Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 614 – 624
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doi:10.3906/mat-1307-24

On density theorems for rings of Krull type with zero divisors

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Received: 10.07.2013 • **Accepted:** 11.02.2014 • **Published Online:** 25.04.2014 • **Printed:** 23.05.2014

Abstract: Let R be a commutative ring and $\mathcal{I}(R)$ denote the multiplicative group of all invertible fractional ideals of R, ordered by $A \leqslant B$ if and only if $B \subseteq A$. If R is a Marot ring of Krull type, then $R_{(P_i)}$, where $\{P_i\}_{i\in I}$ are a collection of prime regular ideals of R, is a valuation ring and $R = \bigcap R_{(P_i)}$. We denote by G_i the value group of the valuation associated with $R_{(P_i)}$. We prove that there is an order homomorphism from $\mathcal{I}(R)$ into the cardinal direct sum $\coprod_{i\in I} G_i$ and we investigate the conditions that make this monomorphism *onto* for R.

Key words: Krull ring, ring of Krull type, valuation Marot ring

1. Introduction

Let R be a commutative ring with zero divisors. We call an element of R *regular* if it is not a zero-divisor. Let $Reg(R)$ denote the monoid of regular elements of R and $Q(R) = Q$ denote the total ring of fractions R. We note that $Q = (Reg(R))^{-1}R$. We say that an ideal I of R is *regular* if I contains a regular element of R. Let $\mathcal{F}(R)$ be the set of all fractional regular ideals of R. The set of all invertible fractional ideals of R is a subgroup of $\mathcal{F}(R)$; this group is denoted by $\mathcal{I}(R)$. The principal fractional regular ideals form a subgroup $\beta(R)$ in $\mathcal{I}(R)$. Furthermore, $Min(R)$ denotes the set of all prime regular ideals, which are minimal among prime regular ideals of R. We note that every invertible fractional ideal of R is finitely generated and regular. For a prime ideal P of R, we set $R_{(P)} = (Reg(R) - P)^{-1}R \subseteq Q$ and $R_{[P]} = \{y \in Q(R) : xy \in R, x \in R - P\}$.

We recall that a valuation is a map ν from a ring K onto a totally ordered group G and a symbol ∞, such that for all x and y in K:

1. $\nu(xy) = \nu(x) + \nu(y)$.
2. $\nu(x + y) \geqslant min\{\nu(x), \nu(y)\}$.
3. $\nu(1) = 0$ and $\nu(0) = \infty$.

The ring $R_\nu = \{x \in Q | \nu(x) \geqslant 0\}$, together with the ideal $P_\nu = \{x \in Q | \nu(x) > 0\}$, denoted (R_ν, P_ν), is called a *valuation pair (of K)*. R_ν is called a valuation ring (of K), and G is called *the value group of* R_ν. We note that given a valuation pair (R, P), $R = R_P$ and that (R, P) is said to be *discrete rank one* if G is isomorphic to the group of integers $\mathbb{Z}$.

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2010 *AMS Mathematics Subject Classification:* 13F05, 13A18.

A ring R is called a *Marot ring* if every regular ideal can be generated by a set of regular elements. This property was defined by Marot [12]. Moreover, every overring of a Marot ring is Marot. Below we see a couple of characterizations of a Marot ring.

Theorem 1.1 *[6, Theorem 3.5] Let R be a ring. Then the following are equivalent:*

1. *R is a Marot ring.*
2. *Any 2-generated ideal (a,b) with b regular can be generated by a finite set of regular elements.*
3. *Every regular fractional ideal of R, that is, every R-submodule M of Q such that $M \cap Reg(Q) \neq \emptyset$, can be generated by a set of regular elements.*

In the presence of the Marot property, valuation rings share some properties of valuation domains. For example, as in the domain case, it is not true, in general, that given a valuation pair (R_ν, P_ν), P_ν is the unique maximal (regular) ideal of R_ν. However, if a valuation ring R_ν is Marot, then we have the following.

Proposition 1.2 *[6, Proposition 4.1] Let R be a Marot ring. Assume that $R \neq Q$. Then the following conditions are equivalent:*

1. *R is a valuation ring.*
2. *For each regular element $x \in Q$, either $x \in R$ or $x^{-1} \in R$.*
3. *R has only one maximal regular ideal and each of its finitely generated regular ideals is principal.*

Lemma 1.3 *Let R be a Marot ring. Assume that $R \neq Q$. Then R is a valuation ring if and only if the set of R-submodules M of Q such that $M \cap Reg(Q) \neq \emptyset$ is totally ordered by inclusion.*

