U.P.B. Sci. Bull., Series A, Vol. 75, Iss. 1, 2013 ISSN 1223-7027

APPLICATION OF THE DIVISION THEOREM TO NONLINEAR PHYSICAL MODELS FOR CONSTRUCTING TRAVELING WAVES

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We extend the so-called first integral method, which is based on the division theorem, to the Sharma-Tasso-Olver equation and the (2+1)-dimensional modified Boussinesq equation. Our approach provides first integrals in polynomial form with a high accuracy for two-dimensional plane autonomous systems. Traveling wave solutions are constructed through the established first integrals.

Keywords: First integral method; Sharma-Tasso-Olver equation; (2+1)-dimensional modified Boussinesq equation; Traveling waves

1. Introduction

A good many modeling problems arising in nonlinear physical sciences deal with nonlinear partial differential equations (NPDEs) which exhibit rich structure. The existence of a special class of explicit solutions called traveling waves is one of the most fundamental questions regarding NPDEs since they can be widely found in many scientific fields such as fluid mechanics, plasma physics, crystal lattice theory, etc. Thus, over the four decades or so, several efficient and powerful analytic methods have been successfully developed by a diverse group of physicists and mathematicians to find such types of exact and explicit solutions for NPDEs. To make mention of a few, sine-cosine method [1], tanh-coth method [2], Adomian decomposition method [3], Wronskian technique [4], variational iteration method [5], Jacobi elliptic function method [6], homotopy perturbation method [7], sinh-Gordon equation expansion method [8], (G'/G)-expansion method [9, 10], Exp-function method [11, 12] etc. However, most of the methods are not enough effective for integrable equations.

Recently, a new method has been proposed by Feng [13, 14] for the implementation of the theory of commutative algebra to NPDEs. The procedure is termed as the first integral method or the algebraic curve method [15]. The method is precise, effective, and reliable by avoiding tedious and complicated algebraic calculations. It can be used as an alternative method for obtaining new analytic solutions of many NPDEs arising in applied physical sciences. For the development of the method, some useful works by others have appeared in the research literature [16-23].

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The core idea of the first integral method is to find the first integrals of nonlinear differential equations in polynomial form. Taking the polynomials with unknown polynomial coefficients into account the method provides an algorithm which is based on the division theorem for two variables in the complex domain.

Our goal in this paper is to stress the power of the first integral method in handling NPDEs arising in applied sciences. To this end, we study two physically important NPDEs, namely, the Sharma-Tasso-Olver (STO) equation and the (2+1)-dimensional modified Boussinesq ((2+1)-mB) equation using the first integral method for the first time. The reason we choose to investigate the STO by our method is that it leads, in a remarkable manner, to a high number of first integrals. Besides, we consider the (2+1)-mB equation for the generalization of our method to higher-dimensional equations.

2. The first integral method

Before proceeding, let us consider a partial differential equation for a function $u(x,t)$ in the form

$$P(u,u_t,u_x,u_{tt},u_{tx},u_{xx},\ldots)=0\,, \tag{1}$$

where P is a polynomial in its arguments while subscripts denote partial derivatives. Via the transformation $u(x,t)=U(\xi)$, $\xi=kx-wt+\xi_0$, where k, w, and ξ_0 are arbitrary constants, Eq. (1) reduces to the ordinary differential equation

$$P(U,-wU',\ kU',\ w^2U'',-kwU'',k^2U'',\ldots)=0\,, \tag{2}$$

where $U=U(\xi)$ and the primes denote ordinary derivatives with respect to ξ. On the other hand, by means of the new variables

$$X(\xi)=U(\xi),\,Y(\xi)=U_\xi(\xi)\,, \tag{3}$$

Eq. (2) can be reduced to a two–dimensional autonomous system of the form

$$\begin{aligned} X_\xi(\xi)&=Y(\xi),\\ Y_\xi(\xi)&=Q\big(X(\xi),Y(\xi)\big), \end{aligned} \tag{4}$$

where the subscript denotes ordinary derivative with respect to ξ. In general, solving a planar autonomous system of ODEs of the form (4) is a challenging and difficult task. Hence, based on the qualitative theory of ODEs [24], if one can derive a single first integral for the system (4), then one may be able to reduce Eq. (2) to a first–order integrable ODE. Then, a class of exact solutions may be obtained by solving the resulting first–order ODE by a quadrature. Let us recall the division theorem for two variables in the complex domain $\mathbb{C}$:

