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Poor and pi-poor Abelian groups

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ABSTRACT
In this paper, poor abelian groups are characterized. It is proved that an abelian group is poor if and only if its torsion part contains a direct summand isomorphic to $\oplus_{p \in P}\mathbb{Z}_p$, where P is the set of prime integers. We also prove that pi-poor abelian groups exist. Namely, it is proved that the direct sum of $U^{(\mathbb{N})}$, where U ranges over all nonisomorphic uniform abelian groups, is pi-poor. Moreover, for a pi-poor abelian group M, it is shown that M can not be torsion, and each p-primary component of M is unbounded. Finally, we show that there are pi-poor groups which are not poor, and vise versa.

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1. Introduction

Let R be a ring with an identity element and *Mod-R* be the category of right R-modules. Recall that a right R-module M is said to be an N-injective (or injective relative to N) if for every submodule K of N and every morphism $f : K \to M$ there exists a morphism $\bar{f} : N \to M$ such that $\bar{f}|_K = f$. For a module M, as in [2], the injectivity domain of M is defined to be the collection of modules N such that M is an N-injective, that is, $\mathfrak{In}^{-1}(M) = \{N \in Mod - R | M \text{ is } N\text{-injective}\}$. Clearly, for any right R-module M, semisimple modules in *Mod- R* are contained in $\mathfrak{In}^{-1}(M)$, and M is an injective if and only if $\mathfrak{In}^{-1}(M)$ =*Mod-R*. Following [1], M is called *poor* if for every right R-module N, M is an N-injective only if N is semisimple, i.e., $\mathfrak{In}^{-1}(M)$ is exactly the class of all semisimple right R-modules. Poor modules exist over arbitrary rings [3, Proposition 1]. Although poor modules exist over arbitrary rings, their structure is not known over certain rings including also the ring of integers.

A right R-module N is *pure-split* if every pure submodule of N is a direct summand. Let K and N be right R-modules. K is an N-*pure-injective* if for each pure submodule L of N every homomorphism $f : L \to K$ can be extended to a homomorphism $g : N \to K$. Following [7], a right R-module M is called *pure-injectively poor (or simply pi-poor)* if whenever M is an N-pure-injective, then N is pure-split. It is not known whether pi-poor modules exist over arbitrary rings. In particular, in [7], some classes of abelian groups that are not pi-poor are given but the authors point out that they do not know whether a pi-poor abelian group exists.

The purpose of this paper is to give a characterization of poor abelian groups and also to prove that pi-poor abelian groups exist.

Namely, in Section 3, we prove that an abelian group G is poor if and only if the torsion part of G contains a direct summand isomorphic to $\oplus_{p \in P}\mathbb{Z}_p$, where P is the set of prime integers (Theorem 3.1).

Section 4 is devoted to the proof of the existence of pi-poor abelian groups. Let $\{A_\gamma | \gamma \in \Gamma\}$ be a complete set of representatives of isomorphism classes of reduced uniform groups. We prove that the

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group $M = \bigoplus_{\gamma\in\Gamma} A_\gamma^{(\mathbb{N})}$ is pi-poor (Theorem 4.1). In addition, it is proved that if G is a pi-poor abelian group, then G is not torsion, and the p-primary component $T_p(G)$ of G is unbounded for each prime p.

2. Definitions and preliminaries

We recall some definitions and results which will be useful in the sequel. For more details, we refer the reader to [5]. By group, we will mean an abelian group throughout the paper. Let $p \in P$ be a prime integer. A group G is called *p-group* if every nonzero element of G has order p^n for some $n \in \mathbb{Z}^+$. For a group G, $T(G)$ denotes the torsion submodule of G. The set $T_p(G) = \{a \in G | p^k a = 0 \text{ for some } k \in \mathbb{Z}^+\}$ is a subgroup of G, which is called the *p-primary component* of G. For every torsion group G, we have $G = \oplus_{p\in P} T_p(G)$. A subgroup A of a group B is *pure in B* if $nA = A \cap nB$ for each integer n. A monomorphism (resp. epimorphism) $\alpha : A \to B$ of abelian groups is called *pure* if $\alpha(A)$ (resp. Ker(α)) is pure in B. For any group G, the subgroups $T(G)$ and $T_p(G)$ are pure in G. A group G is said to be *bounded* if $nG = 0$, for some nonzero integer n. Bounded groups are direct sum of cyclic groups [5, Theorem 17.2]. A group G is called a *divisible group* if $nG = G$ for each positive integer n. A group G is called a *reduced group* if G has no proper divisible subgroup. Note that, since $\mathbb{Z}$ is Noetherian, every group G contains a largest divisible subgroup. Therefore, G can be written as $G = N \oplus D$, where N is reduced and D is divisible subgroup of G.

