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CMMSE-Convergence Analysis for Operator Splitting Methods with Application to Burgers-Huxley Equation

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Abstract: We provide an error analysis of the operator splitting method of the Lie-Trotter type applied to the Burgers-Huxley equation $u_t + \alpha u u_x - \varepsilon u_{xx} = \beta(1-u)(u-\gamma)u$. We show that the Lie-Trotter splitting method converges with the expected rate in $H^s(\mathbb{R})$, where $H^s(\mathbb{R})$ is the Sobolev space and s is an arbitrary nonnegative integer. We split the equation into linear and nonlinear parts and apply numerical methods for these subproblems. We present errors and confirm the theoretical results with the numerical example.

Keywords: Operator splitting method, Burgers-Huxley equation, regularity, local and global error.

1 Introduction

Partial differential equations have great importance in most fields of science. Real-world physical systems, including gas dynamics, fluid mechanics, elasticity, relativity, ecology, neurology, thermodynamics, and many more are modeled by nonlinear partial differential equations (NPDEs). Burgers-Huxley equation (BHE) being a NPDE is a model that describes the interactions between reaction mechanisms, convection effects and diffusion transports.[2], with some special cases BHE reduces to Huxley equation [3] which describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals [4,5]. The other case is Burgers' equation which is a parabolic second order partial differential equation governs nonlinear process. This equation was firstly introduced by Bateman [6], then treated by Burgers [7] in a mathematical modelling of turbulence. These NPDEs are high importance in nonlinear physics.

There are many numerical methods which have been studied to compute the approximate solutions to the BHE such as spectral methods [19], Adomain decomposition method [20] which have been studied to solve the generalized Burgers-Huxley equation.In [8], they apply the operator splitting method to the Burgers-Huxley equation by solving two nonlinear subequations. In this paper, we divide the BHE into linear and nonlinear parts and solve easily. To prove the convergence of the Lie-Trotter splitting in H^s norm we use the local well-posedness of the BHE in H^s norm and boundedness of the exact solution of BHE in Sobolev spaces.

In [9], the KdV equation is studied and they apply Lie-Trotter and Strang splitting in order to have error estimates for convergence. They actively make use of the fact that solutions of KdV equation remain bounded in a Sobolev space and this, together with an bootstrap argument guarantees the existence of a uniform choice of time step Δt that prevents the solution from any Burgers step from blowing up. On the other hand, [10] studies equation with a Burgers type nonlinearity including the KdV equation. They make use of the fact that solutions of Burgers type equations remain bounded in a Sobolev space and perform an analysis which identifies error terms in the local error as quadrature errors which are estimated via Lie commutator bounds. In [11] and [12], similar analyses are studied for linear evolution equations and for nonlinear Schrödinger equations, respectively. In this paper, we follow a similar approach to [10] and [17].

This paper is organized as follows. After this introduction in Section 2, we give the idea of the operator splitting method and apply Lie-Trotter splitting to the BHE. In Section 3, we give two hypotheses which are connected with the local well-posedness and boundedness of the solution of the BHE. Section 4 proves the regularity results for the BHE. Furthermore in Section 5 by using the regularity results we prove the local and global error estimates in time. In identifying the local error terms we use quadrature errors. Finally, in Section 6 numerical

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results are given and the correct convergence rates are proved for Lie-Trotter splitting method.

2 Application the Lie-Trotter Splitting to the Burgers-Huxley Equation

The idea of operator splitting, [14,15,16] is widely used for the approximation of partial differential equations. The basic idea is based on splitting a complex problem into simpler sub-problems, each of which is solved by an efficient method. One of the reasons for the popularity of operator splitting is the use of dedicated special numerical techniques for each of the equations.

We focus our attention on the case of linear and nonlinear operators such as,

$$u_t = Au(t) + B(u(t)),\ with\ t \in [0,T],\ u|_{t=t_0} = u_0 \tag{1}$$

We employ Lie-Trotter splitting method to the one-dimensional Burgers-Huxley equation,

$$u_t + \alpha u u_x - \varepsilon u_{xx} = \beta(1-u)(u-\gamma)u, \tag{2}$$

with the initial condition

$$u|_{t=t_0} = u_0 \tag{3}$$

where $t > 0$, $\alpha, \beta \geq 0$, $0 < \varepsilon \leq 1$ and $0 < \gamma < 1$. When $\alpha = 0$ and $\varepsilon = 1$, equation (2) reduces to Huxley equation and when $\beta = 0$, reduces to Burgers' equation.

With the help of the operator splitting, we break the (1) into linear diffusion equation and nonlinear reaction equation. In this latter type of the operator splitting, the simpler equations are solved and then recoupled over the initial conditions in delicate ways to preserve a certain accuracy. We denote by $u(t) = \Phi^t_{A+B}(u_0)$ is the solution at the time t of (1) with given initial condition and the approximate split solution is denoted by u_n, at $t = n\Delta t \leq T$, as $\Delta t \to 0$, where $u_{n+1} = \Phi^{\Delta t}_A(\Phi^{\Delta t}_B(u_n))$, $n = 0,1,2,\ldots$.