Proof Suppose that R is a valuation ring. Let $A, B \in Q$ such that $A \cap Reg(Q) \neq \emptyset$ and $B \cap Reg(Q) \neq \emptyset$. Assume that $A \nsubseteq B$ and $B \nsubseteq A$. By [8, Theorem 7.1], there are regular elements $r \in A - B$ and $s \in B - A$. Set $x = rs^{-1} \in Reg(Q)$. By Proposition 1.2, either $x \in R$ or $x^{-1} \in R$. This implies that $s \in rR \subseteq A$ or $r \in sR \subseteq B$, which is a contradiction. Conversely, let $x \in Reg(Q)$, so $x = rs^{-1}$ for some regular elements $r, s \in R$. We observe that rR and sR are R-submodules that contain a regular element of Q, hence either $sR \subseteq rR$ or $rR \subseteq sR$. Thus, we have $x \in R$ or $x^{-1} \in R$. By Proposition 1.2, R is a valuation ring. □

A commutative ring R is said to be *additively regular* if for each $z \in Q$, there exists a $u \in R$ such that $z + u$ is a regular element in Q, or, equivalently, for each $a \in R$ and each regular element $b \in R$, there exists a $u \in R$ such that $a + ub$ is regular in R [5, Lemma 7]. The class of additively regular rings is an example of Marot rings [6, Theorem 3.6].

Consider the following conditions on a commutative ring R:

1. There exists a family $\{(V_\alpha, P_\alpha) : \alpha \in I\}$ of valuation pairs, where V_αs are overrings of R with the property that $R = \bigcap\{V_\alpha : \alpha \in I\}$.
2. For each regular element $q \in \mathrm{Reg}(Q)$, q is a nonunit in only finitely many V_αs, and each P_α is a regular ideal of V_α.

3. For each pair (V_α, P_α), V_α is a localization of R at a prime ideal L such that $L = P_\alpha \cap R$.

4. Each pair $\{(V_\alpha, P_\alpha) : \alpha \in I\}$ is rank one discrete.

A ring R is called a *Krull ring* if it satisfies conditions 1–4 and a *ring of Krull type* if it satisfies conditions 1–3. For a valuation pair (V, P), let v be the corresponding valuation. Then the prime ideal L, in condition 3 above, is called the *center of* v. We also note that condition 2 also means that for each regular element $r \in Reg(R)$, r is a unit in V_α for all except finitely many $\alpha \in I$.

Proposition 1.4 *[7, Proposition] Let R be a Marot ring, ν a discrete rank one valuation on Q, and P_ν the ideal generated by $\{x \in Reg(R) | \nu(x) > 0\}$. Then we have the following hold.*

i. P_ν is a prime regular ideal of R.

ii. $R_{(P_\nu)}$ is a discrete rank one valuation ring with the unique prime regular ideal $P_\nu R_{(P_\nu)}$.

Let Γ be the set of valuations on Q. For each $\nu \in \Gamma$, let P_ν be the ideal generated by $\{x \in \text{Reg}(R) | \nu(x) > 0\}$. By Proposition 1.4, if R is a Marot Krull ring, then $R = \bigcap_{\nu\in\Gamma} R_{(P_\nu)}$, and every $x \in \text{Reg}(Q)$, $\nu(x) = 0$ almost for all $\nu \in \Gamma$; that is, x is a unit in almost all $R_{(P_\nu)}$ (see [7] for details). In fact, we can write $R = \bigcap_{\nu\in\Gamma} R_{[P_\nu]}$ for a Marot Krull ring R [8, Theorem 7.6].

Theorem 1.5 *[8, Theorem 8.10] Let R be a Marot Krull ring. Then $R = \bigcap_P R_{(P)}$, where $P \in Min(R)$.*

Let $\mathcal{P}$ be a nonempty set of pairwise incomparable prime ideals of R. We say that a ring R is of $\mathcal{P}$*-finite character* if every regular element of R is contained in at most finitely many prime ideals $P \in \mathcal{P}$. Furthermore, if R is of $\mathcal{P}$-finite character, and if every prime regular ideal of R is contained in at most one prime ideal $P \in \mathcal{P}$, then R is called an $h_\mathcal{P}$*-local ring*. We note that a Krull ring is of $h_\mathcal{P}$-local for the choice $\mathcal{P} = Min(R)$.

If R is a Dedekind domain, then $\mathcal{I}(R)$ is the set of all nonzero fractional ideals of R, and the class group $\mathcal{I}(R)/\beta(R)$ is a measure of unique factorization of elements of R. If the class group is trivial, then R is a unique factorization domain, and hence a principal ideal domain. If R is a Dedekind domain with maximal ideals $\{M_i\}_{i\in I}$, then for a nonzero fractional ideal A, we have $A = M_1^{e_{i_1}} \dots M_n^{e_{i_n}}$, and the mapping $A \to (e_{i_1}, \dots e_{i_n})$ is an order isomorphism from $\mathcal{I}(R)$ onto the cardinal sum $\coprod_{i\in I} \mathbb{Z}_i$, where $\mathbb{Z}_i \cong \mathbb{Z}$ for each i.

