Journal of Algebra 403 (2014) 19–28

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Hereditary rings with countably generated cotorsion envelope

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ARTICLE INFO

Article history:
Received 10 April 2011
Available online 22 January 2014
Communicated by Efim Zelmanov

MSC:
16D90

Keywords:
Cotorsion module
Hereditary ring
Semilocal ring
Indecomposable module

ABSTRACT

Let R be a left hereditary ring. We show that if the left cotorsion envelope $C({}_R R)$ of R is countably generated, then R is a semilocal ring. In particular, we deduce that $C({}_R R)$ is finitely generated if and only if R is a semiperfect cotorsion ring. Our proof is based on set theoretical counting arguments. We also discuss some possible extensions of this result.

1. Introduction and notation

Let R be a unitary ring and R-Mod, the category of unitary left R-modules. We recall that a left R-module C is called *cotorsion* when it has no proper extensions by flat modules. I.e., $\mathrm{Ext}^1(F, C) = 0$ for any flat left R-module F. Cotorsion modules were introduced by Harrison in [13] as a homological generalization of algebraically compact abelian groups and have been recently studied by several authors (see e.g. [7,9–12,19]). In [10] it was proved that flat cotorsion modules enjoy many characteristic properties of pure-injective (equivalently, algebraically compact) modules. In particular, their endomorphism ring is (von Neumann) regular and left self-injective modulo its Jacobson radical and

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1 Partially supported by the DGI (MTM2010-20940-C02-02) and by the Fundación Séneca. Part of the sources of both institutions comes from the FEDER funds of the European Union.
2 Supported by TUBITAK Program 2219.

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.04.002

idempotents lift modulo any two-sided ideal. This means that their endomorphism ring is semiperfect whenever flat cotorsion modules are finite direct sums of indecomposable direct summands.

On the other hand, it has been proved in [11, Section 3] that for any ring R there exists a local homomorphism of rings from R to $S/J(S)$, where S is the endomorphism ring of the left cotorsion envelope of R and $J(S)$, the Jacobson radical of S (see Section 2 or [19] for definitions). Using results in [4], authors have deduced that R is semilocal (i.e., R is semisimple modulo its Jacobson radical) whenever S is semiperfect. The interestingness of this result is that, as the cotorsion envelope of a flat module is always flat (see [19, Theorem 3.4.2]), we can use the above mentioned characterization to deduce that R is semilocal whenever its cotorsion envelope is a direct sum of indecomposable direct summands (see Section 3 in [11]). As pointed out in there, this seems to be a powerful tool to show that a ring R is semilocal in terms of another ring (the endomorphism ring of its cotorsion envelope) which has a milder structure.

The main goal of this paper is to exploit this idea. We consider the special case in which the given ring R is left hereditary and we try to find conditions on its cotorsion envelope which ensure that R is semilocal. These conditions are inspired by [5]. In that paper, authors showed that any left hereditary ring having a countably generated left injective envelope is left noetherian.

We consider the related problem of characterizing left hereditary rings having countably generated left cotorsion envelope. We prove that their left cotorsion envelope is a direct sum of indecomposable direct summands and therefore, they are semilocal. In particular, we deduce that left hereditary rings with finitely generated left cotorsion envelope are just the left hereditary left cotorsion semiperfect rings, thus obtaining the structure of them. We would like to stress that our proof relies on set theoretical arguments which have their origin in the decomposition of infinite sets into almost disjoint subsets obtained by Tarski in [18, Théorème 7] and that were also used in [5,15] to obtain their main results. However, the situation here is much more difficult to handle since we do not have unique cotorsion envelopes of pure submodules of $C = C({}_RR)$ within C. This fact was essential in [5,15], where uniqueness of injective envelopes is a key fact (see [16] for a discussion on this question). Indeed, our result shows one of the first situations in which this kind of techniques is successfully applied in absence of uniqueness of envelopes.

We finish this paper by showing that the decomposition of $C({}_RR)$ into a direct sum of indecomposables obtained in our main result seems to be true under a more general hypothesis. We show that this is the case for any countable ring having countably generated left cotorsion envelope. As any countable ring has left (and right) pure-global dimension bounded by one (and therefore, pure submodules of projective modules are again projective), this fact suggests that the following question may have a positive answer:

Question. Let R be a ring having left pure-global dimension at most 1. Is $C({}_RR)$ a direct sum of indecomposable direct summands provided it is countably (or finitely) generated?

Our proofs do not seem to work in this more general setting and therefore we do not know the answer to the above question.

