CC is the requirement of the structure of the model proposed here, and so the cosmological effects of this term are relevant to the purely affine theory. As we have shown in Sec II, the necessity of nonzero CC is hidden in Eddington's approach and the equations (31) are equivalent to Eddington's equations [13].

The symmetry requirements of isotropy and homogeneity of space lead to Friedmann-Lemaître models for the universe. These models naturally include a de Sitter solution and those incorporating the cosmological constant like the one given in the present work [25]. The spacetime is described by one special case of these models: the flat Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t) d\vec{\mathbf{x}} \cdot d\vec{\mathbf{x}}, \tag{32}$$

where $a(t)$ is the scale factor.

The distribution of the scalar field in the universe may be described by its associated energy density and pressure, respectively

$$\rho(\phi) = \frac{1}{\mathcal{F}(\phi)} \left(\frac{\dot{\phi}^2}{2} + \frac{V(\phi)}{\mathcal{F}(\phi)} \right), \tag{33}$$

$$p(\phi) = \frac{1}{\mathcal{F}(\phi)} \left(\frac{\dot{\phi}^2}{2} - \frac{V(\phi)}{\mathcal{F}(\phi)} \right). \tag{34}$$

Here we see that a quasi-de Sitter solution which requires $p(\phi) = -\rho(\phi)$ is possible for some slowly rolling fields. The CC case we discussed above is implicitly understood here for $\phi = \phi_{\rm min}$.

Now the Hubble parameter H satisfies the following equations that can straightforwardly be derived from the gravitational field equations (20):

$$H^2 = \frac{1}{3M_{\rm Pl}^2 \mathcal{F}(\phi)} \left(\frac{\dot{\phi}^2}{2} + \frac{V(\phi)}{\mathcal{F}(\phi)} \right), \tag{35}$$

and

$$\dot{H} + H^2 = -\frac{1}{3M_{\rm Pl}^2 \mathcal{F}(\phi)} \left(\frac{\dot{\phi}^2}{2} - \frac{V(\phi)}{\mathcal{F}(\phi)} \right). \tag{36}$$

The existence of a quasi-de Sitter solution where the Hubble parameter, Eq. (35), is constant shows that an inflationary regime is possible in the theory.

Theories of inflation driven by scalar fields coupled nonminimally to gravity have been studied in great detail in pure metric gravity [6–11]. The study is usually performed in both the Jordan and Einstein frames where the same predicted results are not guaranteed. Here we will apply the formalism developed so far in this paper to inflation and compare the results with those predicted by GR.

Here we adopt the following potential which satisfies the standard properties discussed in two previous sections:

$$V(\phi) = V_0 + \frac{\lambda}{4}(\phi^2 - v^2)^2, \tag{37}$$

where the v is the vacuum expectation value of ϕ.

The V_0 is nonzero; it saves the affine action Eq. (17) from going singular at $\phi = v$. In fact, assuming that all possible contributions to vacuum energy are incorporated into V_0 and the cosmological constant problem is somehow solved, we take $V_0 \simeq m_\nu^4$ (since V_0 sets the cosmological constant as $\Lambda = V_0/M_{\rm Pl}^2$). It is clear that the nonvanishing of the vacuum energy ensures the nonvanishing nature of the cosmological constant—an observationally known fact. During the inflationary epoch, $V_0 \sim m_\nu^4$ is too tiny to have any observable effect, and it will be dropped in the analyses below. (Of course, in the vacuum $\phi = v$, V_0 is crucial.)

It is easier to study the inflationary regime using the physical field φ given by Eq. (28). To that end, Eq. (28) is integrated straightforwardly to get

$$\phi(\varphi) = \frac{M_{\rm Pl}}{\sqrt{\xi}} \sinh\left(\frac{\sqrt{\xi}}{M_{\rm Pl}} \varphi \right). \tag{38}$$

In spite of using the field φ rather than ϕ, the spacetime metric Eq. (32) remains unchanged, and then the physical field φ satisfies the standard slow roll conditions,

$$\frac{\dot{\varphi}^2}{2} \ll U(\varphi), \qquad \frac{\ddot{\varphi}}{\dot{\varphi}} \ll H, \tag{39}$$

for the following potential

$$U(\varphi) \simeq \frac{\lambda}{4} \left[\frac{M_{\rm Pl}^2 \xi^{-1} \sinh^2(\frac{\sqrt{\xi}}{M_{\rm Pl}}\varphi) - v^2}{1 + \sinh^2(\frac{\sqrt{\xi}}{M_{\rm Pl}}\varphi)} \right]^2. \tag{40}$$

