J. Korean Math. Soc. **51** (2014), No. 5, pp. 971–985
http://dx.doi.org/10.4134/JKMS.2014.51.5.971

ON w-LOCAL MODULES AND *Rad*-SUPPLEMENTED MODULES

Engin Büyükaşık and Rachid Tribak

Abstract. All modules considered in this note are over associative commutative rings with an identity element. We show that a w-local module M is *Rad*-supplemented if and only if $M/P(M)$ is a local module, where $P(M)$ is the sum of all radical submodules of M. We prove that w-local nonsmall submodules of a cyclic *Rad*-supplemented module are again *Rad*-supplemented. It is shown that commutative Noetherian rings over which every w-local *Rad*-supplemented module is supplemented are Artinian. We also prove that if a finitely generated *Rad*-supplemented module is cyclic or multiplication, then it is amply *Rad*-supplemented. We conclude the paper with a characterization of finitely generated amply *Rad*-supplemented left modules over any ring (not necessarily commutative).

1. Introduction

All rings considered in this paper will be commutative with an identity element (except for Section 5) and all modules will be left unitary modules. Unless otherwise stated R denotes an arbitrary commutative ring. Let M be an arbitrary R-module. We will denote by $Rad(M)$ the Jacobson radical of M. A submodule L of M is called *small* in M (notation $L \ll M$) if $M \neq L + N$ for every proper submodule N of M. The *annihilator* of M in R will be denoted by $Ann_R(M) = \{\alpha \in R : \alpha x = 0 \text{ for all } x \in M\}$ and for every element x of M, the *annihilator* of x is denoted by $Ann_R(x) = \{\alpha \in R : \alpha x = 0\}$. A module M is said to be *radical* if $Rad(M) = M$. The sum of all radical submodules of a module M will be denoted by $P(M)$. A module M is said to be *reduced* if $P(M) = 0$. We say that the ring R is reduced if the R-module ${}_RR$ is reduced. For submodules U and V of a module M, the submodule V is said to be a *Rad-supplement* of U in M if $U + V = M$ and $U \cap V \subseteq Rad(V)$. A module M is called *Rad-supplemented* if every submodule of M has a *Rad*-supplement in M. On the other hand, a submodule N of M is said *to have ample Rad-supplements in*

Received October 11, 2013.
2010 *Mathematics Subject Classification.* 16D10, 16D80.
Key words and phrases. w-local modules, *Rad*-supplemented modules, amply *Rad*-supplemented modules.

M if every submodule K of M with $M = N + K$ contains a *Rad*-supplement of N in M. The module M is called *amply Rad-supplemented* if every submodule of M has ample *Rad*-supplements in M. A nonzero module L is called *local* if the sum of all its proper submodules is also a proper submodule. We say that a nonzero module W is *w-local* if it has a unique maximal submodule. w-local modules were first studied by Ware in [18], Gerasimov and Sakhaev in [8]. In [3], Büyükaşik and Lomp showed that this type of modules play a key role in the study of *Rad*-supplemented modules. This role is as important as the role played by local modules for supplemented modules. In fact, it is shown (in [3]) that any *Rad*-supplement of a maximal submodule is w-local and any finitely generated *Rad*-supplemented module is a sum of finitely many w-local submodules. But these modules may have a complicated structure. In Section 2 we will give a brief exposition of some properties of w-local modules.

Section 3 deals with the question: When w-local modules are (*Rad*-)supplemented? We will show that a w-local module M is *Rad*-supplemented if and only if $M/P(M)$ is a local module. It is also shown that w-local nonsmall submodules of a cyclic *Rad*-supplemented module are again *Rad*-supplemented. We conclude this section by showing that commutative Noetherian rings over which every w-local *Rad*-supplemented module is supplemented are Artinian.

It is proved in [14, Corollary 4.6] that finitely generated supplemented modules are amply supplemented. One may ask whether this is true for *Rad*-supplemented modules. In Section 4 we will not solve this question, but we will show that it has an affirmative response for some type of modules. Among other results, we show that if a *Rad*-supplemented module M is a cyclic or a multiplication module, then M is amply *Rad*-supplemented. Moreover, we show that the study of finitely generated *Rad*-supplemented modules over commutative rings can be restricted to the class of finitely generated reduced modules over semilocal reduced rings.

In the last section we characterize finitely generated amply *Rad*-supplemented modules.

2. Some properties of w-local modules

There was little known about the structure of w-local modules. The aim of this section is to shed some light on the structure of w-local modules.

Proposition 2.1. *The following statements are equivalent for an R-module* M:

(i) M *is w-local*;

(ii) (a) *There exists a unique maximal ideal* m *of* R *such that* $M \neq mM$, *and*

(b) *If* m_0 *is the maximal ideal satisfying the condition* (a)*, then* M/m_0M *is indecomposable.*

In this case, m_0M *is the unique maximal submodule of* M.

Proof. (i) $\Rightarrow$ (ii) Assume that M is w-local. Then $Rad(M) \neq M$. So there exists a maximal ideal m of R such that $mM \neq M$ by [7, Lemma 3]. Since $m(M/mM) = 0$, M/mM is semisimple. But M is w-local. Then M/mM has only one maximal submodule. This implies that M/mM is simple. Hence mM is a maximal submodule of M. Assume that there exists a maximal ideal m' of R such that $m' \neq m$ and $M \neq m'M$. As above, we get that $m'M$ is a maximal submodule of M. So $m'M = mM$. Therefore $M = (m + m')M = mM$, a contradiction.