Throughout this paper, all rings will be unitary and associative. By a module we will always mean a unitary left module unless otherwise stated. We will denote by R-Mod the category of left modules over a ring R. If $f : M \to N$ is a homomorphism of modules and M' is a submodule of M, we will denote the restriction of f to M' by $f|_{M'}$. We refer to [2,9,17,19] for any undefined concept used along this paper.

2. Main results

We begin this section by recalling some well-known facts about cotorsion modules. Let R be a unitary ring. A left R-module C is called *cotorsion* if $\mathrm{Ext}^1_R(F, C) = 0$ for any flat left R-module F. A homomorphism $u : M \to C$ from a module M to a cotorsion module C is called a *cotorsion preenvelope* of M if any other morphism from M to a cotorsion module factors through u. A cotorsion preenvelope $u : M \to C$ is called a *cotorsion envelope* if, moreover, u is minimal in the sense that any endomorphism f of C satisfying that $f \circ u = u$ is an isomorphism. The existence of cotorsion envelopes of modules is a consequence of the solution of the so-called Flat Cover Conjecture given in [3] (see also [9]). We will denote the cotorsion envelope of a module M by $C(M)$. As noted in [19, Theorem 3.4.2], any cotorsion envelope $u : M \to C(M)$ is a monomorphism with flat cokernel. In particular, any module is a pure submodule of its cotorsion envelope. Moreover, the cotorsion envelope of a flat module is always flat since flat modules are closed under extensions. The following definitions from [12] reflect the purity associated to cotorsion modules.

Definition 1. A homomorphism of modules $u : N \to M$ is called a *strongly pure monomorphism* if for any cotorsion module C and any morphism $f : N \to C$, there exists $g : M \to C$ such that $g \circ u = f$.

As noted in [12, p. 14], any monomorphism with flat cokernel, as well as any splitted monomorphism, is a strongly pure monomorphism. Therefore, the embedding of a module in its cotorsion envelope is a strongly pure monomorphism.

Definition 2. Let $u : N \to M$ be a strongly pure monomorphism. The morphism u is called *strongly pure-essential* if whenever composed with a morphism $f : M \to L$ gives a strongly pure monomorphism, the morphism f itself must be a monomorphism.

Again, it was observed in [12, p. 15] that the embedding of a module in its cotorsion envelope is always a strongly pure-essential monomorphism.

The following easy lemma will be quite useful in the sequel.

Lemma 3. *Let $\{u_i : M_i \to C_i \mid i \in I\}$ be a family of embeddings of modules into their cotorsion envelopes. Then the induced morphism $\bigoplus_I u_i : \bigoplus_I M_i \to \bigoplus_I C_i$ is a strongly pure-essential monomorphism.*

Proof. Let $f : \bigoplus_I C_i \to L$ be a homomorphism such that $f \circ \bigoplus_I u_i$ is a strongly pure monomorphism. The homomorphism f is induced by morphisms $f_i : C_i \to L$ for each $i \in I$. We must show that f is a monomorphism. Let us assume on the contrary that there exists a nonzero element $x \in \bigoplus_I C_i$ such that $f(x) = 0$. Let $I' \subseteq I$ be a finite subset of I such that $x \in \bigoplus_{I'} C_i$. It is straightforward to show that, as $f \circ \bigoplus_I u_i$ is a strongly pure monomorphism, so is $(f|_{\bigoplus_{I'} C_i}) \circ (\bigoplus_{I'} u_i)$. But, as I' is a finite set, $\bigoplus_{I'} C_i$ is cotorsion and thus, $\bigoplus_{I'} u_i : \bigoplus_{I'} M_i \to \bigoplus_{I'} C_i$ is the cotorsion envelope of $\bigoplus_{I'} M_i$. But this means that $\bigoplus_{I'} u_i$ is strongly pure-essential and, as $(f|_{\bigoplus_{I'} C_i}) \circ \bigoplus_{I'} u_i$ is a strongly pure monomorphism, we deduce that $f|_{\bigoplus_{I'} C_i}$ must be a monomorphism. A contradiction, since $0 \neq x \in \mathrm{Ker}(f|_{\bigoplus_{I'} C_i})$. □

Lemma 4. *Let M be a left R-module, $C = C(M)$, its cotorsion envelope and $S = \mathrm{End}(C)$. If $f \in S$ satisfies that $M \subseteq \mathrm{Ker}(f)$, then $f \in J(S)$.*

Proof. We need to show that $1_C - g \circ f$ is an automorphism for any $g \in S$ (see [2, Theorem 15.3]). Let $u : M \to C$ be the morphism which makes C the cotorsion envelope of M. As $M \subseteq \mathrm{Ker}(f)$, we get that $(1_C - g \circ f) \circ u = u$. So, $1_C - g \circ f$ is an automorphism by the definition of envelope. □

Our next lemma will be used to prove our main result. Given an infinite cardinal number $\aleph$, we will say that a module M is $\aleph$-generated if it has a generator set of cardinality bounded by $\aleph$.

