

Exact quantization of Cauchy-Euler type forced parametric oscillator

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Abstract. Driven and damped parametric quantum oscillator is solved by Wei-Norman Lie algebraic approach, which gives the exact form of the evolution operator. This allows us to obtain explicitly the probability densities, time-evolution of initially Glauber coherent states, expectation values and uncertainty relations. Then, as an exactly solvable model, we introduce the driven Cauchy-Euler type quantum parametric oscillator, which appears as self-adjoint quantization of the classical Cauchy-Euler differential equation. We discuss some typical behavior of this oscillator under the influence of external terms and give a concrete example.

1. Introduction

Quantum harmonic oscillator with explicitly time-dependent Hamiltonian is a fundamental model, which appears in many physical branches such as quantum optics, quantum fluid, ion-traps, and cosmology. It is a useful model also in quantum information and quantum computation.

In the present work, we first solve the evolution problem for quantum harmonic oscillator

$$\begin{cases} i\hbar \frac{\partial}{\partial t} \Psi(q, t) = \hat{H}(t) \Psi(q, t), \\ \Psi(q, t_0) = \Psi_0(q), \quad -\infty < q < \infty, \end{cases} \quad (1)$$

with generalized quadratic Hamiltonian

$$\hat{H}(t) = \frac{\hat{p}^2}{2\mu(t)} + \frac{\mu(t)\omega^2(t)}{2} \hat{q}^2 + \frac{B(t)}{2} (\hat{q}\hat{p} + \hat{p}\hat{q}) + D(t)\hat{q}, \quad (2)$$

where $\mu(t) > 0$, $\omega^2(t)$, $B(t)$ and $D(t)$ are sufficiently smooth, real-valued parameters depending on time. To understand better the behavior of such quantum systems it is always important to have exactly solvable models. The best known one is the Caldirola-Kanai oscillator with an exponentially increasing mass, [1, 2], which is widely used to study dissipation in quantum mechanics. In this article, we introduce the Cauchy-Euler type quantum parametric oscillator as another exactly solvable model, and briefly discuss some of its typical properties.



2. Quantization of the harmonic oscillator

2.1. The evolution operator and wave functions

As known, the evolution operator for system (1) can be found by the Wei-Norman Lie algebraic approach. Indeed, the Hamiltonian (2) can be written as time-dependent linear combination of Lie algebra generators as

$$\hat{H}(t) = -i \left(\frac{\hbar^2}{\mu(t)} \hat{K}_- + \mu(t)\omega^2(t)\hat{K}_+ + 2\hbar B(t)\hat{K}_0 + D(t)\hat{E}_1 \right), \quad (3)$$

where $\hat{E}_1 = iq$, $\hat{E}_2 = \partial/\partial q$, $\hat{E}_3 = i\hat{I}$ are generators of the Heisenberg-Weyl algebra, and $\hat{K}_- = -(i/2)(\partial^2/\partial q^2)$, $\hat{K}_+ = (i/2)q^2$, $\hat{K}_0 = (q(\partial/\partial q) + 1/2)/2$ are generators of the $su(1,1)$ algebra. Then, the evolution operator for the general oscillator can be written as product of exponential operators

$$\hat{U}(t, t_0) = e^{c(t)\hat{E}_3} e^{\frac{a(t)}{\hbar}\hat{E}_1} e^{-b(t)\hat{E}_2} e^{f(t)\hat{K}_+} e^{2h(t)\hat{K}_0} e^{g(t)\hat{K}_-}, \quad (4)$$

where $f(t), g(t), h(t), a(t), b(t), c(t)$ are unknown real-valued functions to be determined from the IVP defining the unitary operator \hat{U} , that is $i\hbar(d\hat{U}(t, t_0)/dt) = \hat{H}(t)\hat{U}(t, t_0)$, $\hat{U}(t_0, t_0) = \hat{I}$. Plugging (3) and (4) in that problem, after some algebra we obtain the exact form of the evolution operator

