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Chiral resonant solitons in Chern–Simons theory and Broer–Kaup type new hydrodynamic systems

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ABSTRACT

New Broer–Kaup type systems of hydrodynamic equations are derived from the derivative reaction–diffusion systems arising in SL(2,R) Kaup–Newell hierarchy, represented in the non-Madelung hydrodynamic form. A relation with the problem of chiral solitons in quantum potential as a dimensional reduction of 2 + 1 dimensional Chern–Simons theory for anyons is shown. By the Hirota bilinear method, soliton solutions are constructed and the resonant character of soliton interaction is found.

1. Introduction

Recently, a modification of the nonlinear Schrödinger (NLS) equation by a quantum potential has been studied in several problems. As a low dimensional gravity model, the Jackiw–Teitelboim model, in a special non-covariant gauge, [1,2]. In plasma physics, as nonlinear equations governing the transmission of uni-axial waves in a cold collisionless plasma subject to a transverse magnetic field [3]. The capillarity model [12], and information theory with Fisher measure of maximal uncertainty, [8]. It was studied recently as integrable deformation of dispersion for generic envelope equation of nonlinear Schrodinger type [7]. Subsequently, the influence of this potential on anyons in 2 + 1 dimensions has been studied [4], and the Abelian Chern–Simons gauge field interacting with NLS has been represented as a planar Madelung fluid [6], where the Chern–Simons Gauss law has the simple physical meaning of creation of the local vorticity for the flow. For the static flow when the velocity of the center-of-mass motion is equal to the quantum velocity, the fluid admits an N-vortex solution. It turns out that in this theory the Chern–Simons coupling constant and the quantum potential strength are quantized.

Reduction of problem to 1 + 1 dimensions leads to JNLS and some versions of Derivative NLS in quantum potential. Hence the chiral solitons appear as solutions of the derivative NLS with quantum potential [5]. The last one by the Madelung transform is represented as the derivative Reaction–Diffusion (DRD) system, arising in SL (2,R) Kaup–Newell hierarchy, and giving rise to the resonant soliton phenomena [7].

In the present paper by using new, the non-Madelung representation, we formulate the problem in terms of novel hydrodynamic systems of the Broer–Kaup type. Then by Hirota's bilinear method we construct chiral solitons for the system and show the resonance character of their interaction.

2. Dimensional reduction of Chern–Simons theory

We consider the Chern–Simons gauged nonlinear Schrödinger model (the Jackiw–Pi model) with nonlinear quantum potential term of strength s [4]:

$$L = \frac{\kappa}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \frac{i}{2}(\bar{\psi}D_0\psi - \psi\overline{D}_0\bar{\psi}) - \overline{\mathbf{D}}\bar{\psi}\mathbf{D}\psi + s\nabla|\psi|\nabla|\psi| + V(\bar{\psi}\psi), \quad (1)$$

where $D_\mu = \partial_\mu + ieA_\mu$ $(\mu = 0,1,2)$. Classical equations of motion are

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$$iD_0\psi + \mathbf{D}^2\psi + V'\psi = s\frac{\Delta|\psi|}{|\psi|}\psi, \quad (2)$$

$$\partial_1 A_2 - \partial_2 A_1 = \frac{e}{\kappa}\bar{\psi}\psi, \quad (3)$$

$$\partial_0 A_j - \partial_j A_0 = -\frac{e}{\kappa} i\epsilon_{jk}(\bar{\psi}D_k\psi - \psi\overline{D}_k\bar{\psi}) \quad (j,k = 1,2). \quad (4)$$

We consider dimensional reduction of this model when all fields are independent of x_2 space variable, so that $\partial_2 = 0$. Then in terms of $\widetilde{A}_0 \equiv A_0 + eA_2^2$, and $B \equiv A_2$, we obtain

$$i(\partial_0 + ieA_0)\psi + (\partial_1 + ieA_1)^2\psi + V'\psi = s\frac{\partial_1^2|\psi|}{|\psi|}\psi, \quad (5)$$

$$\partial_0 A_1 - \partial_1 A_0 = 0, \quad (6)$$

$$\partial_1 B = \frac{e}{\kappa}\bar{\psi}\psi, \quad (7)$$

$$\partial_0 B = i\frac{e}{\kappa}[\bar{\psi}(\partial_1 + ieA_1)\psi - \psi(\partial_1 - ieA_1)\bar{\psi}]. \quad (8)$$

Here and below we skip the tilde sign for A_0. The last two equations are compatible due to (5) and the corresponding continuity equation

$$\partial_0(\bar{\psi}\psi) = i\partial_1[\bar{\psi}(\partial_1 + ieA_1)\psi - \psi(\partial_1 - ieA_1)\bar{\psi}] \quad (9)$$

implies compatibility of Eqs. (7) and (8). Integrating these equations we find B in terms of density $\bar{\psi}\psi$

$$B = \frac{e}{\kappa}\int^x \bar{\psi}\psi dx'. \quad (10)$$

From another side, flatness of connection (6) implies $A_0 = \partial_0\phi$, $A_1 = \partial_1\phi$, and these potentials can be removed by the gauge transformation, $\psi = e^{-e\phi}\Psi$. As a result we obtain the Schrödinger equation with self-interacting nonlinear potential $V(\bar{\psi}\psi)$, and the quantum potential

$$i\partial_0\Psi + \partial_1^2\Psi + V'\Psi = s\frac{\partial_1^2|\Psi|}{|\Psi|}\Psi. \quad (11)$$