In our case we split the equation (2) into two subequations,

$$v_t = Av = \varepsilon v_{xx} \tag{4}$$

and

$$w_t = B(w) = \beta(1-w)(w-\gamma)w - \alpha w w_x \tag{5}$$

acting on appropriate Sobolev spaces.

3 Error Bounds for Lie-Trotter Splitting

In the begining of the analysis, we assume that the solutions to the BHE are locally well-posed and bounded. Thus, the following hypotheses are about the local well-posedness of the solutions to (2) and boundedness of the solution and the initial condition in Sobolev spaces.
Hypothesis 3.1. For a fixed time T, there exists $M > 0$such that for all u_0 in $H^k(\mathbb{R})$ with $\|u_0\| \leq M$, there exists a unique strong solution u in $C([0,T],H^k)$ of (2). In addition, for the initial data u_0 there exists a constant $K(M,T) < \infty$, such that

$$\|\tilde{u}(t) - u(t)\|_{H^k} \leq K(M,T)\|\tilde{u}_0 - u_0\|_{H^k} \tag{6}$$

for two arbitrary solutions u and $\tilde{u}$, corresponding to two different initial data $\tilde{u}_0$ and u_0. This well-posedness result holds, with sufficiently small $t \leq T = T(M)$ for any M.
Hypothesis 3.2. The solution $u(t)$ and the initial data u_0 of (2) are both in $H^k(\mathbb{R})$, and are bounded as

$$\|u(t)\|_{H^k} \leq M < \rho \text{ and } \|u_o\|_{H^k} \leq C < \infty, \tag{7}$$

for $0 \leq t \leq T$.

Let s be a positive integer, we define following set of integers such that,

$$s \geq 1, \quad m = s+3, \quad ,n = s+1 = m-2 \tag{8}$$

We specify for which integers the hypothesis should hold in the lemmas and theorems for the Lie-Trotter splitting method.

4 Regularity results for Burgers-Huxley Equation

We will present and prove several results to estimate the local error for the Lie-Trotter splitting for the Burgers-Huxley equation. We need to show that there exists a small time step Δt for the solutions $\Phi^t_A(v_0)$ and $\Phi^t_B(w_0)$ in a Sobolev spaces. The following results have an importance of proving the convergence rate of Lie-Trotter splitting.

4.1 Results for the Nonlinear Part

Lemma 1.*For m and n in (8) assume the solution $\Phi^t_B(w_0) = w(t)$ of (5) with initial data w_0 in $H^m(\mathbb{R})$, satisfies $\|\Phi^t_B(w_0)\|_{H^n} \leq \alpha$ for $0 \leq t \leq \Delta t$. Then $\Phi^t_B(w_0)$ is in $H^m(\mathbb{R})$ and in particular*

$$\|\Phi^t_B(w_0)\|_{H^m} \leq e^{c\alpha_1 t}\|w_0\|_{H^m}, \tag{9}$$

where $\alpha_1 = (C + 2C\alpha + C\alpha^2), C$ is a general constant and c is independent of w_0.

Proof. From the definition of norm $H^m(\mathbb{R})$, we find that $w(t)$ satisfies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Phi_B^t(w_0)\|_{H^m}^2\\
&=\frac{1}{2}\frac{d}{dt}\|w\|_{H^m}^2=\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{m}\int_{\mathbb{R}}\partial_x^j w\partial_x^j w_t dx\\
&=(w,w_t)_{H^m}=(w,w(1-w)(w-\gamma)-ww_x)_{H^m}\\
&=\beta(1+\gamma)\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{j}{k}\int_{\mathbb{R}}\partial_x^j w\partial_x^k w\partial_x^{j-k} wdx\\
&-\beta\sum_{j=0}^{m}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{j}{k}\binom{k}{l}\int_{\mathbb{R}}\partial_x^j w\partial_x^l w\partial_x^{k-l} w\partial_x^{j-k} wdx\\
&-\beta\gamma\sum_{j=0}^{m}\int_{\mathbb{R}}\partial_x^j w\partial_x^j wdx-\alpha\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{j}{k}\int_{\mathbb{R}}\partial_x^j w\partial_x^{k+1} w\partial_x^{j-k} w
\end{aligned}\tag{10}$$

We investigate the each parts for different cases.

Case 1: For $j<m$ and $k<j$, we obtain for the first term of (10)

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\partial_x^j w\partial_x^k w\partial_x^{j-k} wdx\right|\\
&\le\int_{\mathbb{R}}|\partial_x^j w\partial_x^k w\partial_x^{j-k} w|dx\\
&\le\|\partial_x^j w\|_{L^\infty}\|\partial_x^{\max\{k,j-k\}} w\|_{L^2}\|\partial_x^{\min\{k,j-k\}} w\|_{L^2}\\
&\le C\|\ w\|_{H^m}\|\ w\|_{H^m}\|\ w\|_{H^n}\\
&\le C\alpha\|\ w\|_{H^m}^2
\end{aligned}\tag{11}$$

where we have used Sobolev inequality and the fact that

$$\max\{k,j-k\}\le j+1\le m$$
$$\min\{k,j-k\}\le\frac{j}{2}\le\frac{m}{2}=\frac{s-1}{2}+2\le s+1=m-2=n$$

since $m\ge 4$.