Definition 2.1 (see [5]). Let $p \in P$. A subgroup B of a group A is called a *p-basic subgroup of B* if it satisfies the following three conditions:

(i) B is a direct sum of cyclic p-groups and infinite cyclic groups;
(ii) B is p-pure in A;
(iii) A/B is p-divisible, i.e., $p(A/B) = A/B$.

Lemma 2.2.

(a) *[5, Theorem 32.3] Every group G contains a p-basic subgroup for each $p \in P$.*
(b) *[5, Theorem 27.5] If H is a pure and bounded subgroup of a group G, then H is a direct summand of G.*

For $q \neq p$ q-basic subgroups of p-groups are 0, so only p-basic subgroups of p-groups may be nontrivial. Therefore, they are usually called simply basic subgroups. Clearly, basic subgroups of p-groups are pure. Subgroups of the group of the rational integers $\mathbb{Q}$ are called *rational* groups. Let A be a uniform group. Then, it is easy to see that either A is isomorphic to a rational group or $A \cong \mathbb{Z}_{p^n}$ for some $p \in P$ and $n \in \mathbb{Z}^+$. For a torsion-free group G, we shall denote the (torsion-free) rank (=uniform dimension) of G by $r_0(G)$ [5]. By [5, page 86, Example 3], $r_0(G) = r_0(H) + r_0(G/H)$ for each subgroup H of G. A torsion-free group G is said to be *completely decomposable* if $G = \oplus_{i\in I} K_i$, where I is an index set and each K_i is isomorphic to a rational group, i.e., $r_0(K_i) = 1$ for each $i \in I$.

3. Poor Abelian groups

In this section, we give a characterization of poor groups. The authors prove that the group $\oplus_{p\in P}\mathbb{Z}_p$ is poor [1]. The following result shows that this group is crucial in investigation of poor groups.

Theorem 3.1. *A group is poor if and only if its torsion part has a direct summand isomorphic to $\oplus_{p\in P}\mathbb{Z}_p$.*

Proof. To prove the necessity, let G be a poor group and let p be any prime. If $T_p(G) = 0$, then G is an N-injective for every p-group N, therefore $T_p(G) \neq 0$. If every element of order p of G is divisible by p, then G is $\mathbb{Z}_{p^2}$-injective since $\mathbb{Z}_{p^2}$ has only one nontrivial subgroup: $p\mathbb{Z}_{p^2}$. So there is at least one element a_p with $|a_p| = p$, that is, not divisible by p. Then the cyclic group $< a_p >$ is a p-pure subgroup of $T_p(G)$, therefore a pure subgroup of $T_p(G)$. Since bounded pure subgroups are direct summands, $< a_p >$ is a direct summand of $T_p(G)$. Hence $\oplus_{p\in P} < a_p >$ is a direct summand of $\oplus_{p\in P} T_p(G) = T(G)$. Clearly, $\oplus_{p\in P} < a_p > \cong \oplus_{p\in P}\mathbb{Z}_p$.

Conversely, suppose that $T(G)$ contains a direct summand isomorphic to $\oplus\mathbb{Z}_p$. Let V be a direct summand of $T(G)$ such that $V \cong \mathbb{Z}_p$. Then, V is pure in G because $T(G)$ is pure in G. So V is a direct summand in G by [5, Theorem 27.5]. This implies, for each prime p, G contains a direct summand isomorphic to $\mathbb{Z}_p$. Now, suppose G is an N-injective for some group N. Then $\mathbb{Z}_p$ is an N-injective for each prime p. Suppose that N is not semisimple (not elementary in terminology of [5]). Then, there is an element a of infinite order or with $o(a) = p^n$, where p is a prime and $n > 1$. In first case, $\langle a\rangle = \mathbb{Z}$ and in second case, $\langle a\rangle = \mathbb{Z}_{p^n}$. So $\mathbb{Z}_p$ must be $\mathbb{Z}$-injective or $\mathbb{Z}_{p^n}$-injective by [8, Proposition 1.4]. But the homomorphism $f : p\mathbb{Z} \to \mathbb{Z}_p$ with $f(p) = 1$ cannot be extended to $g : \mathbb{Z} \to \mathbb{Z}_p$ since otherwise $1 = f(p) = g(p) = pg(1) = 0$ and $\mathbb{Z}_p$ is isomorphic to the subgroup $\langle p^{n-1}\rangle$ of $\mathbb{Z}_{p^n}$, which is not a direct summand of $\mathbb{Z}_{p^n}$. So in both cases we get a contradiction, that is, N is semisimple. □

