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Research Article

Noetherian and Artinian Lattices

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It is proved that if L is a complete modular lattice which is compactly generated, then $\mathrm{Rad}(L)/0$ is Artinian if, and only if for every small element a of L, the sublattice $a/0$ is Artinian if, and only if L satisfies DCC on small elements.

1. Introduction

By a *lattice* we mean a partially ordered set $(L, \leq)$ such that every pair of elements a, b in L has a *greatest lower bound* (or a *meet*) $a \wedge b$ and a *least upper bound* (or a *join*) $a \vee b$; that is,

(i) $a \wedge b \leq a$, $a \wedge b \leq b$, and $c \leq a \wedge b$ for all $c \in L$ with $c \leq a$, $c \leq b$,

(ii) $a \leq a \vee b$, $b \leq a \vee b$, and $a \vee b \leq d$ for all $d \in L$ with $a \leq d$, $b \leq d$.

Note that, for given $a, b \in L$, $a \wedge b$ and $a \vee b$ are unique, and

$$a \leq b \Longleftrightarrow a = a \wedge b \Longleftrightarrow b = a \vee b. \tag{1.1}$$

Let $(L, \leq, \wedge, \vee)$ (or just L) be any lattice. Given $a, b \in L$, we set

$$a \leq' b \Longleftrightarrow b \leq a. \tag{1.2}$$

Then $(L, \leq')$ is a partially ordered set; moreover, for any $a, b \in L$, a, b have greatest lower bound $a \vee b$ and least upper bound $a \wedge b$. We call $(L, \leq', \vee, \wedge)$ the *opposite lattice* of L, and denote it by L°.

Let $(L, \leq \wedge, \vee)$ be any lattice. Let $a \leq b$ in L. We define

$$\frac{b}{a} = \{x \in L : a \leq x \leq b\}. \tag{1.3}$$

(Sometimes b frac a is denoted by b/a.)

A lattice $(L, \leq, \wedge, \vee)$ *has a least element* if there exists $z \in L$ such that $z \leq a (a \in L)$. In this case, z is uniquely defined and is usually denoted by 0. The lattice L *has a greatest element* if there exists $u \in L$ such that $a \leq u(a \in L)$. In this case, u is uniquely defined and is usually denoted by 1. A lattice L is called *complete* if every subset of L has a meet and a join, and it is called *modular* if $a \wedge (b \vee c) = b \vee (a \wedge c)$ for all a, b, c in L with $b \leq a$. For more information about lattice theory, refer to [1–3].

Throughout this paper $(L, \leq, \vee, \wedge, 0, 1)$ will be a complete modular lattice. An element $e \in L$ is called an *essential* element if $e \wedge x \neq 0$ for every nonzero element $x \in L$. An element $s \in S$ is said to be *small* if s is an essential element of the opposite lattice L°. Let $E(L)$ denote the set of all essential elements of L. The set of all small elements of L will be denoted by $S(L)$.