For the second term of (10),

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\partial_x^j w\partial_x^l w\partial_x^{k-l} w\partial_x^{j-k} wdx\right|\\
&\le\int_{\mathbb{R}}|\partial_x^j w\partial_x^l w\partial_x^{k-l} w\partial_x^{j-k} w|dx\\
&\le\|\partial_x^j w\|_{L^\infty}\|\partial_x^{j-k} w\|_{L^\infty}\int_{\mathbb{R}}|\partial_x^l w\partial_x^{k-l} w|dx\\
&\le\|w\|_{H^m}\|w\|_{H^m}\|\partial_x^l w\|_{L^2}\|\partial_x^{k-l} w\|_{L^2}\\
&\le C\|w\|_{H^m}^2\|w\|_{H^m}\|w\|_{H^{k-m}}\\
&\le C\alpha^2\|w\|_{H^m}^2
\end{aligned}\tag{12}$$

If we take $l<k\le n$ and $k-l<n$.

For the third term of (10),

$$\begin{aligned}
\left|\int_{\mathbb{R}}\partial_x^j w\partial_x^j wdx\right|&\le\int_{\mathbb{R}}|\partial_x^j w\partial_x^j w|dx\\
&\le\|\partial_x^j w\|_{L^2}|\partial_x^j w\|_{L^2}\\
&\le C\|w\|_{H^m}^2
\end{aligned}\tag{13}$$

The last term of the (10) we have the bound

$$\begin{aligned}
\left|\int_{\mathbb{R}}\partial_x^j w\partial_x^{k+1} w\partial_x^{j-k} wdx\right|&\le\|\partial_x^j w\|_{L^\infty}\|w\|_{H^m}\|w\|_{H^n}\\
&\le C\alpha\|w\|_{H^m}^2
\end{aligned}\tag{14}$$

see [17].

Case 2: For $j=m$, we obtain for the first term of (10)

$$\begin{aligned}
\left|\int_{\mathbb{R}}\partial_x^j w\partial_x^k w\partial_x^{j-k} wdx\right|&\le\|\partial_x^k w\|_{L^\infty}\|\partial_x^m w\|_{L^2}\|\partial_x^{m-k} w\|_{L^2}\\
&\le C\|\partial_x w\|_{H^k}\|w\|_{H^m}\|w\|_{H^{m-k}}\\
&\le C\|w\|_{H^{k+1}}\|w\|_{H^m}^2
\end{aligned}\tag{15}$$

To get a bound we investigate this inequality in two cases; when $k+1\le n$ and when $k=n$. For the first case we obtain

$$\left|\int_{\mathbb{R}}\partial_x^j w\partial_x^k w\partial_x^{j-k} wdx\right|\le C\alpha\|w\|_{H^m}^2\tag{16}$$

For the second case, we get

$$\begin{aligned}
\left|\int_{\mathbb{R}}\partial_x^j w\partial_x^k w\partial_x^{j-k} wdx\right|&\le\|w\|_{H^{n+1}}\|w\|_{H^m}\|w\|_{H^{m-n}}\\
&\le C\alpha\|w\|_{H^m}^2
\end{aligned}\tag{17}$$

here we have used that $n+1\le n+2\le m$, and $m-n=2\le s+1=n$.

We are left with 2 cases; $k\le m$ and $k=m=j$. For the first case we get,

$$\begin{aligned}
\left|\int_{\mathbb{R}}\partial_x^j w\partial_x^k w\partial_x^{j-k} wdx\right|&\le\|\partial_x^m w\|_{L^2}\|\partial_x^k w\|_{L^2}\|\partial_x^{m-k} w\|_{L^\infty}\\
&\le C\|w\|_{H^m}\|w\|_{H^m}\|w\|_{H^{m-k+1}}\\
&\le C\alpha\|w\|_{H^m}^2
\end{aligned}\tag{18}$$

because $m-k+1<m-n\le 2\le n$. For the second case, we have

$$\begin{aligned}
\left|\int_{\mathbb{R}}\partial_x^m w\partial_x^m wwdx\right|&\le\|w\|_{L^\infty}\|\partial_x^m w\|_{L^2}\|\partial_x^m w\|_{L^2}\\
&\le C\|w\|_{H^n}\|w\|_{H^m}^2\\
&\le C\alpha\|w\|_{H^m}^2.
\end{aligned}\tag{19}$$

For the second term of (10),

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\partial_x^m w\partial_x^l w\partial_x^{k-l} w\partial_x^{m-k} wdx\right|\\
&\le\|\partial_x^l w\|_{L^\infty}\|\partial_x^{k-l} w\|_{L^\infty}\|\partial_x^m w\|_{L^2}\|\partial_x^{m-k} w\|_{L^2}\\
&\le C\|w\|_{H^{l+1}}\|w\|_{H^{k-l+1}}\|w\|_{H^m}\|w\|_{H^{m-k}}
\end{aligned}\tag{20}$$

The above inequality is divided in two cases; when $l+1\le n,\ k-l+1\le n$ and $l+1\le n,\ k=n$. For the first case we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\partial_x^m w\partial_x^l w\partial_x^{k-l} w\partial_x^{m-k} wdx\right|\\
&\le C\|w\|_{H^n}\|w\|_{H^n}\|w\|_{H^m}\|w\|_{H^m}\\
&\le C\alpha^2\|w\|_{H^m}^2
\end{aligned}\tag{21}$$

For the second case we have

$$\begin{aligned}&\left|\int_{\mathbb{R}} \partial_x^m w \partial_x^l w \partial_x^{k-l} w \partial_x^{m-k} w dx\right| \\ &\leq C\|w\|_{H^n}\|w\|_{H^m}\|w\|_{H^m}\|w\|_{H^n} \\ &\leq C\alpha^2\|w\|_{H^m}^2 \end{aligned} \quad (22)$$

Since, $n-l+1 \leq m$, and $m-n \leq 2 \leq s+1 = n$.

