RATIONAL AND MULTI-WAVE SOLUTIONS TO SOME NONLINEAR PHYSICAL MODELS

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The Exp-function method is shown to be an effective tool to explicitly construct rational and multi-wave solutions of completely integrable nonlinear evolution equations. The procedure does not require the bilinear representation of the equation. The method is straightforward, concise, and its applications to other types of nonlinear evolution equations are promising.

Key words: Exp-function method, modified Korteweg de Vries equation, fourth order Burgers-like equation, multi-wave solution, rational solution

1. INTRODUCTION

Applications of nonlinear evolution equations (NEEs) can be seen in many areas of nonlinear sciences. Hence, seeking innovative methods to solve and analyze these equations has been an interesting research subject over the four decades or so. Nowadays, many ingenious techniques are available for obtaining exact solutions mainly through analytic studies such as tanh function method [1], Adomian decomposition method [2], homotopy analysis method [3], variational iteration method [4], homotopy perturbation method [5], first integral method [6], Exp-function method [7], (G'/G)-expansion method [8], three-wave method [9], multi-exp function method [10] and so forth. In addition, He *et al.* [11] put forward three standard variational iteration algorithms for dealing with differential equations, fractional differential equations, integro-differential equations, fractal differential equations, fractional/fractal differential-difference equations, as well as differential-difference equations arising in applied mathematical sciences. It is notable that these methods may not work well to tackle a specific nonlinear problem.

On the other hand, special types of exact solutions have been of fundamental importance to our understanding of physical, chemical and biological phenomena modeled by NEEs. Traveling waves of NEEs may be coupled with different frequencies and different velocities. Multi-wave solutions are crucial in the sense that they may sometimes be converted into a single soliton of very high energy that propagates over large domains of space without dispersion. Therefore, an extremely destructive wave may be produced. The *tsunami* is a good example for this kind of phenomena. In fact, multi-wave solutions of completely integrable NEEs can be constructed by three distinct methods; the inverse scattering method [12], the Hirota bilinear method [13] and the Bäcklund transformation method [14]. Though each of these methods has its own features, the Hirota bilinear method is quite heuristic and provides multi-wave solutions for a wide class of NEEs. Moreover, this method is also useful to analyze the integrable properties of such equations. The main point of the Hirota bilinear method is to derive the bilinear form of the equation by means of a proper dependent-variable transformation.

In the recent literature, the elegant method introduced by He *et al.* [7] has been developed by many scientists and become one of the most powerful tools, especially in the area of nonlinear differential equations. In the subsequent papers [15-25], this technique has been extended for NEEs with variable coefficients, multi-dimensional equations, differential-difference equations, coupled NEEs, and stochastic equations as well as for *n*-soliton solutions, double-wave solutions, and rational solutions. However, the generalization of this procedure to higher order NDDEs to find rational and multi-wave solutions in an effective manner is still an interesting and important issue.

In the present paper, we wish is to show the applicability of the Exp-function method to the modified Korteweg de Vries equation (defocusing case) and the fourth order Burgers-like equation for rational and multi-wave solutions. The paper is organized as follows: In the next section, to make the paper self-contained, we summarize the method. In Sections 3 and 4, we analyze our problems. In Section 5, we present a brief conclusion.

2. METHODOLOGY

In this section, in order to present our results in a straightforward manner, we start our study by briefly reviewing the procedure [16-18] for constructing multi-wave and rational solutions. Consider a nonlinear partial differential equation for a function u of two real variables, space x and time t;

$$P\left(u,u_t,u_x,u_{tt},u_{tx},u_{xx},\ldots\right)=0\,, \tag{1}$$

where P is a polynomial in its arguments and subscripts denoting partial derivatives. The Exp-function method is based on the assumption that the solutions of Eq. (1) can be expressed in the form

$$u\left(x,t\right)=\frac{\sum_{i=0}^{m}a_i\exp\left(i\xi\right)}{\sum_{j=0}^{n}b_j\exp\left(j\xi\right)},\ \xi=kx+wt+\delta\,, \tag{2}$$

where m and n are positive integers to be determined by balancing the highest-order terms in Eq. (1); a_i, b_j, k and w are arbitrary constants to be specified at the stage of solving Eq. (1); δ is the phase shift. To search for rational and multi-wave solutions to Eq. (1), the ansatz (2) can be modified as follows:
For a two- wave solution, we set

$$u(x,t)=\frac{\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2} a_{i_1i_2}\exp\left(i_1\xi_1+i_2\xi_2\right)}{\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2} b_{j_1j_2}\exp\left(j_1\xi_1+j_2\xi_2\right)},\ \xi_l=k_lx+w_lt+\delta_l,\ l=1,2\,. \tag{3}$$