A set $\{c_i \mid i \in I\} \subseteq L$ is called a *direct* set if, for all $i, j \in I$, there exists $k \in I$ with $c_i \vee c_j \leq c_k$. The lattice L is said to be *upper continuous* if, for every direct set $\{c_i \mid i \in I\}$ in L and element $a \in L$, we have $a \wedge (\bigvee_{i \in I} c_i) = \bigvee_{i \in I} (a \wedge c_i)$. On the other hand, L is said to be *lower continuous* if for every inverse set $\{c_i \mid i \in I\}$ (i.e., for all i, j in I, there exists $k \in I$ with $c_k \leq c_i \wedge c_j$) and element $a \in L$, $a \vee (\bigwedge_{i \in I} c_i) = \bigwedge_{i \in I} (a \vee c_i)$. We will call an element f in L *finitely generated* element (or *compact* element) if whenever $f \leq \vee S$, for some direct set S in L, then there exists $x \in S$ such that $f \leq x$. Note that 0 is always a finitely generated element of L. It is known that an element f is finitely generated if and only if for every nonempty subset U of L with $f \leq \vee U$ there exists a finite subset F of U such that $f \leq \vee F$. A lattice L is said to be *finitely generated* (or *compact*) if 1 is finitely generated. We call the lattice L *compactly generated* if each of its elements is a join of finitely generated elements (see [2]). Note that every compactly generated lattice is upper continuous (see, e.g., [4, Proposition 2.4]). Moreover, it is shown in [4, Exercises 2.7 and 2.9] that for every element a of a compactly generated lattice L, the sublattices $a/0$ and $1/a$ are again compactly generated. A lattice L is called a *finitely cogenerated* (or *cocompact*) lattice, if for every subset X of L such that $\wedge X = 0$ there is a finite subset F of X such that $\wedge F = 0$. An element $g \in L$ is said to be *finitely cogenerated* (or *cocompact*) if the sublattice $g/0$ is a finitely cogenerated lattice. If $a < b$ and $a \leq c < b$ imply $c = a$, then we say that a is *covered* by b (or b *covers* a). If 0 is covered by an element a of L, then a is called an *atom* element of L. A lattice L is said to be *semiatomic* if 1 is a join of atoms in L (see [4]). The meet of all maximal elements (different from 1) in L is denoted by $\mathrm{Rad}(L)$, and it is called the *radical* of L (see [2]). If L is compactly generated, then $\mathrm{Rad}(L)$ is the join of all small elements of L (see [2, Theorem 8]). The join of all atoms of L, denoted by $\mathrm{Soc}(L)$, is called the *socle* of L. The socle of a compactly generated lattice is equal to the meet of all essential elements (see [4, Theorem 5.1]).

A non-empty subset S of L is called an *independent* set if, for every $x \in S$ and finite subset $T = \{t_1, \ldots, t_n\}$ of S with $x \notin T$, $x \wedge (t_1 \vee \cdots \vee t_n) = 0$. We say that a nonzero lattice L has *finite uniform* (*or Goldie*) *dimension* if L contains no infinite independent sets; equivalently, $\sup\{k \mid L$ contains an independent subset of cardinality equal to $k\} = n < \infty$. In this case L is said to have uniform (or Goldie) dimension n and this is denoted by $u(L)$. We shall say that L has *hollow* (or *dual Goldie*) dimension n, provided the opposite lattice L° has uniform dimension n. The lattice L is said to be *Artinian* (*noetherian*) if L satisfies the descending (ascending) chain condition on its elements. A lattice L will be called an *E-complemented* lattice if, for each $a \in L$, there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b \in E(L)$.

In Section 2 we mainly prove that a lattice L is noetherian if and only if L is E-complemented and every essential element of L is finitely generated (Corollary 2.4). In Section 3 we generalize Theorem 5 in [5] to lattice theory (Theorem 3.7).

2. Noetherian Lattices

The following lemma was given us by Patrick F. Smith from his unpublished notes.

Lemma 2.1. *Let L be a lattice. Consider the following statements.*

(i) *L is noetherian.*

(ii) *L has finite uniform dimension.*

(iii) *L is E-complemented.*

Then (i) ⇒ (ii) ⇒ (iii).

Proof. (i) ⇒ (ii) Suppose L is noetherian but that L does not have finite uniform dimension. Then there exists an infinite independent set of nonzero elements $x_n (n \in \mathbb{N})$. Consider the ascending chain $x_1 \le x_1 \vee x_2 \le \cdots$ in L. Because L is noetherian, there exists a positive integer n such that $x_1 \vee \cdots \vee x_n = x_1 \vee \cdots \vee x_n \vee x_{n+1}$. This implies that $x_{n+1} \le (x_1 \vee \cdots \vee x_n) \wedge x_{n+1} = 0$, a contradiction. Therefore L has finite uniform dimension.

(ii) ⇒ (iii) Let $a \in L$. If $a \in E(L)$, we are done. If $a \notin E(L)$, then there exists $0 \ne b_1 \in L$ such that $a \wedge b_1 = 0$. If $a \vee b_1 \in E(L)$, we are done. Otherwise, there exists $0 \ne b_2 \in L$ such that $(a \vee b_1) \wedge b_2 = 0$. Repeating this argument we produce an independent set $\{a, b_1, b_2, \ldots\}$. Thus this process must stop, so there exists $k \in \mathbb{N}$ such that $a \wedge (b_1 \vee \cdots \vee b_k) = 0$ and $a \vee (b_1 \vee \cdots \vee b_k) \in E(L)$. □

Remark 2.2. Note that if f is a finitely generated element of a lattice L, then for every non-empty set U with $f = \vee U$ there exists a finite subset F of U such that $f = \vee F$.