(ii) $\Rightarrow$ (i) It is clear that M/m_0M is semisimple. Since M/m_0M is nonzero indecomposable, M/m_0M is simple. Thus m_0M is a maximal submodule of M. Let N be a maximal submodule of M. Since M/N is simple, there exists a maximal ideal q of R such that $q(M/N) = 0$. Therefore $qM \subseteq N$. So $qM \neq M$. By hypothesis, we have $q = m_0$ and $N = m_0M$. It follows that M is a w-local module. □

We call a w-local module satisfying the conditions of Proposition 2.1 m_0-w-local.

Corollary 2.2. *Let m be a maximal ideal of R and let M be an m-w-local R-module. Then $P(M) \subseteq \bigcap_{n\geq 1} m^n M$.*

Proof. By Proposition 2.1, $P(M) \subseteq mM$. Since $Rad(P(M)) = P(M)$, we have $mP(M) = P(M)$ by [7, Lemma 3]. So $P(M) \subseteq m^2M$. By induction, we get $P(M) \subseteq m^nM$ for all $n \geq 1$. The result follows. □

Corollary 2.3. *Let m be a maximal ideal of R and let M be an m-w-local R-module. Let n be a positive integer. If $Rad(m^nM) \neq m^nM$, then $Rad(m^nM) = m^{(n+1)}M$.*

Proof. Let n be a positive integer. Assume that $Rad(m^nM) \neq m^nM$. Let q be a maximal ideal of R with $q \neq m$. By Proposition 2.1, we have $qM = M$. So $q(m^nM) = m^nM$. Moreover, [7, Lemma 3] shows that $m(m^nM) \neq m^nM$ and $Rad(m^nM) = m^{(n+1)}M$. □

Corollary 2.4. *Let m be a maximal ideal of R and let M be an m-w-local reduced R-module. If m is principal, then for every positive integer n, $m^nM = 0$ or m^nM is m-w-local.*

Proof. Let n be a positive integer such that $m^nM \neq 0$. Since m is principal, there exists an element $a \in m$ such that $m = Ra$. Since M is reduced, we have $Rad(m^nM) \neq m^nM$. Thus $Rad(m^nM) = m^{(n+1)}M \neq m^nM$ by Corollary 2.3. Consider the map $\varphi : M/mM \to m^nM/m^{(n+1)}M$ defined by $\varphi(x + mM) = a^nx + m^{(n+1)}M$. It is easily seen that φ is well defined and it is an epimorphism. It follows that $m^nM/m^{(n+1)}M$ is a simple module. Therefore $Rad(m^nM)$ is a maximal submodule of m^nM. Hence m^nM is m-w-local. □

Proposition 2.5. *Let M be an m-w-local reduced R-module. Then for every $x \in M \setminus mM$, we have $Ann_R(x) = Ann_R(M)$.*

Proof. Let $x \in M \setminus mM$. Then $M = mM + Rx$. Let $\alpha \in Ann_R(x)$. Thus $\alpha M = \alpha(mM)$. That is, $\alpha M = m(\alpha M)$. Moreover, it is easily seen that for every maximal ideal $m' \neq m$, we have $m'(\alpha M) = \alpha M$. It follows that $Rad(\alpha M) = \alpha M$ by [7, Lemma 3]. Since M is reduced, we have $\alpha M = 0$. It follows that $\alpha \in Ann_R(M)$. Hence $Ann_R(x) = Ann_R(M)$. □

Lemma 2.6. *Let N and K be submodules of an R-module M. If U is a maximal submodule of N, then $U + K = N + K$ or $U + K$ is a maximal submodule of $N + K$.*

Proof. Note that $(N+K)/(U+K) = [N+(U+K)]/(U+K) \cong N/[N \cap (U+K)] \cong N/[U + (N \cap K)]$. If $N \cap K \not\subseteq U$, then $U + (N \cap K) = N$ and hence $U + K = N + K$. Now if $N \cap K \subseteq U$, then $(N+K)/(U+K) \cong N/U$. The result follows. □

Proposition 2.7. *Let M be a nonzero Artinian w-local R-module. If M is reduced, then M is a local module of finite length.*

Proof. Assume that M does not have a composition series. Let U_1 be the maximal submodule of M and let $a \in M$ with $a \notin U_1$. Then $Ra + U_1 = M$. By hypothesis, $Rad(U_1) \neq U_1$ since $U_1 \neq 0$. Therefore U_1 has a maximal submodule U_2. If $Ra + U_2 \neq M$, then $Ra + U_2$ is a maximal submodule of M by Lemma 2.6. Thus $Ra + U_2 = U_1$ and $a \in U_1$, a contradiction. So $Ra + U_2 = M$. By repeating the same reasoning, we construct an infinite descending chain of submodules of M. This contradicts the fact that M is Artinian. So M is of finite length. Now the fact that M is w-local and finitely generated implies that M is local. □

3. When w-local modules are (Rad-)supplemented?

Note that contrary to local modules which are always supplemented, a w-local module (even if it is reduced) need not be Rad-supplemented (see Example 3.1).

Example 3.1. Let p be a prime number and consider the $\mathbb{Z}$-module $\mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} \mid p \text{ does not divide } b\}$. Clearly, $p\mathbb{Z}_{(p)} \neq \mathbb{Z}_{(p)}$. Let q be a prime number with $q \neq p$ and let $a/b \in \mathbb{Z}_{(p)}$ such that p does not divide b. Then $a/b = q(a/qb)$. Since p does not divide qb, we have $a/qb \in \mathbb{Z}_{(p)}$. Hence $\mathbb{Z}_{(p)} \subseteq q\mathbb{Z}_{(p)}$ and $q\mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}$. Moreover, we have $\frac{\mathbb{Z}_{(p)}}{p\mathbb{Z}_{(p)}} = \frac{p\mathbb{Z}_{(p)}+\mathbb{Z}}{p\mathbb{Z}_{(p)}} \cong \frac{\mathbb{Z}}{p\mathbb{Z}_{(p)} \cap \mathbb{Z}} = \frac{\mathbb{Z}}{p\mathbb{Z}}$. By Proposition 2.1, it follows that $\mathbb{Z}_{(p)}$ is w-local. It is easy to see that the module $\mathbb{Z}_{(p)}$ is reduced. On the other hand, note that the module $\mathbb{Z}_{(p)}$ is not Rad-supplemented by [4, Theorem 7.1 and Proposition 7.3].