The fact described in the previous paragraph about Dedekind domains is well known. In [2], the authors dropped both the Noetherian and the one-dimensional assumptions and considered $\mathcal{I}(R)$ when R is a Prüfer domain of $\mathcal{P}$-finite character for the choice $\mathcal{P}$, the set of all maximal ideals of R. It turns out that an analogous fact is true. If R is a Prüfer domain of $\mathcal{P}$-finite character with the same choice for $\mathcal{P}$, then there is an order monomorphism from $\mathcal{I}(R)$ *into* the cardinal direct sum $\coprod_{i\in I} G_i$, where G_i is a value group [2, Theorem 2(3)], and it is *onto* if R is an $h_\mathcal{P}$-local Prüfer domain, where $\mathcal{P}$ is the set of all maximal ideals of R [2, Theorem 5]. In this paper, we prove that if R is a Marot Krull ring, then there is also an order homomorphism from $\mathcal{I}(R)$ *into* the cardinal direct sum $\coprod_{i\in I} G_i$, where G_i is a value group. Moreover, we investigate when this homomorphism restricts to an isomorphism from $\beta(R)$ *onto* the cardinal direct sum $\coprod_{i\in I} G_i$, where G_i is a value group, for an additively regular Krull ring R. Furthermore, we generalize these results to additively regular rings of Krull type.

This paper is organized as follows. In Section 2, we prove that $\mathcal{I}(R)$ maps *into* the cardinal direct sum $\coprod_{i\in I} G_i$, where G_i is a value group, when R is a Marot Krull ring. In Section 3 it is shown that the "Density Theorem" holds for elements in R. Furthermore, we prove that a stronger version of the "Density Theorem" for regular elements holds when R is, *in addition*, additively regular. In Section 4 we generalize our results in the previous section for an additively regular ring of Krull type R, and we investigate when there is a monomorphism from $\mathcal{I}(R)$ *onto* the cardinal sum $\coprod_{i\in I} G_i$, where G_i is a value group.

2. Embedding $\mathcal{I}(R)$ into $\coprod_{i\in I} G_i$

The group $\mathcal{I}(R)$ of all invertible ideals is partially ordered under the order $A \leqslant B$ if and only if $B \subseteq A$. Before we prove the generalization of [2, Proposition 1] for the integrally closed rings with zero divisors, we define a useful tool.

A $*-$*operation* on R is a mapping $F \to F^*$ of $\mathcal{F}(R)$ into $\mathcal{F}(R)$ such that for each $q \in Reg(Q)$ and all $A, B \in \mathcal{F}(R)$:

i. $(qA)^* = qA^*$.

ii. $A \subseteq A^*$; if $A \subseteq B$, then $A^* \subseteq B^*$.

iii. $(A^*)^* = A^*$.

Let $R = \bigcap_{i\in I} V_i$, where $\{V_i\}_{i\in I}$ is a collection of valuation overrings of R and F is a fractional regular ideal of R. Then it is a routine check to see that the mapping $F \to \bigcap_{i\in I} FV_i$ is a $*-$*operation* on R.

Lemma 2.1 *If $F \to F^*$ is a $*-$operation on a commutative ring R, and if A is an invertible fractional ideal of R, then for each $B \in \mathcal{F}(R)$, $(AB)^* = AB^*$. In particular, $A^* = (AR)^* = AR^* = A$.*

Proof The proof is similar to the analogous lemma in the domain case [4, Lemma 32.17]. □

Proposition 2.2 *Let R be a Marot ring with $\{V_i\}_{i\in I}$ a collection of valuation overrings of R such that $R = \bigcap_{i\in I} V_i$. Denote by v_i the valuation associated with V_i, and by G_i the corresponding value group. Let $A = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be an invertible fractional ideal of R. Then the mapping*

$$\Phi : \mathcal{I}(R) \to \prod_{i\in I} G_i$$

defined by

$$\Phi(A) = (v_i(A))_{i\in I} = (min\{v_i(\alpha_j)\}_{1\leqslant j\leqslant n})_{i\in I}$$

is an order-preserving monomorphism from $\mathcal{I}(R)$ into $\prod_{i\in I} G_i$.

Proof It follows from the definition of a valuation that Φ is a well-defined map. Next we claim that Φ is a one-to-one order-preserving group homomorphism.

Let $A = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $B = (\beta_1, \beta_2, \ldots, \beta_m)$ be invertible fractional ideals of R. Since V_i is a Marot valuation ring, every finitely generated regular ideal is principal. Thus, we have $v_i(A) = v_i(\alpha_{j(i)})$, where

$AV_i = \alpha_{j(i)}V_i$ for $v_i(\alpha_{j(i)}) = min\{v_i(\alpha_1), v_i(\alpha_2), \ldots, v_i(\alpha_n)\}$, and $v_i(B) = v_i(\beta_{k(i)})$, where $BV_i = \beta_{k(i)}V_i$ for $v_i(\beta_{k(i)}) = min\{v_i(\beta_1), v_i(\beta_2), \ldots, v_i(\beta_m)\}$.