2.1. Madelung representation and RNLS

If we substitute the Madelung Ansatz $\Psi = \sqrt{\rho}e^{-iS}$ to wave function in (11) then we get the coupled system

$$\partial_0 S - (\partial_1 S)^2 + V'(\rho) + (1-s)\frac{\partial_1^2\sqrt{\rho}}{\sqrt{\rho}} = 0, \quad (12)$$

$$\partial_0\rho - \partial_1(2\rho\partial_1 S) = 0. \quad (13)$$

For velocity field $v = -2\partial_1 S$ it implies the hydrodynamical system

$$\partial_0 v + v\partial_1 v = 2\partial_1\left(V'(\rho) + (1-s)\frac{\partial_1^2\sqrt{\rho}}{\sqrt{\rho}}\right), \quad (14)$$

$$\partial_0\rho + \partial_1(\rho v) = 0, \quad (15)$$

which is the Madelung fluid representation of (11).

First we consider the under-critical case, when the strength of the quantum potential $s < 1$. Then introducing rescaled the time and the phase $\tilde{t} = t\sqrt{1-s}$, $\widetilde{S} = \frac{S}{\sqrt{1-s}}$, for new wave function $\widetilde{\Psi} = \sqrt{\rho}e^{-i\widetilde{S}}$ we get

$$i\partial_{\tilde{0}}\widetilde{\Psi} + \partial_1^2\widetilde{\Psi} + \frac{V'}{1-s}\widetilde{\Psi} = 0. \quad (16)$$

In the over-critical case when $s > 1$, we can't reduce (11)–(16). However, if we introduce pair of real functions

$$e^{(+)}(x,t) = \sqrt{\rho}e^{\widetilde{S}}, \quad e^{(-)}(x,t) = \sqrt{\rho}e^{-\widetilde{S}} \quad (17)$$

instead of one complex wave function, then we get the time reversal pair of reaction–diffusion equations

$$\partial_{\tilde{0}}e^{(+)} + \partial_1^2 e^{(+)} - \frac{V'}{s-1}e^{(+)} = 0, \quad (18)$$

$$-\partial_{\tilde{0}}e^{(-)} + \partial_1^2 e^{(-)} - \frac{V'}{s-1}e^{(-)} = 0, \quad (19)$$

where $\tilde{t} = t\sqrt{s-1}$, $\widetilde{S} = \frac{S}{\sqrt{s-1}}$, $\rho = e^{(+)}e^{(-)}$, $V = V(e^{(+)}e^{(-)})$.

If interaction between material particles is the delta function pair form then potential $V(\rho) = g\rho^2/2$ and Eq. (11) becomes

$$i\partial_0\Psi + \partial_1^2\Psi + g|\Psi|^2\Psi = s\frac{\partial_1^2|\Psi|}{|\Psi|}\Psi. \quad (20)$$

We called this equation the resonant nonlinear Schrödinger equation (RNLS). It appears in the study of low-dimensional gravity model on a line, the Jackiw–Teitelboim model [1,2]. In plasma physics, it governs the transmission of uni-axial waves in a cold collisionless plasma subject to a transverse magnetic field [3]. As an integrable capillarity model in [12]. It was studied recently as an integrable deformation of dispersion for generic envelope equation of nonlinear Schrodinger type [7].

For under-critical case $s < 1$ it reduces to the standard NLS Eq. (16), and is integrable model. In the case $s = 0$ it describes dispersionless limit of NLS which has been studied intensively as descriptive of shock waves in nonlinear optics. However for the over-critical case $s > 1$ it can not be reduced to NLS equation, but to the couple of cubic reaction–diffusion Eqs. (18), (19). This system is integrable as SL(2,R) connection from the second flow of AKNS hierarchy, and admits infinite set of integrals of motions. Moreover in the last case new resonance phenomena for envelope solitons take place [1], which are absent for NLS. In the next section we will discuss another reduction of Chern–Simons theory with dynamical field B, and show that in this case resonant versions of the JNLS and DNLS equations appear.

3. Dynamical BF theory

To do the gauge field component B in Section 2 to be dynamical, following [11] we introduce the corresponding kinetic term so that

$$L = \kappa B(\partial_0 A_1 - \partial_1 A_0) + \theta\partial_0 B\partial_1 B + \frac{i}{2}(\bar{\psi}(\partial_0 + ieA_0)\psi - \psi(\partial_0 - ieA_0)\bar{\psi}) - (\partial_1 - ieA_1)\bar{\psi}(\partial_1 + ieA_1)\psi + s\partial_1|\psi|\partial_1|\psi| + V(\bar{\psi}\psi). \quad (21)$$