Division Theorem. *Suppose that* $P(w,z)$ *and* $Q(w,z)$ *are polynomials in* $\mathbb{C}[w,z]$ *and* $P(w,z)$ *is irreducible in* $\mathbb{C}[w,z]$. *If* $Q(w,z)$ *vanishes at all zero points of*

$P(w,z)$, then there exist a polynomial $G(w,z)$ in $\mathbb{C}[w,z]$ such that $Q(w,z)=P(w,z)G(w,z)$.

The division theorem can be proved either by the theory of functions of several complex variables [15, 25] or by the following Hilbert-Nullstellensatz theorem from the theory of commutative algebra [26].

Hilbert-Nullstellensatz Theorem. *Let k be a field and L an algebraic closure of k. Then*

(i) *Every ideal γ of $k[X_1,\ldots,X_n]$ not containing 1 admits at least one zero in L^n.*

(ii) *Let $\mathbf{x}=(x_1,\ldots,x_n)$ and $\mathbf{y}=(y_1,\ldots,y_n)$ be two elements of L^n. For the set of polynomials of $k[X_1,\ldots,X_n]$ zero at $\mathbf{x}$ to be identical with the set of polynomials of $k[X_1,\ldots,X_n]$ zero at $\mathbf{y}$, it is necessary and sufficient that there exists a k-automorphism s of L such that $y_i=s(x_i)$ for $1\le i\le n$.*

(iii) *For an ideal α of $k[X_1,\ldots,X_n]$ to be maximal, it is necessary and sufficient that there exists an $\mathbf{x}$ in L^n such that α is the set of $k[X_1,\ldots,X_n]$ zero at $\mathbf{x}$.*

(iv) *For a polynomial Q of $k[X_1,\ldots,X_n]$ to be zero on the set of zeros in L^n of an ideal γ of $k[X_1,\ldots,X_n]$, it is necessary and sufficient that there exist an integer $m>0$ such that $Q^m\in\gamma$.*

Remark 1. Since the real field $\mathbb{R}$ is a subfield of the complex field $\mathbb{C}$, we can always extend an equation given in $\mathbb{R}$ to an equation in $\mathbb{C}$. If the extended equation has an algebraic curve solution in $\mathbb{C}$, then the real plane and the intersection of the manifold of this solution must be the algebraic curve solution of the original equation in $\mathbb{R}$. Thus, the division theorem stated in $\mathbb{C}$ can also be stated in $\mathbb{R}$ [15].

3. The Sharma–Tasso–Olver equation

Let us consider the famous STO equation in the form

$$u_t + k(u^3)_x + \frac{3}{2}k(u^2)_{xx} + ku_{xxx} = 0, \tag{5}$$

where $k\neq 0$ is an arbitrary constant, and $u=u(x,t)$. Now, to seek for the traveling wave solutions of Eq. (1), we make the transformation $u(x,t)=U(\xi)$, $\xi=x-ct+\delta$, where c and δ denote the wave speed and the phase shift, respectively. Then, integrating the resulting equation once, we get

$$kU'' + 3kUU' + kU^3 - cU - kD = 0, \tag{6}$$

where the primes denote derivatives with respect to ξ and D is an integration constant. Let $z=U$ and $y=U'$. Hence, Eq. (6) is equivalent to the two-dimensional autonomous system

$$\begin{cases} z' = y, \\ y' = -3zy + \dfrac{c}{k}z - z^3 + D \end{cases} \tag{7}$$

which is not Hamiltonian. Now, suppose that $z=z(\xi)$ and $y=y(\xi)$ are the nontrivial solutions of (7). Also, assume that $q(z,y)=\sum_{i=0}^{m} A_i(z)y^i$ is an irreducible polynomial in the complex domain $\mathbb{C}$ such that

