Article

Transactions of the Institute of Measurement and Control
35(6) 808–823

DOI: 10.1177/0142331213476914
tim.sagepub.com
SAGE

Online time delay identification and control for general classes of nonlinear systems

Alper Bayrak and Enver Tatlicioglu

Abstract

In this study, online identification of state delays is discussed. First, a novel adaptive time delay identification technique is proposed for general classes of nonlinear systems subject to state delays. The stability of the time delay identification algorithm is analyzed via Lyapunov-based techniques. In this work, we consider the time delay as a nonlinear parameter effecting the system which is a seemingly novel departure from the existing literature. As an extension, this technique is modified to design a tracking controller for general classes of nonlinear systems subject to state delays. The main novelty of this controller is that identification of unknown state delays are ensured while output tracking objective is satisfied. Numerical simulations are conducted that demonstrate the efficiency of the time delay identification algorithm and the tracking controller.

Keywords

Adaptive identification, adaptive control, time delay, Lyapunov-based methods

Introduction

Time delay, also named as time difference of arrival or dead time in different disciplines, is an important research area mostly due to its negative effects (such as instability or reduced performance) on systems (Richard, 2003). Time delay may originate from the dynamics of systems, or may be introduced by feedback loops, sensors, and communication lines.

Since time delay is a real problem that occurs in several systems, a significant amount of research has been conducted on its effects on stability, and identification and control methods for time-delayed systems. A broad overview on time delay and its effects on systems may be found in Richard's (2003) work. Gu and Niculescu (2003) also presented a broad overview, that particularly focused on engineering applications and recent progresses of stability and control of time delay systems.

A significant amount of research has been devoted to designing time-delay identification algorithms (Ahmed et al., 2006; Belkoura and Richard, 2006; Drakunov et al., 2006; Wang et al., 2008; Bjorklund and Ljung, 2009; Tang et al., 2009; Tang and Guan, 2009; Loxton et al., 2010; Ni et al., 2010; Selvanathan and Tangirala, 2010; Bayrak and Tatlicioglu, 2011; Tan and Cham, 2011). Most of the past research on time-delay identification were usually presented for linear or linearized systems and review of the relevant past research highlights the fact that there are no time-delay identification algorithms for general classes of nonlinear systems.

Owing to the negative effects of time delay on stability and performance, a significant amount of research was devoted to designing controllers for systems subject to time delays. Gu and Niculescu (2003) and Zhong (2001) investigated robust control and robust stability of time-delay systems. Schoen (1995) investigated the stability of time-delay systems by using Razumikhin theory, Lyapunov–Krasovskii theory, and eigenvalue consideration. Krstic (2009) focused on systems with input delays and converted the problem to boundary control of partial differential equations after introducing a transformation. Niculescu (2001) analyzed effects of time delays on stability of dynamical systems.

For nonlinear systems subject to state delays, accurate knowledge of time delays is advantageous for control development, however time delay is usually unknown. To overcome this problem, estimating time delay while controlling the system may be an effective method. Peng et al. (2004) considered the Smith predictor based controller design for network control systems with time delay identification. Zhang and Li (2003) presented a fuzzy Smith predictor based controller for time-varying processes based on time-delay identification for signal processing applications. Zhang and Li (2006) proposed a control method for master–slave systems based on time-delay identification. A review of the relevant literature highlights the fact that there are no notable control approaches based on time delay identification. The approaches in the literature are usually valid for some special cases, and not for general nonlinear systems.

Department of Electrical and Electronics Engineering, Izmir Institute of Technology, Urla, Izmir, Turkey

Corresponding author:
Alper Bayrak, Department of Electrical and Electronics Engineering, Izmir Institute of Technology, Urla, Izmir 35430, Turkey.
Email: alperbayrak@iyte.edu.tr

In this work, first, general classes of nonlinear systems subject to state delays are considered and a novel time-delay identification technique is proposed. While designing the identification algorithm, the time delay is considered as a nonlinear parameter affecting the system, and the nonlinear parameter identification method of Annaswamy et al. (1998) is utilized as the time-delay identification method. In the design of the time-delay identification algorithm, auxiliary observer-like signals are designed. The stability of the closed-loop system and the convergence of the time-delay identification is proven via Lyapunov-based methods. When compared with the literature, the proposed time-delay identification is designed via Lyapunov-based methods, it works online, and it can be applied to general classes of nonlinear systems without imposing any restrictions. As an extension, general classes of nonlinear systems subject to state delays is considered and a tracking controller is designed. The main novelty of this part is that while the controller ensures tracking of a desired trajectory, state delays can be identified online. The performance of the identification algorithm and tracking controller were evaluated by using MATLAB/simulink simulation program.

System model

The following system is considered

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{m-1} &= x_m \\ \dot{x}_m &= f(x,\tau,t) \end{aligned} \tag{1}$$

where $f(\cdot) \in \mathbb{R}$ is a nonlinear function, $x(t) = [x_1\ x_2\ \cdots\ x_m] \in \mathbb{R}^m$ is state vector, and $\tau \in \mathbb{R}^n$ denotes unknown constant time-delay vector. It is noted that, function $f(\cdot)$ is used in the rest of this work with the same arguments. It is assumed that the structure of $f(\cdot)$ is known and the state vector $x(t)$ is measurable.

Assumption 1. It is assumed that τ, the unknown time-delay vector, is bounded and is in a known hypercube $\Omega \subset \mathbb{R}^n$.

Assumption 2. It is assumed that the function $f(\cdot)$ is either concave or convex on a simplex[1] Ω_s in $\mathbb{R}^n$, and also $\Omega_s \supset \Omega$.

Assumption 3. It is assumed that the state vector $x(t)$ is continuous, bounded, and Lipschitz in time as follows

$$\|x(t_1) - x(t_2)\| \le L_1|t_1 - t_2| \quad \forall t_1,\ t_2 \in \mathbb{R}^+ \tag{2}$$

where $L_1 \in \mathbb{R}$ is a positive Lipschitz constant.

Assumption 4. It is assumed that $f(\tau_0, x)$ is Lipschitz with respect to its arguments in the sense that

$$|f(\tau_0 + \Delta\tau_0, x + \Delta x) - f(\tau_0, x)| \le L_2(\|\Delta x\| + \|\Delta\tau_0\|) \tag{3}$$

where $\Delta x \triangleq x(t_1) - x(t_2)$, $\Delta\tau_0 \triangleq \tau_0(t_1) - \tau_0(t_2)$, and $L_2 \in \mathbb{R}$ is a positive Lipschitz constant.

Time-delay identifier design

In this section, auxiliary observer-like signals will be designed to facilitate the error system design and the time delay identifier will be designed subsequently. Observer-like signals, denoted by $\hat{x}_i(t) \in \mathbb{R}$, $i = 1, \ldots, m$, are updated according to the following rule

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 - k_1\tilde{x}_1 \\ \dot{\hat{x}}_2 &= \hat{x}_3 - k_2\tilde{x}_2 \\ &\vdots \\ \dot{\hat{x}}_{m-1} &= \hat{x}_m - k_{m-1}\tilde{x}_{m-1} \\ \dot{\hat{x}}_m &= \hat{f} - \alpha\tilde{x}_\varepsilon - a^* sat(r) \end{aligned} \tag{4}$$

where $\tilde{x}_i \triangleq \hat{x}_i - x_i \in \mathbb{R}$, $i = 1, \ldots, m$ are the observer errors, $k_i \in \mathbb{R}$, $i = 1, \ldots, (m-1)$ are observer gains, $\hat{f} \triangleq f|_{\tau=\hat{\tau}}$ where $\hat{\tau}(t) \in \mathbb{R}^n$ is the estimate of τ, $\alpha \in \mathbb{R}$ is a positive constant gain, $a^*(t) \in \mathbb{R}$ is the tuning function, $r(t)$, $\tilde{x}_\varepsilon(t) \in \mathbb{R}$ are auxiliary error signals defined as

$$\tilde{x}_\varepsilon \triangleq \tilde{x}_m - \varepsilon sat(r) \tag{5}$$

$$r \triangleq \tilde{x}_m/\varepsilon \tag{6}$$

where $\varepsilon \in \mathbb{R}$ is the desired precision, and $sat(\cdot) \in \mathbb{R}$ is the standard saturation function defined as follows

$$sat(r) = \begin{cases} 1, & r \ge 1 \\ r, & |r| < 1 \\ -1, & r \le -1 \end{cases} \tag{7}$$

Remark 1. It should be noted that from (5) and its time derivative, it is clear that

$$\begin{aligned} \tilde{x}_\varepsilon &= 0 && when\ |\tilde{x}_m| \le \varepsilon \\ \dot{\tilde{x}}_\varepsilon &= \dot{\tilde{x}}_m && when\ |\tilde{x}_m| > \varepsilon \end{aligned} \tag{8}$$

where (6) was utilized.