Then equations of motion are

$$i(\partial_0 + ieA_0)\psi + (\partial_1 + ieA_1)^2\psi + V'\psi = s\frac{\partial_1^2|\psi|}{|\psi|}\psi, \quad (22)$$

$$\partial_0 A_1 - \partial_1 A_0 = \frac{2\theta}{\kappa}\partial_0\partial_1 B, \quad (23)$$

$$\partial_1 B = \frac{e}{\kappa}\rho, \quad (24)$$

$$\partial_0 B = -\frac{e}{\kappa}J, \quad (25)$$

where the particle and the momentum density are

$$\rho = \bar{\psi}\psi, \quad J = \frac{1}{i}[\bar{\psi}(\partial_1 + ieA_1)\psi - \psi(\partial_1 - ieA_1)\bar{\psi}] \\ = j + 2eA_1\rho \tag{26}$$

and $j = -i[\bar{\psi}\partial_1\psi - \psi\partial_1\bar{\psi}]$. Eq. (22) implies the conservation law

$$\partial_0\rho + \partial_1 J = 0. \tag{27}$$

This conservation law is the compatibility condition for the system (24),(25) and allows us to write

$$\partial_0\partial_1 B = \frac{e}{\kappa}[\alpha\partial_0\rho - (1-\alpha)\partial_1 J], \tag{28}$$

where α is an arbitrary real constant. Substituting (28) to (23) and combining terms under the same derivatives we have

$$\partial_0\left(A_1 - \frac{2\theta e}{\kappa^2}\alpha\rho\right) - \partial_1\left(A_0 - \frac{2\theta e}{\kappa^2}(1-\alpha)J\right) = 0. \tag{29}$$

The system (22)–(25) is invariant under the local $U(1)$ gauge transformations

$$\psi \to \psi' = e^{-ie\phi(x,t)}\psi, \quad A_\mu \to A'_\mu = A_\mu + \partial_\mu\phi. \tag{30}$$

Then solving (29) we have

$$A_1 = \frac{2\theta e}{\kappa^2}\alpha\rho + \partial_1\phi, \quad A_0 = \frac{2\theta e}{\kappa^2}(1-\alpha)J + \partial_0\phi \tag{31}$$

and for the gauge invariant field $\Psi = e^{i\,e\phi}\psi$ it gives

$$i\left(\partial_0 + i\frac{2\theta e^2}{\kappa^2}(1-\alpha)J\right)\Psi + \left(\partial_1 + i\frac{2\theta e^2}{\kappa^2}\alpha\rho\right)^2\Psi + V'\Psi \\ = s\frac{\partial_1^2|\psi|}{|\Psi|}\Psi, \tag{32}$$

where $J = j + \frac{4\theta e^2}{\kappa^2}\alpha\rho^2, \quad j = -i[\overline{\Psi}\Psi_x - \Psi\overline{\Psi}_x], \quad \rho = \overline{\Psi}\Psi$. Finally we have

$$i\Psi_t + \Psi_{xx} + i\frac{2\theta e^2}{\kappa^2}\left[(2\alpha+1)|\Psi|^2\Psi_x + (2\alpha-1)\Psi^2\overline{\Psi}_x\right] \\ + 4\frac{\theta^2 e^4}{\kappa^4}\alpha(\alpha-2)|\Psi|^4\Psi + V'\Psi = s\frac{|\Psi|_{xx}}{|\Psi|}\Psi, \tag{33}$$

where partial differentiation notations are evident. The remaining gauge transformation for this equation is just the global $U(1)$ transformation: $\Psi \to e^{i\lambda}\Psi$, $\lambda = const$.

3.1. Reductions of general RDNLS

The behavior of Eq. (33) depends on value of parameter s. If we replace $\Psi = e^{R-iS}$ then we have couple of equations

$$R_t - (S_{xx} + 2R_xS_x) + \frac{2\theta e^2}{\kappa^2}4\alpha R_x e^{2R} = 0,$$

$$S_t - S_x^2 + (1-s)(R_{xx} + R_x^2) + \frac{2\theta e^2}{\kappa^2}2S_x e^{2R} + \frac{4\theta^2 e^4}{\kappa^4}\alpha(\alpha-2)e^{4R} + V' = 0$$

determining the Madelung fluid representation

$$\rho_t + \left(\rho v + \frac{2\theta e^2}{\kappa^2}2\alpha\rho^2\right)_x = 0, \tag{34}$$

$$v_t + vv_x = 2\left[(1-s)\frac{\sqrt{\rho}_{xx}}{\sqrt{\rho}} - \frac{2\theta e^2}{\kappa^2}\rho v + \left(\frac{2\theta e^2}{\kappa^2}\right)^2\alpha(\alpha-2)\rho^2 + V'\right]_x. \tag{35}$$

For $s < 1$ for redefined variables $t\sqrt{1-s} \equiv \tilde{t}$, $S/\sqrt{1-s} \equiv \widetilde{S}$, $\widetilde{\Psi} \equiv e^{R-i\widetilde{S}}$, we have

$$i\widetilde{\Psi}_{\tilde{t}} + \widetilde{\Psi}_{xx} + i\frac{2\theta e^2}{\kappa^2\sqrt{1-s}}\left[(2\alpha+1)|\widetilde{\Psi}|^2\widetilde{\Psi}_x + (2\alpha-1)\widetilde{\Psi}^2\overline{\widetilde{\Psi}}_x\right] \\ + 4\frac{\theta^2 e^4}{\kappa^4(1-s)}\alpha(\alpha-2)|\widetilde{\Psi}|^4\widetilde{\Psi} + \frac{V'}{1-s}\widetilde{\Psi} = 0. \tag{36}$$