The following two results give an answer to the natural question whether a (reduced) w-local module is Rad-supplemented.

Proposition 3.2. *The following statements are equivalent for a reduced w-local R-module M:*

(i) *M is amply Rad-supplemented*;
(ii) *M is Rad-supplemented*;
(iii) *M is a local module*;
(iv) *M is supplemented*;
(v) *M is amply supplemented.*

Proof. (i) $\Rightarrow$ (ii) This is clear.

(ii) $\Rightarrow$ (iii) Let $x \in M \setminus Rad(M)$ and let K be a Rad-supplement of Rx in M. Since $Rx + K = M$, we have $M/K \cong Rx/(Rx \cap K)$. Thus M/K has a maximal submodule. But $Rad(M)$ is the only maximal submodule of M. Then $K \subseteq Rad(M)$. Now, since K is a Rad-supplement submodule of M, it follows that $Rad(K) = K \cap Rad(M)$ by [4, Corollary 4.2]. Therefore $Rad(K) = K$. But M is reduced. Then $K = 0$. This gives that $M = Rx$ is a local module.

(iii) $\Rightarrow$ (i) This is immediate.

(v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (v) These run as before. □

Proposition 3.3. *The following statements are equivalent for a w-local R-module M*:

(i) *M is Rad-supplemented*;
(ii) *$M/P(M)$ is Rad-supplemented*;
(iii) *$M/P(M)$ is supplemented*;
(iv) *$M/P(M)$ is a local module*;
(v) *For every $x \in M \setminus Rad(M)$, $M = P(M) + Rx$ and the ring R/I_x is local, where $I_x = \{r \in R \mid rx \in P(M)\}$*;
(vi) *There exists $x \in M$ such that $M = P(M) + Rx$ and the ring R/I_x is local, where $I_x = \{r \in R \mid rx \in P(M)\}$.*

If the ring $R/P(R)$ is semiperfect, then (i)-(vi) *are equivalent to*:
(vii) *$M = P(M) + Rx$ for every $x \in M \setminus Rad(M)$*;
(viii) *There exists $x \in M$ such that $M = P(M) + Rx$.*

Proof. (i) $\Rightarrow$ (iv) Since M is Rad-supplemented, $M/P(M)$ is Rad-supplemented by [4, Proposition 4.8]. It is easily seen that $M/P(M)$ is a reduced w-local module. Applying Proposition 3.2, we conclude that $M/P(M)$ is a local module.

(iv) $\Rightarrow$ (v) Let $x \in M \setminus Rad(M)$. Since $Rad(M)$ is a maximal submodule of M, we have $M = Rad(M) + Rx$. As $P(M) \subseteq Rad(M)$, we have $M/P(M) = (Rad(M)/P(M)) + ((P(M) + Rx)/P(M))$. Since $M/P(M)$ is local and $M \neq Rad(M)$, we get $M = P(M) + Rx$. Moreover, note that $M/P(M) \cong Rx/(Rx \cap P(M))$ and $Ann_R(M/P(M)) = I_x$. Thus R/I_x is a local ring.

(v) $\Rightarrow$ (vi) This is clear.

(vi) $\Rightarrow$ (iii) This follows from the fact that $M/P(M) \cong Rx/(Rx \cap P(M)) \cong R/I_x$.

(iii) $\Rightarrow$ (ii) This is immediate.

(ii) $\Rightarrow$ (i) By [4, Proposition 4.8].

(v) $\Rightarrow$ (vii) $\Rightarrow$ (viii) These are immediate.

(viii) $\Rightarrow$ (ii) By [4, Theorem 6.5] and the fact that $M/P(M)$ is cyclic. □

Corollary 3.4. *Assume that R is a Dedekind domain. The following statements are equivalent for an R-module M:*

(i) *M is a w-local Rad-supplemented module*;

(ii) *There exist a submodule $E \leq M$ with $Rad(E) = E$ and a local submodule $L \leq M$ such that $M = E \oplus L$.*

Proof. (i) $\Rightarrow$ (ii) By Proposition 3.3, $M/P(M)$ is a local module. Since R is a Dedekind domain, $P(M)$ is injective. Thus there exists a local submodule $L \leq M$ such that $M = P(M) \oplus L$.

(ii) $\Rightarrow$ (i) It is clear that E is *Rad*-supplemented. So M is *Rad*-supplemented by [17, Proposition 2.5]. Moreover, note that if K is the maximal submodule of L, then $E \oplus K$ is the only maximal submodule of M. $\square$

Next we will be concerned with the study of w-local submodules of a finitely generated *Rad*-supplemented module.

Proposition 3.5. *Let M be an R-module with $Rad(M) \ll M$. The following statements are equivalent for a nonsmall submodule N of M:*

(i) *N is w-local*;

(ii) *N is a Rad-supplement of a maximal submodule of M.*

Proof. (i) $\Rightarrow$ (ii) Since N is not small in M, there exists a maximal submodule K of M such that $N + K = M$. Suppose that $N \cap K \not\subseteq Rad(N)$. Then $(N \cap K) + Rad(N) = N$. Therefore $(N \cap K) + Rad(N) + K = M$. Hence $Rad(N) + K = M$. Since $Rad(N) \subseteq Rad(M) \ll M$, we get $K = M$, a contradiction. So $N \cap K \subseteq Rad(N)$ and hence N is a *Rad*-supplement of K in M.