We are left with three cases; $l+1 = k = n$, $l+1 \leq m$, with $m = n$ and $k = m = j = l$. For the first case, we obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}} \partial_x^m w \partial_x^l w \partial_x^{k-l} w \partial_x^{m-k} w dx\right| \\ &\leq \|\partial_x^{m-k} w\|_{L^\infty}\|\partial_x^l w\|_{L^\infty}\|\partial_x^m w\|_{L^2}\|\partial_x^{k-l} w\|_{L^2} \\ &\leq C\|w\|_{H^{m-k+1}}\|w\|_{H^{l+1}}\|w\|_{H^m}\|w\|_{H^{k-l}} \\ &\leq C\|w\|_{H^m}\|w\|_{H^n}\|w\|_{H^m}\|w\|_{H^n} \end{aligned} \quad (23)$$

Since, $n-l \leq n$, $m-k+1 \leq m$. For the second case we get the same result, but now we use that $m-k+1 \leq m-n \leq 2 \leq n$.

For the third case,

$$\begin{aligned}\left|\int_{\mathbb{R}} \partial_x^m w \partial_x^m w w w dx\right| &\leq \int_{\mathbb{R}} |(\partial_x^m w)^2 w^2| dx \leq \|w\|_{L^\infty}^2 \|\partial_x^m w\|_{L^2}^2 \\ &\leq C\|w\|_{H^n}^2\|w\|_{H^m}^2 \\ &\leq C\alpha^2\|w\|_{H^m}^2 \end{aligned} \quad (24)$$

For the third term of (10),

$$\begin{aligned}\left|\int_{\mathbb{R}} \partial_x^m w \partial_x^m w dx\right| &\leq \|\partial_x^m w\|_{L^2}\|\partial_x^m w\|_{L^2} \\ &\leq C\|w\|_{H^m}^2 \end{aligned} \quad (25)$$

Finally, the last term of the (10) we have the bound

$$\left|\int_{\mathbb{R}} \partial_x^m w \partial_x^{k+1} w \partial_x^{m-k} w dx\right| \leq C\alpha\|w\|_{H^m}^2 \quad (26)$$

see [17].

All in all we get, by summing up the estimates, the following inequality

$$\frac{d}{dt}\|w(t)\|_{H^m}^2 = \|w(t)\|_{H^m}\frac{d}{dt}\|w(t)\|_{H^m} \leq c\beta\|w(t)\|_{H^m}^2 \quad (27)$$

which leads to

$$\frac{d}{dt}\|w(t)\|_{H^m} \leq c\alpha_1\|w(t)\|_{H^m} \quad (28)$$

where $\alpha_1 = (C + 2C\alpha + C\alpha^2)$. This result concludes the proof.[10]

Lemma 2. *Assume $\|w_0\|_{H^k} \leq K$ for some $k \geq 1$. Then there exists $\bar{t}(K) > 0$ such that $\|\Phi_B^t(w_0)\|_{H^k} \leq 2K$ for $0 \leq t \leq \bar{t}(K)$.*

Proof. By doing the same calculations as in the proof of the Lemma 1 with k instead of m and using the bound for u_0 in $H^k(\mathbb{R})$, we arrive with the following inequality

$$\|w(t)\|_{H^k}\frac{d}{dt}\|w(t)\|_{H^k} \leq c\|w(t)\|_{H^k}^4, \quad (29)$$

which simplifies to

$$\frac{d}{dt}\|w(t)\|_{H^k} \leq c\|w(t)\|_{H^k}^3. \quad (30)$$

The result follows by comparing with the solution of the differential equation $y' = cy^3$.

Lemma 3. *If $\|w_0\|_{H^{s+2}} \leq C$ for $s \geq 1$, then there exists $\bar{t}$ depending on C, such that the solution $w(t)$ of the (5) is $C^2([0,\bar{t}], H^s)$.*

Proof. Let t be in $[0,\bar{t}]$, with $\bar{t}$ from Lemma 2, and define

$$\tilde{w}(t) = w_0 + tB(w_0) + \int_0^t (t-s) dB(w(s))[B(w(s))] ds, \quad (31)$$

where $dB(.)[.]$ is the Fréchet derivative which is given as follows,

$$\begin{aligned} dB(w(s))[B(w(s))] &= -3\beta w^2 B(w) + 2\beta(1+\gamma) w B(w) \\ &- \beta\gamma B(w) - \alpha w B(w)_x \\ &- \alpha B(w) w_x \end{aligned} \quad (32)$$