$$\begin{aligned} \hat{U}(t, t_0) &= \exp \left(\frac{i}{\hbar} \int_{t_0}^t \left[\frac{-1}{2\mu(s)} p_p^2(s) + \frac{\mu(s)\omega^2(s)}{2} x_p^2(s) \right] ds \right) \\ &\times \exp(ip_p(t)q) \times \exp \left(-x_p(t) \frac{\partial}{\partial q} \right) \times \exp \left(i \frac{\mu(t)}{2\hbar} \left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t) \right) q^2 \right) \\ &\times \exp \left(\ln \left| \frac{x_1(t_0)}{x_1(t)} \right| \left(q \frac{\partial}{\partial q} + \frac{1}{2} \right) \right) \times \exp \left(\frac{i}{2} \hbar x_1^2(t_0) \left(\frac{x_2(t)}{x_1(t)} \right) \frac{\partial^2}{\partial q^2} \right), \end{aligned} \quad (5)$$

where the functions $x_1(t)$ and $x_2(t)$ are two linearly independent homogeneous solutions of the classical equation of motion,

$$\ddot{x} + \frac{\dot{\mu}}{\mu} \dot{x} + \left(\omega^2(t) - \left(\dot{B} + B^2 + \frac{\dot{\mu}}{\mu} B \right) \right) x = -\frac{1}{\mu} D, \quad (6)$$

satisfying the IC's $x_1(t_0) = x_0 \neq 0$, $\dot{x}_1(t_0) = x_0 B(t_0)$, and $x_2(t_0) = 0$, $\dot{x}_2(t_0) = 1/\mu(t_0)x_1(t_0)$, and $x_p(t)$ is the particular solution of (6) satisfying the IC's $x_p(t_0) = \dot{x}_p(t_0) = 0$. Also $p_p(t)$ is the particular solution to the classical equation of motion in momentum space

$$\ddot{p} - \frac{(\mu\dot{\omega}^2)}{\mu\omega^2} \dot{p} + \left(\omega^2(t) + \left(\dot{B} - B^2 - \frac{(\mu\dot{\omega}^2)}{\mu\omega^2} B \right) \right) p = -\dot{D} + \left(\frac{(\mu\dot{\omega}^2)}{\mu\omega^2} + B \right) D, \quad (7)$$

satisfying the IC's $p_p(t_0) = 0$, $\dot{p}_p(t_0) = -D(t_0)$. Next, applying the evolution operator (5) to the normalized eigenstates of the standard Hamiltonian

$$\varphi_k(q) = N_k e^{-\frac{m\omega_0}{2\hbar} q^2} H_k \left(\sqrt{\frac{m\omega_0}{\hbar}} q \right), \quad k = 0, 1, 2, \dots, \quad (8)$$

we obtain the exact wave function in the form

$$\Psi_k(q, t) = N_k \sqrt{R_B(t)} \times \exp \left[i \left(k + \frac{1}{2} \right) \arctan \left(-m\omega_0 x_1^2(t_0) \left(\frac{x_2(t)}{x_1(t)} \right) \right) \right]$$

$$\begin{aligned}
& \times \exp \left\{ \frac{i}{2\hbar} \left[\left(\mu(t) \left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t) \right) - (m\omega_0 x_1(t_0))^2 \frac{x_2(t)}{x_1(t)} R_B^2(t) \right) (q - x_p(t))^2 + 2p_p(t)q \right] \right\} \\
& \times \exp \left[\frac{i}{\hbar} \int_{t_0}^t \left(\frac{-1}{2\mu(s)} p_p^2(s) + \frac{\mu(s)\omega^2(s)}{2} x_p^2(s) \right) ds \right] \\
& \times \exp \left[-\frac{1}{2} \left(\sqrt{\frac{m\omega_0}{\hbar}} R_B(t) (q - x_p(t)) \right)^2 \right] \times H_k \left(\sqrt{\frac{m\omega_0}{\hbar}} R_B(t) (q - x_p(t)) \right). \quad (9)
\end{aligned}$$

Therefore, the probability density is

$$\rho_k(q, t) = N_k^2 R_B(t) \exp \left(- \left(\sqrt{\frac{m\omega_0}{\hbar}} R_B(t) (q - x_p(t)) \right)^2 \right) \times H_k^2 \left(\sqrt{\frac{m\omega_0}{\hbar}} R_B(t) (q - x_p(t)) \right), \quad (10)$$

where

$$R_B(t) = \sqrt{\frac{x_0^2}{x_1^2(t) + (m\omega_0 x_0^2 x_2(t))^2}}. \quad (11)$$

We see that, in (10) $R_B(t)$ is the squeezing (or spreading) coefficient and $x_p(t)$ acts as displacement of the wave packet. Clearly, $R_B(t)$ depends on $B(t)$, but does not depend on $D(t)$. On the other hand, $x_p(t)$, clearly depends on all parameters of the Hamiltonian (2).