(ii) $\Rightarrow$ (i) This follows from [3, Lemma 3.3]. $\square$

Proposition 3.6. *Let M be a finitely generated Rad-supplemented R-module. Consider the following conditions*:

(i) *Every nonsmall w-local submodule of M is Rad-supplemented*;

(ii) *$M/P(M)$ is supplemented.*

Then (i) $\Rightarrow$ (ii).

Proof. Since M is *Rad*-supplemented, $M = \sum_{i=1}^{k} M_i$ is an irredundant sum of w-local submodules M_i $(1 \leq i \leq k)$. Then $M/P(M) = \sum_{i=1}^{k}(M_i + P(M))/P(M)$. Let $1 \leq i \leq k$. We have $(M_i + P(M))/P(M) \cong M_i/(M_i \cap P(M))$ and $P(M_i) \subseteq M_i \cap P(M)$. Thus $(M_i + P(M))/P(M)$ is a factor module of $M_i/P(M_i)$. By (i) and Proposition 3.3, $M_i/P(M_i)$ is supplemented and so is $(M_i + P(M))/P(M)$ (see [19, 41.2(3)]). Therefore $M/P(M)$ is supplemented by [19, 41.2(2)]. $\square$

Theorem 3.7. *Let M be a cyclic module over a commutative ring R. If M is Rad-supplemented, then every nonsmall w-local submodule of M is Rad-supplemented.*

Proof. We first note that there exists an ideal I of R such that $M \cong R/I$. It is clear that ${}_R(R/I)$ is Rad-supplemented if and only if ${}_{(R/I)}(R/I)$ is Rad-supplemented. So without loss of generality we can assume that $I = 0$ and $M = R$. Since ${}_RR$ is Rad-supplemented, $R/P(R)$ is semiperfect by [4, Theorem 6.5]. Let J be the Jacobson radical of R. Then the ring $R/J \cong (R/P(R))/(J/P(R))$ is semisimple. Let W be a nonsmall w-local submodule of ${}_RR$ and let U be a submodule of W. If $U \subseteq Rad(W)$, then it is easy to see that W is a Rad-supplement of U in M. Now assume that $U \nsubseteq Rad(W)$. Since W is nonsmall in ${}_RR$, there exists a maximal ideal K of R such that $W+K = R$. Clearly, we have $R/K \cong W/(W \cap K)$. So $W \cap K = Rad(W)$ is the maximal submodule of W. Since $U+Rad(W) = W$, we have $U+Rad(W)+K = R$. Therefore $U+K = R$ since $Rad(W) \ll R$. Note that the module ${}_RR$ is amply Rad-supplemented by [16, Theorem 3.7]. Then there is a submodule $L \leq K$ such that $U + L = R$ and $U \cap L \subseteq Rad(L)$. Now we have $W \cap L \subseteq W \cap K = Rad(W) = JW$ and $W \cap L \subseteq (Rad(R)) \cap L = Rad(L) = JL$ by [4, Corollary 4.2]. It follows that $W \cap L = JW \cap JL$. Since $W+L = R$, we have $Rad(W \cap L) = J(W \cap L) = W \cap L$ by [4, Lemma 6.3]. Moreover, we have $U + (W \cap L) = W$. Therefore $W \cap L$ is a Rad-supplement of U in M. This completes the proof. □

Remark 3.8. Let M be a cyclic reduced Rad-supplemented R-module. From Theorem 3.7 and Proposition 3.2, it follows that every nonsmall w-local submodule of M is a local module. This is not true, in general, if the module M is not reduced (see the following example).

Example 3.9. Let R be an integral domain with exactly two maximal ideals m_1 and m_2 such that both of m_1 and m_2 are idempotent (see [9, p. 293]). Then the Jacobson radical $Rad(R) = m_1 \cdot m_2$ of R is idempotent. So $Rad(R) = P(R)$. Therefore $R/P(R)$ is semiperfect (even semisimple), but R is not semiperfect since a semiperfect integral domain is local.

(1) Note that ${}_RR = m_1 + m_2$, $Rad(m_1) = Rad(R) \cdot m_1 = Rad(R)$ and $Rad(m_2) = Rad(R) \cdot m_2 = Rad(R)$. Moreover, we have $m_1/m_1 \cdot m_2 \cong R/m_2$ and $m_2/m_1 \cdot m_2 \cong R/m_1$. It follows that m_1 and m_2 are w-local R-modules. Since $R/P(R)$ is semiperfect, ${}_RR$ is Rad-supplemented by [4, Theorem 6.5]. Therefore the R-modules ${}_Rm_1$ and ${}_Rm_2$ are Rad-supplemented by Theorem 3.7.

(2) Since R is not semiperfect, it follows that at least one of the ${}_Rm_i$ $(i = 1, 2)$ is not local.

(3) Note that R is a commutative ring such that ${}_RR$ is Rad-supplemented, but ${}_RR$ is not supplemented (see [11, Corollary 4.42]).

The next example shows that, in general, the converse of Theorem 3.7 need not be true.

Example 3.10. Let K be a field and let $R = \prod_{i=1}^{\infty} K_i$ with $K_i = K$ for $i = 1, 2, \ldots$. It is well known that the ring R is von Neumann regular which is not semisimple. Thus R is a V-ring and hence $Rad(M) = 0$ for any R-module M. Let N be a w-local submodule of ${}_RR$. Since $Rad(N) = 0$, N is simple. So N is Rad-supplemented. On the other hand, the module ${}_RR$ is not Rad-supplemented. For if not, the ring $R/P(R)$ will be semiperfect by [4, Theorem 6.5]. But $P(R) = 0$. Then R is semiperfect and so R is semisimple, a contradiction.

We conclude this section by dealing with the question when a w-local Rad-supplemented module is supplemented. We begin with an example showing that a w-local Rad-supplemented module need not be supplemented, in general.