We have that $B \leqslant A$ if and only if $A \subseteq B$, if and only if $AV_i \subseteq BV_i$, if and only if $v_i(\alpha_{j(i)}) \geqslant v_i(\beta_{k(i)})$, if and only if $v_i(A) \geqslant v_i(B)$ for all i. Therefore, Φ is order-preserving, showing that $\Phi(AB) = \Phi(A) + \Phi(B)$; that is, Φ is a group homomorphism similar to the analogous proposition in the domain case.

It remains to show that Φ is one-to-one. If F is any nonzero fractional regular ideal of R, the mapping $F \to \bigcap_{i\in I} FV_i$ is a $*-$ *operation* on R. Let $A, B \in \mathcal{F}(R)$. Suppose that $\Phi(A) = \Phi(B)$. Then $v_i(A) = v_i(B)$ for each $i \in I$, so that $AV_i = BV_i$ for each $i \in I$. It follows that $\bigcap_{i\in I} AV_i = \bigcap_{i\in I} BV_i$. Since A is invertible, by Lemma 2.1, we have $A^* = (AR)^* = AR^* = A$. Similarly, $B^* = B$. Therefore, $A = \bigcap_{i\in I} AV_i$ and $B = \bigcap_{i\in I} BV_i$, and hence Φ is one-to-one. □

We specialize Proposition 2.2 to Marot rings of Krull type and determine when the embedding defined in Proposition 2.2 maps *into* the cardinal sum of G_is.

Theorem 2.3 *Let R be a Marot ring of Krull type. Denote by v_i the valuation associated with the valuation ring $R_{(P_i)}$, where P_i is the center of v_i, that is, $P_i = M_i \cap R$, where M_i is the corresponding maximal regular ideal M_i of v_i, for each $i \in I$ and by G_i the associated value group. Let Φ be the mapping defined in Proposition 2.2. Then:*

1. *The mapping Φ is an order-preserving monomorphism from $\mathcal{I}(R)$ into $\prod_{i\in I} G_i$, the cardinal product of the G_is.*
2. *Φ maps $\mathcal{I}(R)$ into $\coprod_{i\in I} G_i$, the cardinal direct sum of the G_is.*

Proof

1. It immediately follows from Proposition 2.2 that Φ is an order-preserving monomorphism from $\mathcal{I}(R)$ into $\prod_{i\in I} G_i$.
2. Since R is a ring of Krull type, each of its regular elements is contained in at most finitely many P_is. Thus, $\Phi(A)$ is finitely nonzero, or, in other words, $\Phi(A) \in \coprod_{i\in I} G_i$. Since each invertible fractional ideal of R can be written as AB^{-1} for some invertible ideals A and B, the image of Φ is contained in $\coprod_{i\in I} G_i$ for every invertible fractional ideal A of R.

□

We finish this section with a result that will be important for us in the following sections, but before stating that we prove a helpful proposition.

Proposition 2.4 *Let R be a Marot ring and $P, P_1, \ldots, P_n$ a collection of prime regular ideals such that $P \nsubseteq P_i$ for any i. Then $Reg(P) \nsubseteq \bigcup_{i=1}^{n} P_i$.*

Proof We have that $P \nsubseteq P_i$ for any i, and hence, by [1, Proposition 1.11], $P \nsubseteq \bigcup_{i=1}^{n} P_i$. Since R is a Marot ring and P is a regular ideal, P is generated by a set of regular elements. Thus, there exists at least one regular generator of P that cannot be contained in $\bigcup_{i=1}^{n} P_i$. Therefore, $Reg(P) \nsubseteq \bigcup_{i=1}^{n} P_i$. □

Lemma 2.5 *Let R be a Marot ring of Krull type. Denote v_i the valuation associated with the valuation ring $R_{(P_i)}$, where P_i is the center of v_i, for each i, and by G_i the associated value group. Suppose each nonzero prime regular ideal of R is contained in at most one P_i. Then for every finite collection of centers $P, P_1, P_2, \ldots, P_n$ of R, with corresponding valuations $v, v_1, \ldots, v_n$, and given some nonnegative value g of v, there is a regular element $r \in R$ such that $v(r) > g$ and $v_i(r) = 0$, $1 \leqslant i \leqslant n$.*