2.2. Coherent states, expectation values and uncertainty relation

Coherent states of the standard harmonic oscillator (Glauber CS's), can be written in the form

$$\phi_\alpha(q) = \left(\frac{m\omega_0}{\pi\hbar} \right)^{1/4} e^{\frac{i}{\hbar} \langle \hat{p} \rangle_\alpha q} e^{-\frac{m\omega_0}{2\hbar} (q - \langle \hat{q} \rangle_\alpha)^2}, \quad \alpha \in \mathcal{C}. \quad (12)$$

Using (5) we can find the time-evolution of initially Glauber CS's, that is $\Phi_\alpha(q, t) = \hat{U}_g(t, t_0) \phi_\alpha(q)$, so that the corresponding probability density $\rho_\alpha(q, t) = |\Phi_\alpha(q, t)|^2$ becomes

$$\begin{aligned}
\rho_\alpha(q, t) &= \sqrt{\frac{m\omega_0}{\pi\hbar}} R_B^2(t) \times \exp \left\{ 2 \left[((m\omega_0 x_0) x_2(t) R_B(t))^2 (\alpha_1^2 - \alpha_2^2) \right. \right. \\
&\quad \left. \left. - 2(m\omega_0) x_1(t) x_2(t) R_B^2(t) \alpha_1 \alpha_2 - \alpha_1^2 \right] \right\} \\
&\times \exp \left(2 \sqrt{\frac{2m\omega_0}{\hbar}} R_B^2(t) \left(\alpha_1 \frac{x_1(t)}{x_0} + \alpha_2 (m\omega_0 x_0) x_2(t) \right) (q - x_p(t)) \right) \\
&\times \exp \left(- \left(\frac{m\omega_0}{\hbar} \right) R_B^2(t) (q - x_p(t))^2 \right). \quad (13)
\end{aligned}$$

Using again (5), we compute the position and momentum operators in Heisenberg picture as

$$\hat{q}_H(t) \equiv \hat{U}_g^\dagger(t, t_0) \hat{q}_S \hat{U}_g(t, t_0) = \frac{1}{x_0} x_1(t) \hat{q}_H(t_0) + x_0 x_2(t) \hat{p}_H(t_0) + x_p(t), \quad (14)$$

$$\hat{p}_H(t) \equiv \hat{U}_g^\dagger(t, t_0) \hat{p}_S \hat{U}_g(t, t_0) = \frac{1}{x_0} \mu(t) p_1(t) \hat{q}_H(t_0) + x_0 \mu(t) p_2(t) \hat{p}_H(t_0) + p_p(t). \quad (15)$$

Since $\langle \hat{q} \rangle_\alpha(t) \equiv \langle \Phi_\alpha(q, t) | \hat{q} | \Phi_\alpha(q, t) \rangle = \langle \phi_\alpha(q, t_0) | \hat{q}_H(t) | \phi_\alpha(q, t_0) \rangle$, the expectation value of position at coherent state $\Phi_\alpha(q, t)$ is

$$\langle \hat{q} \rangle_\alpha(t) = \sqrt{\frac{2\hbar}{m\omega_0}} \left(\frac{\alpha_1}{x_0} x_1(t) + \alpha_2 (m\omega_0 x_0) x_2(t) \right) + x_p(t). \quad (16)$$