Example 3.11. Consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus L$ such that $\mathbb{Q}$ is the field of rational numbers and L is a local $\mathbb{Z}$-module. Let K be the maximal submodule of L. Clearly, M is a w-local module with the maximal submodule $\mathbb{Q} \oplus K$. Since $\mathbb{Q}$ and L are Rad-supplemented, so is M by [17, Proposition 2.5]. On the other hand, M is not supplemented since its factor module $M/L \cong \mathbb{Q}$ is not supplemented (see [5, Example 20.12 and Corollary 20.15]).

Proposition 3.12. *Let M be a w-local Rad-supplemented module over a commutative Noetherian ring. The following statements are equivalent*:
(i) *M is supplemented*;
(ii) *$P(M)$ is supplemented.*

Proof. By [12, Proposition 2.6] and Proposition 3.3. □

As in [21], a module M is called *minimax* if it has a finitely generated submodule N such that M/N is Artinian.

Recall that a module M is said to be weakly supplemented if for every submodule $N \leq M$, there exists a submodule $L \leq M$ such that $M = N + L$ and $N \cap L \ll M$ (see [5]).

The following theorem characterizes the class of commutative Noetherian rings over which w-local Rad-supplemented modules are supplemented.

Theorem 3.13. *The following are equivalent for a commutative Noetherian ring R*:
(i) *Every Rad-supplemented R-module is supplemented*;
(ii) *Every Rad-supplemented R-module is weakly supplemented*;
(iii) *Every radical R-module is supplemented*;
(iv) *Every w-local Rad-supplemented R-module is supplemented*;
(v) *R is Artinian.*

Proof. (i) ⇒ (ii) This is obvious.

(ii) ⇒ (v) Assume that the ring R is not Artinian. Then there exists a nonzero module K such that $Rad(K) = K$ by [10, Theorem 1]. Let $M = K^{(\mathbb{N})}$. Since $Rad(M) = M$, M is Rad-supplemented. Hence M is weakly supplemented by (ii). By [12, Proposition 3.4], M_m is an R_m-minimax module for all maximal ideals m of R. Since $K \neq 0$, there exists a maximal ideal m_0

of R such that $K_{m_0} \neq 0$. Thus $M_{m_0} = (K_{m_0})^{(\mathbb{N})}$ is a minimax R_{m_0}-module having an infinite decomposition, a contradiction (see [13, Theorem 2.1]).

(v) $\Rightarrow$ (iv) This follows from the fact that every R-module is supplemented by [11, Theorem 4.41].

(iv) $\Rightarrow$ (iii) Let M be a radical module and let S be any simple module. Clearly, the module $M \oplus S$ is w-local. Moreover, the module $M \oplus S$ is *Rad*-supplemented since it is a direct sum of two *Rad*-supplemented modules. By hypothesis, $M \oplus S$ is supplemented. Therefore M is supplemented by [5, Corollary 20.15].

(iii) $\Rightarrow$ (i) Let M be a *Rad*-supplemented module. By (iii), $P(M)$ is supplemented. Note that $M/P(M)$ is *Rad*-supplemented by [17, Proposition 2.6]. Then $M/P(M)$ is supplemented by [4, Proposition 7.3]. Now [12, Proposition 2.6] shows that M is supplemented. □

4. When finitely generated *Rad*-supplemented modules are amply *Rad*-supplemented?

It is shown in [14, Corollary 4.6] that any finitely generated supplemented R-module is amply supplemented. We begin this section with giving some examples of classes of rings over which every finitely generated *Rad*-supplemented module is amply supplemented. First we prove the following proposition.

Proposition 4.1. *Let M be a finitely generated Rad-supplemented R-module. Assume that every w-local submodule of M is local. Then M is amply supplemented.*

Proof. By [3, Corollary 3.8], $M = W_1 + \cdots + W_n$ is a sum of w-local submodules W_i $(1 \leq i \leq n)$. Since each W_i $(1 \leq i \leq n)$ is a local module, M is supplemented by [19, 41.6(1)]. So M is amply supplemented by [14, Corollary 4.6]. □

Recall that a ring R is said to be a *left max ring* if every left R-module has a maximal submodule, equivalently $Rad(M) \ll M$ for every left R-module M.

Example 4.2. (1) Let R be a commutative Noetherian ring or a commutative max ring. Let M be a finitely generated *Rad*-supplemented R-module. If W is a w-local submodule of M, then W is local since $Rad(W) \ll W$. Therefore M is amply supplemented by Proposition 4.1.

(2) It is well known that over a semiperfect ring, every finitely generated module is amply supplemented (see [11, Theorem 4.41]).

Recall that a module M is called *hereditary* if every submodule of M is projective.

Example 4.3. Let M be a finitely generated hereditary R-module. Let W be a w-local submodule of M. Since W is projective, W is local by [18, Proposition 4.11]. Hence M is amply supplemented by Proposition 4.1.

Recall that a module M is called *Rad*-$\oplus$-supplemented if every submodule has a *Rad*-supplement that is a direct summand of M.

Remark 4.4. Let M be a finitely generated *Rad*-$\oplus$-supplemented R-module. By [6, Corollary 2.20], M is a finite sum of local submodules. Thus M is amply supplemented by [19, 41.6(1)] and [14, Corollary 4.6].

Lemma 4.5 (See [16, Corollary 3.6])**.** *Let M be a finitely generated R-module such that every cyclic submodule of M is Rad-supplemented. Then M is amply Rad-supplemented.*

A module M is called a *multiplication* R-module provided for each submodule $N \leq M$, there exists an ideal I of R such that $N = IM$.