Proof Let $b \in Q$ such that $v(b) = g$. By [8, Theorem 7.7] and [8, Theorem 7.9], we can choose b to be a regular element. Thus, $b = \dfrac{s}{l}$, where $s, l \in Reg(R)$. Hence, $v(s) \geqslant v(b) = g$. Suppose that there is a minimal prime regular ideal L in P. Since R is a Marot ring, L is generated by a set of regular elements. By assumption, $L \nsubseteq \bigcup_{i \neq 1} P_i$. By Proposition 2.4, we can choose a regular element c of L such that $c \in L - \bigcup_{i=1}^{n} P_i$. Since R is a Marot ring, $R_{[L]} = R_{(L)}$, and hence locally at L, $R_{(L)}$ is a rank one valuation, and therefore has an Archimedean value group by [3, Proposition 2.1, page 61]. Since $v'(c) > 0$, where v' is the valuation corresponding to L, there exists a positive integer t such that $v'(c^t) = t.v'(c) > v'(s)$. This implies that $c^t R_{[P]} = c^t R_{(P)} \subsetneq sR_{(P)}$, and hence that $v(c^t) > v(s) \geqslant v(b) = g$. But $c \notin P_i$, $1 \leqslant i \leqslant n$. This implies that $c^t \notin P_i$, $1 \leqslant i \leqslant n$. Hence, $v_i(c^t) = t.v_i(c) = 0$ for $1 \leqslant i \leqslant n$. Thus, c^t meets the requirements of the claim, that is, takes $r = c^t$.

Suppose now that there is no minimal prime regular ideal contained in P. Let I be the intersection of prime regular ideals of $R_{(P)}$. By Lemma 1.3, it follows that prime regular ideals of $R_{(P)}$ are totally ordered by inclusion, and so I is a prime ideal. We note that I cannot be regular since otherwise I would become a minimal prime regular ideal. Thus, s cannot be contained in I, and hence there must be a prime regular ideal contained in P, say L, such that $s \notin L$. As in the first case, $L \nsubseteq \bigcup_{i=1}^{n} P_i$, and so, by Proposition 2.4, we can choose a regular element d of L such that $d \in L - \bigcup_{i=1}^{n} P_i$. By choice of L, we have $v(d) > g$ and $v_i(d) = 0$, $1 \leq i \leq n$. □

3. Density and strong density theorems for Krull rings

One of our goals in this section is to prove that, for a Krull ring, the "Density Theorem" holds for its elements. In addition, it is shown that the "Density Theorem" holds for *regular* elements in a Krull ring that is additively regular. Furthermore, for the latter class of rings, we prove that the "Strong Density Theorem" holds for finitely generated regular ideals.

Proposition 3.1 *Let R be a Marot Krull ring with $\{P_i\}_{i \in I} \in Min(R)$. Denote by v_i the valuation associated with the valuation ring $R_{(P_i)}$, and by G_i the associated value group. Then the "Density Theorem" holds for elements in R; that is, for every finite collection of minimal prime regular ideals $P_1, P_2, \ldots, P_n$ of R, and every choice of nonnegative elements $g_i \in G_i$, there is an element $r \in R$ such that $v_i(r) = g_i$ for $1 \leq i \leq n$.*

Proof

Let $g_1, g_2, \ldots, g_n$ be nonnegative elements of $G_1, G_2, \ldots, G_n$ respectively. By Lemma 2.5 we can choose, for each i, a regular element $r_i \in R$ such that $v_i(r_i) > g_i$ and $v_j(r_i) = 0$ for all $j \neq i$. Let $t_1, t_2, \ldots, t_n \in R$ be such that $v_i(t_i) = g_i, 1 \leq i \leq n$, and set

$$s_i = t_i(r_1 \cdots r_{i-1} \cdot r_{i+1} \cdots r_n).$$

Then for $1 \leq i \leq n$, we get

$$v_i(s_i) = v_i(t_i) + v_j(r_1) + \ldots + v_i(r_{i-1}) + v_i(r_{i+1}) + \ldots + v_i(r_n) = g_i,$$

since $v_i(r_j) = 0$ for $j \neq i$. On the other hand, for $j \neq i$, we have that

$$\begin{aligned} v_j(s_i) &= v_j(t_i) + v_j(r_1) + \ldots + v_j(r_{i-1}) + v_j(r_{i-1}) + \ldots + v_j(r_n) \\ &= v_j(t_i) + v_j(r_j) \geq v_j(r_j) > g_j. \end{aligned}$$

Finally, if we set $s = s_1 + \ldots + s_n$, then for $1 \leq i \leq n$, it follows that $v_i(s) = v_i(s_i) = g_i$, since $v_i(s_j) > g_i$ for $j \neq i$. □

The element s constructed in the proof of Proposition 3.1 is not necessarily regular. If R is an additively regular Krull ring, it follows from the next result that the element s can be chosen to be regular. We recall that an additively regular ring is Marot.