Lemma 5. *Let P be a finitely generated projective left R-module, $C = C(P)$, its cotorsion envelope, $S = \mathrm{End}_R(C)$, $J = J(S)$ the Jacobson radical of S, and $\aleph$, an infinite cardinal number. If N is a left ideal of S such that ${}_RCN$ is an $\aleph$-generated module, then so is ${}_S(N + J)/J$.*

Proof. Let us denote by $w : N \to S$ the inclusion of N in S and let $\pi : S^{(I)} \to N$ be an epimorphism for some index set I. Tensorizing by $C \otimes_S -$, we get a morphism $1_C \otimes w : C \otimes_S N \to C \otimes S \cong C$ which factors as $1_C \otimes w = u \circ q$, where $u : CN \to C$ is the inclusion and $q : C \otimes N \to CN$ is the evaluation. We are assuming that ${}_R CN$ is $\aleph$-generated, so there exists a subset $I' \subseteq I$ of cardinality bounded by $\aleph$ such that, if we denote by $v : C^{(I')} \to C^{(I)}$ the natural embedding, then $q \circ (1_C \otimes \pi) \circ v$ is an epimorphism. Note that, as the tensor functor $C \otimes_S -$ commutes with direct sums, and ${}_R C \cong_R C \otimes_S \mathrm{Hom}_R(C, C)$, $v = 1_C \otimes v'$, where $v' : S^{(I')} \to S^{(I)}$ is also the natural embedding. Applying now the functor $\mathrm{Hom}_R(C, -)$, we obtain a commutative diagram in S-Mod as follows:

$$\begin{array}{ccccccc}
S^{(I')} & \xrightarrow{\ v'\ } & S^{(I)} & \xrightarrow{\ \pi\ } & N & \xhookrightarrow{\ \omega\ } & S \\
\downarrow{\scriptstyle \eta_{S^{(I')}}} & & \downarrow{\scriptstyle \eta_{S^{(I)}}} & & \downarrow{\scriptstyle q_* \circ \eta_N} & & \simeq\ \downarrow{\scriptstyle \eta_S} \\
\mathrm{Hom}(C, C^{(I')}) & \xrightarrow[(1_C \otimes v')_*]{} & \mathrm{Hom}(C, C^{(I)}) & \xrightarrow[q_* \circ (1_C \otimes \pi)_*]{} & \mathrm{Hom}(C, CN) & \xrightarrow[u_*]{} & \mathrm{Hom}(C, C)
\end{array}$$

where $(-)_* \equiv \mathrm{Hom}_R(C, -)$ and $\eta : 1_{S\text{-Mod}} \to \mathrm{Hom}_R(C, C \otimes_S -)$ is the arrow of the adjunction. Let us write $u_* = \beta \circ \alpha$, where

$$\alpha : \mathrm{Hom}(C, CN) \to \mathrm{Im}\, u_*$$

is an epimorphism and

$$\beta : \mathrm{Im}\, u_* \to \mathrm{Hom}_R(C, C),$$

a monomorphism. Let us note that $\alpha \circ q_* \circ \eta_N$ is a monomorphism as so $\eta_S \circ w$ is. And this means that $N \cong \mathrm{Im}(\alpha \circ q_* \circ \eta_N) \subseteq \mathrm{Im}\, u_*$. Let us choose any element $x \in \mathrm{Im}\, u_*$. There exists an $f \in \mathrm{Hom}_R(C, CN)$ such that $x = \alpha(f)$. We have a diagram in R-Mod:

$$\begin{array}{ccccccc}
0 & \longrightarrow & P & \xrightarrow{\ \epsilon\ } & C & & \\
 & & & & \downarrow{\scriptstyle f} & & \\
 & & C^{(I')} & \xrightarrow{\ q \circ (1_C \otimes \pi) \circ (1_C \otimes v')\ } & CN & \longrightarrow & 0
\end{array}$$

where $\epsilon : P \to C$ denotes the morphism that makes C the cotorsion envelope of P. As P is projective, there exists a $\delta : P \to C^{(I')}$ such that

$$q \circ (1_C \otimes \pi) \circ \left(1_C \otimes v'\right) \circ \delta = f \circ \epsilon.$$