The following is a consequence of Theorem 3.1.

Corollary 3.2. *For a group G, the following are equivalent.*
(1) *G is poor.*
(2) *The reduced part of G is poor.*
(3) *$T(G)$ is poor.*
(4) *For each prime p, G has a direct summand isomorphic to $\mathbb{Z}_p$.*

4. Pi-poor Abelian groups

The authors investigate the notion of pi-poor module and study properties of these modules over various rings [7]. In particular, they give some classes of groups that are not pi-poor and point out that they do not know whether a pi-poor group exists or not. In this section, we shall prove that pi-poor groups exist.

Theorem 4.1. *Let $\{A_\gamma | \gamma \in \Gamma\}$ be a complete set of representatives of isomorphism classes of uniform groups. Then the group*

$$M = \bigoplus_{\gamma\in\Gamma} A_\gamma^{(\mathbb{N})}$$

is pi-poor.

Before proving the theorem, we will first give some lemmas. Throughout this section, M denotes the group given in Theorem 4.1.

The following result is well known. We include it for completeness.

Lemma 4.2. *Let R be a ring and L, N be right R-modules. Let K be a pure submodule of N. If L is an N-pure-injective, then L is both K-pure-injective and N/K-pure-injective.*

Proof. Let A be a pure submodule of K and $f : A \to L$ be a homomorphism. Then A is pure in N, and so f extends to a map $g : N \to L$. Clearly, $g|_K : K \to L$ is an extension of f to K. Hence L is K-pure-injective. Now, let X/K be a pure submodule of N/K and $f : X/K \to L$ be a homomorphism. Since K is pure in N and X/K is pure in N/K, X is pure in N. Therefore, there is a homomorphism $g : N \to L$ such that $f\pi' = gi$, where $i : X \to N$ is the inclusion and $\pi' : X \to X/K$ is the usual epimorphism. Since $g(K) = 0$, $\mathrm{Ker}(\pi) \subseteq \mathrm{Ker}(g)$, where $\pi : N \to N/K$ is the usual epimorphism. Therefore, there is a homomorphism $h : N/K \to L$ such that $h\pi = g$. Then for each $x \in X$, $h(x + K) = h(\pi(x)) = g(x) = (f\pi')(x) = f(x + K)$. That is, h extends f. Hence, L is an N/K-pure-injective. □

Lemma 4.3. *Let G be a reduced torsion group. The following are equivalent.*
(1) *M is G-pure-injective.*
(2) *$T_p(G)$ is bounded for each $p \in P$.*
(3) *G is pure-split.*

Proof.
(1) $\Rightarrow$ (2) Write $G = \oplus_{p \in P} T_p(G)$. Let $B_p(G)$ be a basic subgroup of $T_p(G)$. Then $B_p(G)$ is pure in $T_p(G)$, and so in G and $T_p(G)/B_p(G)$ is divisible. We claim that $B_p(G)$ is bounded. Suppose the contrary that $B_p(G)$ is not bounded. Then for every positive integer n, $B_p(G)$ contains an element of order p^n. In this case, since $B_p(G)$ is a direct sum of cyclic p-groups, there is an epimorphism

$$B_p(G) \xrightarrow{g} \mathbb{Z}_{p^\infty} \to 0,$$

where the restrictions of g to the cyclic summands of $B_p(G)$ are monic. It can be proved as in [5, Lemma 30.1] that g is a pure epimorphism, i.e., $K = \text{Ker}(g)$ is a pure submodule of $B_p(G)$. Now, K is pure in $B_p(G)$ and is a direct sum of cyclic p-groups. Since M contains a direct summand isomorphic to K, and $B_p(G)$ is a pure subgroup of G, K is $B_p(G)$-pure-injective. Therefore $B_p(G) \cong K \oplus \mathbb{Z}_{p^\infty}$. This contradicts with the fact that $B_p(G)$ is reduced. Hence $B_p(G)$ is bounded, and so $B_p(G)$ is a direct summand of G. The fact that G is reduced and $T_p(G)/B_p(G)$ divisible implies that $B_p(G) = T_p(G)$.