Similarly, using that $\langle \hat{p} \rangle_\alpha(t) = \langle \phi_\alpha(q, t_0) | \hat{p}_H(t) | \phi_\alpha(q, t_0) \rangle$, the expectation value of the momentum becomes

$$\langle \hat{p} \rangle_\alpha(t) = \sqrt{\frac{2\hbar}{m\omega_0}} \mu(t) \left(\frac{\alpha_1}{x_0} p_1(t) + \alpha_2 (m\omega_0 x_0) p_2(t) \right) + p_p(t), \quad (17)$$

where $p_1(t) = \mu(t)(\dot{x}_1(t) - B(t)x_1(t))$ and $p_2(t) = \mu(t)(\dot{x}_2(t) - B(t)x_2(t))$ are two linearly independent solutions of the homogeneous part of the classical oscillator (7) in momentum space. This explicitly shows that, the expectation values at coherent states satisfy the classical equations of motion. After some calculations, we obtain also the uncertainties

$$(\Delta \hat{q})_\alpha(t) = \sqrt{\frac{\hbar}{2m\omega_0}} \frac{1}{R_B(t)}, \quad (18)$$

$$(\Delta \hat{p})_\alpha(t) = \sqrt{\frac{m\omega_0 \hbar}{2}} R_B(t) \sqrt{1 + \frac{\mu^2(t)}{(m\omega_0 R_B^2(t))^2} \left(\frac{\dot{R}_B(t)}{R_B(t)} + B(t) \right)^2}. \quad (19)$$

We note that the expectation values depend on all parameters of the Hamiltonian, however the uncertainties does not depend on the external linear term, which contributes only to the displacement of the wave packet. Finally, the uncertainty relation for the generalized harmonic oscillator with time dependent parameters is found

$$(\Delta \hat{q})_\alpha (\Delta \hat{p})_\alpha(t) = \frac{\hbar}{2} \sqrt{1 + \frac{\mu^2(t)}{(m\omega_0 R_B^2(t))^2} \left(\frac{\dot{R}_B(t)}{R_B(t)} + B(t) \right)^2}, \quad (20)$$

where clearly $(\Delta \hat{q})_\alpha (\Delta \hat{p})_\alpha \geq \hbar/2$. Summarizing, coherent states of the time-dependent parametric oscillator are displaced Gaussian wave packets, see (13), and follow the classical trajectory as can be seen from (16) and (17). However, they are spreading or squeezing in time, since $(\Delta \hat{q})_\alpha(t)$ given by (18) depends explicitly on time, and are no longer minimum uncertainty states, as we can see from (20).

3. Cauchy-Euler type quantum parametric oscillator

We introduce the Cauchy-Euler type quantum parametric oscillator and study its behavior under the influence of the mixed and external terms. Precisely, we consider the Hamiltonian

$$\hat{H}(t) = \frac{1}{2} t^{-\gamma} \hat{p}^2 + \frac{1}{2} w_0^2 t^{\gamma-2} \hat{q}^2 + B(t) \frac{(\hat{q}\hat{p} + \hat{p}\hat{q})}{2} + D(t) \hat{q}, \quad t > 0, \quad (21)$$

with variable mass $\mu(t) = t^\gamma$, variable frequency $\omega^2(t) = w_0^2/t^2$, $\gamma > 0$, $w_0^2 > 0$, mixed term parameter $B(t)$ and linear driving $D(t)$. When $B(t) = D(t) = 0$, the corresponding classical oscillator is the Cauchy-Euler type ordinary differential equation

$$\ddot{x} + (\gamma/t)\dot{x} + (w_0^2/t^2)x = 0, \quad t > 0,$$

which solutions depend on the sign of $\Omega_0^2 = w_0^2 - (\gamma - 1)^2/4$. Accordingly, we have the cases: (i) $\Omega_0^2 = 0$ critical damping, (ii) $\Omega_0^2 < 0$ over damping, and (iii) $\Omega_0^2 > 0$ under damping. Then, we choose $B(t) = \dot{y}(t)/y(t)$, where $y(t)$ satisfies the Cauchy-Euler equation $\ddot{y} + (\gamma/t)\dot{y} + (w_1^2/t^2)y = 0$, with the same γ , but w_1 possibly different than w_0 . This special choice of $B(t)$ modifies the original frequency w_0^2/t^2 , but does not change the Cauchy-Euler type structure of the oscillator. Indeed, in that case the classical oscillator corresponding to the Hamiltonian (21) takes the form

$$\ddot{x} + \frac{\gamma}{t}\dot{x} + \frac{(w_0^2 + w_1^2)}{t^2}x = -\frac{1}{t^\gamma}D(t), \quad t > 0, \quad (22)$$

with time-variable damping $\Gamma(t) = \gamma/t$, $\gamma > 0$, modified frequency $w_g^2(t) = (w_0^2 + w_1^2)/t^2$, and driving force. In this short article, some typical behavior of the quantum oscillator is illustrated by an example.