$$q(z(\xi),y(\xi)) = \sum_{i=0}^{m} A_i(z)y^i = 0, \tag{8}$$

where $A_i(z)$ $(0 \le i \le m)$ are polynomials of z and they are relatively prime in $\mathbb{C}$, and $A_m(z) \not\equiv 0$. Here, Eq. (8) is called a first integral to the system (7). Since $dq/d\xi$ is a polynomial in z and y, we note that $P(z(\xi),y(\xi))=0$ implies $dq/d\xi=0$. Then, by the division theorem, there exists a polynomial $B(z)+C(z)y$ in the complex domain $\mathbb{C}$ such that

$$\frac{dq}{d\xi} = \frac{\partial q}{\partial z}\frac{dz}{d\xi} + \frac{\partial q}{\partial y}\frac{dy}{d\xi} = \left(B(z)+C(z)y\right)\left[\sum_{i=0}^{m} A_i(z)y^i\right]. \tag{9}$$

We consider the case $m=2$ of (8). Hence, taking Eq. (7) and Eq. (9) into account, we get

$$\sum_{i=0}^{2}\left[A_i'(z)y^{i+1}\right] + \sum_{i=0}^{2}\left[iA_i(z)y^{i-1}\left(-3zy+\frac{c}{k}z-z^3+D\right)\right] = \left[B(z)+C(z)y\right]\left[\sum_{i=0}^{2}A_i(z)y^i\right]. \tag{10}$$

Equating the coefficients of y^i $(0 \le i \le 3)$ of both sides of Eq. (10), we obtain

$$y^3: \ A_2'(z) = C(z)A_2(z), \tag{11}$$

$$y^2: \ A_1'(z) = C(z)A_1(z) + \left[B(z)+6z\right]A_2(z), \tag{12}$$

$$y^1: \ A_0'(z) = C(z)A_0(z) + \left[B(z)+3z\right]A_1(z) + \left[2z^3 - \frac{2c}{k}z - 2D\right]A_2(z), \tag{13}$$

$$y^0: \ B(z)A_0(z) = \left[\frac{c}{k}z - z^3 + D\right]A_1(z). \tag{14}$$

From Eq. (11), we obtain that $A_2(z) = c_0\exp\left(\int C(z)dz\right)$, where c_0 is integration constant. Since $A_2(z)$ and $C(z)$ are polynomials of z, we deduce that $C(z)=0$ and $A_2(z)$ must be a constant. For simplicity, we can take $A_2(z)=1$. Thus, (12) and (13) reduces to the following equations

$$A_1'(z) = B(z) + 6z, \tag{15}$$

$$A_0'(z) = \left[B(z) + 3z\right]A_1(z) + 2z^3 - \frac{2c}{k}z - 2D. \tag{16}$$

Balancing the degrees of $A_0(z)$, $A_1(z)$, and $B(z)$ in (15) and (16), we conclude that $\deg B(z) = 1$ and $\deg A_1(z) = 2$. Letting $B(z) = b_1 z + b_0$ $(b_1 \neq 0)$ and $A_1(z) = a_2 z^2 + a_1 z + a_0$ $(a_2 \neq 0)$ in Eq. (15), we obtain $b_1 = 2a_2 - 6$ and $b_0 = a_1$. Then, integrating Eq. (16) once leads to

$$A_0(z) = \frac{2a_2^2 - 3a_2 + 2}{4}z^4 + a_1(a_2 - 1)z^3 + \frac{a_1^2 + 2a_0 a_2 - 3a_0 - \frac{2c}{k}}{2}z^2 + (a_0 a_1 - 2D)z + E, \tag{17}$$

where E is an integration constant. By substituting $A_0(z)$, $A_1(z)$, and $B(z)$ into Eq. (14) and equating all coefficients of z^i $(0 \le i \le 5)$ to zero, we obtain the following system of nonlinear algebraic equations