This remark will later be utilized in the stability analysis.

The following expressions can be obtained for the time derivatives of the observer errors

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 - k_1\tilde{x}_1 \\ \dot{\tilde{x}}_2 &= \tilde{x}_3 - k_2\tilde{x}_2 \\ &\vdots \\ \dot{\tilde{x}}_{m-1} &= \tilde{x}_m - k_{m-1}\tilde{x}_{m-1} \\ \dot{\tilde{x}}_m &= \hat{f} - f - \alpha\tilde{x}_\varepsilon - a^* sat(r) \end{aligned} \tag{9}$$

where (1) and (4) were utilized.

The following update law is proposed

$$\dot{\hat{\tau}} = Proj\{-\Gamma\tilde{x}_\varepsilon\phi^*\} \tag{10}$$

where the projection strategy $Proj\{\cdot\} \in \mathbb{R}^n$ guarantees that $\hat{\tau}(t)$ always belongs to the hypercube Θ and defined as

$$\hat{\tau}_j = \begin{cases} \hat{\tau}_j, & \text{if } \hat{\tau}_j \in [\tau_{j,min}, \tau_{j,max}] \\ \tau_{j,min}, & \text{if } \hat{\tau}_j < \tau_{j,min} \\ \tau_{j,max}, & \text{if} \hat{\tau}_j > \tau_{j,max} \end{cases} \tag{11}$$

where the subscript j denotes the jth element of the corresponding vector $\forall j = 1, 2, \ldots n$, $\tau_{j,min}$, $\tau_{j,max} \in \mathbb{R}$ are the minimum and maximum values of the jth component of τ, respectively, $\phi^*(t) \in \mathbb{R}^n$ is the sensitivity function, and $\Gamma \in \mathbb{R}^{n\times n}$ is a positive definite diagonal gain matrix. The solutions for $\phi^*(t)$ and $a^*(t)$ are obtained from a min–max optimization problem of the following form (Annaswamy et al., 1998)

$$a^* = \min_{\phi\in\mathbb{R}^n} \max_{\tau\in\tau_s} J(\phi, \tau) \tag{12}$$

$$\phi^* = \arg \min_{\phi\in\mathbb{R}^n} \max_{\tau\in\tau_s} J(\phi, \tau) \tag{13}$$

where $J(\cdot) \in \mathbb{R}$ is a performance index defined as follows

$$J(\cdot) = sat(r)[\hat{f} - f - (\Gamma\tilde{\tau})^{\mathrm{T}}\phi] \tag{14}$$

where $\tilde{\tau}(t) \in \mathbb{R}^n$ is the identification error defined as follows

$$\tilde{\tau} \triangleq \hat{\tau} - \tau \tag{15}$$

The solutions for $\phi^*(t)$ and $a^*(t)$ are obtained as:
when $\tilde{x}_m(t) < 0$

$$a^* = \begin{cases} 0 & \text{if } f \text{ is concave on } \Theta_s \\ A_1 & \text{if } f \text{ is convex on } \Theta_s \end{cases} \tag{16}$$

$$\phi^* = \begin{cases} \nabla f(\hat{\tau}) & \text{if } f \text{ is concave on } \Theta_s \\ A_2 & \text{if } f \text{ is convex on } \Theta_s \end{cases} \tag{17}$$

when $\tilde{x}_m(t) \geq 0$

$$a^* = \begin{cases} A_1 & \text{if } f \text{ is concave on } \Theta_s \\ 0 & \text{if } f \text{ is convex on } \Theta_s \end{cases} \tag{18}$$

$$\phi^* = \begin{cases} \boldsymbol{A}_2 & \text{if } f \text{ is concave on } \Theta_s \\ \nabla f(\hat{\tau}) & \text{if } f \text{ is convex on } \Theta_s \end{cases} \tag{19}$$

where $A(t) \in \mathbb{R}^{(n+1)}$ is given as follows

$$\boldsymbol{A} = [A_1 \quad \boldsymbol{A}_2]^{\mathrm{T}} = G^{-1}b \tag{20}$$

where $A_1(t) \in \mathbb{R}$, $\boldsymbol{A}_2(t) \in \mathbb{R}^n$ and $\boldsymbol{G}(t) \in \mathbb{R}^{(n+1)\times(n+1)}$, $\boldsymbol{b}(t) \in \mathbb{R}^{(n+1)}$ are obtained as follows

$$G = \begin{bmatrix} -1 & \beta\Gamma(\hat{\tau} - \tau_{s1})^{\mathrm{T}} \\ -1 & \beta\Gamma(\hat{\tau} - \tau_{s2})^{\mathrm{T}} \\ \vdots & \vdots \\ -1 & \beta\Gamma(\hat{\tau} - \tau_{s(n+1)})^{\mathrm{T}} \end{bmatrix} \tag{21}$$

$$\boldsymbol{b} = \begin{bmatrix} \beta(\hat{f} - f_{s1}) \\ \beta(\hat{f} - f_{s2}) \\ \vdots \\ \beta(\hat{f} - f_{s(n+1)}) \end{bmatrix} \tag{22}$$

where $\beta \in \mathbb{R}$ is defined as follows

$$\beta = \begin{cases} 1 & \text{if } f \text{ is convex on } \Omega_s \\ -1 & \text{if } f \text{ is concave on } \Omega_s \end{cases} \tag{23}$$

In (22), $f_{sh} \triangleq f(\tau_{sh}, x)\ \forall h = 1, 2, \ldots, n+1$ where $\tau_{sh} \in \mathbb{R}^n$ are the vertices of the simplex Ω_s. In (17) and (19), $\nabla f(\hat{\tau}) \in \mathbb{R}^n$ is the gradient of $f(\cdot)$ defined as follows

$$\nabla f(\hat{\tau}) = (\delta f/\delta\tau)|_{\tau=\hat{\tau}} \tag{24}$$

Remark 2. The tuning error $\tilde{x}_\varepsilon(t)$ and the saturation function $sat(r)$ assure that the estimator is continuous even if a discontinuous solution of the min–max algorithm is obtained (Annaswamy et al., 1998).

Remark 3. The projection strategy in (11) assures the boundedness of the $\hat{\tau}(t)$; thus, $\phi^*(t)$ can be upper bounded as follows

$$\|\phi^*(t)\| \leq L_\phi \quad \forall t \geq t_0 \tag{25}$$

where $L_\phi \in \mathbb{R}$ is a positive constant.

Theorem 1. The observer dynamics in (4) and the adaptive update law in (10) guarantee stability and global boundedness of the closed-loop system, and $|\tilde{x}_\varepsilon(t)| \to 0$ as $t \to \infty$.

Proof 1. The proof of this theorem can be found in Appendix A. In Appendix A, it is proven that $\tilde{x}_\varepsilon(t) \in L_2 \cap L_\infty$ and $\dot{\tilde{x}}_\varepsilon(t) \in L_\infty$; thus, $|\tilde{x}_\varepsilon(t)| \to 0$ as $t \to \infty$. From its definition in (5), it is easy to see that $|\tilde{x}_m(t)|$ is ultimately bounded in the sense that $|\tilde{x}_m(t)| \leq \varepsilon$ as $t \to \infty$. Linear analysis tools can then be utilized to prove that $|\tilde{x}_i(t)| \leq \varepsilon$ as $t \to \infty$, $i = 1, 2, \ldots, (m-1)$; thus, proving ultimate boundedness of the observer errors.