Similar to the Chern–Simons 2 + 1 dimensional case [4] we have effective result of the quantum potential in the rescaling of the statistical parametr $\kappa^2 \to \kappa^2\sqrt{1-s}$, but in contrast no quantization of this parameter now appears. Transformation between wave functions has nonlinear form

$$\Psi(x,t) = |\widetilde{\Psi}|\left(\frac{\widetilde{\Psi}}{|\widetilde{\Psi}|}\right)^{\sqrt{1-s}}(x, t\sqrt{1-s}). \tag{37}$$

For $s > 1$ it is impossible to reduce the system to the Schrödinger type form. However for redefined parameters $t\sqrt{s-1} \equiv \tilde{t}$, $S/\sqrt{s-1} \equiv \widetilde{S}$ and two real functions $E^+ = e^{R+S}$, $E^- = e^{R-S}$ we get

$$\mp E^\pm_{\tilde{t}} + E^\pm_{xx} \mp \frac{2\theta e^2}{\kappa^2\sqrt{s-1}}\left[(2\alpha+1)E^+E^-E^\pm_x + (2\alpha-1){E^\pm}^2E^\mp_x\right] \\ - 4\frac{\theta^2 e^4}{\kappa^4(s-1)}\alpha(\alpha-2)(E^+E^-)^2E^\pm - \frac{V'}{s-1}\widetilde{E}^\pm = 0. \tag{38}$$

3.2. Gauge transformation

We notice that in the gauge potential representation (31), the gauge function $\phi = \phi^{(\alpha)}$ depends on α:

$$A_1 = \frac{2\theta e}{\kappa^2}\alpha\rho + \partial_1\phi^{(\alpha)}, \quad A_0 = \frac{2\theta e}{\kappa^2}(1-\alpha)J + \partial_0\phi^{(\alpha)}. \tag{39}$$

Comparison with the case $\alpha = 0$

$$A_1 = \partial_1\phi^{(0)}, \quad A_0 = \frac{2\theta e}{\kappa^2}J + \partial_0\phi^{(0)} \tag{40}$$

gives relations

$$\partial_1(\phi^{(0)} - \phi^{(\alpha)}) = \frac{2\theta e}{\kappa^2}\alpha\rho, \quad \partial_0(\phi^{(0)} - \phi^{(\alpha)}) = -\frac{2\theta e}{\kappa^2}\alpha J. \tag{41}$$

Compatibility of this system is ensured by the continuity Eq. (27). Then corresponding gauge transformed wave functions $\Psi^{(\alpha)}$ and $\Psi^{(0)}$

$$\psi = e^{-ie\phi^{(\alpha)}}\Psi^{(\alpha)} = e^{-ie\phi^{(0)}}\Psi^{(0)} \tag{42}$$

are related by

$$\Psi^{(\alpha)} = e^{-ie(\phi^{(0)} - \phi^{(\alpha)})}\Psi^{(0)}. \tag{43}$$

Integrating (41) and substituting to (43) we have gauge transformation between Eq. (33) and the same equation with $\alpha = 0$:

$$\Psi^{(\alpha)} = \exp\left(-i\frac{2\theta e^2}{\kappa^2}\alpha\int^x \rho\,dx'\right)\Psi^{(0)}. \tag{44}$$

From this relation we can connect two samples of Eq. (33) with different constants α and β

$$\Psi^{(\alpha)} = \exp\left(-i\frac{2\theta e^2}{\kappa^2}(\alpha-\beta)\int^x \rho dx'\right)\Psi^{(\beta)}. \quad (45)$$

Indeed one can check easily from

$$\overline{\Psi}^{(\alpha)}\Psi^{(\alpha)} = \overline{\Psi}^{(\beta)}\Psi^{(\beta)}, \overline{\Psi}^{(\alpha)}(\partial_1 + i\nu\alpha\rho)\Psi^{(\alpha)} = \overline{\Psi}^{(\beta)}(\partial_1 + i\nu\alpha\rho)\Psi^{(\beta)} \quad (46)$$

that $\rho^{(\alpha)} = \rho^{(\beta)}$, $J^{(\alpha)} = J^{(\beta)}$, and

$$(\partial_1 + i\nu\alpha\rho)\Psi^{(\alpha)} = e^{-i\nu(\alpha-\beta)\int^x \rho dx'}(\partial_1 + i\nu\beta\rho)\Psi^{(\beta)}, \quad (47)$$

$$(\partial_0 + i\nu(1-\alpha)J)\Psi^{(\alpha)} = e^{-i\nu(\alpha-\beta)\int^x \rho dx'}(\partial_0 + i\nu(1-\beta)J)\Psi^{(\beta)}, \quad (48)$$

where $\nu \equiv \frac{2\theta e^2}{\kappa^2}$

The gauge transformation (45) for the Madelung representation implies

$$S^{(\alpha)} - S^{(\beta)} = \nu(\alpha-\beta)\int^x \rho dx' + 2\pi n, \quad R^{(\alpha)} = R^{(\beta)}. \quad (49)$$

For $s < 1$ it gives $U(1)$ gauge transformation for (36) in the form

$$\widetilde{\Psi}^{(\alpha)} = \widetilde{\Psi}^{(\beta)} e^{-i\frac{\nu}{\sqrt{1-s}}(\alpha-\beta)\int^x \rho dx'} e^{-i\frac{2\pi n}{\sqrt{1-s}}} \quad (50)$$

The last multiplier can be absorbed by the global phase transformation on Ψ.