For a three- wave solution, we consider

$$u(x,t)=\frac{\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2}\sum_{i_3=0}^{m_3} a_{i_1i_2i_3}\exp\left(i_1\xi_1+i_2\xi_2+i_3\xi_3\right)}{\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}\sum_{j_3=0}^{n_3} b_{j_1j_2j_3}\exp\left(j_1\xi_1+j_2\xi_2+j_3\xi_3\right)},\ \xi_l=k_lx+w_lt+\delta_l,\ l=1,2,3\,, \tag{4}$$

and so on.
For a rational solution, we take

$$u(x,t)=\frac{\sum_{i=0}^{m} a_i\left(\mu_1\exp(\xi)+\mu_2\xi\right)^i}{\sum_{j=0}^{n} b_j\left(\mu_1\exp(\xi)+\mu_2\xi\right)^j},\ \xi=kx+wt+\delta\,, \tag{5}$$

where μ_1 and μ_2 are two embedded constants. We remark that when $\mu_1=1$ and $\mu_2=0$, the ansatz (5) turns out to be the ansatz (2).

3. THE MODIFIED KORTEWEG DE VRIES EQUATION

First, we consider the modified Korteweg de Vries equation (defocusing case)

$$u_t-6u^2u_x+u_{xxx}=0\,, \tag{6}$$

where $u=u(x,t)$. We assume that Eq. (6) admits a solution of the form

$$u(x,t)=\frac{a_1\exp(\xi)}{1+b_1\exp(\xi)+b_2\exp(2\xi)},\ \xi=kx+wt+\delta\,, \tag{7}$$

which is embedded in (2). Substituting (7) into Eq. (6), we get a relation of the form

$$\left(1+b_1\exp(\xi)+b_2\exp(2\xi)\right)^{-4}\sum_{1\le i\le 7,i\ne 4} A_i\exp(i\xi)=0\,, \tag{8}$$

where

$$A_1 = a_1 k^3 + a_1 w\,,$$
$$A_2 = -4a_1 b_1 k^3 + 2a_1 b_1 w\,,$$
$$A_3 = -6a_1^3 k + a_1 b_1^2 k^3 - 23a_1 b_2 k^3 + a_1 b_1^2 w + a_1 b_2 w\,,$$
$$A_5 = 6a_1^3 b_2 k - a_1 b_1^2 b_2 k^3 + 23a_1 b_2^2 k^3 - a_1 b_1^2 b_2 w - a_1 b_2^2 w\,,$$
$$A_6 = 4a_1 b_1 b_2^2 k^3 - 2a_1 b_1 b_2^2 w\,,$$
$$A_7 = -a_1 b_2^3 k^3 - a_1 b_2^3 w\,.$$

Thus, solving the system $A_i = 0 \; \left(1 \le i \le 7,\ i \ne 4\right)$ simultaneously, we obtain the solution set

$$w = -k^3,\ b_1 = 0,\ b_2 = -a_1^2 / 4k^2\,, \tag{9}$$

which yields a one-wave solution to Eq. (6) as

$$u_1\left(x,t\right) = \frac{4k^2 a_1 \exp\left(kx - k^3 t + \delta\right)}{4k^2 - a_1^2 \exp\left(2\left(kx - k^3 t + \delta\right)\right)}\,, \tag{10}$$

where k , a_1 and δ remain arbitrary. In fact, taking $a_1 = 2k\exp(\delta_0)$ into account, (10) can be written as

$$u_1\left(x,t\right) = -\frac{k}{\sinh\left(\eta\right)},\ \eta = kx - k^3 t + \delta + \delta_0\,, \tag{11}$$

where k , δ and δ_0 are arbitrary constants. Clearly, (11) has a singularity at $\eta = 0$.