(2) $\Rightarrow$ (3) Let H be a pure subgroup of G. Since $G = \oplus_{p \in P} T_p(G)$ and $H = \oplus_{p \in P} T_p(H)$, $T_p(H)$ is a pure subgroup of $T_p(G)$. Then, $T_p(H)$ is a direct summand of $T_p(G)$ by [5, Theorem 27.5]. Let $T_p(G) = T_p(H) \oplus N_p$, where $N_p \leq G$. Then $G = \oplus_{p \in P}[T_p(H) \oplus N_p] = (\oplus_{p \in P} T_p(H)) \oplus (\oplus_{p \in P} N_p) = H \oplus (\oplus_{p \in P} N_p)$. Hence G is pure-split.

(3) $\Rightarrow$ (1) Clear by the definition. □

Remark 4.4. Pure-split groups are completely characterized in [4]. The implications (2) $\Leftrightarrow$ (3) in Lemma 4.3 also can be found in [4].

Lemma 4.5. *Let B be a p-group. Suppose that M is B-pure-injective. Then B is pure-split.*

Proof. Let D be the divisible subgroup of B and A be a pure subgroup of B. Then $B = C \oplus D$ for some reduced group C. Let D_A be the divisible subgroup of A. Then $D_A \leq D$ and $D = D_1 \oplus D_A$ for some $D_1 \leq D$. So $B = C \oplus D_1 \oplus D_A = E \oplus D_A$, where $E = C \oplus D_1$. By modular law, $A = (E \cap A) \oplus D_A$. Then $L = E \cap A$ is a pure submodule of B. Hence, M is L-pure-injective, and $L \cong A/D_A$ is reduced. Therefore, L is bounded by Lemma 4.3. Since L is pure in B, L is also pure in E. Then, $E = K \oplus L$ for some $K \leq E$ by [5, Theorem 27.5]. Then $B = E \oplus D_A = K \oplus L \oplus D_A = K \oplus A$. So A is a direct summand in B. Hence B is pure-split. □

Lemma 4.6. *If N is a reduced torsion-free group such that M is an N-pure-injective then N is pure-split. Moreover, N is completely decomposable with finite rank.*