Proposition 3.2 *Let R be an additively regular Krull ring with $\{P_i\}_{i\in I} \in Min(R)$. Denote v_i the valuation associated with the valuation ring $R_{(P_i)}$, and by G_i the associated value group. Then the "Density Theorem" holds for regular elements in R; that is, for every finite collection of minimal prime regular ideals $P_1, P_2, \ldots, P_n$ of R, and every choice of nonnegative elements $g_i \in G_i$, there is a regular element $r \in R$ such that $v_i(r) = g_i$ for $1 \leq i \leq n$.*

Proof

Let us use the same notation as in the proof of Proposition 3.1. R is additively regular and $Reg(R)$ is multiplicatively closed, so we have that $s' = s + ur_1 \cdot r_2 \cdot \ldots \cdot r_n \in Reg(R)$ for some $u \in R$. Since $v_i(r_i) > g_i, v_i(u) \geqslant v_i(r_j) = 0$, it follows that $v_i(u) + v_i(r_1) + v_i(r_2) + \ldots + v_i(r_i) + \ldots + v_i(r_n) > g_i$. Thus, $v_i(s') = min\{v_i(s), v_i(u) + v_i(r_1) + v_i(r_2) + \ldots + v_i(r_n)\} = g_i$, for $1 \leqslant i \leqslant n$. □

Theorem 3.3 *Let R be an additively regular Krull ring with $\{P_i\}_{i\in I} \in Min(R)$. Denote by v_i the valuation associated with the valuation ring $R_{(P_i)}$, and by G_i the associated value group. Then the "Strong Density Theorem" holds for finitely generated regular ideals of R; that is, for every finite collection of minimal prime regular ideals $P_1, P_2, \ldots, P_n$ of R, and every choice of nonnegative elements $g_i \in G_i$, there is a finitely generated regular ideal A of R such that $v_i(A) = g_i$ for $1 \leqslant i \leqslant n$, and $v_j(A) = 0$ for all other minimal prime regular ideals P_j for R. Moreover, the ideal A can be chosen to be 2-generated.*

Proof By Proposition 3.2, the Density Theorem holds for regular elements in R, so we can find a regular element $r \in R$ such that $v_i(r) = g_i$ for $1 \leq i \leq n$. Since R is a Krull ring, it is of $\mathcal{P}$-finite character for the choice $\mathcal{P} = Min(R)$. Therefore, there are at most finitely many minimal prime regular ideals $Q_1, Q_2 \ldots, Q_t$ with corresponding valuations $w_1, \ldots, w_t$ at which r is positive. By the Density Theorem again, we can find a regular element $s \in R$ such that $v_i(s) = g_i$, for $1 \leqslant i \leqslant n$, and $w_j(s) = 0$, for $1 \leqslant j \leqslant t$. Then the ideal (r, s) has desired properties. □

As a consequence of Theorem 3.3, we get a result for rings whose finitely generated regular ideals are principal. Furthermore, such rings are Prüfer; that is, every finitely generated regular ideal is invertible. We thus note that for such a ring S, $\mathcal{I}(S) = \beta(S)$. Next we get the following result concerning when each finitely generated regular ideal of an additively regular Krull ring R is principal and, hence, R has a trivial class group.

Corollary 3.4 *Let R be an additively regular Krull ring with $\{P_i\}_{i\in I} \in Min(R)$. Denote by v_i the valuation associated with the valuation ring $R_{(P_i)}$, and by G_i the associated value group. The following are equivalent:*

1. *Each finitely generated regular ideal of R is principal.*

2. *The "Strong Density Theorem" holds for regular elements of R; that is, for every finite collection of minimal prime regular ideals $P_1, P_2, \ldots, P_n$ of R, and every choice of nonnegative elements $g_i \in G_i$, there is a regular element a of R such that $v_i(a) = g_i$ for $1 \leqslant i \leqslant n$, and $v_j(a) = 0$ for all other minimal prime regular ideals P_j for R.*

3. *The mapping Φ defined in Proposition 2.2 restricts to an isomorphism from the group $\beta(R)$ onto the cardinal direct sum $\coprod_{i\in I} G_i$.*

Proof

(2) $\Rightarrow$ (1) : Let A be a finitely generated regular ideal of R. Since R is of $\mathcal{P}$-finite character for $\mathcal{P} = Min(R)$, A is contained in at most finitely many minimal prime regular ideals of R. Hence, (2) implies that there is a regular element $a \in Reg(R)$ such that $v_i(a) = v_i(A)$ for every regular minimal ideal P_i of R. So, $A = Ra$; that is, A is principal.

(1) $\Rightarrow$ (3) : Since every finitely generated regular ideal of R is principal, $\mathcal{I}(R) = \beta(R)$, and hence it follows from Theorem 3.3 that Φ maps $\beta(R)$ onto $\coprod_{i\in I} G_i$.

(3) $\Rightarrow$ (2) : It follows immediately from the definitions. □

4. Density and strong density theorems for rings of Krull type

Let R be an additively regular ring of Krull type. In this section we study "Density" and "Strong Density" theorems for R. Moreover, we prove that the mapping Φ defined in Proposition 2.2 becomes an *isomorphism* on $\mathcal{I}(R)$ under a certain condition.

We need the following definition. Two valuation rings V and W with the same total ring of fractions Q are said to be *independent* if and only if V and W generate Q. Since any overring of a Marot valuation ring is a valuation ring [8, Corollary 7.8], this is equivalent to saying that there does not exist a valuation ring $U \subseteq Q$ such that $V \subseteq U$ and $W \subseteq U$.