Let us note that P is finitely generated and thus, $\mathrm{Im}\, \delta$ embeds in a finite subsum of $C^{(F)}$ of $C^{(I')}$. Therefore, $C^{(F)}$ is cotorsion and, as $\mathrm{Coker}(\epsilon)$ is flat, there exists a $\varphi : C \to C^{(F)} \subseteq C^{(I')}$ such that $\varphi \circ \epsilon = \delta$. But this means that

$$P \subseteq \mathrm{Ker}\bigl(f - q \circ (1_C \otimes \pi) \circ \left(1_C \otimes v'\right) \circ \varphi\bigr)$$

and hence

$$\beta\bigl[\alpha(f) - \alpha \circ q_* \circ (1_C \otimes \pi)_* \circ \left(1_C \otimes v'\right)_*(\varphi)\bigr] \in J(S)$$

by the above lemma. Thus, $\mathrm{Im}\, u_* + J \cong N + J$.

On the other hand, as $\varphi : C \to C^{(I')}$ factors through the finite direct subsum $C^{(F)}$, and $\eta_{S^{(F)}} : S^{(F)} \to \operatorname{Hom}_R(C, C \otimes_S S^{(F)})$ is an isomorphism (since functor $\operatorname{Hom}_R(C, -)$ commutes with finite direct sums), there exists an element $y \in S^{(I')}$ such that $\eta_{S^{(I')}}(y) = \varphi$. Therefore,

$$\begin{aligned} u_*(f) + J &= u_* \circ q_* \circ (1_C \otimes \pi)_* \circ \left(1_C \otimes v'\right)_*(\varphi) + J \\ &= u_* \circ q_* \circ (1_C \otimes \pi)_* \circ \left(1_C \otimes v'\right)_* \circ \eta_{S^{(I')}}(y) + J \\ &= u_* \circ q_* \circ \eta_N \circ \pi \circ v'(y) + J \end{aligned}$$

and we deduce that $\operatorname{Im} u_* + J = \operatorname{Im}(u_* \circ q_* \circ \eta_N \circ \pi \circ v') + J$. Therefore, $(\operatorname{Im} u_* + J)/J$ is $|I'|$-generated and this means that $(N + J)/J \cong (\operatorname{Im} u_* + J)/J$ is also $\aleph$-generated. $\square$

We are now ready to prove our main result. Recall that a left R-module P is called a *progenerator* if it is a finitely generated projective generator in R-Mod (see [2, Chapter 6, §22, p. 262]). A set $\{M_i \mid i \in I\}$ of independent submodules of a module M is called a *local direct summand* of M [6, p. 66] when $\bigoplus_F M_i$ is a direct summand of M for every finite subset $F \subseteq I$. If $\bigoplus_I M_i$ is a direct summand of M, then we will say that the local direct summand $\{M_i \mid i \in I\}$ is a summand of M.

Theorem 6. *Let R be a left hereditary ring, P, a progenerator in R-Mod and $u : P \to C(P)$, the cotorsion envelope of P. If $C(P)$ is countably generated, then $C(P)$ is a finite direct sum of indecomposable cotorsion modules.*

Proof. Let us call $C = C(P)$ and assume on the contrary that C is not a finite direct sum of indecomposable cotorsion modules. As P is a finitely generated module which is strongly pure-essential in C, any decomposition of C into indecomposable direct summands would only have finitely many nonzero direct summands. So C cannot be a direct sum of indecomposable direct summands. By [14, Theorem 10.17], there must exist a local direct summand $\bigoplus_I C_i \subseteq C$ which is not a direct summand. Let us note that $\bigoplus_I C_i \subseteq C$ is a strongly pure submodule of C, since it is a pure submodule of a flat module. Thus, the cotorsion envelope of $\bigoplus_I C_i$ is a direct summand of C, say $C = C(\bigoplus_I C_i) \oplus C'$. Adding the direct summand C' to $\bigoplus_I C_i$ if necessary, we may assume that C is the cotorsion envelope of $\bigoplus_I C_i$ and thus, the embedding $v : \bigoplus_I C_i \to C$ is a strongly pure-essential monomorphism.