Theorem 2. The estimator assures that $\|\tilde{\tau}(t)\| \leq \sqrt{\gamma}$ as $t \to \infty$ provided the following nonlinear persistent excitation condition holds

$$\beta(x(t_2))(f(\hat{\tau}(t_1), x(t_2)) - f(\tau, x(t_2))) \geq \varepsilon_u \|\hat{\tau}(t_1) - \tau\| \tag{26}$$

where

$$\gamma = \frac{8\varepsilon c_1}{\varepsilon_u^2} \quad ; \quad c_1 = 4L_1L_2 + 2\nu L_2 L_\phi + \nu L_\phi^2 \tag{27}$$

where ν is maximum eigenvalue of Γ, $t_2 \in [t_1, t_1 + T_0]$, $t_1 > t_0$, and $T_0, \varepsilon_u \in \mathbb{R}$ are positive constants.

Proof 2. The proof of this theorem can be found in Appendix B.

Remark 4. From the definition of γ in (27), it is clear that γ can be made smaller by choosing a smaller ε. It should be noted that, as the desired precision $\varepsilon \to 0$, then $\gamma \to 0$; thus, the observer errors and the time delay identification error is driven to zero.

Remark 5. This algorithm can be applied to systems subject to input delay. In this case, the system model can be described as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{m-1} &= x_m \\ \dot{x}_m &= f(x, \tau, t, u) \end{aligned} \tag{28}$$

where $u(t) \in \mathbb{R}$ is the control input. In the case of $u(t)$ being exposed to time delay(s), this time delay can also be considered as a member of time-delay vector τ and can be estimated along with the state delays.

Tracking controller while identifying time delays

In this section, we design a controller for the following general nonlinear systems

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{m-1} &= x_m \\ \dot{x}_m &= f(\boldsymbol{x}, \tau, t) + u(t) \end{aligned} \tag{29}$$

where $u(t) \in \mathbb{R}$ is the control input. Model in Equation (29) is an extension of the model in Equation (1) by adding an input signal and all assumptions given for model in Equation (1) is valid for this model. The control objective is to design $u(t)$ to guarantee that $x_1(t)$ tracks a desired trajectory, while identifying time delays. We can achieve this objective by redefining the error signal $\tilde{x}_1(t)$ as follows

$$\tilde{x}_1 \triangleq x_d - x_1 \tag{30}$$

where $x_d(t) \in \mathbb{R}$ is a desired trajectory. Auxiliary filtered error signals, denoted by $\tilde{x}_i(t) \in \mathbb{R}$, $i = 2, \ldots, m$, are defined as follows

$$\begin{aligned} \tilde{x}_2 &\triangleq \dot{\tilde{x}}_1 + k_1\tilde{x}_1 \\ \tilde{x}_3 &\triangleq \dot{\tilde{x}}_2 + k_2\tilde{x}_2 \\ &\vdots \\ \tilde{x}_m &\triangleq \dot{\tilde{x}}_{m-1} + k_{m-1}\tilde{x}_{m-1} \end{aligned} \tag{31}$$

where $k_i \in \mathbb{R}$, $i = 1, \ldots, (m-1)$ are control gains. To facilitate the control design the time derivative of $\tilde{x}_m(t)$ can be obtained as follows

$$\dot{\tilde{x}}_m = x_d^{(m)} - f - u + \sum_{i=1}^{m-1} k_i\tilde{x}_i^{(m-i)} \tag{32}$$

where (29), the mth-order time derivative of (30), and (31) were utilized. The control input $u(t)$ is designed as follows

$$u = \alpha x_\varepsilon - \hat{f} + a^* sat(r) + x_d^{(m)} + \sum_{i=1}^{m-1} k_i\tilde{x}_i^{(m-i)} \tag{33}$$

After substituting (33) into (32), we obtain the following closed-loop error system

$$\dot{\tilde{x}}_m = -\alpha x_\varepsilon + \hat{f} - f - a^* sat(r) \tag{34}$$

The rest of the development is considered to continue from (10).

Remark 6. It can be seen that the expression in (34) is exactly same as that in (9), and since the rest of the development is same, the stability analysis is valid and the proofs of Theorems 1 and 2 are applicable for this case as well. Thus, the proof of Theorem 1 ensures ultimate boundedness of the output tracking error $\tilde{x}_1(t)$, and the proof of Theorem 2 guarantees convergence of the time delay identification algorithm.

Numerical simulation results

The performance of the proposed technique was evaluated by conducting numerical simulations using Matlab/Simulink. Numerical simulation section was divided into two subsections: (i) time-delay identification; (ii) control with time-delay identification.

Time-delay identification

The following model was considered

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -(1 + x_2(t))x_2(t - \tau) \end{aligned} \tag{35}$$

where τ is the time delay.

The performance of the proposed technique was evaluated with and without additive noise. In noisy case, additive white Gaussian noise with a 20 dB signal-to-noise ratio (SNR) was injected into $f(\cdot)$ to demonstrate robustness against measurement noise.

During the simulations, the lower and upper bounds of unknown time delay τ were chosen as 0.1 and 1.1 seconds, respectively, the initial values of $x(t)$ and $\hat{x}(t)$ were set to $[0, 0.1]^{\mathrm{T}}$ and $[0.3, 0.3]^{\mathrm{T}}$, respectively, and the initial value of $\hat{\tau}(t)$ was set to 1.1 seconds. The time delay τ was chosen as 0.4 seconds. The update law in (10) was utilized with the desired precision $\varepsilon = 10^{-6}$, and the control gains were chosen as $\alpha = 6$ and $\Gamma = 0.7$, and k_1 was chosen as 56 for the noise-free case and as 52 for the noisy case.[2]

In Figures 1 and 2, the estimation performances are presented for noise-free and noisy cases, respectively.

Control with time-delay identification

The following model (Sharma et al., 2012) belongs to a chattering phenomenon during a metal cutting operation was considered

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= m^{-1}(-c\dot{y} - ky + k_c b(f - y + y(t - \tau)) + u) \\ y &= x_1 \end{aligned} \tag{36}$$

where m, c, k, k_c, b, f are the effective mass, damping coefficient, stiffness constant, cutting stiffness, width of cut, and feed rate, respectively, and τ is the time delay. Model parameters were taken from Sharma et al. (2012) as $m = 1.16$ kg, $\tau = 60/\omega$, $\omega = 550$ rpm, $k = m\omega^2$, $c = 2m\eta\omega$, $\eta = 0.1$, $\omega = 83\pi$, $k_c/k = 0.5$, $b = 2$ mm, and $f = 0.25$ mm per revolution.

The performance of the proposed technique was evaluated with and without additive noise. In noisy case, additive white Gaussian noise with a 20 dB SNR was injected to $f(\cdot)$ to demonstrate robustness against measurement noise.

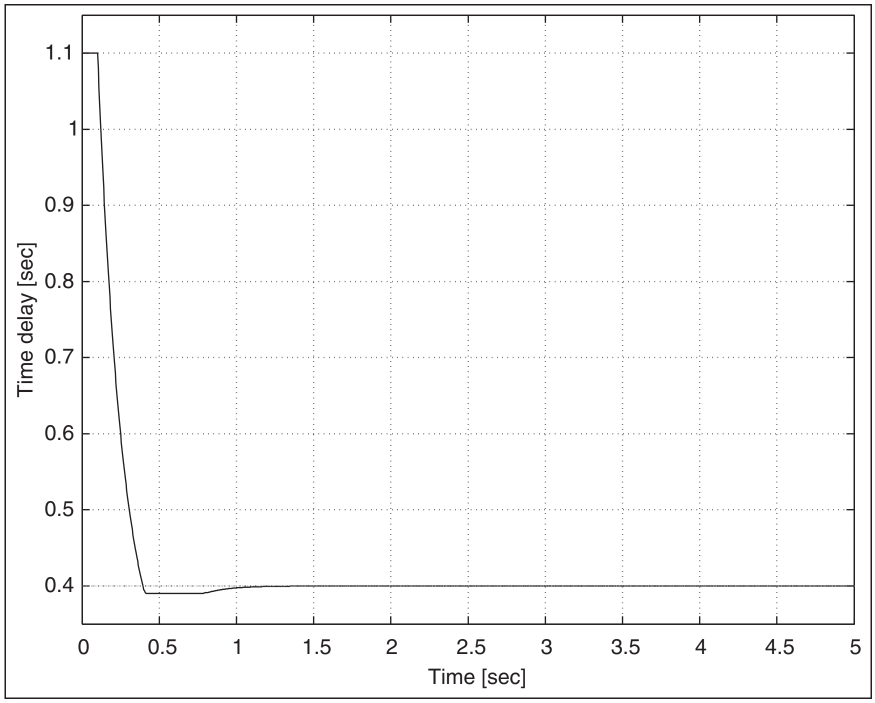

Figure 1 The estimate of τ for the noise-free case.