Example: We take the Hamiltonian

$$\hat{H}(t) = \frac{t^{-4}}{2}\hat{p}^2 + \frac{t^2}{2}\hat{q}^2 + B(t)(\hat{q}\hat{p} + \hat{p}\hat{q}) + D(t)\hat{q}, \quad (\gamma = 4, w_0 = 1), \quad (23)$$

with mixed term parameter $B(t) = -|\Omega_1|[\tan(|\Omega_1|\ln t - \alpha) + \tan \alpha]/t$, where $\Omega_1^2 = 247/4$, $\alpha = \tan^{-1}(3/\sqrt{247})$, and external parameter $D(t) = \sqrt{t}$. The corresponding classical equation of motion for (23) becomes

$$\ddot{x} + (4/t)\dot{x} + (65/t^2)x = t^{-7/2}, \quad t > 0. \quad (24)$$

Note that, when $B(t) = 0$, the oscillator is in over damping case, but with above choice of $B(t)$ we have $w_1 = 8$, so that the modified frequency is $w_g^2(t) = (w_0^2 + w_1^2)/t^2 = 65/t^2$, and then oscillator becomes in under damping case. For $t_0 = 1$, two linearly independent solutions of (24) satisfying the initial conditions $x_1(1) = 1$, $\dot{x}_1(1) = 0$, and $x_2(1) = 0$, $\dot{x}_2(1) = 1$, are

$$x_1(t) = \frac{\sqrt{65}}{|\Omega_g|}t^{-3/2}\cos(|\Omega_g|\ln t - \beta), \quad x_2(t) = \frac{1}{|\Omega_g|}t^{-3/2}\sin(|\Omega_g|\ln t), \quad t > 0,$$

where $\beta = \tan^{-1}(3/\sqrt{251})$ and $\Omega_g^2 = 251/4$. The particular solution satisfying the initial conditions $x_p(1) = \dot{x}_p(1) = 0$, is $x_p(t) = -t^{-3/2}(1 - \cos(|\Omega_g|\ln t))/\Omega_g^2$. Thus, we obtain

$$R_B(t) = |\Omega_g|\sqrt{\frac{t^3}{65\cos^2(|\Omega_g|\ln t - \beta) + 25\sin^2(|\Omega_g|\ln t)}}, \quad t > 0, \quad (25)$$

which is smooth, shows oscillatory behavior, and $R_B(t) \rightarrow \infty$ as $t \rightarrow \infty$. Then the probability density becomes

$$\begin{aligned} \rho_k(q, t) &= N_k^2 R_B(t) \exp\left(-\left(5R_B(t)\left(q + \frac{1}{\Omega_g^2}t^{-(\frac{\gamma-1}{2})}(1 - \cos(|\Omega_g|\ln t))\right)\right)^2\right) \\ &\quad \times H_k^2\left(5R_B(t)\left(q + \frac{1}{\Omega_g^2}t^{-(\frac{\gamma-1}{2})}(1 - \cos(|\Omega_g|\ln t))\right)\right), \end{aligned} \quad (26)$$

and in Fig. 1 we plot it for $k = 1, 2$ respectively. We observe that, probability density amplitudes increase without bound, and wave packets are squeezing as time increases, since $R_B(t) \rightarrow \infty$ as $t \rightarrow \infty$. Also, probability is oscillatory in time, and is localized near $q = 0$ as $t \rightarrow \infty$, since $x_p(t) \rightarrow 0$. Knowing (25), the uncertainties and uncertainty relation take the form