Now the Hubble parameter and the equation of motion of ϕ are written as

$$H^2 \simeq \frac{U(\varphi)}{3M_{\rm Pl}^2}, \quad \text{and} \quad 3H\dot{\varphi} \simeq -U'(\varphi). \tag{41}$$

For large field $\varphi > M_{\rm Pl}/\sqrt{\xi}$, the slow roll parameters take the following forms:

$$\epsilon = \frac{M_{\rm Pl}^2}{2} \left(\frac{U'}{U} \right)^2 \simeq 128\xi \exp\left(-4\frac{\sqrt{\xi}}{M_{\rm Pl}}\varphi \right), \tag{42}$$

$$\eta = M_{\rm Pl}^2 \left(\frac{U''}{U} \right) \simeq -32\xi \exp\left(-2\frac{\sqrt{\xi}}{M_{\rm Pl}}\varphi \right), \tag{43}$$

$$\zeta^2 = M_{\rm Pl}^4 \frac{U'''U'}{U^2} \simeq (32\xi)^2 \exp\left(-4\frac{\sqrt{\xi}}{M_{\rm Pl}}\varphi \right). \tag{44}$$

These are equivalent to the results obtained from Palatini formalism [26].

The number of e-foldings is given by

$$N = \frac{1}{M_{\rm Pl}^2} \int_{\varphi_f}^{\varphi_i} \frac{U(\varphi)}{U'(\varphi)} d\varphi \simeq \frac{1}{32\xi} \left[\exp\left(2\frac{\sqrt{\xi}}{M_{\rm Pl}}\varphi_i \right) - \exp\left(2\frac{\sqrt{\xi}}{M_{\rm Pl}}\varphi_f \right) \right]. \tag{45}$$

Here the final field φ_f corresponds to the end of inflation where the slow roll conditions Eq. (39) break down, or $\epsilon \simeq 1$, and the initial field φ_i is determined from the number of e-foldings N.

The slow roll parameters are evaluated at the value φ when the scale of interest crosses the horizon during the inflationary phase, and they should remain smaller than one; then deviations of the spectrum of perturbations from a scale-invariant spectrum are small. The smallness of the parameter ϵ is shown in Fig. 1 in terms of ξ. The parameter behaves as in GR only for large ξ.

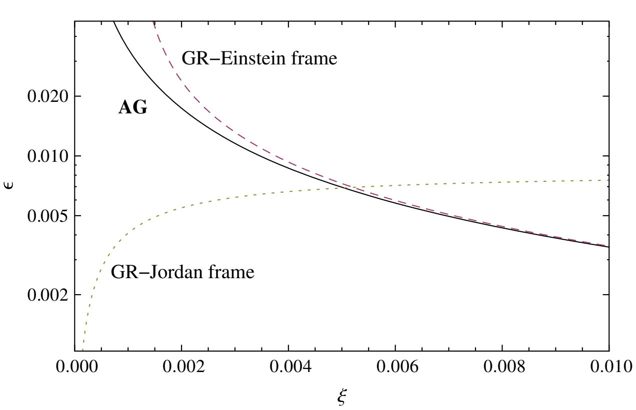

FIG. 1. The slow roll parameter ϵ as a function of ξ.

Now at first order the spectral index $n_s = 1 - 6\epsilon + 2\eta$ is written as

$$n_s \simeq 1 - \frac{3}{4\xi N^2} - \frac{2}{N}. \tag{46}$$

It has been shown that to first order, the nonminimal coupling in GR yields the following spectral index [8]:

$$n_s \simeq \begin{cases} 1 - \frac{32\xi}{16\xi N - 1}, & \text{for } \phi_f^2 \gg v^2 \\ 1 - \frac{16\xi(1+\delta^2)}{8\xi N(1+\delta^2)+\delta^2} & \text{for } \phi_f^2 \simeq v^2 \end{cases} \tag{47}$$

where $\delta^2 = \xi v^2 / M_{\mathrm{Pl}}^2$.

The first-order spectral index predicted by AG and GR for larger fields is depicted in Fig. 2 in terms of the parameter ξ, for $N = 60$. This comparison is made for $\phi \gg v$, where the potential behaves like ϕ^4.