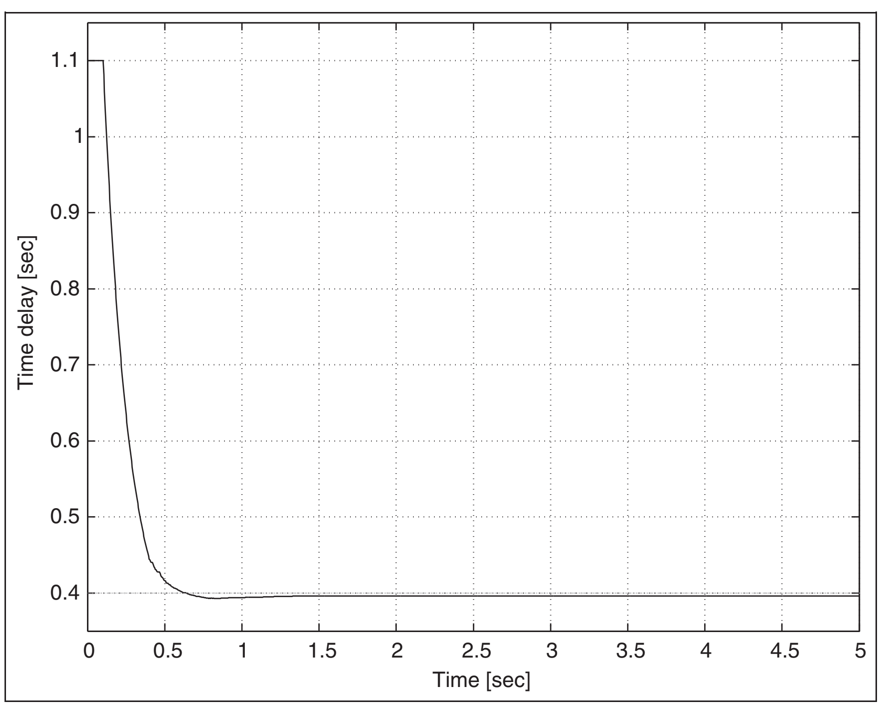

Figure 2 The estimate of τ for the noisy case.

During the simulations, the lower and upper bounds of unknown time delay τ were chosen as 0.05 and 0.3 seconds, respectively, the initial values of both $x(t)$ and $\hat{x}(t)$ were set to $[2, 1]^{\mathrm{T}}$, and the initial value of $\hat{\tau}(t)$ was set to 0.3 seconds. The update law in (10) was utilized with the desired precision $\varepsilon = 10^{-6}$, and the gains α, Γ, and k_1 were chosen as 150, 550, and 180 for the noise-free case and as 310, 105, and 90 for the noisy case, respectively. The time delay was considered as

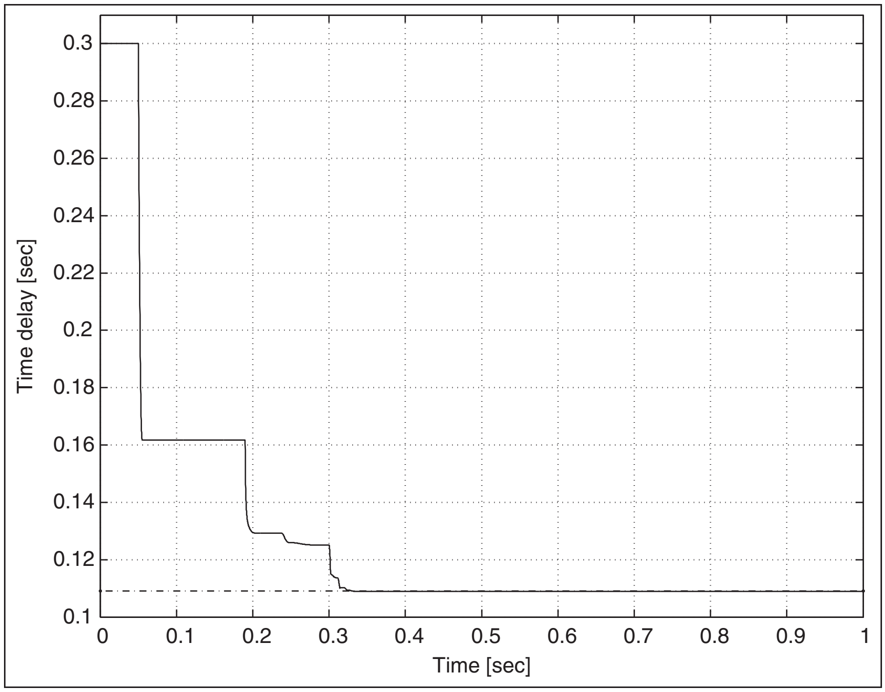

Figure 3 The estimate of τ for the noise-free case.

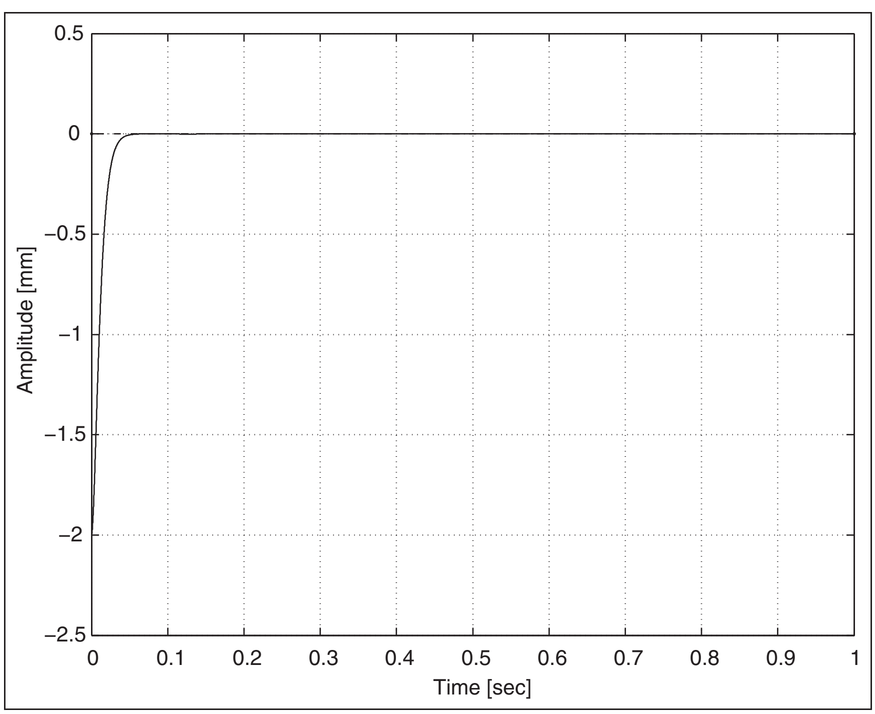

Figure 4 The tracking error for the noise-free case.

constant and selected as $\tau = 60/550$ seconds. In Figures 3 and 6, 4 and 7, and 5 and 8, the estimation performances, tracking errors, and control efforts are presented for noise-free and noisy cases, respectively.