3.1. TWO-WAVE SOLUTIONS

Now, suppose that Eq. (6) admits a solution of the form

$$u\left(x,t\right) = \frac{v_1\left(\xi_1,\xi_2\right)}{v_2\left(\xi_1,\xi_2\right)}\,, \tag{12}$$

where $\xi_l = k_l x + w_l t + \delta_l,\ l = 1,2$, and

$$v_1\left(\xi_1,\xi_2\right) = a_{10}\exp\left(\xi_1\right) + a_{01}\exp\left(\xi_2\right) + a_{11}\exp\left(\xi_1+\xi_2\right) + a_{21}\exp\left(2\xi_1+\xi_2\right) + a_{12}\exp\left(\xi_1+2\xi_2\right),$$
$$v_2\left(\xi_1,\xi_2\right) = 1 + b_{10}\exp\left(\xi_1\right) + b_{01}\exp\left(\xi_2\right) + b_{11}\exp\left(\xi_1+\xi_2\right) + b_{20}\exp\left(2\xi_1\right) + b_{02}\exp\left(2\xi_2\right)$$
$$+ b_{21}\exp\left(2\xi_1+\xi_2\right) + b_{12}\exp\left(\xi_1+2\xi_2\right) + b_{22}\exp\left(2\xi_1+2\xi_2\right).$$

Clearly, the ansatz (12) is embedded in (3). Substituting (12) into Eq. (6), we obtain the relation

$$\left(v_2\left(\xi_1,\xi_2\right)\right)^{-4}\sum_{i=0}^{8}\sum_{j=0}^{8}A_{ij}\exp\left(i\xi_1+j\xi_2\right)=0\,, \tag{13}$$

where $v_2\left(\xi_1,\xi_2\right)$ is as in (12). Thus, solving the system $A_{ij}=0\ \left(0\le i,j\le 8\right)$ simultaneously, we obtain the solution set

$$w_1=-k_1^3,\ w_2=-k_2^3,\ b_{20}=-\frac{a_{10}^2}{4k_1^2},\ b_{22}=\frac{a_{01}^2a_{10}^2}{16k_1^2k_2^2}\left(\frac{k_1-k_2}{k_1+k_2}\right)^4,\ b_{21}=0,\ b_{12}=0,\ b_{01}=0,\ b_{10}=0\,, \tag{14}$$

$$a_{11}=0,\ b_{11}=-\frac{2a_{01}a_{10}}{\left(k_1+k_2\right)^2},\ a_{12}=-\frac{a_{01}^2a_{10}}{4k_2^2}\left(\frac{k_1-k_2}{k_1+k_2}\right)^2,\ a_{21}=-\frac{a_{01}a_{10}^2}{4k_1^2}\left(\frac{k_1-k_2}{k_1+k_2}\right)^2,\ b_{02}=-\frac{a_{01}^2}{4k_2^2}\,, \tag{15}$$

which gives rise a two- wave solution to Eq. (6) as

$$u_2\left(x,t\right)=\frac{v_3\left(x,t\right)}{v_4\left(x,t\right)}, \tag{16}$$

where

$$v_3\left(x,t\right)=16\left(\begin{array}{l}a_{10}\exp\left(k_1x-k_1^3t+\delta_1\right)-\dfrac{a_{01}^2a_{10}}{4k_2^2}\left(\dfrac{k_1-k_2}{k_1+k_2}\right)^2\exp\left(k_1x-k_1^3t+\delta_1+2k_2x-2k_2^3t+2\delta_2\right)\\ +a_{01}\exp\left(k_2x-k_2^3t+\delta_2\right)-\dfrac{a_{01}a_{10}^2}{4k_1^2}\left(\dfrac{k_1-k_2}{k_1+k_2}\right)^2\exp\left(2k_1x-2k_1^3t+2\delta_1+k_2x-k_2^3t+\delta_2\right)\end{array}\right),$$