Proof. Take any $0 \neq a_1 \in N$ and let $G_1 = \{x \in N \mid mx \in \langle a_1 \rangle \text{ for some } 0 \neq m \in \mathbb{Z}\}$ (that is, G_1 is the subgroup purely generated by a_1). Clearly, G_1 is a pure subgroup of N and isomorphic to a rational group, so M has a direct summand isomorphic to G_1. Therefore, G_1 is a direct summand of N, that is, $N = G_1 \oplus N_1$ for some $N_1 \leq N$. If $N_1 \neq 0$, we can find in similar way a pure subgroup G_2 of N_1 purely generated by an element a_2. Clearly, M is an N_1-pure-injective, so $N_1 = G_2 \oplus N_2$. The same can be done for N_2 if $N_2 \neq 0$ and so on. If this process continues infinitely, then N contains a subgroup $\oplus_{i=1}^{\infty} G_i$ which is pure as a direct limit of pure subgroups. Therefore, M is $\oplus_{i=1}^{\infty} G_i$-pure-injective. For each a_i, $i = 1, 2, \ldots$, there is a homomorphism $f_i : \langle a_i \rangle \to \mathbb{Q}$ with $f(a_i) = \frac{1}{i}$. Since $\mathbb{Q}$ is an injective, there is a homomorphism $f : \oplus_{i=1}^{\infty} G_i \to \mathbb{Q}$ with $f(a_i) = f_i(a_i) = \frac{1}{i}$. Clearly, f is an epimorphism. Since $\mathbb{Q}$ is torsion-free, $K = \mathrm{Ker}(f)$ is a pure subgroup of $\oplus_{i=1}^{\infty} G_i$. Let Γ be the set of all completely decomposable pure subgroups of K and R be the set of all subgroups of K of rank 1. Define order $\preceq$ on Γ as follows: $\oplus_{S\in I} S \preceq \oplus_{S\in J} S$ if $I \subseteq J \subseteq R$. If P is any chain in Γ, then $\cup_{X\in P} X$ is clearly a completely decomposable and pure subgroup of K, since the direct limit of pure subgroups is pure. So by Zorn's Lemma, there is a maximal element $B = \oplus_{S\in T} S$ in Γ. Since K is countable T is also countable, so B is a direct summand of K, that is, $K = B \oplus C$ for some $C \leq K$. If $C \neq 0$, then as at the beginning of the proof, we can find a pure subgroup of X of C of rank 1. Clearly, $B \oplus X \in \Gamma$. Contradiction with maximality of B. So $C = 0$. Then, K is a direct summand of $\oplus_{i=1}^{\infty} G_i$. So $\oplus_{i=1}^{\infty} G_i \cong K \oplus \mathbb{Q}$. But $\oplus_{i=1}^{\infty} G_i$ is reduced. Contradiction. Thus, the process must be finite, that is, $N = G_1 \oplus G_2 \oplus \cdots \oplus G_n$ for some $n \in \mathbb{Z}^+$. To show that N is pure-split, let L be a pure subgroup of N. Then M is L-pure-injective, so it is the direct sum of groups of rank one of finite number as we have proved above. Then, L is a direct summand of N, because N-pure-injectiveness of M implies that the inclusion $L \to N$ is splitting. Hence, N is pure-split and completely decomposable with finite rank. This completes the proof. □

Lemma 4.7. *Let N be a torsion-free group. If M is an N-pure-injective, then N is pure-split.*

Proof. Let K be a pure subgroup of $N = A \oplus D$, where D is the divisible subgroup of N. Let D_K be the divisible subgroup of K. Then $D_K \leq D$, and so $D = D_1 \oplus D_K$ for some $D_1 \leq D$. So $N = A \oplus D_1 \oplus D_K = E \oplus D_K$, where $E = A \oplus D_1$. By modular law, $K = (E \cap K) \oplus D_K$. Denote $E \cap K = L$. Then, $L \cong K/D_K$ is reduced and pure in N. Hence, M is an L-pure-injective, and so $L \cong \oplus_{i=1}^{n} R_i$ for some rational groups $R_1, \ldots R_n$, by Lemma 4.6. Then, M contains a direct summand isomorphic to L. So the inclusion $L \to N$ splits, i.e., $N = L \oplus H$ for some $H \leq N$. Since L is reduced, $D_K \leq H$. Then $N = L \oplus D_K \oplus H' = K \oplus H'$. This implies that N is pure-split. □

Definition 4.8 (See, [6]). Let G be a torsion-free group and $a \in G$. Given a prime p, the largest integer k such that $p^k | a$ holds is called the *p-height* $h_p(a)$ of a; if no such maximal integer k exists, then we set $h_p(a) = \infty$. The sequence of p-heights

$$\chi(a) = (h_{p_1}(a), h_{p_2}(a), \ldots, h_{p_n}(a), \ldots)$$

is said to be the characteristic of a. Two characteristics $(k_1, k_2, \ldots)$ and $(l_1, l_2, \ldots)$ are *equivalent* if $k_n \neq l_n$ holds only for a finite number of n such that in case $k_n \neq l_n$ both k_n and l_n are finite. An equivalence class of characteristics is called a *type*. G is called *homogeneous* if all nonzero elements of G are of the same type.

Corollary 4.9. *Let N be a torsion-free reduced group. The following are equivalent.*
(1) *M is an N-pure-injective.*
(2) *N is pure-split.*
(3) *N is a completely decomposable homogeneous group of finite rank.*

Proof.
(1) ⇔ (2) By Lemma 4.6.

(2) ⇔ (3) See [4] or [6, Example 8, page 116]. □

Now, we can prove our theorem.