For $s > 1$ the above $U(1)$ gauge transformation give rise to the local $SO(1,1)$ scale transformation (the Weyl transformation) for Eq. (38)

$$E^{\pm(\alpha)} = E^{\pm(\beta)} e^{\pm\frac{\nu}{\sqrt{s-1}}(\alpha-\beta)\int^x \rho dx'} e^{\pm\frac{2\pi n}{\sqrt{s-1}}}. \quad (51)$$

4. Integrable DRD systems

It was shown above that the one dimensional problem of anyons in quantum potential with a specific form of the three-body interaction, can be reduced to the general resonant DNLS equation.

4.1. General resonant DNLS

This equation

$$i\Psi_{\tilde{t}} + \Psi_{xx} + i\tilde{\nu}\left[(2\alpha+1)|\Psi|^2\Psi_x + (2\alpha-1)\Psi^2\overline{\Psi}_x\right] + 4\tilde{\nu}^2\left(\alpha-\frac{1}{2}\right)\left(\alpha-\frac{3}{2}\right)|\Psi|^4\Psi = s\frac{|\Psi|_{xx}}{|\Psi|}\Psi \quad (52)$$

is integrable for any values of parameter α.

4.2. The resonant case

For special case $s > 1$, by the Madelung transformation $\Psi = e^{R-iS}$ and introduction of two new real functions $E^+ = e^{R+S}$, $E^- = e^{R-S}$ we get the general DRD system

$$\mp E_t^\pm + E_{xx}^\pm \mp \frac{2\theta e^2}{\kappa^2\sqrt{s-1}}\left[(2\alpha+1)E^+E^-E_x^\pm + (2\alpha-1)E^{\pm 2}E_x^\mp\right] - 4\frac{\theta^2 e^4}{\kappa^4(s-1)}\left(\alpha-\frac{1}{2}\right)(\alpha-\frac{3}{2})(E^+E^-)^2E^\pm = 0, \quad (53)$$

where θ is the statistical parameter.

This system has particular reductions.

1. DRD-I ($\alpha = 3/2$)

$$-E_t^+ + E_{xx}^+ - 2\nu(E^+E^-E^+)_x = 0, \quad (54)$$
$$+E_t^- + E_{xx}^- + 2\nu(E^+E^-E^-)_x = 0. \quad (55)$$

2. DRD-II ($\alpha = 1/2$)

$$-E_t^+ + E_{xx}^+ - 2\nu E^+E^-E_x^+ = 0, \quad (56)$$
$$+E_t^- + E_{xx}^- + 2\nu E^+E^-E_x^- = 0. \quad (57)$$

3. DRD-III ($\alpha = -1/2$)

$$-E_t^+ + E_{xx}^+ + 2\nu E^{+2}E_x^- - 2\nu^2(E^+E^-)^2E^+ = 0, \quad (58)$$
$$+E_t^- + E_{xx}^- - 2\nu E^{-2}E_x^+ - 2\nu^2(E^+E^-)^2E^- = 0. \quad (59)$$

4. JRD ($\alpha = 0$)

$$\mp E_t^\pm + E_{xx}^\pm - \nu[E_x^+E^- - E^+E_x^-]E^\pm - \frac{3\nu^2}{4}(E^+E^-)^2E^\pm = 0. \quad (60)$$

5. Resonant hydrodynamic systems

To find hydrodynamic form of the above equations we introduce velocity variables according to the Cole-Hopf transformation

$$v^+ = (\ln E^+)_x, \quad v^- = (\ln E^-)_x \quad (61)$$

and density

$$\rho = E^+E^-. \quad (62)$$

Then by identity

$$\rho_x = \rho v^+ + \rho v^-, \quad (63)$$

we can rewrite the DRD system in a closed form for only one of the couples of hydrodynamic variables (ρ, v^+) or (ρ, v^-).

5.1. Hydrodynamic form for DRD-I

For DRD-I case it gives the new hydrodynamic system

$$v_t^+ = \left[v_x^+ + (v^+)^2 - 2\nu(\rho_x + \rho v^+)\right]_x,$$
$$\rho_t + \rho_{xx} = [2\rho v^+ - 3\nu\rho^2]_x. \quad (64)$$

5.2. Hydrodynamic form for DRD-II

For DRD-II case first we get the coupled heat equation with transport

$$-E_t^+ + E_{xx}^+ - 2\nu\rho E_x^+ = 0,$$
$$\rho_t + \rho_{xx} = (2\rho(\ln E^+)_x - \nu\rho^2)_x. \quad (65)$$

Then the hydrodynamic form for this system is

$$v_t^+ = \left[v_x^+ + (v^+)^2 - 2\nu\rho v^+\right]_x,$$
$$\rho_t + \rho_{xx} = [2\rho v^+ - \nu\rho^2]_x. \quad (66)$$

5.3. Hydrodynamic form for DRD-III

For DRD-III case it gives the new hydrodynamic system

$$v_t^+ = \left[v_x^+ + (v^+)^2 + 2\nu(\rho - \rho v^+) - 2\nu^2\rho^2\right]_x,$$
$$\rho_t + \rho_{xx} = [2\rho v^+ + \nu\rho^2]_x. \tag{67}$$

5.4. Hydrodynamic form for JRD

For JRD case it gives the new hydrodynamic system

$$v_t^+ = \left[v_x^+ + (v^+)^2 - \nu(2\rho v^+ - \rho_x) - \frac{3}{4}\nu^2\rho^2\right]_x,$$
$$\rho_t + \rho_{xx} = [2\rho v^+]_x. \tag{68}$$

In all above cases for v^- we have the system with replaced $t \to -t$, $\nu \to -\nu$.