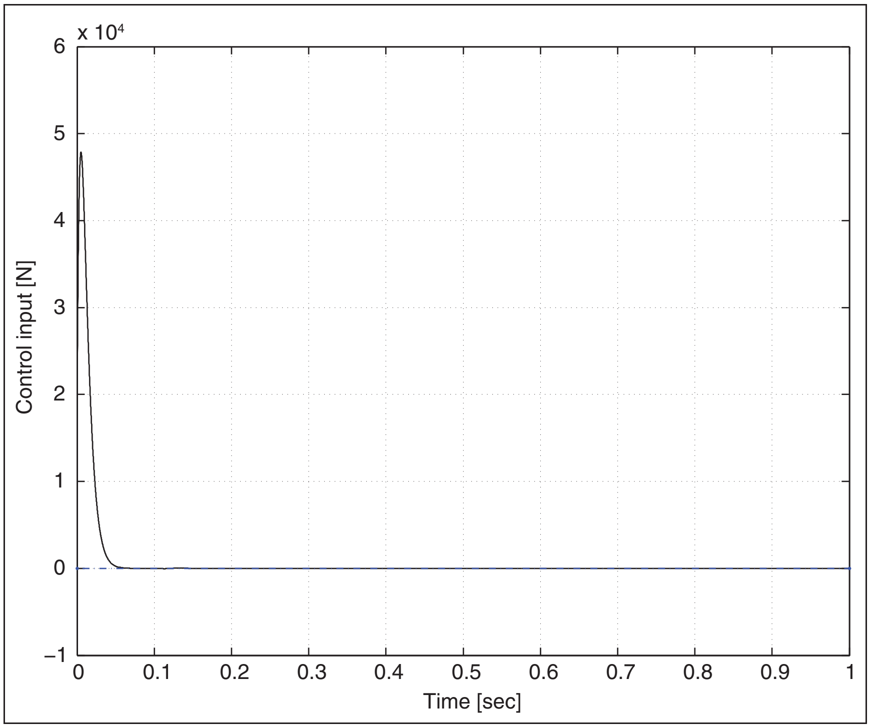

Figure 5 The control effort for the noise-free case.

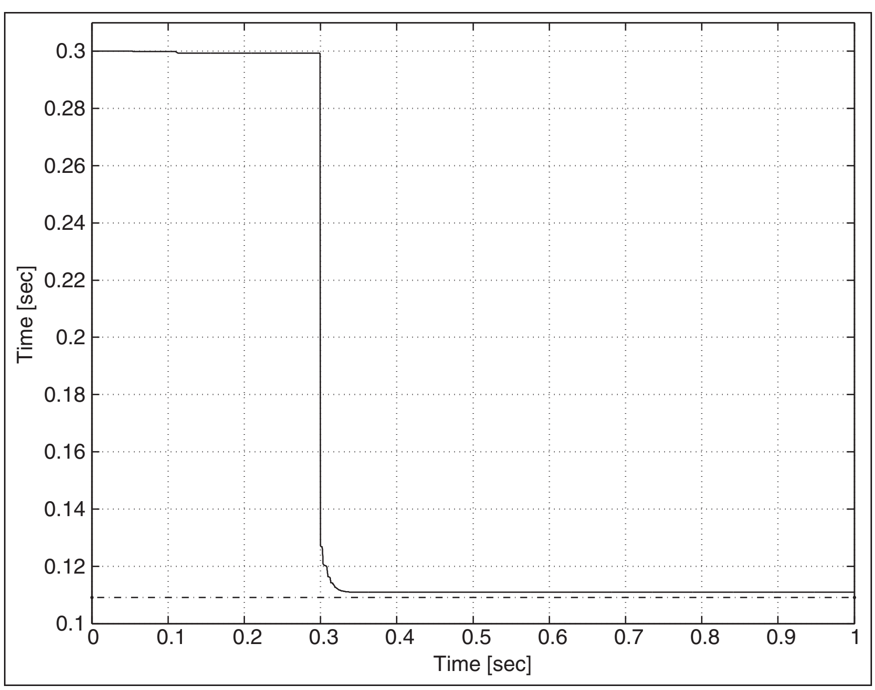

Figure 6 The estimate of τ for noisy case.

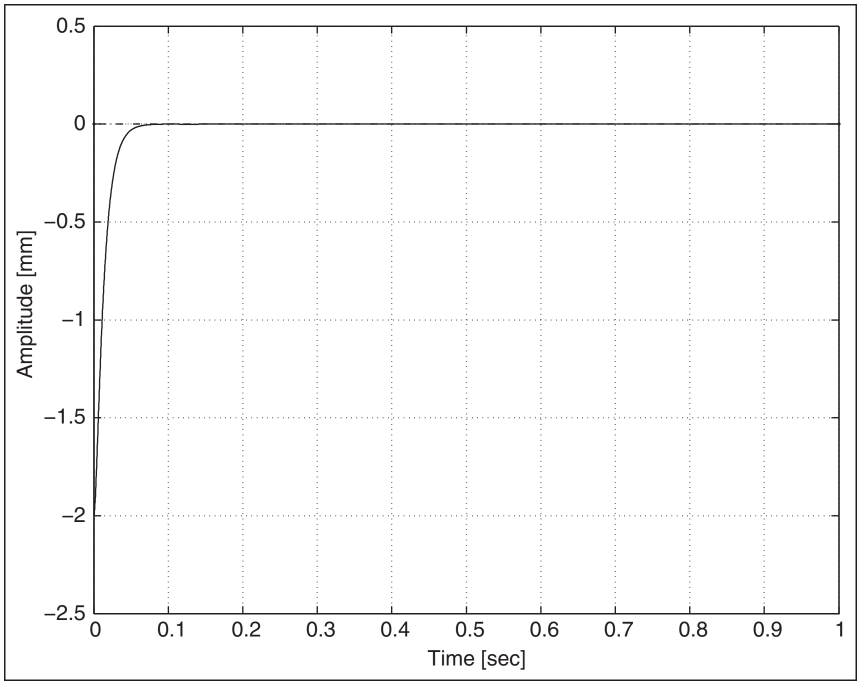

Figure 7 The tracking error for the noisy case.

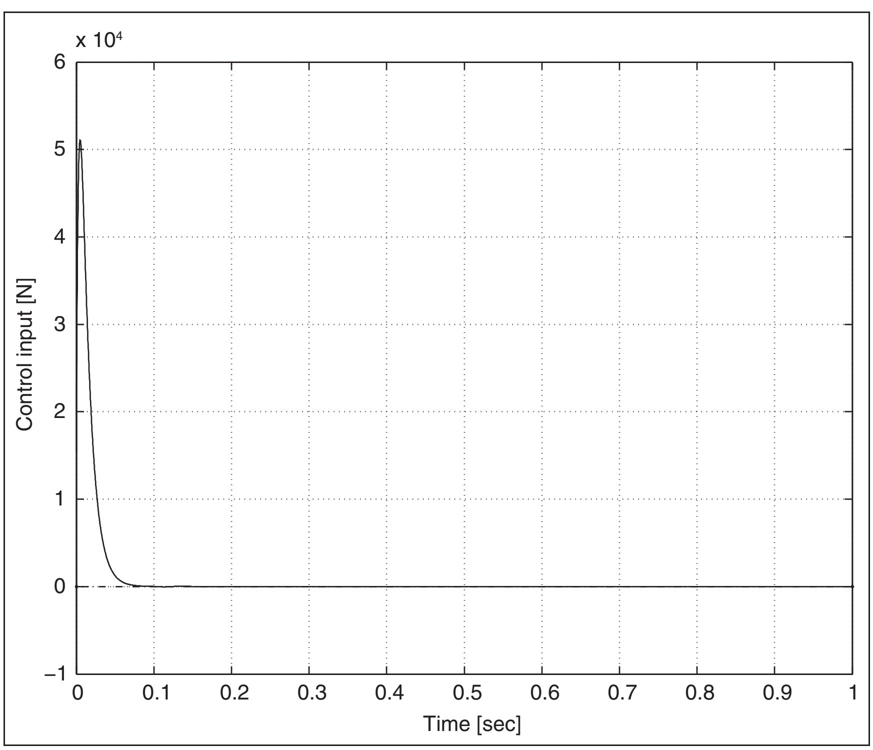

Figure 8 The control effort for the noisy case.

Conclusion

In this work, a novel time-delay identification algorithm was proposed for general classes of nonlinear systems subject to state delays. While designing the identification algorithm, different from most of the studies in the literature, the time delay was considered as a nonlinear parameter, and the nonlinear parameter identification method of Annaswamy et al. (1998) was utilized as the time-delay identification method. Auxiliary observer-like signals were utilized when designing the time-delay identification algorithm. As an extension, the time-delay identification algorithm was modified to be applicable to general classes of nonlinear systems subject to state delays by designing a tracking controller. The main novelty of this design is that while the controller ensured tracking of a desired trajectory, state delays were identified online.

The performance of the identification algorithm and tracking controller were evaluated by using MATLAB/Simulink. To numerically verify the time-delay identification, a second-order dynamical system was considered, while the model of a chattering phenomenon during a metal cutting operation system studied by Sharma et al. (2012) was considered to numerically verify the tracking controller with time-delay identification.