Proof of Theorem 4.1. Let M be G-pure-injective for some group G. We have $G = N \oplus D$ for some reduced group N and a divisible group D. Then M is an N-pure-injective, and since $T(N)$ is a pure subgroup of N, M is $T(N)$-pure-injective and M is an $N/T(N)$-pure-injective. Then, by Lemmas 4.3 and 4.6, $T(N) = \oplus_{p\in P} B_p(N)$ and $N/T(N) = \oplus_{i\in I} K_i$, where for each $p \in P$, $B_p(N)$ is a bounded

p-group, I is a finite index set, and each K_i is isomorphic to a rational group. We claim that $T(N)$ is a direct summand in N, that is, the short exact sequence:

$$\mathbb{E} : 0 \to T(N) \to N \to N/T(N) \to 0$$

is splitting. By [5, Theorem 52.2], there is a natural isomorphism

$$\text{Ext}(N/T(N), T(N)) = \text{Ext}\left(\bigoplus_{i\in I} K_i, T(N)\right) \cong \prod_{i\in I} \text{Ext}(K_i, T(N))$$

induced by the inclusions $\alpha_j : K_j \to \oplus_{i\in I} K_i$. Therefore, it is sufficient to prove that each short exact sequence:

$$\mathbb{E}\alpha_j : 0 \to T(N) \to N' \xrightarrow{f} K_j \to 0$$

is splitting. We have the following commutative diagram with exact columns and rows.

$$\begin{array}{cccccccccc}
 & & & & & 0 & & 0 & & \\
 & & & & & \downarrow & & \downarrow & & \\
\mathbb{E} : & 0 & \longrightarrow & T(N) & \longrightarrow & N' & \xrightarrow{f} & K_j & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle \alpha_j} & & \\
\mathbb{E}\alpha_j : & 0 & \longrightarrow & T(N) & \longrightarrow & N & \longrightarrow & \oplus_{i\in I} K_i & \longrightarrow & 0 \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & \oplus_{i\neq j} K_i & = & \oplus_{i\neq j} K_i & & \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & 0 & & 0 & &
\end{array}$$

Since $\oplus_{i\in I} K_i$ is torsion free, N' is a pure subgroup of N, therefore M is an N'-pure-injective. There is a countable set $\{n_k | k = 1, 2, \ldots\}$ in N' such that the elements $f(n_k)$ generate K_j. By [5, Proposition 26.2], there is a countable pure subgroup L of N' containing the subgroup $\sum_{k=1}^{\infty} \mathbb{Z} n_k$. Then, M is an L-pure-injective as well. Clearly, $f(L) = K_j$ and $\text{Ker}(f|_L) = T(L)$. Since L is countable, $T(L)$ is a countable subgroup of $T(N)$. But $T(N)$ is a direct sum of cyclic primary groups, therefore $T(L)$ is a countable direct sum of cyclic primary groups and hence is isomorphic to a direct summand of M. Since $T(L)$ is a subgroup of L and M is an L-pure-injective, $T(L)$ is a direct summand of L. We have the following commutative diagram with exact rows:

$$\begin{array}{cccccccccc}
\mathbb{E}' : & 0 & \longrightarrow & T(L) & \longrightarrow & L & \longrightarrow & K_j & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \beta} & & \downarrow & & \| & & \\
\mathbb{E}\alpha_j : & 0 & \longrightarrow & T(N) & \longrightarrow & N' & \longrightarrow & K_j & \longrightarrow & 0
\end{array}$$

where β is the inclusion. Since $\mathbb{E}'$ is splitting $\mathbb{E}\alpha_j = \beta\mathbb{E}$ is also splitting. So $N = T(N) \oplus K$, where $T(N)$ and K are groups as in Lemmas 4.3 and 4.6, respectively. This proves our claim.

To prove that G is pure-split, take a pure subgroup A of G. By the first part of the proof, we have

$$G = N \oplus D = T(N) \oplus K \oplus T(D) \oplus D' = (T(N) \oplus T(D)) \oplus (K \oplus D') = T(G) \oplus G'.$$