We are assuming that C is countably generated. So there exists an epimorphism $\pi : R^{(\mathbb{N})} \to C$. $\operatorname{Ker} \pi$ is a submodule of ${}_R R^{(\mathbb{N})}$ and, in particular, a projective module since R is left hereditary. Moreover, as C is flat, the inclusion $\operatorname{Ker} \pi \hookrightarrow R^{(\mathbb{N})}$ has flat cokernel and thus, it is a strongly pure monomorphism.

On the other hand, we know by [1] that $\operatorname{Ker} \pi$ is a direct sum of finitely generated projective modules, say $\operatorname{Ker} \pi = \bigoplus_T P_t$. We claim that there exists an $n \in \mathbb{N}$ such that R^n contains a local direct summand $\bigoplus_H X_h$ with $|H| \geqslant |T|$ and $X_h \neq 0$ for every $h \in H$. Let us note that, if $|T| \leqslant |I|$, this is obvious since we are assuming that the progenerator P contains the local direct summand $\bigoplus_I P_i$. So let us assume that $|T| \gneqq |I|$. In particular, T is uncountable. As each P_t is finitely generated, there exists a finite subset $n_t \subset \mathbb{N}$ such that $P_t \subseteq R^{(n_t)} \subseteq R^{(\mathbb{N})}$. Therefore, as T is an uncountable subset but the set of all finite subsets of $\mathbb{N}$ is countable, there must exist a finite number n such that the set

$$T' = \left\{t \in T \;\middle|\; P_t \in R^{(n)}\right\}$$

has cardinality $|T|$, and this proves our claim. Let us note that, replacing now, if necessary, P by R^n, $\bigoplus_I C_i$ by $\bigoplus_{T'} C_t$ and C by C^n, we may assume that C contains a strongly pure-essential local direct summand $\bigoplus_I C_i$ with $|I| \geqslant |T|$.

Let us apply [18, Théorème 7] (see also the proof of [15, Theorem 1] and [8]) to construct a family $\mathcal{K}$ of infinite subsets of I satisfying:

1. $|I| \lneqq |\mathcal{K}|$,
2. $|K \cap K'| < |K| = |K'|$ for every $K, K' \in \mathcal{K}$ with $K \neq K'$.

Let us call $S = \mathrm{End}_R(C)$ and let $J = J(S)$ be its Jacobson radical. We know from [10] that the ring S/J is left self-injective, (von Neumann) regular and idempotents in S/J lift modulo J. As each C_i is a direct summand of C, there exists a set $\{e_i \mid i \in I\}$ of pairwise orthogonal idempotents in S such that $C_i = Ce_i$ for every $i \in I$. Let us also fix, for any subset $A \subseteq I$, injective envelopes $E_A = E(\bigoplus_{i\in A} Se_i/Je_i)$ within S/J. As idempotents lift modulo J, there exists an idempotent $e_A \in S$ such that $E_A = Se_A/Je_A$.

We claim that, for any $A \subseteq I$, Ce_A is a cotorsion envelope of $\bigoplus_{i\in A} Ce_i$ within ${}_RC$. Note that Ce_A is flat and cotorsion since it is a direct summand of C and that $\bigoplus_{i\in A} Ce_i$ is a strongly pure submodule of Ce_A. So $Ce_A = C' \oplus C''$, where C' is the cotorsion envelope of $\bigoplus_{i\in A} Ce_i$. This means that

$$Se_A \cong \mathrm{Hom}_R(C, Ce_A) = \mathrm{Hom}_R\big(C, C'\big) \oplus \mathrm{Hom}_R\big(C, C''\big).$$

But clearly $Se_i \subset \mathrm{Hom}_R(C, C')$ for each $i \in A$. Therefore,

$$\bigoplus_A Se_i/Je_i \subseteq \big(\mathrm{Hom}_R\big(C, C'\big) + J\big)/J$$

and this means that $\mathrm{Hom}_R(C, C'') + J/J = 0$ since Se_A/Je_A is the injective envelope of $\bigoplus_A Se_i/Je_i$ in S/J-Mod. Thus, $\mathrm{Hom}_R(C, C'') = 0$. In particular,

$$C'' \cong C \otimes \mathrm{Hom}_R\big(C, C''\big) = 0$$

and $Ce_A = C'$ is the cotorsion envelope of $\bigoplus_{i\in A} Ce_i$.