The observed value, $n_s \simeq 0.9655 \pm 0.0062$ [5], is reached quickly, i.e., for smaller ξ in GR than in AG. For large ξ, AG is closer to the observed values. Possible larger values of ξ in AG may give rise to smaller ratios φ/M_{Pl}, even when $\sqrt{\xi}\varphi/M_{\mathrm{Pl}}$ is large as is required for the inflationary regime.

To second order, the spectral index n_s depends explicitly on the third slow roll parameter ζ^2 as follows [27,28]:

$$n_s = 1 - 6\epsilon + 2\eta + \frac{1}{3}(44 - 18c)\epsilon^2 + (4c - 14)\epsilon\eta + \frac{2}{3}\eta^2 + \frac{1}{6}(13 - 3c)\zeta^2, \tag{48}$$

where $c = 4(\ln 2 + \gamma) \simeq 5.081$ and γ is Euler's constant.

Detailed analysis in both the Jordan and Einstein frames showed that at second order, the spectral index takes different forms in different frames [8,29]. This is a consequence of the metrical theory where the Friedmann-Robertson-Walker (FRW) metric is conformally transformed from the Jordan to the Einstein frame.

The form of the spectral index Eq. (48) shows that deviations from first order are tiny for small slow roll parameters; this is a case of affine inflation where these parameters are decaying exponentially. However, significant deviation from first order appears in GR as illustrated in Fig. 3. Unlike the first-order case, AG does not show much difference from GR.

Last but not least, the tensor-to-scalar ratio is given by $r \equiv \Delta_t^2/\Delta_s^2 = 16\epsilon$, where Δ_t^2 and Δ_s^2 are the power spectra of the tensor and scalar fluctuations, respectively, created by inflation. In AfI, this quantity takes the form

$$r \simeq \frac{2}{\xi N^2}. \tag{49}$$

It is clear that this ratio is very small. For the range given above, $\xi \gtrsim 3.12 \times 10^{-2}$ for 60 e-foldings; this ratio has an upper bound

$$r \lesssim 1.7 \times 10^{-2}, \tag{50}$$

showing a small amount of tensor perturbations which is in the range of the observed value [5]. However, a large ξ produces a very tiny ratio.

The Planck data constraint on the power spectrum of the primordial perturbations generated during inflation is given by [5]

$$\frac{H^2}{8\pi^2 \epsilon M_{\mathrm{Pl}}^2} \simeq 2.4 \times 10^{-9}, \tag{51}$$

which leads to

$$\frac{\lambda}{\xi} \simeq 7.8 \times 10^{-11}. \tag{52}$$

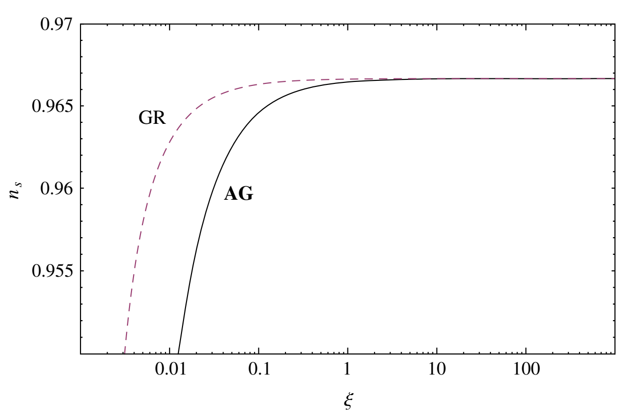

FIG. 2. First-order spectral index n_s in GR and AG for e-foldings $N = 60$. The Planck result [5], $0.960 \lesssim n_s \lesssim 0.970$, corresponds to $\xi \gtrsim 6.25 \times 10^{-3}$ for GR and $\xi \gtrsim 3.12 \times 10^{-2}$ for AG.

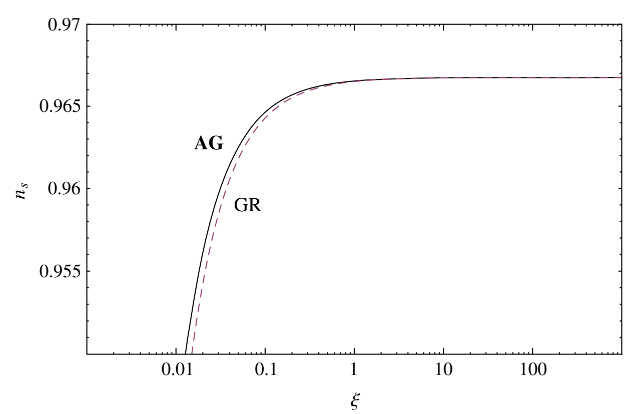

FIG. 3. The spectral index n_s to second order in the GR and the AG for $N = 60$ e-foldings. Deviation from the first order is significant in GR. In the AG, the slow roll parameters are small and corrections are tiny.