$$z^5 : \ 4ka_2^3 - 18ka_2^2 + 26ka_2 - 12k = 0, \tag{18}$$

$$z^4 : \ 30ka_1 - 35ka_1 a_2 + 10ka_1 a_2^2 = 0, \tag{19}$$

$$z^3 : \ 24c + 40ka_0 - 16ka_1^2 - 12ca_2 - 36ka_0 a_2 + 8ka_1^2 a_2 + 8ka_0 a_2^2 = 0, \tag{20}$$

$$z^2 : \ 48kD - 8ca_1 - 30ka_0 a_1 + 2ka_1^3 - 20kDa_2 + 12ka_0 a_1 a_2 = 0, \tag{21}$$

$$z^1 : \ -24Ek - 4ca_0 - 12kDa_1 + 4ka_0 a_1^2 + 8Eka_2 = 0, \tag{22}$$

$$z^0 : \ -4kDa_0 + 4Eka_1 = 0. \tag{23}$$

Solving (18)-(23) simultaneously, we get the solution sets

$$E = \frac{c^2}{4k^2},\ D = 0,\ a_2 = 1,\ a_0 = -\frac{c}{k},\ a_1 = 0, \tag{24}$$

$$E = 0,\ D = 0,\ a_2 = \frac{3}{2},\ a_0 = -\frac{c}{2k},\ a_1 = \mp\sqrt{\frac{c}{k}}, \tag{25}$$

$$E = \frac{16c^2 - 8cka_1^2 + k^2 a_1^4}{16k^2},\ D = \frac{ka_1^3 - 4ca_1}{8k},\ a_2 = 2,\ a_0 = \frac{ka_1^2 - 4c}{2k}, \tag{26}$$

$$E = a_1^4 - \frac{ca_1^2}{k},\ D = \frac{ca_1 - ka_1^3}{k},\ a_2 = 2,\ a_0 = -a_1^2, \tag{27}$$

$$E = -\frac{ca_0}{2k},\ D = 0,\ a_2 = 2,\ a_1 = 0, \tag{28}$$

$$E = \frac{50c^2}{147k^2},\ D = \mp\frac{20c}{21k}\sqrt{\frac{c}{21k}},\ a_2 = \frac{3}{2},\ a_0 = -\frac{15c}{14k},\ a_1 = \pm\sqrt{\frac{3c}{7k}}, \tag{29}$$

where all other constants remain arbitrary. Now, we make the following observations:

(i) Using the relation (24) in (8), we get the first integral

$$y = \frac{c - kz^2}{2k}. \tag{30}$$

Combining the first equation of (7) with Eq. (30), solving the resulting equations by quadratures, and changing to the original variables, we obtain a solution of Eq. (5) in the form

$$u(x,t) = \sqrt{\frac{c}{k}} \tanh\left(\frac{1}{2}\sqrt{\frac{c}{k}}\left(x - ct + \delta\right)\right), \tag{31}$$

where c and δ are free parameters.

(ii) Using the relation (25) in (8), we obtain the first integral

$$y = \pm\sqrt{\frac{c}{k}z - z^2}. \tag{32}$$

Combining the first equation of (7) with Eq. (32), solving the resulting equations by quadratures, and changing to the original variables, we get a solution of Eq. (5) in the form

$$u(x,t) = \frac{\sqrt{c}}{\cosh\sqrt{\frac{c}{k}}\left(x - ct + \delta\right) \mp \sinh\sqrt{\frac{c}{k}}\left(x - ct + \delta\right) \pm \sqrt{k}}, \tag{33}$$

where c and δ are free parameters.

(iii) Using the relation (26) in (8), we get the first integral

$$y = \frac{-4kz^2 - 2ka_1 z - ka_1^2 + 4c}{4k}. \tag{34}$$

Combining the first equation of (7) with Eq. (34), solving the resulting equations by quadratures, and changing to the original variables, we obtain a solution of Eq. (5) in the form

$$u(x,t) = -\frac{a_1}{4} - \frac{1}{4}\sqrt{\frac{3ka_1^2 - 16c}{k}} \tan\frac{1}{4}\sqrt{\frac{3ka_1^2 - 16c}{k}}\left(x - ct + \delta\right), \tag{35}$$

where a_1, c and δ are free parameters.