$$\begin{aligned}v_4\left(x,t\right)=16&-\frac{4a_{10}^2}{k_1^2}\exp\left(2\left(k_1x-k_1^3t+\delta_1\right)\right)+\frac{a_{01}^2a_{10}^2}{k_1^2k_2^2}\left(\frac{k_1-k_2}{k_1+k_2}\right)^4\exp\left(2\left(k_1x-k_1^3t+\delta_1+k_2x-k_2^3t+\delta_2\right)\right)\\ &-\frac{32a_{01}a_{10}}{\left(k_1+k_2\right)^2}\exp\left(k_1x-k_1^3t+\delta_1+k_2x-k_2^3t+\delta_2\right)-\frac{4a_{01}^2}{k_2^2}\exp\left(2\left(k_2x-k_2^3t+\delta_2\right)\right),\end{aligned}$$

in which a_{01}, a_{10}, k_1, k_2, δ_1, and δ_2 remain arbitrary. However, letting $a_{10}=2k_1\exp\left(\delta_{01}\right)$, $a_{01}=2k_2\exp\left(\delta_{02}\right)$, and $\eta_i=k_ix-k_i^3t+\delta_i+\delta_{0i}\ \left(i=1,2\right)$ in (16), the expressions $v_3\left(x,t\right)$ and $v_4\left(x,t\right)$ can be reformulated as

$$v_3\left(x,t\right)=32\left(k_1\exp\left(\eta_1\right)+k_2\exp\left(\eta_2\right)-k_1\left(\frac{k_1-k_2}{k_1+k_2}\right)^2\exp\left(\eta_1+2\eta_2\right)-k_2\left(\frac{k_1-k_2}{k_1+k_2}\right)^2\exp\left(2\eta_1+\eta_2\right)\right),$$

$$v_4\left(x,t\right)=16\left(1-\exp\left(2\eta_1\right)-\exp\left(2\eta_2\right)+\left(\frac{k_1-k_2}{k_1+k_2}\right)^4\exp\left(2\left(\eta_1+\eta_2\right)\right)-\frac{8k_1k_2}{\left(k_1+k_2\right)^2}\exp\left(\eta_1+\eta_2\right)\right),$$

where k_1, k_2, δ_{01}, δ_{02}, δ_1, and δ_2 are arbitrary constants. Clearly, $u_2(x,t)$ has a singular behavior for $v_4(x,t)=0$.

Remark 1

Due to its complexity, we skip the investigation of $N(\geq 3)$-wave solutions for Eq. (6). So far, we have observed that the implementation of the Exp-function method to Eq. (6) for multi-wave solutions leads to singular solutions, of very little physical significance as far as we could verify.

3.2. RATIONAL SOLUTIONS

Suppose that Eq. (6) admits a solution of the form

$$u(x,t)=\frac{a_1(\mu_1\exp(\xi)+\mu_2\xi)+a_0+a_{-1}(\mu_1\exp(\xi)+\mu_2\xi)^{-1}}{b_1(\mu_1\exp(\xi)+\mu_2\xi)+b_0+b_{-1}(\mu_1\exp(\xi)+\mu_2\xi)^{-1}},\ \xi=kx+wt+\delta. \quad (17)$$

By the same procedure, we obtain the solution set of the resultant algebraic system as

$$a_0=\mp\frac{A}{b_1},\ b_0=\mp\frac{A}{a_1},\ A=\sqrt{b_1\left(4a_1^2b_{-1}+k^2b_1^3\right)},\ a_{-1}=\frac{a_1^2b_{-1}+k^2b_1^3}{a_1b_1},\ w=\frac{6ka_1^2}{b_1^2},\ \mu_1=0,\ \mu_2=1 \quad (18)$$

which provide a rational solution to Eq. (6) in the form

$$u(x,t)=\frac{a_1\xi^2+a_0\xi+a_{-1}}{b_1\xi^2+b_0\xi+b_{-1}},\ \xi=kx+\frac{6ka_1^2}{b_1^2}t+\delta, \quad (19)$$

where a_0, b_0, and a_{-1} are as given in (18) and all other involved constants remain arbitrary. It is interesting that the solution (19) can be non-singular if $b_0^2-4b_1b_{-1}<0$.