Calculating the second derivative of $\tilde{w}$, gives

$$\begin{aligned}\tilde{w}_{tt} &= dB(w(s))[B(w(s))] \\ &= -3\beta w^2 B(w) + 2\beta(1+\gamma) w B(w) - \beta\gamma B(w) - \alpha w B(w)_x \\ &- \alpha B(w) w_x \end{aligned} \quad (33)$$

from which we have that $\tilde{w}$ is in $C^2([0,\bar{t}], H^s)$. To prove that $\tilde{w} = w$, we must show that the two functions satisfies the same differential equation and the same initial conditions. By differentiating (5) with respect to t, we get

$$\begin{aligned} w_{tt} &= B(w)_t = (-\beta w^3 + \beta(1+\gamma) w^2 - \beta\gamma w - \alpha w w_x)_t \\ &= -3\beta w^2 w_t + 2\beta(1+\gamma) w w_t - \beta\gamma w_t - \alpha w_t w_x - \alpha w w_{xt} \\ &= \tilde{w}_{tt}, \end{aligned}$$

From definition of $\tilde{w}$ we see that $\tilde{w}(0) = u_0$ and $\tilde{w}_t(0) = B(u_0) = w_t$. Thus we have shown that $w = \tilde{w}$.

5 Local and Global errors in H^s space

Lemma 4. *Let $s \geq 1$ be an integer and (7) holds for $k = s+2$ for the solution $u(t) = \Phi_{A+B}^{\Delta t}(u_0)$ of (2). If the initial data u_0 is in $H^{s+2}(\mathbb{R})$, then the local error of the Lie-Trotter splitting is bounded in $H^s(\mathbb{R})$ by*

$$\|\Phi_A^{\Delta t}(\Phi_B^{\Delta t}(u_0)) - \Phi_{A+B}^{\Delta t}(u_0)\|_{H^s} \leq C\Delta t^2, \quad (34)$$

where C only depends on $\|u_0\|_{H^{s+2}}$.

Proof. We write $e^{tA}v = \Phi_A^t(v)$ to denote the linearity of the operator A. We start with

$$B(\varphi(s)) - B(\varphi(0)) = \int_0^s dB(\varphi(\rho))[\dot{\varphi}(\rho)] d\rho. \quad (35)$$

where

$$\varphi(\rho) = \Phi_A^{(s-\rho)}(u(\rho)), \quad (36)$$

$$\dot{\varphi}(\rho) = \Phi_A^{(s-\rho)}(B(u(\rho))). \quad (37)$$

Hence we get,

$$B(u(s)) = B(\Phi_A^s(u_0)) + \int_0^s dB(\Phi_A^{(s-\rho)}(u(\rho)))[\Phi_A^{(s-\rho)}(B(u(\rho)))]d\rho \quad (38)$$

where we have used that $\varphi(0) = \Phi_A^s(u(0)) = \Phi_A^s(u_0)$. From the variation of constants formula we have the exact solution of (2) such that,

$$\Phi_{A+B}^t(u_0) = \Phi_A^t(u_0) + \int_0^t \Phi_A^{(t-s)}(B(u(s)))ds \quad (39)$$

To find the exact solution after one step, we insert (38) into (39) and evaluate $t = \Delta t$,

$$u(\Delta t) = \Phi_A^{\Delta t}(u_0) + \int_0^{\Delta t} \Phi_A^{(\Delta t-s)}(B(\Phi_A^s(u_0)))ds + E_1 \quad (40)$$

where

$$E_1 = \int_0^{\Delta t}\int_0^s \Phi_A^{(\Delta t-s)}(dB(\Phi_A^{(s-\rho)}u(\rho))[\dot{\varphi}(\rho)])d\rho ds. \quad (41)$$

One step with Lie-Trotter splitting is

$$u_1 = \Phi_A^{\Delta t}(\Phi_B^{\Delta t}(u_0)). \quad (42)$$

Using the Taylor series expansion we get,

$$u_1 = \Phi_A^{\Delta t}(u_0) + \Delta t \Phi_A^{\Delta t}(B(u_0)) + E_2 \quad (43)$$

with

$$E_2 = \Delta t^2 \int_0^1 (1-\theta)\Phi_A^{\Delta t} dB(\Phi_B^{\Delta t\theta}(u_0))[B(\Phi_B^{\Delta t\theta}(u_0))]d\theta. \quad (44)$$

The error between the exact and the split solution, after one step becomes,

$$\begin{aligned} u_1 - u(\Delta t) &= \Delta t \Phi_A^{\Delta t}(B(u_0)) - \int_0^{\Delta t} \Phi_A^{(\Delta t-s)}(B(\Phi_A^s(u(s))))ds \\ &+ (E_2 - E_1). \end{aligned} \quad (45)$$

by defining,

$$h(s) = \Phi_A^{(\Delta t-s)}(B(\Phi_A^s(u_0))), \quad (46)$$

we can rewrite equation (45) as follows,

$$u_1 - u(\Delta t) = \int_0^{\Delta t} K_R(t)h'(t)dt + (E_2 - E_1) \quad (47)$$

By using the substitution $\theta = t/\Delta t$, the integral is transformed to

$$\begin{aligned} \int_0^{\Delta t} K_R(t)h'(t)dt &= (\Delta t)^2 \int_0^1 (\theta - 1)h'(\theta\Delta t)d\theta \\ &= (\Delta t)^2 K_R(\theta)h'(\theta\Delta t)d\theta. \end{aligned} \quad (48)$$