5.5. Generic case

For the generic case of arbitrary α firstly we have the system

$$-E_t^+ + E_{xx}^+ - \nu\left[2\rho E_x^+ + (2\alpha - 1)\rho_x E^+\right] - \nu^2\left(\alpha - \frac{1}{2}\right)\left(\alpha - \frac{3}{2}\right)\rho^2 E^+ = 0,$$
$$\rho_t = [2\rho(\ln E^+)_x - \rho_x - 2\nu\alpha\rho^2]_x. \tag{69}$$

It gives the new hydrodynamic system

$$v_t^+ = \left[v_x^+ + (v^+)^2 - \nu(2\rho v^+ + (2\alpha - 1)\rho_x) - \nu^2\left(\alpha - \frac{1}{2}\right)\left(\alpha - \frac{3}{2}\right)\rho^2\right]_x,$$
$$\rho_t + \rho_{xx} = [2\rho v^+ - 2\nu\alpha\rho^2]_x. \tag{70}$$

6. RNLS and Broer–Kaup system

The RNLS for $s > 1$ can be transformed to the reaction–diffusion system

$$R_t^+ = R_{xx}^+ + 2\nu R^+ R^- R^+, \tag{71}$$
$$-R_t^- = R_{xx}^- + 2\nu R^+ R^- R^-. \tag{72}$$

By substitution $v^+ = (\ln E^+)_x$, $\rho = E^+E^-$, it can be transformed to the the hydrodynamic form as the Broer–Kaup system, [9], [10],

$$v_t^+ = (v_x^+ + (v^+)^2)_x + 2\nu\rho_x,$$
$$\rho_t + \rho_{xx} = (2\rho v^+)_x. \tag{73}$$

If $v^- = (\ln E^-)_x$, $\rho = E^+E^-$, then we have

$$-v_t^- = (v_x^- + (v^-)^2)_x + 2\nu\rho_x,$$
$$-\rho_t + \rho_{xx} = (2\rho v^-)_x. \tag{74}$$

7. Relation with Broer–Kaup system

Given $E^+(x,t)$, $E^-(x,t)$ satisfying general DRD system (53), then real functions

$$R^+ = E^+ e^{-(\alpha+\frac{1}{2})\nu\int^x E^+E^-},$$
$$R^- = \left[E_x^- + \left(\alpha - \frac{1}{2}\right)\nu E^+E^-E^+\right] e^{(\alpha+\frac{1}{2})\nu\int^x E^+E^-} \tag{75}$$

or

$$R^+ = \left[-E_x^+ + \left(\alpha - \frac{1}{2}\right)\nu E^+E^-E^+\right] e^{-(\alpha+\frac{1}{2})\nu\int^x E^+E^-},$$
$$R^- = E^- e^{(\alpha+\frac{1}{2})\nu\int^x E^+E^-} \tag{76}$$

satisfy the reaction–diffusion (RD) system

$$R_t^+ = R_{xx}^+ + 2\nu R^+R^-R^+, \tag{77}$$
$$-R_t^- = R_{xx}^- + 2\nu R^+R^-R^-. \tag{78}$$

From this fact we can get next result.

If v_E^+ and ρ_E satisfy (70) then v_R^+ and ρ_R determined by

$$v_R^+ = v_E^+ - \left(\alpha + \frac{1}{2}\right)\nu\rho_E, \tag{79}$$
$$\rho_R = (\rho_E)_x - \rho_E v_E^+ + \left(\alpha - \frac{1}{2}\right)\nu\rho_E^2 \tag{80}$$

is solution of the Broer–Kaup system (73). For v_E^- and ρ_E satisfying the analog of system (70),

$$v_R^- = v_E^- + (\alpha + \frac{1}{2})\nu\rho_E + \left[\ln\left(v_E^- + \left(\alpha - \frac{1}{2}\right)\right)\nu\rho_E\right]_x, \tag{81}$$
$$\rho_R = \rho_E v_E^- + \left(\alpha - \frac{1}{2}\right)\nu\rho_E^2 \tag{82}$$

is solution of (74).

Similar way we can get result.

If v_E^+ and ρ_E satisfy (70) then v_R^+ and ρ_R determined by

$$v_R^+ = v_E^+ - \left(\alpha + \frac{1}{2}\right)\nu\rho_E + \left[\ln\left(-v_E^+ + \left(\alpha - \frac{1}{2}\right)\right)\nu\rho_E\right]_x, \tag{83}$$
$$\rho_R = -\rho_E v_E^+ + \left(\alpha - \frac{1}{2}\right)\nu\rho_E^2 \tag{84}$$

is solution of the Broer–Kaup system (73). For v_E^- and ρ_E satisfying the analog of system (70),

$$v_R^- = v_E^- + \left(\alpha + \frac{1}{2}\right)\nu\rho_E, \tag{85}$$
$$\rho_R = -(\rho_E)_x + \rho_E v_E^- + \left(\alpha - \frac{1}{2}\right)\nu\rho_E^2 \tag{86}$$

is solution of (74).