Proposition 2.3. *Let L be a lattice such that x is finitely generated for every $x \in E(L)$. Then the following are equivalent.*

(i) *L is noetherian.*

(ii) *L has finite uniform dimension.*

(iii) *L is E-complemented.*

Proof. We only need to prove (iii) ⇒ (i) by Lemma 2.1. Let a be a nonzero element in L. By (iii), there exists an element b of L such that $a \wedge b = 0$ and $a \vee b \in E(L)$. By hypothesis, $a \vee b$ is finitely generated. Let $a = \vee S$ for a nonempty set S in L. Then $a \vee b = \vee(S \cup \{b\})$. Since $a \vee b$ is finitely generated, $a \vee b = \vee F \vee b$ for a finite subset F of S. Since L is modular, we have $a = \vee F$. Therefore every element in L is finitely generated. Hence L is noetherian by [4, Proposition 2.3]. □

Corollary 2.4. *A lattice L is noetherian if and only if L is E-complemented and every essential element of L is finitely generated.*

Lemma 2.5. *Every upper continuous lattice L is E-complemented.*

Proof. Let $a \in L$. Let $S = \{b \in L \mid a \wedge b = 0\}$. Clearly, $0 \in S$. Let $\{c_i \mid i \in I\}$ be a chain in S and let $c = \bigvee_{i \in I} c_i$. Then $a \wedge c = a \wedge (\bigvee_{i \in I} c_i) = \bigvee_{i \in I}(a \wedge c_i) = 0$. By Zorn's lemma, S contains a maximal member u. Then $a \wedge u = 0$. Suppose that $(a \vee u) \wedge x = 0$ for some $x \in L$. Then $a \wedge (u \vee x) = 0$, and hence $u \vee x \in S$. Since $u \leq u \vee x$, we have $u = u \vee x$ and $x \leq u$. Thus $x = (a \vee u) \wedge x = 0$. It follows that $a \vee u \in E(L)$. Therefore L is E-complemented. □

Corollary 2.6. *Let L be an upper continuous lattice. Then L is noetherian if and only if every essential element in L is finitely generated.*

Lemma 2.7 (see [4, Lemmas 7.3 and 7.5]). *Let L be a lattice and k a positive integer. Then*

(i) *if $t \in S(L)$, then $s \in S(L)$ for every $s \leq t$;*

(ii) *if $s_1, s_2, \ldots, s_k \in S(L)$, then $s_1 \vee s_2 \vee \cdots \vee s_k \in S(L)$.*

As an easy observation of Lemma 2.7, we can give the following two results.

Proposition 2.8 (see cf. [5, Proposition 2]). *Let L be a compactly generated lattice. Then* $\mathrm{Rad}(L)/0$ *is noetherian if and only if L satisfies ACC on small elements.*

Proof. (⇒) By [2, Theorem 8].

(⇐) By assumption, L contains a maximal small element x. Since x is small in L, $x \leq \mathrm{Rad}(L)$. Suppose that $x \neq \mathrm{Rad}(L)$. Then there exists a small element s of L such that $s \notin x/0$. On the other hand, $s \vee x$ is a small element of L by Lemma 2.7(ii). By the maximality of x, we have $s \vee x = x$. This gives $s \in x/0$, a contradiction. Thus $x = \mathrm{Rad}(L)$. By Lemma 2.7(i), $\mathrm{Rad}(L)/0 \subseteq S(L)$. Consequently, $\mathrm{Rad}(L)/0$ is noetherian. □

Proposition 2.9 (see cf. [5, Proposition 3]). *Let L be a compactly generated lattice. Then the following are equivalent.*

(i) $\mathrm{Rad}(L)/0$ *has finite uniform dimension.*

(ii) *There exists a positive integer k such that for every small element s of L we have $u(s/0) \leq k$.*

(iii) *L does not contain an infinite independent set of nonzero small elements.*

Proof. (i) ⇒ (ii) Let s be a small element of L. By [2, Theorem 8], $s \leq \mathrm{Rad}(L)$. Since $u(s/0) \leq u(\mathrm{Rad}(L)/0)$, $s/0$ has finite uniform dimension. The rest is clear.