To demonstrate the robustness of the time-delay identification and the tracking controller, both the numerical simulations were run in the presence of additive noise that were artificially added to some of the signals. Successful results were obtained for both the time-delay identification algorithm and the tracking controller. Specifically, as presented in the figures, estimation and tracking objectives were achieved.

There is much to be considered for future work. One of the major assumption of this work is that the knowledge of nonlinear function f is required. While this is a restrictive assumption, to the best of the authors knowledge there are no notable identification methods that identify both time delays and linear parameters. As a result, future time-delay identification strategies should relax the requirement for the exact knowledge of nonlinear function. Currently, work is under way to design time-delay identification algorithms that does not require the knowledge of the model parameters.

Notes

1 A simplex in $\mathbb{R}^n$ is a convex polyhedron with $n + 1$ vertices.
2 We would like to note that, as highlighted in Remark 4, the desired precision effects the ultimate bound that the time-delay identification error reaches, thus, we chose it very small. The other gains were chosen via trial error. Specifically, first conservative (i.e. bigger) values of the gains were chosen and when satisfactory performance was achieved, the gains were decreased to obtain satisfactory performance with smaller gains.
3 A similar proof can be given if $\beta(\Pi(t_2)) = -1$, i.e. $q(\cdot)$ is concave on Ω_s; $\Pi(\cdot)$ is a measurable function including known and measurable parameters.

Funding

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

References

Ahmed S, Huang B and Shah S (2006) Parameter and delay estimation of continuous-time models using a linear filter. *Journal of Process Control* 16(4): 323–331.
Annaswamy AM, Skantze FP and Loh AP (1998) Adaptive control of continuous time systems with convex/concave parametrization. *Automatica* 34(1): 33–49.
Bayrak A and Tatlicioglu E (2011) A novel online adaptive time delay identification technique. In: *2011 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, Orlando, FL, USA.
Belkoura L and Richard JP (2006) A distribution framework for the fast identification of linear systems with delays. In: *Proceedings of 6th IFAC Workshop on Time Delay Systems*, L'Aquila, Italy.
Bjorklund S and Ljung L (2009) An improved phase method for time-delay estimation. *Automatica* 45(10): 2467–2470.
Cao C, Annaswamy A and Kojic A (2003) Parameter convergence in nonlinearly parameterized systems. *IEEE Transactions on Automatic Control* 48(3): 397–412.
Drakunov S, Perruquetti W, Richard JP and Belkoura L (2006) Delay identification in time-delay systems using variable structure observers. *Annual Reviews in Control* 30(2): 143–158.
Gu K and Niculescu SI (2003) Survey on recent results in the stability and control of time-delay systems. *Transactions of the ASME* 125(2): 158–165.
Ioannou P and Sun J (1996) *Robust Adaptive Control*. Englewood Cliffs, NJ: Prentice-Hall.
Khalil H (2002) *Nonlinear Systems*. New York: Prentice-Hall.
Krstic M (2009) *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*. Boston, MA: Birkhäuser.
Loxton R, Teo K and Rehbock V (2010) An optimization approach to state-delay identification. *IEEE Transactions on Automatic Control* 55(9): 2113–2119.
Ni B, Xiao D and Shah SL (2010) Time delay estimation for MIMO dynamical systems - with time-frequency domain analysis. *Journal of Process Control* 20(1): 83–94.
Niculescu SI (2001) *Delay Effects on Stability: A Robust Control Approach*. Berlin: Springer.
Peng C, Yue D and Sun J (2004) The study of smith prediction controller in ncs based on time-delay identification. In: *Proceedings of the Control, Automation, Robotics and Vision Conference*, vol. 3, pp. 1644–1648.
Richard JP (2003) Time-delay systems: an overview of some recent advances and open problems. *Automatica* 39(10): 1667–1694.
Schoen GM (1995) *Stability and Stabilization of Time-Delay Systems*. Zurich, Switzerland: Swiss Federal Institute of Technology.
Selvanathan S and Tangirala A (2010) Time-delay estimation in multivariate systems using hilbert transform relation and partial coherence functions. *Chemical Engineering Science* 65(2): 660–674.
Sharma N, Bhasin S, Qiang W and Dixon W (2012) Rise-based adaptive control of a control affine uncertain nonlinear system with unknown state delays. *IEEE Tran. on Automatic Control* 57(1): 255–259.
Tan AH and Cham CL (2011) Continuous-time model identification of a cooling system with variable delay. *IET Control Theory and Applications* 5(7): 913–922.
Tang Y, Cui M, Li L, Peng H and Guan X (2009) Parameter identification of time-delay chaotic system using chaotic ant swarm. *Chaos, Solitons and Fractals* 41(4): 2097–2102.

Tang Y and Guan X (2009) Parameter estimation of chaotic system with time-delay: A differential evolution approach. *Chaos, Solitons and Fractals* 42(5): 3132–3139.
Wang Y, Zhao Y, Hao X and Cheng P (2008) Time-delay identification for linear multi-input multi-output systems. In: *Proceedings of the 7th World Congress on Intelligent Control and Automation*, Chongqing, China, pp. 4762–4766.
Zhang T and Li Y (2006) A control scheme for bilateral teleoperation systems based on time-varying communication delay identification. In: *Proceedings of Systems and Control in Aerospace and Astronautics*, Harbin, China, pp. 273–278.
Zhang T and Li YC (2003) A fuzzy smith control of time-varying delay systems based on time delay identification. In: *Proceedings of the International Conference on Machine Learning and Cybernetics*, vol. 1, Changchun, China, pp. 614–619.
Zhong QC (2001) *Robust Control of Time-delay Systems*. Berlin: Springer.

Appendix

A Proof of Theorem 1

Proof 3 To facilitate the proof, a non-negative Lyapunov function, denoted by $V(t) \in \mathbb{R}$ is, defined as follows

$$V = \frac{1}{2}\tilde{x}_\varepsilon^2 + \frac{1}{2}\tilde{\tau}^{\mathrm{T}}\tilde{\tau} \tag{37}$$

After utilizing the time derivative of (15), the time derivative of (37) can be obtained as follows

$$\dot{V} = \tilde{x}_\varepsilon\dot{\tilde{x}}_\varepsilon + \tilde{\tau}^{\mathrm{T}}Proj\{-\Gamma\tilde{x}_\varepsilon\phi^*\} \tag{38}$$

where (10) was utilized. It should be noted that an adaptive law with the projection algorithm defined on a convex set retains all of the properties of the adaptive law without the projection algorithm (Ioannou and Sun, 1996). The projection strategy given in (11) is on the hypercube Ω which is a convex set; hence, the expression given in (38) can be rewritten as follows

$$\dot{V} = \tilde{x}_\varepsilon(\dot{\tilde{x}}_\varepsilon - (\Gamma\tilde{\tau})^{\mathrm{T}}\phi^*) \tag{39}$$

To further facilitate the proof, two different cases are considered: case I when $|\tilde{x}_m| \le \varepsilon$ and case II when $|\tilde{x}_m| > \varepsilon$.

For case I, from Remark 1, it is clear that

$$\dot{V} = 0 \qquad \forall |\tilde{x}_m| \le \varepsilon \tag{40}$$

For case II, also, from Remark 1 and (39), the following expression can be obtained

$$\dot{V} = \tilde{x}_\varepsilon(\dot{\tilde{x}}_m - (\Gamma\tilde{\tau})^{\mathrm{T}}\phi^*) \qquad \forall |\tilde{x}_m| > \varepsilon \tag{41}$$

After substituting (9) into (41), the following expression is obtained

$$\dot{V} = -\alpha\tilde{x}_\varepsilon^2 + \tilde{x}_\varepsilon(\hat{f} - f - (\Gamma\tilde{\tau})^{\mathrm{T}}\phi^* - a^* sat(r)) \tag{42}$$

It should be noted that $|\tilde{x}_m| > \varepsilon$ is satisfied when either $\tilde{x}_m > \varepsilon$ or $\tilde{x}_m < -\varepsilon$. These two distinct sub-cases will be investigated separately.