Then, applying the H^s norm and using the triangle inequality,

$$\begin{aligned} &\|u_1 - u(\Delta t)\|_{H^s} \\ &\le (\Delta t)^2 \int_0^1 \|K_R(\theta)h'(\theta\Delta t)\|_{H^s} + \|(E_2 - E_1)\|_{H^s} \\ &\le (\Delta t)^2 \int_0^1 \|K_R(\theta)h'(\theta\Delta t)\|_{H^s} + \|E_2\|_{H^s} + \|E_1\|_{H^s}. \end{aligned} \quad (49)$$

where K_R is bounded kernel. Here $h'(s) = -\Phi_A^{(\Delta t-s)}[A,B](\Phi_A^s(u_0))$ with double Lie commutator

$$[A,B] = dA(v)[B(v)] - dB(v)[A(v)] \quad (50)$$

We know that $\Phi_A^{\Delta t}(u_0)$ do not increase the Sobolev norm, and therefore it is sufficient to consider the commutator for a general vector v. Using (4) and (5), we write

$$\begin{aligned} [A,B](v) &= -6vv_x^2 - 3v^2v_{xx} + 2(1+\gamma)v_x^2 + 2(1+\gamma)vv_{xx} \\ &- \gamma v_{xx} - 2v_xv_{xx} - v_xv_{xx} - vv_{xxx}(-3v^2v_{xx} \\ &+ 2(1+\gamma)vv_{xx} - \gamma v_{xx} - vv_{xxx} - v_{xx}v_x) \end{aligned} \quad (51)$$

Hence we get,

$$\begin{aligned} &\|h'(s)\|_{H^s} = \| - 6vv_x^2 + 2(1+\gamma)v_x^2 - 2v_xv_{xx}\|_{H^s} \\ &\le 6\|v\|_{H^s}\|\partial_x v\|_{H^s}^2 + 2(1+\gamma)\|\partial_x v\|_{H^s}^2 + 2\|\partial_x v\|_{H^s}^2\|\partial_x^2 v\|_{H^s}^2 \\ &\le 6\|v\|_{H^s}\|v\|_{H^{s+1}}^2 + (4+2\gamma)\|v\|_{H^{s+1}}^2 + \|v\|_{H^{s+1}}\|v\|_{H^{s+2}} \\ &\le 6\|v\|_{H^{s+2}}^3 + (4+2\gamma)\|v\|_{H^{s+2}}^2 \le \eta\|v\|_{H^{s+2}}^3 \end{aligned} \quad (52)$$

Thus, by using $v = \Phi_A^s(u_0)$, we get

$$\|h'(s)\|_{H^s} \le \eta\|\Phi_A^s(u_0)\|_{H^{s+2}}^3 \le \eta\|u_0\|_{H^{s+2}}^3. \quad (53)$$

Next, we will find the error bound for E_1 in (41),

$$\begin{aligned} &\|E_1\|_{H^s} \\ &\le \int_0^{\Delta t}\int_0^s \|\Phi_A^{(\Delta t-s)}(dB(\Phi_A^{(s-\rho)})(u(\rho)))[\tilde{B}(u(\rho))]\|_{H^s} d\rho ds \\ &\le \int_0^{\Delta t}\int_0^s \|dB(\Phi_A^{(s-\rho)})(u(\rho))[\Phi_A^{(s-\rho)}(B(u(\rho)))]\|_{H^s} d\rho ds \\ &\le \int_0^{\Delta t}\int_0^s \beta\| - 3(\Phi_A^{(s-\rho)}(u(\rho)))^2(\Phi_A^{(s-\rho)}(B(u(\rho))))\|_{H^s} d\rho ds \\ &+2\beta(1+\gamma)\|(\Phi_A^{(s-\rho)}(u(\rho)))(\Phi_A^{(s-\rho)}(B(u(\rho))))\|_{H^s} d\rho ds \\ &+\gamma\beta\int_0^{\Delta t}\int_0^s \|(\Phi_A^{(s-\rho)}(B(u(\rho))))\|_{H^s} d\rho ds \\ &+\int_0^{\Delta t}\int_0^s \|(\Phi_A^{(s-\rho)}(u(\rho))\Phi_A^{(s-\rho)}B(u(\rho)))_x\|_{H^s} d\rho ds \end{aligned} \quad (54)$$