(ii) ⇒ (iii) Let $\{s_1, s_2, \ldots\}$ be an infinite independent set of nonzero small elements of L. By Lemma 2.7(ii), $s_1 \vee s_2 \vee \cdots \vee s_{k+1} \in S(L)$, and $u((s_1 \vee s_2 \vee \cdots \vee s_{k+1})/0) \geq k+1$, a contradiction.

(iii) ⇒ (i) Suppose that $\mathrm{Rad}(L)/0$ does not have finite uniform dimension. Then there exists an infinite independent set of nonzero elements $\{x_1, x_2, \ldots\}$ of $\mathrm{Rad}(L)/0$. Let $i \geq 1$. Since $\mathrm{Rad}(L)/0$ is compactly generated, there exists a nonzero finitely generated element k_i of $\mathrm{Rad}(L)/0$ such that $k_i \leq x_i$. So by Lemma 2.7, $k_i \in S(L)$. Therefore $\{k_1, k_2, \ldots\}$ is an infinite independent set of nonzero small elements of L, a contradiction. Thus $\mathrm{Rad}(L)/0$ has finite uniform dimension. □

3. Artinian Lattices

Lemma 3.1. *Let L be a compactly generated semiatomic lattice. Then the following are equivalent.*

(i) *L is finitely generated.*

(ii) *L is finitely cogenerated.*

(iii) *1 is a finite independent join of atoms.*

(iv) *L is Artinian.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) By [4, Theorem 11.1].

(iv) $\Rightarrow$ (ii) By [4, Proposition 11.2].

(iii) $\Rightarrow$ (iv) Note that if a is an atom in L, then $a/0$ is Artinian. Assume that $1 = a_1 \vee a_2 \vee \cdots \vee a_n$ such that the join is independent and each a_i is atom in L. Since each $a_i/0$ is Artinian, $(a_1 \vee a_2 \vee \cdots \vee a_n)/0$ is Artinian, and hence L is Artinian. □

Lemma 3.2. *Let L be a compactly generated lattice which satisfies DCC on small elements. If f is a finitely generated element of $\mathrm{Rad}(L)/0$, then $f/0$ is Artinian.*

Proof. Let f be a finitely generated element of $\mathrm{Rad}(L)/0$. Then $f \leq \mathrm{Rad}(L) = \bigvee_I \{s_i \mid s_i \in S(L)\}$ implies that $f \leq \bigvee_F \{s_i \mid s_i \in S(L)\}$ for some finite subset F of I. By Lemma 2.7, $f \in S(L)$. By assumption and Lemma 2.7(i), $f/0$ is Artinian. □

Lemma 3.3. *Let L be a compactly generated lattice which satisfies DCC on small elements. Then, for every $k < \mathrm{Rad}(L)$, $\mathrm{Soc}(\mathrm{Rad}(L)/k)$ is an essential element of $\mathrm{Rad}(L)/k$.*

Proof. Let $k < \mathrm{Rad}(L)$, and let $\mathrm{Soc}(\mathrm{Rad}(L)/k) = t$. Let $k \leq h \leq \mathrm{Rad}(L)$ such that $t \wedge h = k$. Assume that $k < h$. Since $\mathrm{Rad}(L)/0$ is compactly generated, there exists a nonzero finitely generated element x in $\mathrm{Rad}(L)/0$ such that $x \leq h$ but $x \notin k/0$. By Lemma 3.2, $x/0$ is Artinian. Then $x/(x \wedge k) \cong (k \vee x)/k$ implies that $(k \vee x)/k$ is a nonzero Artinian sublattice. By [4, Proposition 1.4], $(k \vee x)/k$ has an atom element p'. Note that $k < p' \leq x \vee k \leq h$. Since p' is atom in $\mathrm{Rad}(L)/k$, we have $p' \leq t$. Thus $k < p' \leq t \wedge h$. This contradicts the fact that $t \wedge h = k$. Therefore $k = h$ and $t \in E(\mathrm{Rad}(L)/k)$. This completes the proof. □