Theorem 4.1 *Let R be an additively regular ring of Krull type. Denote by v_i the valuation associated with the valuation ring $R_{(P_i)}$ where P_i is the center of v_i for each i and by G_i the associated value group. Then the following are equivalent:*

(1) The valuation rings $\{R_{(P_i)}\}_{i\in I}$ are pairwise independent.

(2) Each nonzero prime regular ideal of R is contained in at most one P_i.

(3) The "Density Theorem" holds for regular elements in R; that is, for every finite collection of centers $P_1, \ldots, P_n$ of $R_{(P_1)}, \ldots, R_{(P_n)}$ respectively, and every choice of nonnegative elements $g_i \in G_i$, there is a regular element $d \in R$ such that $v_i(d) = g_i$ for $1 \leqslant i \leqslant n$.

Proof (1) ⇒ (2) If (2) fails, then R has a nonzero prime regular ideal P contained in 2 distinct P_1 and P_2, which are centers of $R_{(P_1)}$ and $R_{(P_2)}$, respectively, so that the valuation ring $R_(P)$ contains the valuation rings $R_{(P_1)}$ and $R_{(P_2)}$. Thus, $R_{(P_1)}$ and $R_{(P_2)}$ cannot be pairwise independent.

(2) ⇒ (3) Let $g_1, g_2, \ldots, g_n$ be nonnegative elements of $G_1, G_2, \ldots, G_n$ respectively. By Lemma 2.5, we can choose, for each i, a regular element $r_i \in R$ such that $v_i(r_i) > g_i$ and $v_j(r_i) = 0$ for all $j \neq i$. Let $t_1, t_2, \ldots, t_n \in R$ be such that $v_i(r_i) = g_i, 1 \leq i \leq n$, and set

$$s_i = t_i(r_1 \cdot \ldots \cdot r_{i-1}.r_{i+1} \cdot \ldots \cdot r_n).$$

Then for $1 \leq i \leq n$, we get

$$v_i(s_i) = v_i(t_i) + v_j(r_1) + \ldots + v_i(r_{i-1}) + v_i(r_{i+1}) + \ldots + v_i(r_n) = g_i,$$

since $v_j(r_j) = 0$ for $j \neq i$. On the other hand, for $j \neq i$, we have that

$$\begin{aligned} v_j(s_i) &= v_j(t_i) + v_j(r_1) + \ldots + v_j(r_{i-1}) + v_j(r_{i-1}) + \ldots + v_j(r_n) \\ &= v_j(t_i) + v_j(r_j) \geq v_j(r_j) > g_j. \end{aligned}$$

Finally, set $s = s_1 + \ldots + s_n$ and $s' = s + ur_1 \cdot r_2 \cdot \ldots \cdot r_n \in Reg(R)$ for some $u \in R$. Since $v_i(r_i) > g_i, v_i(u) \geqslant v_i(r_j) = 0$, it follows that $v_i(u) + v_i(r_1) + v_i(r_2) + \ldots + v_i(r_i) + \ldots + v_i(r_n) > g_i$. Thus, $v_i(s) = min\{v_i(s), v_i(u) + v_i(r_1) + v_i(r_2) + \ldots + v_i(r_n)\} = g_i$, for $1 \leqslant i \leqslant n$.

(3) ⇒ (1) Let P_1 and P_2 be centers of $R_{(P_1)}$ and $R_{(P_2)}$, respectively, and let γ be a regular element of $Q(R)$. If $v_1(\gamma) \geqslant 0$ or $v_2(\gamma) \geqslant 0$, then $\gamma \in R_{(P_1)}$ or $\gamma \in R_{(P_2)}$. So, suppose that $v_1(\gamma) < 0$ and $v_2(\gamma) < 0$. By (3) there exists a regular element $r \in R$ such that $v_1(r) = -v_1(\gamma)$ and $v_2(\gamma) = 0$. Then we can write $\gamma = (\gamma r)r^{-1}$. Since $v_1(\gamma r) = v_1(\gamma) + v_1(r) = 0$, $\gamma r \in R_{(P_1)}$. Also, $v_2(r^{-1}) = -v_2(r) = 0$ implies that $r^{-1} \in R_{(P_2)}$. □

Theorem 4.2 *Let R be an additively regular ring of Krull type. Denote by v_i the valuation associated with the valuation ring $R_{(P_i)}$, where P_i is the center of v_i for each i and by G_i the associated value group. Then the following are equivalent:*

(1) Each nonzero prime regular ideal of R is contained in at most one P_i.

(2) The "Strong Density Theorem" holds for finitely generated regular ideals of R; that is, for every finite collection of $P_1, \ldots, P_n$, and every choice of nonnegative elements $g_i \in G_i$, there is a finitely generated regular ideal A of R such that $v_i(A) = g_i$ for $1 \leq i \leq n$, and $v_j(A) = 0$ for all other P_j. Moreover, the finitely generated regular ideal A can be chosen to be 2-generated.