Let us now call $M = \sum_{K\in\mathcal{K}} Se_K$. We claim that M cannot be $|I|$-generated. Assume on the contrary that this is the case. Then, there would exist a subset $\mathcal{K}' \subseteq \mathcal{K}$ with $|\mathcal{K}'| \leqslant |I|$ such that $M = \sum_{K\in\mathcal{K}'} Se_K$. In particular,

$$(M + J)/J = \sum_{K\in\mathcal{K}'} Se_K/Je_K.$$

As $|\mathcal{K}| \gneq |I|$, there exists a $K_0 \in \mathcal{K} \setminus \mathcal{K}'$. By assumption,

$$e_{K_0} + J \in \sum_{K\in\mathcal{K}'} Se_K + J,$$

so there exists a finite set $K_1, \ldots, K_n \in \mathcal{K}'$ such that

$$e_{K_0} + J \in (Se_{K_1} + \cdots + Se_{K_n}) + J.$$

As S/J is regular, it is nonsingular and thus, injective envelopes inside ${}_{S/J}S/J$ are unique. Therefore, if we call $A = K_1 \cup \cdots \cup K_n$, we have

$$Se_{K_1}/Je_{K_1} + \cdots + Se_{K_n}/Je_{K_n} \subseteq Se_A/Je_A.$$

And this means

$$Se_{K_0}/Je_{K_0} \subseteq Se_A/Je_A.$$

In particular, we deduce that

$$e_i + J_i \in Se_A/Je_A \quad \text{for each } i \in K_0$$

and thus, $K_0 \subseteq A$. But then,

$$\begin{aligned}|K_0| &= \big|K_0 \cap (K_1 \cup \cdots \cup K_n)\big| = \big|(K_0 \cap K_1) \cup \cdots \cup (K_0 \cap K_n)\big| \\ &\leqslant |K_0 \cap K_1| + \cdots + |K_0 \cap K_n| \lneq |K_0|\end{aligned}$$

since $|K_0 \cap K_l| \lneq |K_0|$ for each $l = 1, \ldots, n$ (as $K_0 \neq K_l$). A contradiction, which proves our claim.

Let us now call $N = \sum_{K \in \mathcal{K}} Ce_K = CM$. Lemma 5 shows that M cannot be $|I|$-generated since neither $(M + J)/J$ is. Let us call $C_K = Ce_K$ for every $K \in \mathcal{K}$. We have a diagram:

$$\begin{array}{ccccc} & & & & \operatorname{Ker}\pi \\ & & & & \big\downarrow \\ & & & & R^n \\ & & & & \big\downarrow \pi \\ \bigoplus_{i \in I} C_i & \hookrightarrow & \sum_{K \in \mathcal{K}} C_K & \overset{\nu}{\hookrightarrow} & C \end{array}$$

Let us set

$$P = \pi^{-1}\Bigg(\sum_{K \in \mathcal{K}} C_K\Bigg) \quad \text{and} \quad Q = \pi^{-1}\Bigg(\bigoplus_{i \in I} C_i\Bigg).$$

As ${}_R R$ is hereditary, both P, Q are projective and therefore, they are direct sums of finitely generated projective modules by [1], say,

$$P = \bigoplus_{\beta \in B} P_\beta \quad \text{and} \quad Q = \bigoplus_{\lambda \in \Lambda} Q_\lambda.$$

Moreover, both $\operatorname{Ker}\pi$ and $\bigoplus_I C_i$ are $|I|$-generated. So we get that Q is also $|I|$-generated. But P cannot be $|I|$-generated since nor it is its homomorphic image $\sum_{K \in \mathcal{K}} C_K$. And this means that $|B| \gneq |I|$. Therefore, there exists a subset $B' \subseteq B$ with $|B'| = |I| \lneq |B|$ such that $Q \subseteq \bigoplus_{\beta \in B'} P_\beta$. Then, $\operatorname{Ker}\pi \subseteq Q \subseteq \bigoplus_{\beta \in B'} P_\beta \subseteq P$, and this means that

$$\begin{aligned}\sum_{K \in \mathcal{K}} C_K = \pi(P) = \frac{P}{\operatorname{Ker}\pi} &= \frac{\bigoplus_{\beta \in B'} P_\beta}{\operatorname{Ker}\pi} \oplus \frac{\bigoplus_{\beta \in B \setminus B'} P_\beta + \operatorname{Ker}\pi}{\operatorname{Ker}\pi} \\ &\cong \frac{\bigoplus_{\beta \in B'} P_\beta}{\operatorname{Ker}\pi} \oplus \Bigg(\bigoplus_{\beta \in B \setminus B'} P_\beta\Bigg).\end{aligned}$$

And, as $|B \setminus B'| = |B| \gneq |B'| = |I|$, we get that, up to a "small direct summand" (in the sense that it is $|I|$-generated), $\sum_{K \in \mathcal{K}} C_K$ is a direct sum of projective modules.