Proposition 4.6. *The following are equivalent for a commutative ring R:*

(i) *The ring $R/P(R)$ is semiperfect;*

(ii) *The module ${}_RR$ is Rad-supplemented;*

(iii) *The module ${}_RR$ is amply Rad-supplemented;*

(iv) *Every finitely generated R-module is Rad-supplemented;*

(v) *Every finitely generated R-module is amply Rad-supplemented;*

(vi) *R has a finitely generated faithful multiplication Rad-supplemented R-module M.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) By [4, Theorem 6.5] and Lemma 4.5.

(ii) $\Rightarrow$ (vi) It suffices to take $M = {}_RR$.

(vi) $\Rightarrow$ (ii) We first note that $Rad(M) = JM$, where J is the Jacobson radical of R by [1, Theorem 2.13]. Let A be an ideal of R. By hypothesis, there exists an ideal B of R such that $AM + BM = M$ and $AM \cap BM \subseteq Rad(BM)$. So $Rad(BM) = BM \cap JM$ by [4, Theorem 4.1]. Now [1, Lemma 2.6] shows that $AM \cap BM = (A \cap B)M$ and $BM \cap JM = (B \cap J)M$. According to [1, Theorem 2.9], we have $A + B = R$ and $A \cap B \subseteq J \ll R$. Therefore the module ${}_RR$ is weakly supplemented. By [5, Corollary 18.7], the ring R is semilocal. It follows that $Rad(BM) = JBM$ by [2, Corollary 15.18]. So $A \cap B \subseteq JB = Rad(B)$ by [1, Theorem 2.9] and [2, Corollary 15.18]. This implies that B is a *Rad*-supplement of A in ${}_RR$. Consequently, ${}_RR$ is *Rad*-supplemented. □

Proposition 4.7. *Let M be a finitely generated multiplication R-module. If M is Rad-supplemented, then M is amply Rad-supplemented.*

Proof. Assume that M is *Rad*-supplemented. Consider the ring

$$R' = R/Ann_R(M).$$

Note that M can be regarded as an R'-module and the submodules of M are the same whether it is regarded as an R-module or as an R'-module. Then M is a faithful multiplication *Rad*-supplemented R'-module. It follows from Proposition 4.6 that M is amply *Rad*-supplemented. □

Proposition 4.8. *Let $M = Rx$ be a cyclic R-module. Then:*

(1) *If M is Rad-supplemented, then M is amply Rad-supplemented.*

(2) *If M is reduced and Rad-supplemented, then M is amply supplemented.*

Proof. Note that $M \cong R/I$, where $I = Ann_R(x)$.

(1) By hypothesis, ${}_{(R/I)}(R/I)$ is Rad-supplemented. By Proposition 4.6, ${}_{(R/I)}(R/I)$ is amply Rad-supplemented. This shows that ${}_RM$ is amply Rad-supplemented.

(2) By Proposition 3.6 and Theorem 3.7, $M/P(M)$ is supplemented. Since M is reduced, M is supplemented. So M is amply supplemented by [14, Corollary 4.6]. □

Proposition 4.9. *Let $M = \sum_{i=1}^{n} Rx_i$ be an R-module which is a sum of cyclic Rad-supplemented submodules Rx_i $(1 \le i \le n)$. Let $I_i = Ann_R(x_i)$ $(1 \le i \le n)$. Then*:

(1) *If $I_i + I_j = R$ for all $1 \le i < j \le n$, then M is amply Rad-supplemented.*

(2) *If M is reduced, then M is amply supplemented.*

Proof. (1) Let $I = \bigcap_{i=1}^{n} I_i$. Note that M is an (R/I)-module and its submodules over R and over R/I are the same. Since $I_i + I_j = R$ for all $1 \le i < j \le n$, we have $R/I \cong \bigoplus_{i=1}^{n} R/I_i$ by the Chinese Remainder Theorem. Since the R-modules R/I_i $(1 \le i \le n)$ are Rad-supplemented, R/I is a Rad-supplemented R-module by [17, Proposition 2.5]. So R/I is a Rad-supplemented (R/I)-module. It follows from Proposition 4.6 that M is amply Rad-supplemented.

(2) By Proposition 4.8, [14, Corollary 4.6] and [5, 20.14]. □

Let a be an ideal of R. An R-module M is called a-local if $Ann_R(x) \subseteq m$ (where $x \in M$ and m is a maximal ideal of R) implies $a \subseteq m$ (see [20, p. 52]).

Proposition 4.10. *Let M be a finitely generated Rad-supplemented R-module. Then there exist maximal ideals $m_1, \ldots, m_n$ of R such that M is an a-local module, where $a = m_1 \cdots m_n$.*

Proof. By [3, Corollary 3.8], $M = \sum_{i=1}^{n} W_i$ is the sum of w-local submodules W_i $(1 \le i \le n)$. For each $1 \le i \le n$, let m_i be the maximal ideal of R such that W_i is m_i-w-local. By rearranging the submodules W_i $(1 \le i \le n)$, if necessary, we can suppose that there exists an integer k, with $1 \le k \le n$, such that m_1, …, m_k are the distinct members of the set $\{m_i : 1 \le i \le n\}$. Note that by Proposition 2.1, we have $m_i M_j = M_j$ for each $i \ne j$. Let $a = m_1 \cdots m_k$ and for each $1 \le i \le k$, let $M_i = \sum\{W_j : 1 \le j \le n \text{ and } m_j = m_i\}$. Therefore, for each $1 \le i \le k$, we have $m_i M_i = \sum\{m_i W_j : 1 \le j \le n \text{ and } m_j = m_i\} = \sum\{Rad(W_j) : 1 \le j \le n \text{ and } m_j = m_i\} \subseteq Rad(M)$ (see Proposition 2.1). It follows that $aM = \sum_{i=1}^{k} aM_i = \sum_{i=1}^{k} m_i M_i \subseteq Rad(M) \ll M$. By [20, Lemma 2.1], the module M is a-local. □

The next result shows that the study of finitely generated Rad-supplemented modules over commutative rings can be reduced to modules over semilocal rings.