Lemma 3.4. *Let a be an element of a compactly generated lattice L. If a is a finitely generated element of $a/0$, then a is a finitely generated element of L.*

Proof. Since L is compactly generated, $a = \vee U$ where U is a set of finitely generated elements in L. Since a is a finitely generated element of $a/0$, $a = \bigvee_{(1 \leq i \leq n)} a_i$ for some elements $a_i (1 \leq i \leq n)$ of U. Therefore a is a finitely generated element of L. □

Lemma 3.5. *Let L be a compactly generated lattice which satisfies DCC on small elements. Suppose that the set*

$$\Omega = \left\{ a_i \mid 0 \leq a_i \leq \mathrm{Rad}(L) \text{ and } \frac{\mathrm{Rad}(L)}{a_i} \text{ is not finitely cogenerated} \right\} \tag{3.1}$$

is nonempty. Then:

(1) *the set Ω has a minimal member p which is a small element of L;*

(2) *if $\mathrm{Soc}(\mathrm{Rad}(L)/p) = s$, then s is not a finitely generated element of $\mathrm{Rad}(L)/p$ and s is a small element of L.*

Proof. (1) Let Γ be any chain in Ω. Let $c = \bigwedge_{c_i \in \Gamma} c_i$. If $c \notin \Omega$, then $\mathrm{Rad}(L)/c$ is finitely cogenerated. Therefore $c = c_i$ for some $c_i \in \Gamma$, a contradiction. By Zorn's Lemma, Ω has a minimal member p. Let $\mathrm{Soc}(\mathrm{Rad}(L)/p) = s$. By Lemma 3.3, $s \in E(\mathrm{Rad}(L)/p)$. Thus s is not a finitely generated element of $\mathrm{Rad}(L)/p$ by [4, Theorem 11.2]. Let $q \in L$ with $1 = p \vee q$. Then $s = s \wedge 1 = s \wedge (p \vee q) = p \vee (s \wedge q)$. It follows that $s/p = [p \vee (s \wedge q)]/p \cong (s \wedge q)/(p \wedge q)$. Suppose that $p \wedge q \neq p$. Then $\mathrm{Rad}(L)/(p \wedge q)$ is finitely cogenerated. Let $\mathrm{Soc}(\mathrm{Rad}(L)/(p \wedge q)) = \alpha$. Then α is finitely generated in $\mathrm{Rad}(L)/(p \wedge q)$ by [4, Theorem 11.2]. Therefore $\alpha/(p \wedge q)$ is Artinian by Lemma 3.1. Since $\mathrm{Rad}(L)/p$ is a sublattice of $\mathrm{Rad}(L)/(p \wedge q)$, we have $s \leq \alpha$. Thus $s \wedge q \leq \alpha \leq \mathrm{Rad}(L)$. Since $\alpha/(p \wedge q)$ is Artinian, $(s \wedge q)/(p \wedge q)$ is also Artinian by [4, Proposition 1.5]. This implies that s/p is Artinian, and hence s is a finitely generated element of s/p by Lemma 3.1. Since $\mathrm{Rad}(L)/p$ is compactly generated, s is a finitely generated element of $\mathrm{Rad}(L)/p$ (see Lemma 3.4), a contradiction. So $p \wedge q = p$ and hence $q \vee p = q = 1$. This gives $p \in S(L)$.

(2) Note that s is not a finitely generated element of $\mathrm{Rad}(L)/0$ as we prove in (1). Let $v \in L$ such that $1 = s \vee v$. Note that s/p is a semiatomic lattice. Then $s/[p \vee (s \wedge v)]$ is also semiatomic by [4, Corollary 6.3]. Therefore,

$$\frac{1}{p \vee v} = \frac{s \vee v}{p \vee v} = \frac{[s \vee (p \vee v)]}{p \vee v} \cong \frac{s}{[s \wedge (p \vee v)]} = \frac{s}{[p \vee (s \wedge v)]}. \tag{3.2}$$

This implies that $1/(p \vee v)$ is semiatomic. Suppose that $1 \neq p \vee v$. By [4, Lemma 6.12], there exists a maximal element w of $1/(p \vee v)$. Clearly, w is a maximal element of L and $v \leq w$. Thus $1 = s \vee v \leq s \vee w$. But $s \leq \mathrm{Rad}(L) \leq w$. Then $w = 1$, a contradiction. It follows that $1 = p \vee v$. Since $p \in S(L)$, we have $v = 1$. Thus $s \in S(L)$. □