We can rewrite the above inequality for simplicity,

$$\|E_1\|_{H^s} \le I_1 + I_2 + I_3 + I_4 \quad (55)$$

We obtain the following bounds by using the Banach algebra property of $H^s(\mathbb{R})$ and non-increasing of the solution of (4),

$$\begin{aligned} I_1 &\le \int_0^{\Delta t}\int_0^s \|u(\rho)\|_{H^s}^2\|B(u(\rho))\|_{H^s} d\rho ds \\ &\le \int_0^{\Delta t}\int_0^s \|u(\rho)\|_{H^s}^2(\|u(\rho)\|_{H^s}^3 + (1+\gamma)\|u(\rho)\|_{H^s}^2 \\ &+ \int_0^{\Delta t}\int_0^s \gamma\|u(\rho)\|_{H^s} + \|u(\rho)\|_{H^s}\|u(\rho)_x\|_{H^s})d\rho ds \\ &\le \|u(\rho)\|_{H^s}^5 + (1+\gamma)\|u(\rho)\|_{H^s}^5 + \gamma\|u(\rho)\|_{H^s}^3 \\ &+ \|u(\rho)\|_{H^s}^3\|u(\rho)\|_{H^{s+1}} \\ &\le C\int_0^{\Delta t}\int_0^s R^5 d\rho ds = CR^5\int_0^{\Delta t} s ds = CR^5(\Delta t)^2, \end{aligned} \quad (56)$$

$$I_2 \leq \int_0^{\Delta t}\int_0^s \|u(\rho)\|_{H^s}\|B(u(\rho))\|_{H^s} d\rho ds \\ \leq CR^4(\Delta t)^2, \tag{57}$$

$$I_3 \leq \int_0^{\Delta t}\int_0^s \|B(u(\rho))\|_{H^s} d\rho ds \\ \leq CR^3(\Delta t)^2, \tag{58}$$

For the last integral, we can write the bound as, (see [17]).

$$I_4 \leq CR^3(\Delta t)^2. \tag{59}$$

Finally, we get

$$\|E_1\|_{H^s} \leq C(R^5+R^4+2R^3)(\Delta t)^2 \leq M(\Delta t)^2. \tag{60}$$

The third and the last term is estimated similarly as the second term.

$$\|E_2\|_{H_s} \leq (\Delta t)^2\int_0^1 \|dB(\Phi_B^{\Delta t\theta}(u_0))[B(\Phi_B^{\Delta t\theta}(u_0))]\|_{H_s} d\theta \\ \leq (\Delta t)^2\int_0^1 \|3(\Phi_B^{\Delta t\theta}(u_0))^2(B(\Phi_B^{\Delta t\theta}(u_0)))\|_{H_s} d\theta \tag{61}$$

By doing the similar approach for E_1 we find following bound for E_2. The only difference is the use of the regularity result for the nonlinear part. The bound for E_2 is given as follows,

$$\|E_2\|_{H_s} \leq C(\Delta t)^2(M_1+M_2+M_3), \tag{62}$$

where $M_1 = (R^5+R^4+2R^3)$, $M_2 = (R^4+2R^3+R^2)$ and $M_3 = (R^3+2R^2+R)$.

Hence, by combining the estimates, we obtain the following bound for the local error,

$$\|u_1 - u(\Delta t)\|_{H^s} \leq c(\Delta t)^2, \tag{63}$$

where c depends only on the initial condition and Δt is sufficiently small.

Theorem 1.*Suppose that the exact solution $u(\cdot,t)$ of Equation (2) is in H^{s+2} for $0 \leq t \leq T$. Then Lie-Trotter splitting solution u_n has first order global error for $\Delta t < \Delta t$ and $t_n = n\Delta t \leq T$,*

$$\|u_n - u(\cdot,t_n)\|_{H^s} \leq G\Delta t, \tag{64}$$

where G only depends on $\|u_0\|_{H^{s+2}}$ and T.

Proof. The "Lady Windermere's Fan" is used in the proof see [13]. Regularity result and local error are given in Lemma 1 and Lemma 2. By using these results we prove the global convergence of the Lie-Trotter splitting with the help of an induction argument. Let us take the exact solution $u(t_n) = \Phi_{A+B}^{(n-k)\Delta t}(u(t_k))$ of (2) and Lie-Trotter solution be $u_n = \Psi^{\Delta t}(u_{n-1}) = \Phi_A^{\Delta t} \circ \Phi_B^{\Delta t}(u_{n-1})$, $n = 1,2,\ldots$ By using the same approach in [10] we get the following estimate,

$$\|u_n - u(\cdot,t_n)\|_{H^s} \\ \leq \sum_{k=0}^{n-1} \|\Phi^{(n-k-1)\Delta t}(\Psi^{\Delta t}(u(t_k)) - \Phi^{\Delta t}(u(t_k)))\|_{H^s} \\ \leq \sum_{k=0}^{n-1} K(R,T)\|\Psi^{\Delta t}(u(t_k)) - \Phi^{\Delta t}(u(t_k))\|_{H^s} \\ \leq nK(R,T)c_1(C_0)(\Delta t)^2 \\ \leq TK(R,T)c_1(C_0)(\Delta t) \tag{65}$$

by using the previous results and $n\Delta t \leq T$. This completes the proof.

6 Numerical results

In order to illustrate the efficiency and accuracy of the operator splitting method, we work on the Burgers-Huxley equation in the form (2) for $\alpha = \beta = 1$, $\gamma = 0.5$, with initial and boundary conditions as follows [18],

$$u(x,0) = sin(\pi x),\ \ 0 \leq x \leq 1 \\ u(0,t) = u(1,t) = 0,\ \ 0 \leq t \leq T. \tag{66}$$

When we apply the operator splitting method on (2), we obtain the two subequations,

$$v_t = A(v) = \varepsilon v_{xx} \\ w_t = B(w) = \beta(1-w)(w-\gamma)w - \alpha w w_x$$

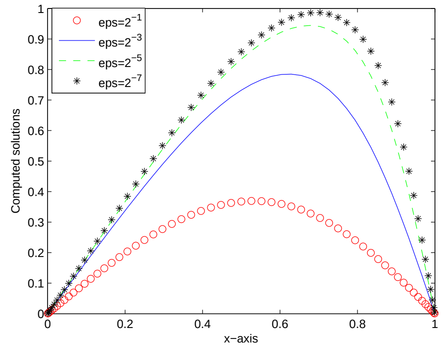
which are solved subsequently for small time steps Δt.