$$(\Delta\hat{q})_\alpha(t) = \sqrt{\frac{1}{10}}\frac{1}{R_B(t)}, \quad (27)$$

$$(\Delta\hat{p})_\alpha(t) = \sqrt{\frac{5}{2}}R_B(t)\sqrt{1 + \frac{t^8}{25R_B^4(t)}\left(\frac{\dot{R}_B(t)}{R_B(t)} - \frac{|\Omega_1|}{t}\left(\tan(|\Omega_1|\ln t - \alpha) + \tan \alpha\right)\right)^2}, \quad (28)$$

$$(\Delta\hat{q})_\alpha(\Delta\hat{p})_\alpha(t) = \frac{1}{2}\sqrt{1 + \frac{t^8}{25R_B^4(t)}\left(\frac{\dot{R}_B(t)}{R_B(t)} - \frac{|\Omega_1|}{t}\left(\tan(|\Omega_1|\ln t - \alpha) + \tan \alpha\right)\right)^2}. \quad (29)$$

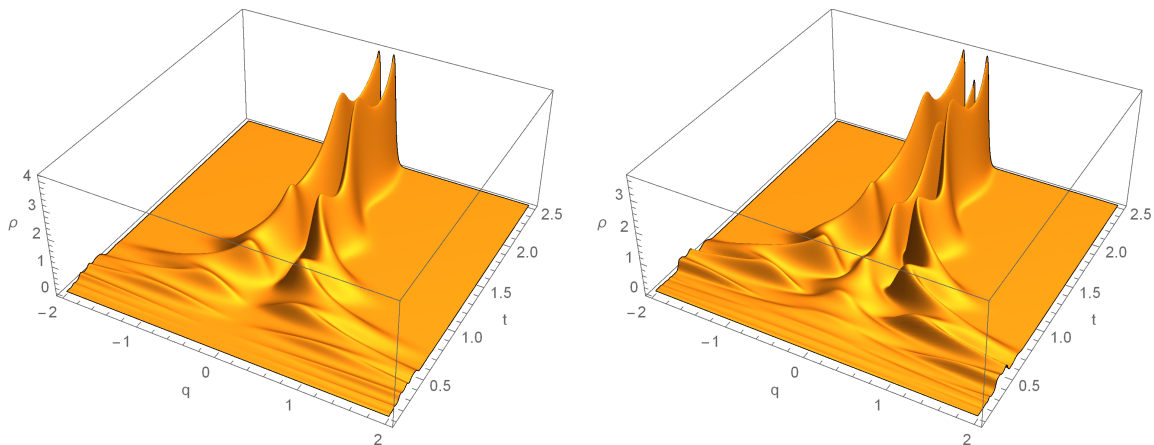


Figure 1. Probability density $\rho_k(q, t) = |\Psi_k(q, t)|^2$ for Cauchy-Euler type generalized oscillator when $B(t) = -|\Omega_1|[\tan(|\Omega_1| \ln t - \alpha) + \tan \alpha]/t$, $D(t) = \sqrt{t} w_0 = 1$, $w_1 = 8$, $\gamma = 4$, and $k = 1, 2$, respectively.

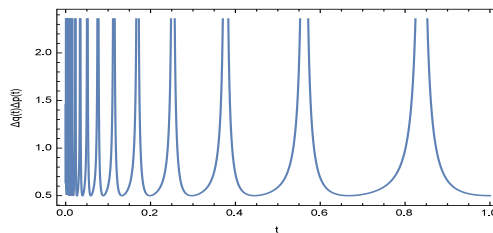


Figure 2. Uncertainty relation for driven Cauchy-Euler oscillator, $w_0 = 1$, $w_1 = 8$, $\gamma = 4$.

As a result, we see that by a special choice of the mixed term parameter $B(t)$, which modifies the original frequency, we are able to preserve the structure of the original oscillator. However, this parameter $B(t)$ has finite time singularities. In that case, the probability densities and expectations of position remain smooth, but singularities of $B(t)$ are reflected in the expectations of momentum and the uncertainty relation. Precisely, momentum is not defined at these points, and when time approaches these singularities, uncertainty relation tends to infinity, as one can see in Fig. (2).

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