Case II(i): When $\tilde{x}_m > \varepsilon$, from (7) and (5), it follows that $\tilde{x}_\varepsilon > 0$ and $sat(r) = sgn(\tilde{x}_m) = 1$, thus from (42), we obtain

$$\dot{V} = -\alpha\tilde{x}_\varepsilon^2 + \tilde{x}_\varepsilon(\hat{f} - f - (\Gamma\tilde{\tau})^{\mathrm{T}}\phi^* - a^*) \tag{43}$$

From which it is clear that $\dot{V}(t) \le 0$ is satisfied if the following inequality holds

$$a^* \ge \hat{f} - f - (\Gamma\tilde{\tau})^{\mathrm{T}}\phi^* \quad \forall\tau \in \Omega_s \tag{44}$$

Therefore, we choose to maximize $a^*(t)$ as follows

$$a^* = \max_{\tau\in\Omega_s}(\hat{f} - f - (\Gamma\tilde{\tau})^{\mathrm{T}}\phi^*) \quad \text{for any } \phi^* \tag{45}$$

Note that the tuning function $a^*(t)$ is like a gain in (9) so it being smaller will be preferred; thus, we seek to find $\phi^*(t)$ so that $a^*(t)$ is minimized:

$$a^* = \min_{\phi\in\mathbb{R}^n}\max_{\tau\in\Omega_s}(\hat{f} - f - (\Gamma\tilde{\tau})^{\mathrm{T}}\phi^*) \tag{46}$$

Case II(ii): When $\tilde{x}_m < -\varepsilon$, from (7) and (5), it follows that $\tilde{x}_\varepsilon < 0$ and $sat(r) = sgn(\tilde{x}_m) = -1$, thus from (42), $\dot{V}(t)$ can be written as

$$\dot{V} = -\alpha\tilde{x}_\varepsilon^2 + \tilde{x}_\varepsilon(\hat{f} - f - (\Gamma\tilde{\tau})^{\mathrm{T}}\phi^* + a^*) \tag{47}$$

From which it is clear that $\dot{V}(t) \le 0$ is satisfied if the following inequality holds

$$a^* \ge f - \hat{f} + (\Gamma\tilde{\tau})^{\mathrm{T}}\phi^* \qquad \forall\tau \in \Omega_s \tag{48}$$

Following along the same lines as in Case II-ii, the following expression can be obtained for $a^*(t)$

$$a^* = \min_{\phi\in\mathbb{R}^n}\max_{\tau\in\Omega_s}(f - \hat{f} + (\Gamma\tilde{\tau})^{\mathrm{T}}\phi^*) \tag{49}$$

The conditions in (46) and (49) for cases II(i) and II(ii), respectively, can be combined to obtain the following expression for the tuning function $a^*(t)$

$$a^* = \min_{\phi\in\mathbb{R}^n}\max_{\tau\in\Omega_s} sat(r)(\hat{f} - f - (\Gamma\tilde{\tau})^{\mathrm{T}}\phi^*) \tag{50}$$

Similarly, from (44) and (48), the following inequality can be obtained

$$sat(r)(\hat{f} - f - (\Gamma\tilde{\tau})^{\mathrm{T}}\phi^*) - a^* \le 0 \tag{51}$$

The expression given in (43) can be rewritten as follows

$$\dot{V} = -\alpha\tilde{x}_\varepsilon^2 + \tilde{x}_\varepsilon sat(r)\{sat(r)(\hat{f} - f - (\Gamma\tilde{\tau})^{\mathrm{T}}\phi^*) - a^*\} \tag{52}$$

After utilizing (51), and the fact that $\tilde{x}_\varepsilon sat(r) \ge 0$ when $|\tilde{x}_m| > \varepsilon$, the right-hand side of (52) can be upper bounded as follows

$$\dot{V} \le -\alpha\tilde{x}_\varepsilon^2 \tag{53}$$

From (37), (40), and (53), it can be concluded that $V(t) \in \mathcal{L}_\infty$, thus $\tilde{x}_\varepsilon(t), \tilde{\tau}(t) \in L_\infty$. After integrating (52), it is easy to see that $\tilde{x}_\varepsilon(t) \in \mathcal{L}_2$; since $sat(\cdot)$ in (7) produces bounded outputs, from

(5), it can be concluded that $\tilde{x}_m(t) \in \mathcal{L}_\infty$. From (15), it is clear that $\hat{\tau}(t) \in \mathcal{L}_\infty$. Since the tuning function $a^*(t)$ is a function of bounded signals, and $f(\cdot)$ is considered to be a bounded signal, from (9), it follows that $\dot{\tilde{x}}_m(t) \in \mathcal{L}_\infty$. Since the projection strategy given in (11) ensures that $\dot{\hat{\tau}}(t) \in \mathcal{L}_\infty$; from the time derivative of (15), it can be concluded that $\dot{\tilde{\tau}}(t) \in \mathcal{L}_\infty$.

B Proof of Theorem 2

To facilitate the proof, without loss of generality, we assume that $\beta(\Pi(t_2)) = 1$, i.e. $f(\tau, \Pi(t_2))$ is convex on Ω_s.[3] Thus, the expression given in (26) can be rewritten as follows

$$f(\hat{\tau}(t_1), \Pi(t_2)) - f(\tau, \Pi(t_2)) \geq \bar{\varepsilon} \tag{54}$$

where $\bar{\varepsilon} \triangleq \varepsilon_u \|\hat{\tau}(t_1) - \tau\|$. To further facilitate the proof, a region of convergence is defined as

$$\Omega_\varepsilon = \{d : V(d) \leq \gamma\} \tag{55}$$

where $d(t) \in \mathbb{R}^{n+1}$ is the combined error signal defined as

$$d \triangleq [\tilde{x}_\varepsilon \quad \tilde{\tau}^{\mathrm{T}}]^{\mathrm{T}} \tag{56}$$

and $V(\cdot)$ is the Lyapunov function previously defined in (37). From the region of convergence, we know that if $d(t_1) \in \Omega_\varepsilon$, then $d(t)$ stays in Ω_ε. Since, $V(\cdot)$ is a Lyapunov function and its time derivative is always non-positive; it is assumed that $d(t_1) \notin \Omega_\varepsilon$. The proof is facilitated by showing that $V(\cdot)$ decreases by a finite amount over every interval of time until the trajectories reach Ω_ε for all $t \geq t_1$. When $d(t_1) \notin \Omega_\varepsilon$, from (55), it is clear that

$$V = \frac{1}{2}\tilde{x}_\varepsilon^2 + \frac{1}{2}\tilde{\tau}^{\mathrm{T}}\tilde{\tau} > \gamma \tag{57}$$

where (37) and (56) were utilized. From the above expression, it is clear that the following inequalities are not satisfied simultaneously

$$|\tilde{x}_\varepsilon(t_1)| < \sqrt{\gamma} \tag{58}$$

$$\|\tilde{\tau}(t_1)\| < \sqrt{\gamma} \tag{59}$$

It can be seen that if the inequalities given in (58) and (59) are satisfied simultaneously, then $V(\cdot) \leq \gamma$, which is not true. Thus, three possible cases arise: 1. $|\tilde{x}_\varepsilon(t_1)| > \sqrt{\gamma}$; 2. $\|\tilde{\tau}(t_1)\| > \sqrt{\gamma}$; or 3. $|\tilde{x}_\varepsilon(t_1)| > \sqrt{\gamma}$ and $\|\tilde{\tau}(t_1)\| > \sqrt{\gamma}$. If case 1 or case 3 holds, since $|\tilde{x}_\varepsilon(t_1)| > \sqrt{\gamma}$, from Property 1 (see Appendix C), it is clear that $V(\cdot)$ decreases. If case 2 holds, in the following analysis it will be shown that $|\tilde{x}_\varepsilon(t)|$ becomes large for some $t > t_1$ and $V(\cdot)$ decreases. Since for case 2, $\|\tilde{\tau}(t)\| > \sqrt{\gamma}$, from its definition, following expression can be obtained

$$\bar{\varepsilon}^2 \geq \varepsilon_u^2 \gamma \tag{60}$$

After substituting (27) into (60), following expression is obtained

$$\bar{\varepsilon}^2 \geq 8\varepsilon c_1 \tag{61}$$

We show that if (61) holds, then there exists a time $t_3 \in [t_2, t_2 + T_1]$ such that

$$|\tilde{x}_\varepsilon(t_3)| > \min\{1, \bar{\delta}\} \tag{62}$$

where $\bar{\delta} \in \mathbb{R}$ is defined as

$$\bar{\delta} \triangleq \min\left\{\frac{\bar{\varepsilon}}{2c_2}, \frac{\bar{\varepsilon}^2 - 4\varepsilon c_1}{2\bar{\varepsilon}c_2 + 4c_1}\right\} \tag{63}$$

with $c_2, T_1 \in \mathbb{R}$ being defined as

$$c_2 \triangleq \upsilon L_2 L_\phi T_0 + \alpha, \quad T_1 \triangleq \frac{\bar{\varepsilon} - \bar{\delta}c_2}{c_1} \tag{64}$$