Let us call $q : \bigoplus_{K \in \mathcal{K}} C_K \twoheadrightarrow \sum_{K \in \mathcal{K}} C_K$ the epimorphism induced by the canonical projections $q_L : \bigoplus_{K \in \mathcal{K}} C_K \twoheadrightarrow C_L$ for any $L \in \mathcal{K}$. As C_K is the cotorsion envelope of $\bigoplus_{i \in K} C_i$ in C, we get that the inclusion $\gamma_K : \bigoplus_{i \in K} C_i \hookrightarrow C_K$ is a strongly pure essential monomorphism. Call $\gamma : \bigoplus_{K \in \mathcal{K}} (\bigoplus_{i \in K} C_i) \hookrightarrow \bigoplus_{K \in \mathcal{K}} C_K$ the morphism induced by the set $\{\gamma_L\}_{L \in \mathcal{K}}$. Lemma 3 shows that γ is also a strongly pure essential monomorphism. Let us note that, by construction, $q(\bigoplus_{K \in \mathcal{K}} (\bigoplus_{i \in K} C_i)) \subseteq \bigoplus_{i \in I} C_i \subseteq \sum_{K \in \mathcal{K}} C_K$. So,

$$q\left(\bigoplus_{K\in\mathcal{K}}\left(\bigoplus_{i\in K}C_i\right)\right)\subseteq\frac{\bigoplus_{\beta\in B'}P_\beta}{\operatorname{Ker}\pi}$$

which is a direct summand of $\sum_{K\in\mathcal{K}}C_K$. Let us call

$$\varphi:\sum_{K\in\mathcal{K}}C_K\to\bigoplus_{\beta\in B\setminus B'}P_\beta$$

the structural projection respect to the decomposition

$$\sum_{K\in\mathcal{K}}C_K=\frac{\bigoplus_{\beta\in B'}P_\beta}{\operatorname{Ker}\pi}\oplus\left(\bigoplus_{\beta\in B\setminus B'}P_\beta\right).$$

As $\bigoplus_{\beta\in B\setminus B'}P_\beta$ is projective and $\varphi\circ q:\bigoplus_{K\in\mathcal{K}}C_K\to\bigoplus_{\beta\in B\setminus B'}P_\beta$ is an epimorphism, it splits. In particular, we get that

$$\bigoplus_{K\in\mathcal{K}}C_K=Z\oplus Z'$$

where $Z=\operatorname{Ker}(\varphi\circ q)$ and $Z'\cong\bigoplus_{\beta\in B\setminus B'}P_\beta$. Let

$$\delta:\bigoplus_{K\in\mathcal{K}}C_K\twoheadrightarrow Z$$

be the associated projection. We claim that $\delta|_{\bigoplus_{K\in\mathcal{K}}(\bigoplus_{i\in K}C_i)}$ is a strongly pure monomorphism. To check this, it is enough to note that the inclusion

$$\bigoplus_{K\in\mathcal{K}}\left(\bigoplus_{i\in K}C_i\right)\hookrightarrow\bigoplus_{K\in\mathcal{K}}C_K$$

is a pure monomorphism by construction and, since it factors as

$$\bigoplus_{K\in\mathcal{K}}\left(\bigoplus_{i\in K}C_i\right)\hookrightarrow Z\hookrightarrow\bigoplus_{K\in\mathcal{K}}C_K,$$

we get that δ is also a pure monomorphism. On the other hand, Z is a flat module since it is a direct summand of the flat module $\bigoplus_{K\in\mathcal{K}}C_K$. So $\operatorname{Coker}\delta$ is flat and thus, δ is a strongly pure monomorphism. But then, as γ is a strongly pure essential monomorphism and $\delta\circ\gamma=\delta|_{\bigoplus_{K\in\mathcal{K}}(\bigoplus_{i\in K}C_i)}$ is a strongly pure monomorphism, we get that δ must be a monomorphism. This means that $Z'=0$ and therefore, $\bigoplus_{\beta\in B\setminus B'}P_\beta=0$ since it is isomorphic to Z'. But then,

$$\sum_{K\in\mathcal{K}'}C_K=\frac{\bigoplus_{\beta\in B'}P_\beta}{\operatorname{Ker}\pi}$$

is $|I|$-generated and we get the desired contradiction. □

The following corollary is an immediate consequence of the above theorem.