8. Bäcklund transformation

When $\rho \equiv 0$, both systems (70) and (73) reduce to the Burgers equation. Then the above Miura type transformations reduce to the auto-Bäcklund transformations

$$v_R^+ = v_E^+ + (\ln v_E^+)_x, \quad v_R^- = v_E^- + (\ln v_E^-)_x \tag{87}$$

for the Burgers and anti-Burgers equations correspondingly.

9. Classical Bousinesque systems

If in (73) we change variables

$$p^+ = v_x^+ + 2\nu\rho, \tag{88}$$

then we get the classical Bousinesque system

$$v_t^+ = ((v^+)^2 + p^+)_x, \quad (89)$$
$$p_t^+ = (v_{xx}^+ + 2p^+v^+)_x. \quad (90)$$

Similar way in (74) by variable change

$$p^- = v_x^- + 2v\rho, \quad (91)$$

we get

$$-v_t^- = ((v^-)^2 + p^-)_x, \quad (92)$$
$$-p_t^- = (v_{xx}^- + 2p^-v^-)_x. \quad (93)$$

10. Bilinear form and solitons

By substitution $E^\pm = g^\pm/f^\pm$ to (69) we have bilinear representation

$$\left(\mp D_{\tilde{t}} + D_x^2\right)(g^\pm \cdot f^\pm) = 0, \quad (94)$$
$$D_x^2(f^+ \cdot f^-) + \frac{1}{2}D_x(g^+ \cdot g^-) = 0, \quad (95)$$
$$D_x(f^+ \cdot f^-) + \alpha g^+ g^- = 0, \quad (96)$$

where $\alpha = \frac{1}{2}$ (DRD-II case), or $\alpha = -\frac{1}{2}$ (DRD-III case). We note that only in these two cases the Hirota substitution has simple bilinear form. Then for solution of the hydrodynamics systems (66) and (67) we have

$$v^+ = (\ln E^+)_x = \frac{g_x^+}{g^+} - \frac{f_x^+}{f^+}, \quad (97)$$
$$\rho = E^+E^- = \left(\ln\frac{f^+}{f^-}\right)_x. \quad (98)$$

Bilinearization for arbitrary α can be derived by the gauge transformation, so that

$$E^+ = \frac{g^+}{(f^+)^{\frac{1}{2}+\alpha}(f^-)^{\frac{1}{2}-\alpha}}, \quad E^- = \frac{g^-}{(f^+)^{\frac{1}{2}-\alpha}(f^-)^{\frac{1}{2}+\alpha}}. \quad (99)$$

It implies next substitution for Eq. (70)

$$v^+ = (\ln E^+)_x = \frac{g_x^+}{g^+} - \left(\frac{1}{2}+\alpha\right)\frac{f_x^+}{f^+} - \left(\frac{1}{2}-\alpha\right)\frac{f_x^-}{f^-}, \quad (100)$$
$$\rho = E^+E^- = \left(\ln\frac{f^+}{f^-}\right)_x. \quad (101)$$

10.1. One soliton solution

For one soliton solution we have

$$g^\pm = e^{\eta_1^\pm}, \quad f^\pm = 1 + e^{\phi_{11}^\pm}e^{\eta_1^+ + \eta_1^-}, \quad (102)$$

where $e^{\phi_{11}^\pm} = \mp\frac{k_1^\pm}{(k_1^+ + k_1^-)^2}$, $\eta_1^\pm = k_1^\pm x \pm (k_1^\pm)^2 t + \eta_1^{\pm(0)}$. For regularity of this solution we choose conditions $k_1^- > 0$ and $k_1^+ < 0$, then $-\tilde{v} < k < \tilde{v}$, where $k = k_1^+ + k_1^-$, $\tilde{v} = k_1^- - k_1^+$, $-kx_0^\pm = \eta_1^{+(0)} + \eta_1^{-(0)} + \phi_{11}^\pm$. Then velocity is positive $\tilde{v} > 0$, so that our dissipaton is chiral. For the density we have soliton solution

$$\rho = E^+E^- = \frac{k^2}{\sqrt{\tilde{v}^2 - k^2}\cosh k(x - \tilde{v}t - x_0) + \tilde{v}}, \quad (103)$$

where $2x_0 = x_0^+ + x_0^-$, and for velocity field

$$v^+ = \frac{k_1^+ - k_1^- e^{\phi_{11}^+}e^{\eta_1^+ + \eta_1^-}}{1 + e^{\phi_{11}^+}e^{\eta_1^+ + \eta_1^-}}, \quad (104)$$

the kink solution

$$v^+ = -\frac{\tilde{v}}{2} - \frac{k}{2}\tanh\frac{k}{2}(x - \tilde{v}t - x_0). \quad (105)$$

10.2. Integrals of motion

The particle number, momentum and energy integrals are given respectively

$$N = \int_{-\infty}^{\infty}\rho dx = -\frac{1}{v}\ln\frac{f^+}{f^-}\Big|_{-\infty}^{\infty}, \quad (106)$$
$$P = -\int_{-\infty}^{\infty}\rho v^+ dx = \frac{1}{2v}\ln(f^+f^-)_x\Big|_{-\infty}^{\infty}, \quad (107)$$
$$E = -\int_{-\infty}^{\infty}[\rho(v^+)^2 - \rho_x v^+ - v\rho^2 v^+]dx. \quad (108)$$

Then substituting for one soliton solution we find

$$N = \frac{1}{v}\ln\frac{\tilde{v}+|k|}{\tilde{v}-|k|}, \quad P = \frac{|k|}{v}, \quad E = \frac{\tilde{v}|k|}{2v}. \quad (109)$$

The mass of soliton $M = |k|/(v\tilde{v})$ in terms of particle number becomes $M = \frac{1}{v}\tanh\frac{Nv}{2}$, and for the momentum and the energy we have non-relativistic free particle form $P = M\tilde{v}$, $E = \frac{M\tilde{v}^2}{2}$.