TABLE III. Inflationary regime predictions from AG and GR (Einstein frame) that correspond to the first-order spectral index $0.960 \lesssim n_s \lesssim 0.970$. The function $\varphi(\phi)$ is given here for $\xi > 1$; in this case and for larger ξ, the fields are below the Planck mass in AG.

	Einstein frame (GR)	AG
Parameter		
ξ	$\xi \gtrsim 6.25 \times 10^{-3}$	$\xi \gtrsim 3.12 \times 10^{-2}$
$\phi(\varphi)$	$\frac{M_{\rm Pl}}{\sqrt{\xi}} \exp\left(\sqrt{\frac{\xi}{1+6\xi}} \frac{\varphi}{M_{\rm Pl}}\right)$	$\frac{M_{\rm Pl}}{\sqrt{\xi}} \sinh\left(\frac{\sqrt{\xi}}{M_{\rm Pl}} \varphi\right)$
$\varphi_i/M_{\rm Pl}$	$\sqrt{\frac{1+6\xi}{\xi}} \ln\left(\sqrt{\frac{8\xi N}{1+6\xi}}\right)$	$\ln(32\xi N)/2\sqrt{\xi}$
$\varphi_f/M_{\rm Pl}$	$\sqrt{\frac{1+6\xi}{16\xi}} \ln\left(\frac{8\xi}{1+6\xi}\right)$	$\ln(128\xi)/4\sqrt{\xi}$

For small parameter ξ, this small ratio requires a very small λ leading to extreme fine-tuning. However, a natural value of λ can be obtained here for a significantly large ξ. This case is permitted in our model where the spectral index takes the value $n_s \simeq 0.97$. The results which arise from a large nonminimal coupling parameter ξ are equivalent to the those obtained in Ref. [26]. We believe that future observational constraints on the parameter ξ will lead to precise differences between standard inflation based on GR and AfI based on AG.

We conclude this section by summarizing our results in Table III. It describes the inflationary regime driven by a nonminimally coupled inflaton in the frameworks of affine gravity and general relativity.

V. CONCLUSION

Affine gravity, since its first formulation by Eddington, Schrödinger, and Einstein [13], has remained for decades a mathematical formulation that lacks concrete physical and cosmological interpretations. The present work may therefore be considered as a modest attempt to utilize affine gravity for the inflationary phase of the Universe. It turns out that the theory provides a viable setup for inflation even if it is equivalent to GR in certain cases (the minimal coupling). This feature stems from the structure of the invariant actions, which requires scalar fields to take nonzero values. This feature, which is necessary to drive inflation, is a useful aspect of AG. Another important feature of AG is that it provides a geometric frame in which the generated metric tensor is unique (no Einstein or Jordan frames). This makes the minimal and nonminimal coupling theories in the AG equivalent descriptions.

In this work, we first studied minimally and nonminimally coupled scalar fields comparatively in GR and in AG. We have revealed a number of interesting features in both cases. The scalar field is required to have nonvanishing potential energy density in the AG. In effect, the energy-momentum tensor of the vacuum is found to define a metric tensor *a posteriori* as an integration constant of the equations of motion.

Another point concerns transition from minimal to nonminimal coupling. It turns out that, unlike the minimal case, the nonminimal coupling in AG differs from that in GR. The differences stem from both the improved energy-momentum tensor and the modified equation of the field ϕ. We have shown that the improved energy-momentum tensor depends on the potential of the scalar field rather than derivatives of the field ϕ as in GR. This is a consequence of the linearity of the Ricci tensor in the first derivative of the affine connection.

We have also shown that the transformation from nonminimal to minimal coupling is simply obtained through the scalar field redefinition. This shows that there is only one frame in which affine gravity is formulated. This is arguably clear since there is only one generated metric tensor. This means that the Jordan and Einstein frames of GR have no correspondent in AG.

In the final stage of this paper, we have presented a detailed study of the primordial inflaton in a unique FRW spacetime metric, and we have shown that an inflationary regime arises naturally for slowly moving fields. We have discussed the possible values of the nonminimal coupling parameter ξ based on the measured spectral index. The study also showed that, unlike the standard inflation based on GR, AfI produces a small tensor-to-scalar ratio. Future observations may reach the sensitivity to distinguish between these models.

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