4. THE FOURTH ORDER BURGERS-LIKE EQUATION

Next, we consider the fourth order Burgers-like equation

$$u_t+\alpha u_{xxxx}+10\alpha u_xu_{xx}+4\alpha uu_{xxx}+12\alpha uu_x^2+6\alpha u^2u_{xx}+4\alpha u^3u_x=0, \quad (20)$$

where α is a nonzero constant, and $u=u(x,t)$. As is well known, the study of integrable hierarchies is a significant and interesting topic in wave theory. Equation (20) appears to be a member of Burgers hierarchy in applications. Now, we suppose that Eq. (20) admits a solution of the form

$$u(x,t)=\frac{a_1\exp(\xi)}{1+b_1\exp(\xi)},\ \xi=kx+wt+\delta\,, \tag{21}$$

which is embedded in (2). We will omit technical details here because the procedure is similar to the scheme used in Section 3.
Substituting (21) into Eq. (20) and solving the resultant algebraic system for the unknowns a_1, b_1, k, and w, we obtain the solution set

$$w=-\alpha k^4,\ b_1=a_1/k\,, \tag{22}$$

which yields a one- wave solution to Eq. (20) as

$$u_1(x,t)=\frac{a_1\exp\left(kx-\alpha k^4 t+\delta\right)}{1+(a_1/k)\exp\left(kx-\alpha k^4 t+\delta\right)}\,, \tag{23}$$

where k, a_1, and δ remain arbitrary. Indeed, setting $a_1=k\exp(\delta_0)$, (23) can be stated as

$$u_1(x,t)=\frac{\partial}{\partial x}\mathrm{In}\left(1+\exp(\eta)\right),\ \eta=kx-\alpha k^4 t+\delta+\delta_0\,, \tag{24}$$

where k, δ and δ_0 are arbitrary constants.

4.1. TWO-WAVE SOLUTIONS

Assume that Eq. (20) admits a solution of the form

$$u(x,t)=\frac{a_{10}\exp(\xi_1)+a_{01}\exp(\xi_2)+a_{11}\exp(\xi_1+\xi_2)}{1+b_{10}\exp(\xi_1)+b_{01}\exp(\xi_2)+b_{11}\exp(\xi_1+\xi_2)},\ \xi_l=k_l x+w_l t+\delta_l,\ l=1,2\,. \tag{25}$$

It is clear that the ansatz (25) is embedded in (3). Substituting (25) into Eq. (20) and solving the resultant algebraic system for the unknowns a_{10}, a_{01}, a_{11}, b_{10}, b_{01}, b_{11}, k_1, k_2, w_1 and w_2, we get the solution set

$$w_1=-\alpha k_1^4,\ w_2=-\alpha k_2^4,\ b_{11}=0,\ a_{11}=0,\ b_{01}=a_{01}/k_2,\ b_{10}=a_{10}/k_1\,, \tag{26}$$

which gives a two-wave solution to Eq. (20) as

$$u_2(x,t)=\frac{a_{01}\exp\left(k_2x-\alpha k_2^4 t+\delta_2\right)+a_{10}\exp\left(k_1x-\alpha k_1^4 t+\delta_1\right)}{1+(a_{01}/k_2)\exp\left(k_2x-\alpha k_2^4 t+\delta_2\right)+(a_{10}/k_1)\exp\left(k_1x-\alpha k_1^4 t+\delta_1\right)}\,, \tag{27}$$

where a_{01}, a_{10}, k_1, k_2, δ_1, and δ_2 remain arbitrary. However, setting $a_{10}=k_1\exp(\delta_{01})$, $a_{01}=k_2\exp(\delta_{02})$, and $\eta_i=k_i x-\alpha k_i^4 t+\delta_i+\delta_{0i}\ (i=1,2)$, the expression (27) can be formulated as

$$u_2\left(x,t\right)=\frac{\partial}{\partial x}\mathrm{In}\left(1+\exp\left(\eta_1\right)+\exp\left(\eta_2\right)\right), \tag{28}$$

where k_1, k_2, δ_{01}, δ_{02}, δ_1, and δ_2 are arbitrary constants.