Corollary 4.11. *Let M be a finitely generated Rad-supplemented R-module. Then the ring $R/Ann_R(M)$ is semilocal.*

Proof. Assume $M = \sum_{i=1}^{n} Rx_i$. Then $Ann_R(M) = \bigcap_{i=1}^{n} Ann_R(x_i)$. Let m be a maximal ideal of R such that $Ann_R(M) \subseteq m$. Then $Ann_R(x_i) \subseteq m$ for some $1 \leq i \leq n$. By Proposition 4.10, there exist maximal ideals $m_1, \dots, m_k$ of R, for some $k \geq 1$, such that M is $(m_1 m_2 \cdots m_k)$-local. So $m_1 m_2 \cdots m_k \subseteq m$. Therefore $m = m_j$ for some $1 \leq j \leq k$. It follows that the ring $R/Ann_R(M)$ has finitely many maximal ideals. This completes the proof. □

Remark 4.12. Assume that R is a commutative ring having infinitely many maximal ideals. Let M be a finitely generated *Rad*-supplemented R-module. By Proposition 4.10, there exist maximal ideals $m_1, \dots, m_n$ of R such that M is $(m_1 \cdots m_k)$-local. So for every nonzero element $x \in M$, we have $Ann_R(x) \neq 0$.

A commutative ring R is called a *Gelfand* ring (or a *pm ring*) if each prime ideal is contained in exactly one maximal ideal (see [20, p. 49]).

Corollary 4.13. *Let M be a finitely generated Rad-supplemented R-module. Then*:

(1) *If the ring R/p is local for every prime ideal p of R with $Ann_R(M) \subseteq p$, then M is amply supplemented.*

(2) *If $Rad(R/Ann_R(M))$ is idempotent, then M is amply Rad-supplemented.*

(3) *If m is a maximal ideal of R such that M is a sum of finitely many m-w-local submodules, then M is amply supplemented.*

Proof. (1) By Corollary 4.11, the ring $R/Ann_R(M)$ is semilocal. By hypothesis, $R/Ann_R(M)$ is Gelfand. Thus $R/Ann_R(M)$ is semiperfect by [20, Folgerung p. 50]. So M is amply supplemented by [11, Theorem 4.41].

(2) By Corollary 4.11, the ring $R/Ann_R(M)$ is semilocal. By [3, Proposition 2.1], the module ${}_{(R/Ann_R(M))}(R/Ann_R(M))$ is *Rad*-supplemented. Therefore M is amply *Rad*-supplemented by Proposition 4.6.

(3) By the proof of Proposition 4.10, the module M is m-local. Thus m is the only maximal ideal containing $Ann_R(M)$ (see the proof of Corollary 4.11). It follows that the ring $R/Ann_R(M)$ is local. Therefore M is amply supplemented (see [11, Theorem 4.41]). □

Proposition 4.14. *Let R be a semilocal ring with $J = Rad(R)$ and $A = P(R)$. If M is a reduced R-module, then M is an (R/A)-module and its submodules over R and over R/A are the same.*

Proof. Since R is semilocal, we have $JA = A$ and $Rad(AM) = J(AM) = (JA)M = AM$ by [2, Corollary 15.18]. But M is reduced. So $AM = 0$. That is, $A \subseteq Ann_R(M)$. This completes the proof. □

Remark 4.15. Combining [4, Proposition 4.8], Corollary 4.11 and Proposition 4.14, we conclude that the study of finitely generated *Rad*-supplemented modules over commutative rings can be restricted to the class of finitely generated reduced modules over semilocal reduced rings.

5. Amply *Rad*-supplemented modules

In this section we consider modules over an arbitrary ring R (not necessarily commutative). We conclude this paper with Theorem 5.2 which characterizes finitely generated amply *Rad*-supplemented modules. We shall require the following lemma which is taken from [15, Lemma 3.5]. Note that [16, Theorem 3.5] showed that the statements (i) and (ii) of Theorem 5.2 are equivalent.

Lemma 5.1. *Let M be a finitely generated R-module. Let $\mathcal{C}$ be a family of submodules of M. Suppose that M has a submodule N such that $M \neq N + A$ for every $A \in \mathcal{C}$. Then M has a submodule U such that U is maximal in the set of submodules $\Omega = \{L \leq M \mid N \subseteq L$ and $M \neq L + A$ for every $A \in \mathcal{C}\}$.*

Proof. Let $M = Rm_1 + \cdots + Rm_k$. By hypothesis, $N \in \Omega$. let $(K_\lambda)_{\lambda\in\Lambda}$ be a chain in Ω. Let $K = \bigcup_{\lambda\in\Lambda} K_\lambda$. Suppose that $K \notin \Omega$. Then $M = K + B$ for some element $B \in \mathcal{C}$. So for each $1 \leq i \leq k$, there exist elements $x_i \in K$ and $y_i \in B$ such that $m_i = x_i + y_i$. It follows that $M = [\sum_{i=1}^{k} Rx_i] + B$. On the other hand, for each $1 \leq i \leq k$, there is $\lambda_i \in \Lambda$ such that $x_i \in K_{\lambda_i}$. Since $(K_\lambda)_{\lambda\in\Lambda}$ is a chain, there exists an integer j, with $1 \leq j \leq k$, such that $K_{\lambda_i} \leq K_{\lambda_j}$ for each $1 \leq i \leq k$. Therefore $M = K_{\lambda_j} + B$, a contradiction. This shows that $K \in \Omega$. It follows by Zorn's Lemma that Ω possesses a maximal member U. □

Theorem 5.2. *Let M be a finitely generated R-module. The following statements are equivalent*:

(i) *M is amply Rad-supplemented*;

(ii) *Every maximal submodule of M has ample Rad-supplements in M*;

(iii) *For every submodules N and L of M such that $M = N+L$ and $N \neq M$, we have $M = N + W_1 + \cdots + W_n$, where n is a positive integer and each W_i is a w-local submodule of L.*

Proof. (i) ⇒ (ii) This is clear.