Corollary 7. *Let R be a left hereditary ring and P, a progenerator in R-*Mod*. If $C = C(P)$ is countably generated, then any pure submodule of P is finitely generated.*

Proof. Assume on the contrary that P contains a pure submodule N which is not finitely generated. As R is left hereditary, N is a projective module and, by [1], it must be a direct sum of finitely generated projective modules. Say that $N = \bigoplus_I N_i$ with I infinite and $N_i \neq 0$ for all $i \in I$. Then $\bigoplus_{i\in I} C(N_i)$ is a local direct summand of C consisting of infinitely many direct summands. But this means that it cannot be a direct summand, since the finitely generated module P is strongly pure-essential in C. We can now apply the same arguments as in the proof of Theorem 6 to get a contradiction. □

Let us now apply the above theorem to left hereditary rings with countably generated left cotorsion envelope.

Corollary 8. *Let R be a left hereditary ring and $C = C({}_R R)$, its left cotorsion envelope. If C is countably generated, then R is a semilocal ring.*

Proof. By Theorem 6, C is a direct sum of indecomposables. As $S = \mathrm{End}({}_R C)$ is (von Neumann) regular and left self-injective modulo its Jacobson radical and idempotents lift modulo any two-sided ideal (see [10]), we deduce that S is semiperfect. On the other hand, it has been proved in [11, Section 3] that there exists a local homomorphism of rings $\varphi : R \to S/J(S)$. Therefore, R is semilocal by the main result in [4]. □

We can finally characterize left hereditary rings with finitely generated left cotorsion envelope.

Corollary 9. *Let R be a left hereditary ring and $C = C({}_R R)$, its left cotorsion envelope. The following conditions are equivalent*:

1. *C is finitely generated.*
2. *R is a left cotorsion semiperfect ring.*

Proof. (1) ⇒ (2) Let us first show that R is left cotorsion. As we are assuming that $C = C({}_R R)$ is finitely generated, there exists an epimorphism $\pi : R^n \to C$ for some $n \in \mathbb{N}$. Let $P = \mathrm{Ker}\,\pi$. P is a pure submodule of the progenerator R^n since C is flat. Therefore, it is finitely generated by Lemma 7. This means that C is finitely presented and therefore, so is the flat module C/R. Then, C/R is projective (as it is flat and finitely presented) and we deduce that the embedding of ${}_R R$ in C splits. Thus, ${}_R R$ is cotorsion. In particular, $R/J(R)$ is (von Neumann) regular and left self-injective and idempotents lift modulo $J(R)$ (see [10]). As R is also semilocal by the above corollary, R must be semiperfect.

(2) ⇒ (1) If ${}_R R$ is cotorsion, then ${}_R R$ coincides with its cotorsion envelope $C({}_R R)$ and thus, $C({}_R R)$ is finitely generated. □

We would like to finish this paper by discussing the following question. Carefully checking the proof of Theorem 6, one may observe that the hypothesis "R is left hereditary" is mainly needed to assure that certain pure submodules of projective modules are again projective. This fact suggests that the following question might have a positive answer:

Question 10. Let R be a ring with left pure-global dimension bounded by 1 (or in which pure submodules of projective left R-modules are projectives). If $C({}_R R)$ is countably (or finitely) generated, is it a finite direct sum of indecomposable direct summands?

We do not know the answer to this question. The reason is that, following the notation of Theorem 6, we cannot assure that the constructed module $\pi^{-1}(\sum_{K\in\mathcal{K}} C_K)$ is a pure submodule of a projective module and therefore, it may not be projective under the hypotheses given in the above question. However, the following proposition shows that this is the case for countable rings. Let us recall that countable rings are one of the main sources of rings with pure-global dimension at most 1.

Proposition 11. *Let R be a countable ring. If $C({}_R R)$ is countably generated, then it is a (finite) direct sum of indecomposable direct summands.*

Proof. Assume on the contrary that C is not a (finite) direct sum of indecomposable direct summands. By [14, Theorem 10.17], $C = C({}_R R)$ must have a countable local direct summand $\bigoplus_{\mathbb{N}} C_i$ which is not a direct summand. We can now use the same arguments as in Theorem 6 to construct a submodule $N = \sum_{K \in \mathcal{K}} C_K$ of C that cannot be countably generated. But, as R is a countable ring and C is a countably generated module, the underlying set of C is countable and therefore, any submodule of it is trivially countably generated. A contradiction, which shows that C is a finite direct sum of indecomposable modules. □

Acknowledgment

This paper was written during the visit of the second author at the University of Murcia. The second author would like to thank Tubitak for their financial support.

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