4.2. THREE-WAVE SOLUTIONS

Assume that Eq. (20) admits a solution of the form

$$u\left(x,t\right)=\frac{\begin{array}{l}a_{100}\exp\left(\xi_1\right)+a_{010}\exp\left(\xi_2\right)+a_{001}\exp\left(\xi_2\right)+a_{110}\exp\left(\xi_1+\xi_2\right)+a_{101}\exp\left(\xi_1+\xi_3\right)\\ +a_{011}\exp\left(\xi_2+\xi_3\right)+a_{111}\exp\left(\xi_1+\xi_2+\xi_3\right)\end{array}}{\begin{array}{l}1+b_{100}\exp\left(\xi_1\right)+b_{010}\exp\left(\xi_2\right)+b_{001}\exp\left(\xi_2\right)+b_{110}\exp\left(\xi_1+\xi_2\right)+b_{101}\exp\left(\xi_1+\xi_3\right)\\ +b_{011}\exp\left(\xi_2+\xi_3\right)+b_{111}\exp\left(\xi_1+\xi_2+\xi_3\right)\end{array}}, \tag{29}$$

where $\xi_l=k_l x+w_l t+\delta_l,\ l=1,2,3$.
Obviously, the ansatz (29) is embedded in (4). After Substituting (29) into Eq. (20) and solving the resultant algebraic system for the unknowns a_{100}, a_{010}, a_{001}, a_{110}, a_{101}, a_{011}, a_{111}, b_{100}, b_{010}, b_{001}, b_{110}, b_{101}, b_{011}, b_{111}, k_1, k_2, k_3, w_1, w_2, and w_3, we get the solution set

$$\begin{array}{l}w_1=-\alpha k_1^4,\ w_2=-\alpha k_2^4,\ w_3=-\alpha k_3^4,\ b_{100}=a_{100}/k_1,\ b_{010}=a_{010}/k_2,\ b_{001}=a_{001}/k_3,\\ b_{101}=0,\ b_{011}=0,\ a_{011}=0,\ a_{101}=0,\ b_{111}=0,\ a_{110}=0,\ b_{110}=0,\ a_{111}=0,\end{array} \tag{30}$$

which leads a three-wave solution to Eq. (20) as

$$u_3\left(x,t\right)=\frac{a_{001}\exp\left(k_3x-\alpha k_3^4t+\delta_3\right)+a_{010}\exp\left(k_2x-\alpha k_2^4t+\delta_2\right)+a_{100}\exp\left(k_1x-\alpha k_1^4t+\delta_1\right)}{1+\left(a_{001}/k_3\right)\exp\left(k_3x-\alpha k_3^4t+\delta_3\right)+\left(a_{010}/k_2\right)\exp\left(k_2x-\alpha k_2^4t+\delta_2\right)+\left(a_{100}/k_1\right)\exp\left(k_1x-\alpha k_1^4t+\delta_1\right)}, \tag{31}$$

where a_{001}, a_{010}, a_{100}, k_1, k_2, k_3, δ_1, δ_2, and δ_3 remain arbitrary. By the same token, setting $a_{100}=k_1\exp\left(\delta_{01}\right)$, $a_{010}=k_2\exp\left(\delta_{02}\right)$, $a_{001}=k_3\exp\left(\delta_{03}\right)$ and $\eta_i=k_ix-\alpha k_i^4t+\delta_i+\delta_{0i}\ \left(i=1,2,3\right)$, the function (31) can be modified as

$$u_3\left(x,t\right)=\frac{\partial}{\partial x}\mathrm{In}\left(1+\exp\left(\eta_1\right)+\exp\left(\eta_2\right)+\exp\left(\eta_3\right)\right), \tag{32}$$

where k_1, k_2, k_3, δ_{01}, δ_{02}, δ_{03}, δ_1, δ_2, and δ_3 are arbitrary constants.