Proof (1) ⇒ (2) By Theorem 4.1, the "Density Theorem holds for regular elements in R, so we can find a regular element $r \in R$ such that $v_i(r) = g_i$ for $1 \leqslant i \leqslant n$. Since R is of Krull type, it is of finite character, and therefore there are at most finitely many centers $M_1, \ldots, M_t$ with corresponding valuations $w_1, \ldots, w_t$ at which r is positive. By the "Density Theorem, we can find a regular element $s \in R$ such that $v_i(s) = g_i$, for $1 \leqslant i \leqslant n$, and $w_j(s) = 0$, for $1 \leqslant j \leqslant t$. Then the ideal (r, s) has the desired properties.

(2) $\Rightarrow$ (1) Suppose that (1) fails; then there are prime regular ideals P_1 and P_2, centers of $R_{(P_1)}$ and $R_{(P_2)}$, respectively, containing P, where P is a nonzero prime regular ideal of R. We choose regular elements $a_1 \in P_1 - P$ and $a_2 \in P_2 - P$, and a regular element $b \in P$. Thus, we have $v_1(a_1) < v_1(b)$ and $v_2(a_2) < v_2(b)$. If there is a finitely generated regular ideal A of R such that $v_1(b) = v_1(A)$, then $AR_{(P_1)} = bR_{(P_1)} \subseteq PR_{(P_1)}$, which implies that $A \subseteq AR_{(P_1)} \cap R \subseteq PR_{(P)} \cap R = P$, so $AR_{(P_2)} \subseteq PR_{(P_2)} \subsetneq a_2R_{(P_2)}$. Thus, $v_2(A) > v_2(a_2)$, which contradicts (2). Therefore, (1) holds. □

If R is an additively regular ring of Krull type, then Φ (defined in Proposition 2.2) is a monomorphism from the group of $\mathcal{I}(R)$ of invertible fractional ideals of R *into* the cardinal direct sum $\coprod_{i\in I} G_i$, where G_is are the value groups. This monomorphism could become onto if and only if each nonzero prime regular ideal of R is contained in at most one P_i and the 2-generated regular ideal A, found in Theorem 4.2, is invertible. In fact, the following result shows that A is invertible.

Proposition 4.3 *Let R be an additively regular ring of Krull type. Denote by v_i the valuation associated with the valuation ring $R_{(P_i)}$, where P_i is the center of v_i for each i and by G_i the associated value group. The 2-generated regular ideal A, found in Theorem 4.2, is invertible.*

Proof The ring R is of finite character, and there are at most finitely many centers $P_1, \ldots, P_n$ with corresponding valuations v_i, $1 \leqslant i \leqslant n$, at which A is positive. By Theorem 4.1, we can choose a regular element $x \in R$ such that $v_i(x) = v_i(A)$ for all i. Let $Q_1, \ldots, Q_t$ with corresponding valuations w_j, $1 \leqslant j \leqslant t$, be the set of centers, other than P_i, at which $w_j(x)$ is positive. By Theorem 4.1, we can choose a regular element $y \in R$ such that $v_i(y) = 0$, $1 \leq i \leqslant n$, and $w_j(y) = w_j(x)$, $1 \leqslant j \leqslant t$. Let $M_1, \ldots, M_l$ with the corresponding valuations u_k, $1 \leqslant k \leqslant l$, be the set of centers, other than P_i and Q_j, at which $u_k(y)$ is positive. Again by Theorem 4.1, there exists a regular element $z \in R$ such that $v_i(z) = 0$, $1 \leqslant i \leqslant n$, $u_k(z) = 0$, and $w_j(z) = w_j(x)$, $1 \leqslant j \leqslant t$. We claim that $(x^{-1}y, x^{-1}z)$ is the inverse of A in R. Consider the ideal $B = A(x^{-1}y, x^{-1}z)$. We observe that, locally at each center P with the corresponding valuation v_P, $v_P(B_{(P)}) = 0$, implying that $B_{(P)} = R_{(P)}$, and hence $B = A(x^{-1}y, x^{-1}z) = R$. □

Corollary 4.4 *Let R be an additively regular ring of Krull type. Denote by v_i the valuation associated with the valuation ring $R_{(P_i)}$, where P_i is the center of v_i for each i and by G_i the associated value group. Let Φ be the mapping defined in Proposition 2.2. Then Φ is a monomorphism from the group $\mathcal{I}(R)$ of invertible fractional ideals of R onto the cardinal direct sum $\coprod_{i\in I} G_i$, where G_is are the value groups, if and only if each nonzero prime regular ideal of R is contained in at most one P_i.*

Proof By Theorem 2.3, the mapping Φ embeds $\mathcal{I}(R)$ into $\coprod_{i\in I} G_i$, and by Theorem 4.2, Φ maps onto if and only if each nonzero prime regular ideal of R is contained in at most one P_i. □

Acknowledgment

The author would like to thank the anonymous referee for a careful reading and numerous suggestions, which greatly improved this paper.

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