For the space discretization, we consider the Chebyshev differentiation matrices for the derivatives u_x and u_{xx}. Third order Semi-implicit Runge-Kutta method is used for the time integration, which is well-known for the numerical stability and less computational cost. Since there is no exact solution to (2), we compare the results to the higher order exponential method to prove convergence of the Lie-Trotter splitting and show the correct convergence rates.

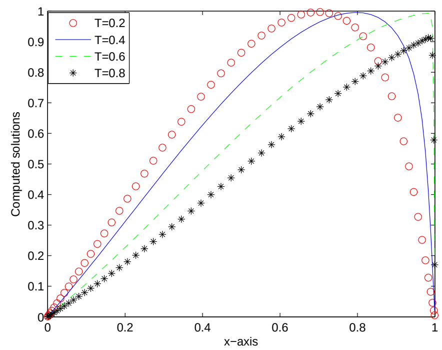
The time step length $\Delta t = 0.001$ is used for the numerical experiment. The Figure 1(a) and Figure 1(b) show the layer behaviour of the problem at different values of time t and ε.

Table 1: Estimated errors and convergence rates for $\varepsilon = 2^{-3}$ at fixed time T. (SR=Splitting Runtime, NR=Nonsplitting Runtime)

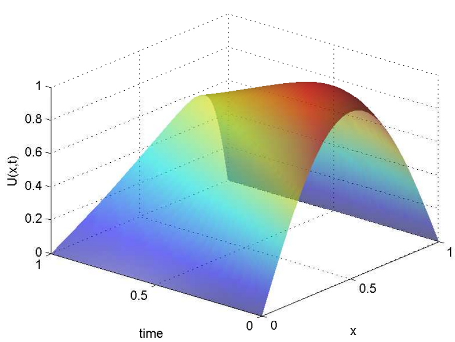
time step	L_1	L_2	L_∞	*SR*	*NR*
0.02	0.0566	0.0113	0.0035	0.5858	2.0430
0.01	0.0284	0.0057	0.0018	0.7763	4.0540
0.002	0.0057	0.0011	$3.5230e-04$	1.1576	5.3499
0.001	0.0028	$5.7345e-04$	$1.7619e-04$	2.1738	15.5342
0.0005	0.0014	$2.8577e-04$	$8.8101e-05$	4.0955	16.1621



(a) Computed solutions of BHE for different values of ε at T$=0.2$.



(b) Computed solutions of BHE for different values of time at $\varepsilon = 2^{-9}$.



(c) Computed solutions of BHE for $\Delta t = 0.001$ and $\varepsilon = 2^{-5}$.

Fig. 1: Computed solutions of BHE

The errors are given in Table 2 and in the Figure 2, we give the expected orders. We observe that Lie-Trotter splitting obtain numerical convergence results which is correct with the theoretical results. We also check the running times for Lie-Trotter splitting and nonsplit solution in Table 1. We observe that, Lie-Trotter splitting results in faster CPU runtimes.

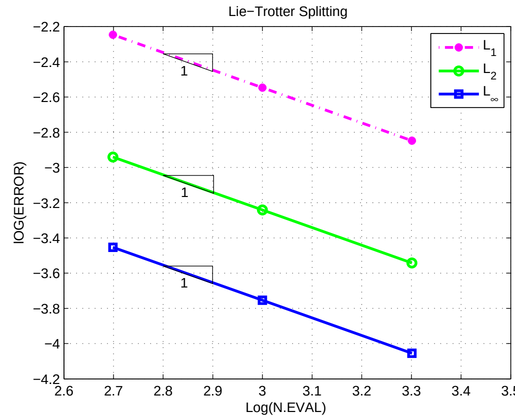


Fig. 2: Order of L_1, L_2 and L_∞ errors.

Table 2: Estimated errors and convergence rates for $\varepsilon = 2^{-3}$ and $\varepsilon = 2^{-7}$.

	$\varepsilon = 2^{-3}$			$\varepsilon = 2^{-7}$		
T	$\Delta t = 0.001$	$\Delta t = 0.002$	*Order*	$\Delta t = 0.001$	$\Delta t = 0.002$	*Order*
0.2	$8.0990e-04$	0.0016	0.9823	0.0016	0.0031	0.9542
0.4	$8.2993e-04$	0.0017	1.0345	0.0109	0.0212	0.9597
0.6	$5.3939e-04$	0.0011	1.0281	0.0137	0.0263	0.9409
0.8	$3.1277e-04$	$6.2554e-04$	1	0.0096	0.0186	0.9542
1	$1.7619e-04$	$3.5230e-04$	0.9997	0.0065	0.0126	0.9549

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