(iv) Using the relation (27) in (8), we obtain the first integral

$$y = -\frac{2kz^2 + ka_1 z \pm \sqrt{-k}\left(z + a_1\right)\sqrt{3ka_1^2 - 4c} - ka_1^2}{2k}. \tag{36}$$

Combining the first equation of (7) with Eq. (36), solving the resulting equations by quadratures, and changing to the original variables, we get a solution of Eq. (5) in the form

$$u(x,t)=\frac{ka_1+\sqrt{k}\sqrt{4c-3ka_1^2}\mp a_1\exp\left(\frac{-\left(3\sqrt{k}a_1+\sqrt{4c-3ka_1^2}\right)}{2\sqrt{k}}\left(x-ct+\delta\right)\right)}{2k\pm\exp\left(\frac{-\left(3\sqrt{k}a_1+\sqrt{4c-3ka_1^2}\right)}{2\sqrt{k}}\left(x-ct+\delta\right)\right)},\quad (37)$$

where a_1, c and δ are free parameters.

(v) Using the relation (28) in (8), we get the first integral

$$y=\frac{-2kz^2\pm\sqrt{\left(k\left(2z^2+a_0\right)\left(2c+ka_0\right)\right)}-ka_0}{2k}.\quad (38)$$

Combining the first equation of (7) with Eq. (38), solving the resulting equations by quadratures, and changing to the original variables, we obtain a solution of Eq. (5) in the form

$$u\left(x,t\right)=\frac{\sqrt{ca_0}\sinh\left(\sqrt{\frac{c}{k}}\left(x-ct+\delta\right)\right)\left(\pm\sqrt{2c+ka_0}-\sqrt{ka_0}\cosh\left(\sqrt{\frac{c}{k}}\left(x-ct+\delta\right)\right)\right)}{2c-ka_0\sinh^2\left(\sqrt{\frac{c}{k}}\left(x-ct+\delta\right)\right)},\quad (39)$$

where a_0, c and δ are free parameters.

(vi) Using the relation (29) in (8), we obtain the first integral

$$y=\frac{45ck+6\sqrt{21}\sqrt{c}k^{3/2}z-63k^2z^2\pm\sqrt{100\sqrt{21}c^{3/2}k^{5/2}z+630ck^3z^2-252\sqrt{21}\sqrt{c}k^{7/2}z^3+441k^4z^4-375c^2k^2}}{84k^2}.\quad (40)$$

Combining the first equation of (7) with Eq. (40), solving the resulting equations by quadratures, and changing to the original variables, we get a solution of Eq. (5) in the form

$$u\left(x,t\right)=\frac{\sqrt{c}\left(20-133\cosh\left(\sqrt{\frac{3c}{7k}}\left(x-ct+\delta\right)\right)+123\sinh\left(\sqrt{\frac{3c}{7k}}\left(x-ct+\delta\right)\right)\right)}{\sqrt{21}\sqrt{k}\left(4+31\cosh\left(\sqrt{\frac{3c}{7k}}\left(x-ct+\delta\right)\right)-33\sinh\left(\sqrt{\frac{3c}{7k}}\left(x-ct+\delta\right)\right)\right)},\quad (41)$$

where c and δ are free parameters.

Remark 2. By assigning special values to the parameters, we can construct various types of traveling waves to the STO. As a special example, if $c/k<0$ in (31) or $a_0=-2c/k$ in (39) then the solutions turns out to be, respectively

$$u(x,t) = -\sqrt{\frac{-c}{k}}\tan\left(\frac{1}{2}\sqrt{\frac{-c}{k}}(x-ct+\delta)\right), \tag{42}$$

$$u(x,t) = \sqrt{\frac{c}{k}}\tanh\left(\sqrt{\frac{c}{k}}(x-ct+\delta)\right). \tag{43}$$