Then for each $p \in P$, $T_p(A)$ is a pure subgroup of $T_p(G)$. Therefore, $T_p(A)$ is a direct summand of $T_p(G)$ by Lemma 4.5. Then, $T(A)$ is a direct summand of $T(G)$. We have a homomorphism $f : A/T(A) \to G/T(G)$ defined by $f(a + T(A)) = a + T(G)$. If $f(a + T(A)) = 0$, then $a \in T(G) \cap A = T(A)$, hence $a + T(A) = 0$, so f is a monomorphism. Now claim that $Im(f)$ is a pure subgroup of $G/T(G)$. To show this, let $a + T(G) = m(b + T(G))$ for some $a \in A$, $b \in G$, $0 \neq m \in \mathbb{Z}$. Then $a - mb \in T(G)$, therefore $ka = kmb$ for some $0 \neq k \in \mathbb{Z}$. Since A is pure in G, $ka = kma'$ for some $a' \in A$. Then $a - ma' \in T(A)$, hence $a + T(A) = m(a' + T(A))$. So $Im(f)$ is pure. Since $G/T(G) \cong G'$ is pure-split by Lemma 4.7, f is splitting. As A is a pure subgroup of G, M is A-pure-injective. So again by the first part of the proof $A = T(A) \oplus K'$ for some $K' \leq A$ with $K' \cong A/T(A)$. Then the inclusion map $A = T(A) \oplus K' \to G = T(G) \oplus G'$ is splitting, that is, A is a direct summand in G. This completes the proof.

5. Structure of pi-poor Abelian groups

In this section, we prove some results concerning a possible structure of pi-poor groups.

Proposition 5.1. *If G is pi-poor group, then $T_p(G)$ is unbounded for each $p \in P$.*

Proof. Suppose G is pi-poor and $T_p(G)$ is bounded for some $p \in P$. Then $T_p(G)$ is pure-injective and $T_p(G)$ is a direct summand of G, because $T_p(G)$ is also pure in G. Consider the group $\oplus_{n=1}^{\infty}\mathbb{Z}_{p^n}$. We claim that G is $\oplus_{n=1}^{\infty}\mathbb{Z}_{p^n}$-pure-injective. Let H be a pure subgroup of $\oplus_{n=1}^{\infty}\mathbb{Z}_{p^n}$ and $f : H \to G$ be a homomorphism. Since H is a p-group, $f(H) \subseteq T_p(G)$. So that f extends to a homomorphism $h : \oplus_{n=1}^{\infty}\mathbb{Z}_{p^n} \to G$ because $T_p(G)$ is pure-injective. This proves our claim.

We shall see that $\oplus_{n=1}^{\infty}\mathbb{Z}_{p^n}$ is not pure-split. There is an exact sequence:

$$\mathbb{E} : 0 \to K \to \oplus_{n=1}^{\infty}\mathbb{Z}_{p^n} \xrightarrow{g} \mathbb{Z}_{p^\infty} \to 0.$$

By the same arguments as in the proof of Lemma 4.3, $\mathbb{E}$ is pure, i.e., K is pure in $\oplus_{n=1}^{\infty}\mathbb{Z}_{p^n}$. Since $\oplus_{n=1}^{\infty}\mathbb{Z}_{p^n}$ is reduced, $\mathbb{E}$ does not split. Hence $\oplus_{n=1}^{\infty}\mathbb{Z}_{p^n}$ is not pure-split. This contradicts with the fact that G is pi-poor. Therefore, $T_p(G)$ can not be bounded. □

Let $\mathbb{Q}_p$ be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$. Note that the elements of $\mathbb{Q}_p$ are of the form ab^{-1}, where $a, b \in \mathbb{Z}$, $b \neq 0$, and $\gcd(b, p) = 1$

Lemma 5.2. *Let p be a prime integer and N be a reduced torsion group. Then for every homomorphism $f : \mathbb{Q}_p \to N$, Imf is bounded.*