(ii) ⇒ (iii) Let L and N be submodules of M such that $M = N + L$ and $N \neq M$. Let $\mathcal{S}$ be the collection of submodules $X \leq M$ such that $X \subseteq L$ and $X = 0$ or X is a finite sum of w-local submodules. Suppose that $M \neq N + A$ for every $A \in \mathcal{S}$. By Lemma 5.1, there is a submodule U of M such that $N \subseteq U$ and U is maximal with respect to the property $M \neq U + A$ for every $A \in \mathcal{S}$. Since M is finitely generated and $U \neq M$, there is a maximal submodule K of M such that $U \subseteq K$. Thus $K + L = M$. By (ii), there exists a submodule E of L such that E is a *Rad*-supplement of K in M. By [3, Lemma 3.3], E is a w-local submodule of L. Note that $U \neq U + E$, since otherwise we have $E \subseteq U \subseteq K$ and $K = K + E = M$. It follows that $M = U + E + F$ for some element $F \in \mathcal{S}$. But $E + F \in \mathcal{S}$, a contradiction. The result follows.

(iii) ⇒ (i) By [6, Proposition 2.14]. □

Remark 5.3. It is easily seen that Lemma 2.6 and Propositions 2.7, 3.2, 3.5 and 4.14 remain true if the ring R is not commutative.

Acknowledgments. Some of this work was done during a visit to the Department of Mathematics, İzmir Institute of Technology, Turkey in August 2012 by the second author who wishes to thank the first author and his family for their hospitality. The authors would like to thank the referee for carefully reading this paper and for the valuable suggestions and comments.

References

[1] R. Ameri, *On the prime submodules of multiplication modules*, Int. J. Math. Math. Sci. **2003** (2003), no. 27, 1715–1724.
[2] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, New-York, Springer-Verlag, 1974.
[3] E. Büyükaşik and C. Lomp, *On a recent generalization of semiperfect rings*, Bull. Aust. Math. Soc. **78** (2008), no. 2, 317–325.
[4] E. Büyükaşik, E. Mermut, and S. Özdemir, *Rad-supplemented modules*, Rend. Semin. Mat. Univ. Padova **124** (2010), 157–177.
[5] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting Modules*, Supplements and Projectivity in Module Theory. Basel: Frontiers in Mathematics, Birkhäuser Verlag, 2006.
[6] Ş. Ecevit, M. T. Koşan, and R. Tribak. *Rad-⊕-supplemented modules and cofinitely Rad-⊕-supplemented modules*, Algebra Colloq. **19** (2012), no. 4, 637–648.
[7] A. I. Generalov, *The w-cohigh purity in a category of modules*, Math. Notes **33** (1983), 402–408; translation from Mat. Zametki **33** (1983), no. 5, 785–796.
[8] V. N. Gerasimov and I. I. Sakhaev. *A counterexample to two conjectures on projective and flat modules*, Sibirsk. Mat. Zh. **25** (1984), no. 6, 31–35.
[9] R. W. Gilmer, *Multiplicative Ideal Theory*, New York, Marcel Dekker, 1972.
[10] R. M. Hamsher, *Commutative Noetherian rings over which every module has a maximal submodule*, Proc. Amer. Math. Soc. **17** (1966), 1471–1472.
[11] S. H. Mohamed and B. J. Müller, *Continuous and Discrete Modules*, Cambridge, UK: London Math. Soc. Lecture Note Ser 147, Cambridge University Press, 1990.
[12] P. Rudlof, *On the structure of couniform and complemented modules*, J. Pure Appl. Algebra **74** (1991), no. 3, 281–305.
[13] ———, *On minimax and related modules*, Canad. J. Math. **44** (1992), no. 1, 154–166.
[14] P. F. Smith, *Finitely generated supplemented modules are amply supplemented*, Arab. J. Sci. Eng. Sect. C Theme Issues **25** (2000), no. 2, 69–79.
[15] R. Tribak, *On δ-local modules and amply δ-supplemented modules*, J. Algebra Appl. **12** (2013), no. 2, 1250144, 14 pages.
[16] E. Türkmen and A. Pancar, *On cofinitely Rad-supplemented modules*, Int. J. Pure Appl. Math. **53** (2009), no. 2, 153–162.
[17] Y. Wang and N. Ding, *Generalized supplemented modules*, Taiwanese J. Math. **10** (2006), no. 6, 1589–1601.
[18] R. Ware, *Endomorphism rings of projective modules*, Trans. Amer. Math. Soc. **155** (1971), no. 1, 233–256.
[19] R. Wisbauer, *Foundations of Module and Ring Theory*, Philadelphia: Gordon and Breach Science Publishers, 1991.
[20] H. Zöschinger, *Gelfandringe und koabgeschlossene Untermoduln*, Bayer. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. **3** (1982), 43–70.
[21] ———, *Minimax-moduln*, J. Algebra **102** (1986), no. 1, 1–32.

Engin Büyükaşik
İzmir Institute of Technology
Department of Mathematics
35430, Urla, İzmir-Turkey
E-mail address: enginbuyukasik@iyte.edu.tr

Rachid Tribak
Centre Régional des Métiers de l'Education
et de la Formation (CRMEF)-Tanger
Avenue My Abdelaziz, BP 3117 Souani
Tanger, Morocco
E-mail address: tribak12@yahoo.com