4. The (2+1)-mB equation

Now, we consider the (2+1)-mB equation which reads

$$u_{tt} = u_{xx} + 8\left(u^3\right)_{xx} + u_{xxxx} + u_{yy}. \tag{44}$$

The wave transformation $u(x,y,t) = U(\xi),\ \ \xi = x + y - ct + \delta$ reduces (44) to the equation

$$c^2 U'' = U'' + 8\left(U^3\right)'' + U'''' + U'' \tag{45}$$

where the primes denote ordinary derivatives with respect to ξ. Then, integrating the resulting equation (45) twice and setting the integration constants to zero, we get

$$U'' + 8U^3 - (c^2-2)U = 0. \tag{46}$$

Letting $z = U$ and $y = U'$ in Eq. (46), we obtain the equivalent two-dimensional autonomous Hamiltonian system

$$\begin{cases} z' = y, \\ y' = \left(c^2-2\right)z - 8z^3, \end{cases} \tag{47}$$

with Hamiltonian function $H(z,y) = \frac{1}{2}\left(y^2 + \left(2-c^2\right)z^2 + 4z^4\right) = h$, where h is a constant. We consider the case $m = 2$ of (8). Hence, taking Eq. (47) and Eq. (9) into account, we get

$$\sum_{i=0}^{2}\left[A_i'(z)y^{i+1}\right] + \sum_{i=0}^{2}\left[iA_i(z)y^{i-1}\left(\left(c^2-2\right)z - 8z^3\right)\right] = \left[B(z) + C(z)y\right]\left[\sum_{i=0}^{2}A_i(z)y^i\right]. \tag{48}$$

Equating the coefficients of y^i $(0 \le i \le 3)$ of both sides of Eq. (48), we obtain

$$y^3:\ A_2'(z) = C(z)A_2(z), \tag{49}$$

$$y^2:\ A_1'(z) = C(z)A_1(z) + B(z)A_2(z), \tag{50}$$

$$y^1:\ A_0'(z) = C(z)A_0(z) + B(z)A_1(z) - 2\left(\left(c^2-2\right)z - 8z^3\right)A_2(z), \tag{51}$$

$$y^0:\ B(z)A_0(z) = \left(\left(c^2-2\right)z - 8z^3\right)A_1(z). \tag{52}$$

From Eq. (49), we obtain that $A_2(z)=c_0\exp\left(\int C(z)dz\right)$, where c_0 is integration constant. Since $A_2(z)$ and $C(z)$ are polynomials of z, we deduce that $C(z)=0$ and $A_2(z)$ must be a constant. For simplicity, we can take $A_2(z)=1$. Thus, (50) and (51) reduces to the following equations

$$A_1'(z)=B(z) \tag{53}$$

$$A_0'(z)=B(z)A_1(z)-2\left(\left(c^2-2\right)z-8z^3\right). \tag{54}$$

Balancing the degrees of $A_0(z)$, $A_1(z)$, and $B(z)$ in (53) and (54), we conclude that $\deg B(z)=1$ and $\deg A_1(z)=2$. Letting $B(z)=b_1z+b_0\ (b_1\neq 0)$ and $A_1(z)=a_2z^2+a_1z+a_0\ (a_2\neq 0)$ in Eq. (53), we obtain $b_1=2a_2$ and $b_0=a_1$. Then, integrating Eq. (54) once leads to

$$A_0(z)=\frac{2a_2^2+16}{4}z^4+a_1a_2z^3+\frac{2a_0a_2+a_1^2-2\left(c^2-2\right)}{2}z^2+a_0a_1z+D \tag{55}$$

where D is an integration constant. By substituting $A_0(z)$, $A_1(z)$, and $B(z)$ into Eq. (52) and equating all coefficients of $z^i\ (0\le i\le 5)$ to zero, we obtain the following system of nonlinear algebraic equations