Proof. For every prime $q \neq p$, it is clear that $q\mathbb{Q}_p = \mathbb{Q}_p$, i.e., $\mathbb{Q}_p$ is q-divisible, and $T_q(N)$ is reduced. Then for $\pi_q \circ f : \mathbb{Q}_p \to T_q(N)$, where $\pi_q : N \to T_q(N)$ is the natural projection, $(\pi_q \circ f)(\mathbb{Q}_p)$ is a q-divisible subgroup of $T_q(N)$. Therefore, $(\pi_q \circ f)(\mathbb{Q}_p)$ is divisible, and so $\pi_q \circ f = 0$ because $T_q(N)$ is reduced. Thus Im$f = f(\mathbb{Q}_p) \subseteq T_p(N)$. Put $a = f(1)$ and $o(a) = p^n$, where $o(a)$ the order of a. Let bc^{-1} be any element of $\mathbb{Q}_p$ with $\gcd(c, p) = 1$. Then $\gcd(c, p^n) = 1$, therefore $cy + p^n z = 1$ for some $y, z \in \mathbb{Z}$. Now $b = bcy + bp^n z$, so $bc^{-1} = by + bp^n zc^{-1}$. Note that $cf(bp^n zc^{-1}) = bzp^n f(1) = zp^n a = 0$. Let $x = f(bp^n zc^{-1})$ and $o(x) = p^m$. Since $\gcd(c, p^m) = 1$, we have $cu + p^m v = 1$ for some $u, v \in \mathbb{Z}$. Then $x = ucx + vp^m x = ucx = 0$, and so $f(bc^{-1}) = f(by) + x = f(by) = byf(1) \in \langle f(1)\rangle$. Hence Im$f$ is contained in $\langle f(1)\rangle$, and so it is bounded. □

A *cotorsion* group G is a group satisfying $\text{Ext}(\mathbb{Q}, G) = 0$.

Theorem 5.3. *There is a group G such that G is not pure-split and every reduced torsion group N is G-pure-injective. Hence a pi-poor group can not be torsion.*

Proof. Fix any prime p. Since $\mathbb{Q}_p$ is not cotorsion, $\text{Ext}(\mathbb{Q}, \mathbb{Q}_p) \neq 0$ (see [5], page 226, Example 15). So there is a nonsplitting pure sequence:

$$0 \to \mathbb{Q}_p \to G \to \mathbb{Q} \to 0.$$

Hence, G is not pure-split. For every prime $q \neq p$, $\mathbb{Q}_p$ and $\mathbb{Q}$ are q-divisible, therefore G is also q-divisible. We claim that N is G-pure injective. Without loss of generality, we can assume that $\mathbb{Q}_p$ is a subgroup of G and $G/\mathbb{Q}_p = \mathbb{Q}$. Let K be any nonzero pure subgroup of G and $f : K \to N$ be any homomorphism, where N is a torsion reduced group. Then, K is q-divisible for every prime $q \neq p$ since K is a pure subgroup of G and G is q-divisible. Clearly, the rank of K is at most 2. So have two cases:

Case I: $r_0(K) = 1$. If K is also p-divisible, then K is divisible. So $K \cong \mathbb{Q}$, and the inclusion $K \to G$ splits, so f can be extended to a homomorphism $f' : G \to N$. Now, let K be not p-divisible. K and $\mathbb{Q}_p$ are of the same type, and so $K \cong \mathbb{Q}_p$ (see [5, Theorem 85.1]). Therefore, Imf is bounded by Lemma 5.2. Then, Imf is pure-injective, hence $f : K \to N$ can be extended to a homomorphism $f' : G \to \text{Im}f \leq N$.

Case II: $r_0(K) = 2$: We claim that $K = G$. Otherwise, since G/K is a nonzero torsion-free group, $r_0(G/K) \geq 1$. Then $2 = r_0(G) = r_0(K) + r_0(G/K) > 2$, a contradiction. Hence $G = K$.

As a consequence, N is G-pure-injective. This implies that N is not pi-poor. □

Corollary 5.4. *Let M be a pi-poor group. Then $M \neq T(M)$ and $T_p(M)$ is unbounded for every $p \in P$.*

Lemma 5.5. *Let M and N be right R-modules. Assume that N is (pure-)injective. Then, $M \oplus N$ is (pi-)poor if and only if M is (pi-)poor.*

Proof. For a right R-module B, it is clear that $M \oplus N$ is B-(pure-)injective if and only if M is B-(pure-)injective. □

Example 5.6. Let $G = \oplus_{p \in P} \mathbb{Z}_p$. Then G is poor by Theorem 3.1. On the other hand, since $T_p(G) = \mathbb{Z}_p$ is bounded, G is not pi-poor by Proposition 5.1.

Example 5.7. Let M be as in Theorem 4.1 and let V be the sum of all direct summands isomorphic to $\mathbb{Z}_p$. If $M = V \oplus K$, then K is pi-poor by Lemma 5.5. But K is not poor by Theorem 3.1, since K does not contain a direct summand isomorphic to $\mathbb{Z}_p$. So pi-poor modules need not be poor.

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