For the process of fusion or fission of two solitons then the next conditions should be valid

$$N = N_1 + N_2, \quad P = P_1 + P_2, \quad E = E_1 + E_2. \quad (110)$$

Using (109) after some algebraic manipulations we get the resonance condition

$$|\tilde{v}_1 - \tilde{v}_2| = |k_1| + |k_2|, \quad (111)$$

where $\tilde{v}_a = k_a^- - k_a^+$, $k_a = k_a^- + k_a^+$, $a = 1, 2$.

10.3. Two soliton solution

For two soliton solution we have

$$g^\pm = e^{\eta_1^\pm} + e^{\eta_2^\pm} + \alpha_1^\pm e^{\eta_2^+ + \eta_2^- + \eta_1^\pm} + \alpha_2^\pm e^{\eta_1^+ + \eta_1^- + \eta_2^\pm}, \quad (112)$$
$$f^\pm = 1 + \sum_{i,j=1}^{2} e^{\phi_{ij}^\pm}e^{\eta_i^+ + \eta_j^-} + \beta^\pm e^{\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^-}, \quad (113)$$

where $\eta_i^\pm = k_i^\pm x \pm \left(k_i^\pm\right)^2 t + \eta_{i0}^\pm$, $k_{ij}^{nm} \equiv \left(k_i^n + k_j^m\right)$ and

$$\alpha_1^\pm = \pm\frac{1}{2}\frac{k_2^\mp\left(k_1^\pm - k_2^\pm\right)^2}{\left(k_{22}^{+-}\right)^2\left(k_{12}^{\pm\mp}\right)^2}, \quad \alpha_2^\pm = \pm\frac{1}{2}\frac{k_1^\mp\left(k_1^\pm - k_2^\pm\right)^2}{\left(k_{11}^{+-}\right)^2\left(k_{21}^{\pm\mp}\right)^2}, \quad (114)$$
$$\beta^\pm = \frac{(k_1^+ - k_2^+)^2(k_1^- - k_2^-)^2}{4\left(k_{11}^{+-}k_{12}^{+-}k_{21}^{+-}k_{22}^{+-}\right)^2}k_1^\pm k_2^\pm, \quad (115)$$
$$e^{\phi_{ii}^\pm} = \mp\frac{k_i^\pm}{2\left(k_{ii}^{+-}\right)^2}, \quad e^{\phi_{ij}^+} = \frac{-k_i^+}{2\left(k_{ij}^{+-}\right)^2}, \quad e^{\phi_{ij}^-} = \frac{k_j^-}{2\left(k_{ij}^{+-}\right)^2}. \quad (116)$$

By regularity we have $k_i^+ \leqslant 0$, $k_i^- \geqslant 0$ in the Case 1, and $k_i^+ \geqslant 0$, $k_i^- \leqslant 0$ in the Case 2. Then solving the resonance condition (111) we find that for every solution of this

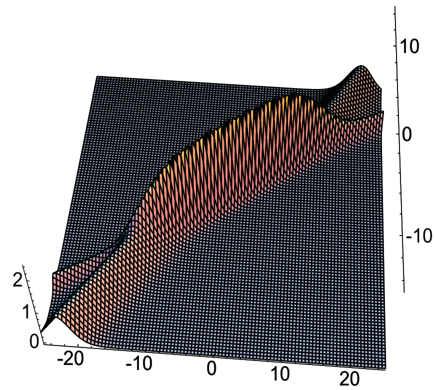

Fig. 1. 3D plot of typical soliton resonant state with one soliton resonance.

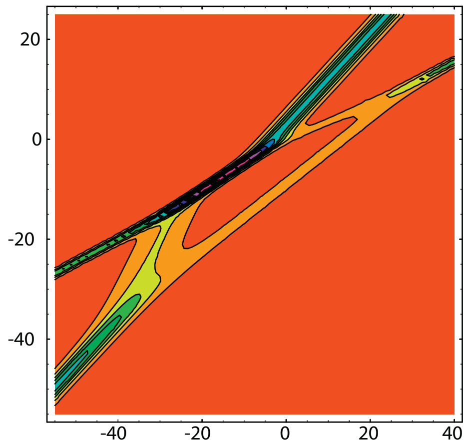

Fig. 2. Contour plot of four soliton resonances.

algebraic equation, the coefficient β vanishes or becomes infinite. In both cases two soliton solution reduces to the one soliton solution. Hence the solution describes a collision of two solitons propagating in the same direction and at some value of parameters creating the resonance states (see Figs. 1 and 2).

11. Conclusions

The problem of chiral solitons in quantum potential, as a reduction of 2 + 1 dimensional Chern–Simons theory, was formulated in terms of family of integrable derivative NLS equations by the Madelung fluid representation. By using new, non-Madelung fluid representation we constructed integrable family of hydrodynamical systems of the Kaup–Broer type. By bilinear method we found resonance character of corresponding chiral soliton mutual interaction.

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