$$z^5:\ 32a_2+2a_2^3=0 \tag{56}$$

$$z^4:\ 24a_1+5a_1a_2^2=0 \tag{57}$$

$$z^3:\ 16a_0+12a_2-6c^2a_2+4a_1^2a_2+4a_0a_2^2=0 \tag{58}$$

$$z^2:\ 8a_1-4c^2a_1+a_1^3+6a_0a_1a_2=0, \tag{59}$$

$$z^1:\ 4a_0-2c^2a_0+2a_0a_1^2+4Da_2=0, \tag{60}$$

$$z^0:\ 2Da_1=0 \tag{61}$$

Solving (56)-(61) simultaneously, we get the solution sets

$$D=\frac{1}{16}\left(4c^2-c^4-4\right),\ a_0=-\frac{1}{2}i\left(c^2-2\right),\ a_1=0,\ a_2=4i, \tag{62}$$

$$D=\frac{1}{16}\left(4c^2-c^4-4\right),\ a_0=\frac{1}{2}i\left(c^2-2\right),\ a_1=0,\ a_2=-4i. \tag{63}$$

Now, we make the following observations:

(i) Using the relation (62) in (8), we get the first integral

$$y=\frac{1}{4}i\left(c^2-2-8z^2\right). \tag{64}$$

Combining the first equation of (7) with Eq. (64), solving the resulting equations by quadratures, and changing to the original variables, we obtain a solution of Eq. (44) in the form

$$u(x,y,t)=-\frac{1}{2}i\sqrt{\frac{2-c^2}{2}}\tanh\sqrt{\frac{2-c^2}{2}}\left(x+y-ct+\delta\right),\quad |c|<\sqrt{2}, \tag{65}$$

$$u(x,y,t)=-\frac{1}{2}i\sqrt{\frac{c^2-2}{2}}\tan\sqrt{\frac{c^2-2}{2}}\left(x+y-ct+\delta\right),\quad |c|>\sqrt{2}\,, \tag{66}$$

where c and δ are free parameters.

(ii) Using the relation (63) in (8), we get the first integral

$$y=-\frac{1}{4}i\left(c^2-2-8z^2\right). \tag{67}$$

Combining the first equation of (7) with Eq. (67), solving the resulting equations by quadratures, and changing to the original variables, we obtain a solution of Eq. (44) in the form

$$u(x,y,t)=\frac{1}{2}i\sqrt{\frac{2-c^2}{2}}\tanh\sqrt{\frac{2-c^2}{2}}\left(x+y-ct+\delta\right),\quad |c|<\sqrt{2}\,, \tag{68}$$

$$u(x,y,t)=\frac{1}{2}i\sqrt{\frac{c^2-2}{2}}\tan\sqrt{\frac{c^2-2}{2}}\left(x+y-ct+\delta\right),\quad |c|>\sqrt{2}\,, \tag{69}$$

where c and δ are free parameters.

Remark 3. We note that our results in Sections 3 and 4 are based on the assumption of $m=2$ in (8). The discussion becomes more complicated for the cases $m=3,4$ since the hyper-elliptic integrals, the irregular singular point theory and the elliptic integrals of the second kind are involved. Also, we do not need to consider the case $m\geq 5$ since the fact that an algebraic equation with the degree greater than or equal to five is generally not solvable is well known.

Remark 4. In the theory of nonlinear differential equations, searching for the first integrals of the nonlinear ordinary differential equations is one of the most important problem since they permit us to get the general solution of a nonlinear differential equation in the form of quadratures. We observe that the first integral method, for discovering first integrals, can be applied to NPDEs which can be converted to the following forms through the traveling wave transformation

$u''(\xi)-\alpha P(u,u')-R(u)=0$,

$u''(\xi)-Q(u,u')u'(\xi)-R(u)=0$

where α is real, $R(u)$ is a polynomial with real coefficients, $P(w,z)$ and $Q(w,z)$ are polynomials in w and z.

5. Conclusions

We used the first integral method to derive a wide class of traveling wave solutions for the STO equation and the (2+1)-mB equation. The obtained solutions may be important for the explanation of some practical physical problems. Being easier and quicker than other traditional techniques, our method provides results with high accuracy. We predict that the first integral method can be found widely applicable in mathematical and physical sciences.

Acknowledgement

The author is grateful to the anonymous referee for useful and valuable suggestions.

R E F E R E N C E S

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