Remark 2

It seems that the properties (24), (28), and (32) are general. Thus, we conclude that the $N\left(\geq 4\right)$- wave solution for Eq. (20) can be constructed in a similar way. It is also worth to mention here that all class of these solutions seem to be kink-type; for example

$$u_1(\eta \to -\infty) = 0\,,\; u_1(\eta \to \infty) = k\,.$$

4.3. RATIONAL SOLUTIONS

Suppose that Eq. (20) admits a solution of the form (17). Using the same procedure, we obtain the solution set of the resultant algebraic system as

$$a_0 = \frac{a_1 b_0}{b_1} + kb_1,\; a_{-1} = \frac{kb_0}{2} + \frac{a_1 b_{-1}}{b_1} \mp \frac{1}{2}k\sqrt{b_0^2 - 4b_{-1}b_1},\;\; w = -\frac{4k\alpha a_1^3}{b_1^3},\; \mu_1 = 0,\; \mu_2 = 1, \tag{33}$$

which provide a rational solution to Eq. (20) in the form

$$u_{\mp}(x,t) = \frac{a_1\xi^2 + a_0\xi + a_{-1}}{b_1\xi^2 + b_0\xi + b_{-1}},\; \xi = kx - \frac{4k\alpha a_1^3}{b_1^3}t + \delta\,, \tag{34}$$

where a_0 and a_{-1} are as given in (33) and all other involved constants remain arbitrary. The expression (34) represent non-singular solutions if $4b_1b_{-1} - b_0^2 > 0$.

Remark 3

It is well known that the Burgers equation

$$u_t + uu_x = \delta u_{xx}\,, \tag{35}$$

can be linearized by the Cole-Hopf transform [26, 27]

$$u = -2\delta(\ln F)_x,\; F = F(x,t)\,, \tag{36}$$

where F satisfies the diffusion/heat equation

$$F_t = \delta F_{xx}\,. \tag{37}$$

We observe that all the solutions (24), (28), and (32) of Eq. (20) are of the form of a Cole-Hopf transform (36). This fact suggests that even this member of the Burgers hierarchy is linearizable *via* a Cole-Hopf transform. Indeed, substituting $u = R(\ln F)_x$ into Eq. (20), we have $R = 1$ and

$$u_t + \alpha u_{xxxx} + 10\alpha u_x u_{xx} + 4\alpha u u_{xxx} + 12\alpha u u_x^2 + 6\alpha u^2 u_{xx} + 4\alpha u^3 u_x = \frac{\partial}{\partial x}\left(\frac{F_t + \alpha F_{xxxx}}{F}\right) = 0\,. \tag{38}$$

Evidently, if $F = F(x,t)$ solves the linear equation

$$F_t + \alpha F_{xxxx} = 0 \tag{39}$$

then $u = (\ln F)_x$ solves Eq. (20).

5. CONCLUSION

All NEEs can be mainly separated as integrable and non-integrable ones. The first type, namely, the integrable ones has infinite number of exact solutions. NEEs with some exact solutions or without exact solutions are assumed to be in the class of non-integrable ones and they may require specific treatment to obtain their solutions due to the form of the equation. From our point of view, there is no single best method to find exact solutions of NEEs of both type and each method have its merits and deficiencies. Searching exact solutions with multi-velocities and multi-frequencies for NEEs is an important research area in the applied physical sciences. It becomes one of the most exciting and extremely active areas but the progress achieved is not adequate.

As is well known, the Hirota bilinear method [13] is a very powerful method which works perfectly in the case of completely integrable systems. However, all methods are problem dependant, namely some methods work well with certain problems but others not. Hence, it is quite significant to implement some well-known methods (such as the Exp-function method) in the literature to NEEs which are not solved with that method to search possibly new exact solutions or to verify the existing solutions with different approach.

This paper shows that multi-wave solutions, as well as rational solutions, can still be constructed straightforwardly using the Exp-function method with the help of a computer algebra system such as MATHEMATICA. The main advantage of our approach is that the bilinear representation for the equation studied becomes superfluous. We observed that, for some problems, the Exp-function method is nothing else but a variant of the Hirota bilinear method. We used two different kinds of equations to demonstrate the applicability of the Exp-function method. The results are tested by back substitution into the original equation; this provides an extra measure in the results.

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