Remark 3.6. By dualizing [6, Theorem 3.4], we have the fact that if L is upper continuous and $a/0$ is Artinian for every small element a of L, then $\vee S(L)/0$ is Artinian. Therefore for compactly generated lattices (ii) ⇒ (i) in Theorem 3.7 holds, but our aim is to give a proof in a different way. We should call attention to the fact that $\vee S(L)$ need not to be the radical of any upper continuous lattice L.

Theorem 3.7 (see cf. [5, Theorem 5]). *Let L be a compactly generated lattice. Then the following are equivalent.*

(i) $\mathrm{Rad}(L)/0$ *is Artinian.*

(ii) *For every small element a of L the sublattice $a/0$ is Artinian.*

(iii) *L satisfies DCC on small elements.*

Proof. (i) ⇒ (ii) Clear by [2, Theorem 8].

(ii) ⇒ (iii) This is immediate.

(iii) ⇒ (i) Suppose that $\mathrm{Rad}(L)/0$ is not Artinian. By [4, Proposition 11.2], there exists an element g in L with $g \leq \mathrm{Rad}(L)$ such that $\mathrm{Rad}(L)/g$ is not finitely cogenerated. By Lemma 3.5, the set

$$\Omega = \left\{ a_i \mid 0 \leq a_i \leq \mathrm{Rad}(L) \text{ and } \frac{\mathrm{Rad}(L)}{a_i} \text{ is not finitely cogenerated} \right\} \tag{3.3}$$

has a minimal member p such that $\mathrm{Soc}(\mathrm{Rad}(L)/p) = s \in S(L)$ and s is not a finitely generated element of $\mathrm{Rad}(L)/p$. By (iii) and Lemma 2.7(i), $s/0$ is Artinian. By Lemma 3.1, $s/0$ is finitely generated. Therefore s is a finitely generated element of $\mathrm{Rad}(L)/p$ by Lemma 3.4. This is a contradiction. Therefore $\mathrm{Rad}(L)/0$ is Artinian. □

Corollary 3.8. *Let L be a compactly generated lattice. If $1/s$ is finitely cogenerated for every small element s of L, then* $\mathrm{Rad}(L)/0$ *is Artinian.*

Proof. Consider the descending chain

$$x_1 \geq x_2 \geq \cdots \tag{3.4}$$

of small elements of L. Put $x = \bigwedge_{i\geq 1} x_i$. Thus x is small in L. By assumption, $1/x$ is finitely cogenerated. So there exists an integer n such that $x = \bigwedge_{i=1}^{n} x_i = x_n$. Hence L has DCC on small elements. By Theorem 3.7, $\mathrm{Rad}(L)/0$ is Artinian. □

Let a and b be elements of L. Then b is called a *supplement* of a in L if b is minimal with respect to $a \vee b = 1$. Equivalently, b is a supplement of a if and only if $a \vee b = 1$ and $a \wedge b \in S(a/0)$ (see [4, Proposition 12.1]). The lattice L is said to be *supplemented* if every element a of L has a supplement in L.

The following result may be proved in much the same way as [5, Lemma 6], and $1/\mathrm{Rad}(L)$ is a semiatomic lattice by [4, Proposition 12.3] already.

Lemma 3.9. *Let L be a compactly generated supplemented lattice with DCC on supplement elements. Then* $1/\mathrm{Rad}(L)$ *is a finitely generated semiatomic lattice.*

By using Theorem 3.7 and Lemma 3.9, we get the following theorem.

Theorem 3.10. *Let L be a compactly generated lattice. Then L is Artinian if and only if L is supplemented and L satisfies DCC on supplement elements and small elements.*

Proof. The necessity is clear. Conversely, suppose that L is a supplemented lattice which satisfies DCC on supplement elements and small elements. By Theorem 3.7, $\mathrm{Rad}(L)/0$ is Artinian, and by Lemmas 3.1 and 3.9, $1/\mathrm{Rad}(L)$ is Artinian. Thus L is Artinian. □

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