Proof by contradiction will be utilized to show that (62) holds. To facilitate the proof, the following inequality is considered

$$|\tilde{x}_\varepsilon(t_2 + \lambda)| < \min\{1, \bar{\delta}\} \quad \forall \lambda \in [0, T_1] \tag{65}$$

From (9), the following inequality may be obtained

$$\begin{aligned}\dot{\tilde{x}}_m(t_2 + \lambda) \geq -\alpha \min\{1, \bar{\delta}\} + f(\hat{\tau}, \Pi(t_2 + \lambda)) \\ - f(\tau, \Pi(t_2 + \lambda)) - a^* sat(r)\end{aligned} \tag{66}$$

where (65) was utilized. To prove that $\tilde{x}_\varepsilon(t)$ becomes large over $[t_2, t_2 + T_1]$, we seek to establish lower bounds for $[f(\hat{\tau}, \Pi(t_2 + \lambda)) - f(\tau, \Pi(t_2 + \lambda))]$ and $-a^* sat(r)$ in (66) in order. From Assumption 4, it follows that

$$|f(\tau + \Delta\tau, \Pi(t_2)) - f(\tau, \Pi(t_2))| \leq L_2 \|\Delta\tau\| \tag{67}$$

After integrating (10) from t_1 to t_2, the following expression is obtained

$$\hat{\tau}(t_2) - \hat{\tau}(t_1) = \int_{t_1}^{t_2} -\Gamma \tilde{x}_\varepsilon(\sigma) \phi^*(\sigma)\, d\sigma \tag{68}$$

where the fact that the projection algorithm retains all of the properties of the adaptive law without the projection algorithm was utilized. From (68), triangle inequality can be utilized to obtain the following expression

$$\|\hat{\tau}(t_2) - \hat{\tau}(t_1)\| \leq \int_{t_1}^{t_2} \upsilon \|\tilde{x}_\varepsilon(\sigma)\| \|\phi^*(\sigma)\|\, d\sigma \tag{69}$$

The left-hand side of (69) can be upper bounded as follows

$$\|\hat{\tau}(t_2) - \hat{\tau}(t_1)\| \leq \upsilon \min\{1, \bar{\delta}\} L_\phi T_0 \tag{70}$$

where (65), Remark 3, and the fact that $T_0 \geq t_2 - t_1$ were utilized. After utilizing (67) and (70), the following inequality can be obtained

$$|f(\hat{\tau}(t_2), \Pi(t_2)) - f(\hat{\tau}(t_1), \Pi(t_2))| \leq L_2 \upsilon \min\{1, \bar{\delta}\} L_\phi T_0 \tag{71}$$

from which, it follows that

$$-\upsilon L_2 \min\{1, \bar{\delta}\} L_\phi T_0 \leq f(\hat{\tau}(t_2), \Pi(t_2)) - f(\hat{\tau}(t_1), \Pi(t_2)) \tag{72}$$

After adding (54) and (72), the following expression is obtained

$$\bar{\varepsilon} - \upsilon L_2 \min\{1, \bar{\delta}\} L_\phi T_0 \le f(\hat{\tau}(t_2), \Pi(t_2)) - f(\tau, \Pi(t_2)) \quad (73)$$

After utilizing Assumptions 3 and 4, the following inequalities can be obtained

$$|f(\tau, \Pi(t_2 + \lambda)) - f(\tau, \Pi(t_2))| \le L_2(\|\Pi(t_2 + \lambda) - \Pi(t_2)\|) \quad (74)$$

$$\le L_2 L_1 \lambda \quad (75)$$

from which the following expressions may be obtained

$$f(\tau, \Pi(t_2 + \lambda)) - f(\tau, \Pi(t_2)) \le L_2 L_1 \lambda \quad (76)$$

$$-L_2 L_1 \lambda \le f(\tau, \Pi(t_2)) - f(\tau, \Pi(t_2 + \lambda)) \quad (77)$$

After combining (70), Assumptions 3 and 4, the following expression can be obtained

$$|f(\hat{\tau}(t_2 + \lambda), \Pi(t_2 + \lambda)) - f(\hat{\tau}(t_2), \Pi(t_2))| \le L_2 L_1 \lambda + L_2 \upsilon L_\phi \lambda \quad (78)$$

where the fact that $\min(a, b) \le a$ and $\min(a, b) \le b$ was utilized. From (78), it follows that

$$-L_2 L_1 \lambda - L_2 \upsilon L_\phi \lambda \le f(\hat{\tau}(t_2 + \lambda), \Pi(t_2 + \lambda)) - f(\hat{\tau}(t_2), \Pi(t_2)) \quad (79)$$

After adding (77) and (79), the following expression is obtained

$$-L_2(2L_1 + \upsilon L_\phi)\lambda \le f(\hat{\tau}(t_2 + \lambda), \Pi(t_2 + \lambda)) - f(\tau, \Pi(t_2 + \lambda)) + f(\tau, \Pi(t_2)) - f(\hat{\tau}(t_2), \Pi(t_2)) \quad (80)$$

which can be rearranged to obtain the following expression

$$-L_2(2L_1 + \upsilon L_\phi)\lambda + f(\hat{\tau}(t_2), \Pi(t_2)) - f(\tau, \Pi(t_2)) \le f(\hat{\tau}(t_2 + \lambda), \Pi(t_2 + \lambda)) - f(\tau, \Pi(t_2 + \lambda)) \quad (81)$$

After utilizing (73), Equation (81) can be rewritten as

$$\bar{\varepsilon} - \upsilon L_2 \min\{1, \bar{\delta}\} L_\phi T_0 - L_2(2L_1 + \upsilon L_\phi)\lambda \le f(\hat{\tau}(t_2 + \lambda), \Pi(t_2 + \lambda)) - f(\tau, \Pi(t_2 + \lambda)) \quad (82)$$

where the lower bound on the term $[f(\hat{\tau}, \Pi(t_2 + \lambda)) - f(\tau, \Pi(t_2 + \lambda))]$ in (66) is established. Now, we seek to find a lower bound on the term $-a^* sat(r)$ in (66). After changing the variable t_2 to $t_2 + \lambda$ and t_1 to t_2, the expression given in (70) can be rewritten as follows

$$\|\hat{\tau}(t_2 + \lambda) - \hat{\tau}(t_2)\| \le \upsilon \min\{1, \bar{\delta}\} L_\phi \lambda \quad (83)$$

After pre-multiplying (68) with $\phi^{*\mathrm{T}}(t_2)$ and then utilizing similar manipulations as those in (68)–(70), the following expression is obtained

$$|\phi^{*\mathrm{T}}(t_2)(\hat{\tau}(t_2 + \lambda) - \hat{\tau}(t_2))| \le \upsilon \min\{1, \bar{\delta}\} L_\phi^2 \lambda \quad (84)$$

where Remark 3 was utilized. When $\beta(\Pi(t_2)) = 1$, Property 3 (see Appendix E) can be utilized to show that

$$a^*_+(\hat{\tau}(t_2), \Pi(t_2)) = 0 \quad (85)$$

where $a^*_+(\cdot)$ denotes $a^*(t)$ when $\tilde{x}_\varepsilon > 0$ (see Appendix D). From (12), the following expression is obtained

$$a^*_+(\hat{\tau}(t_2), \Pi(t_2)) = \max\{\hat{f}_2 - \phi^{*\mathrm{T}}(t_2)(\hat{\tau}(t_2) - \tau)\} \quad (86)$$

where $\hat{f}_2(\cdot) \in \mathbb{R}$ is an auxiliary signal defined as

$$\hat{f}_2 \triangleq f(\hat{\tau}(t_2), \Pi(t_2)) - f(\tau, \Pi(t_2)) \quad (87)$$

At time instant $t_2 + \lambda$, the expression given in (86) can be written as follows

$$a^*_+(\hat{\tau}(t_2 + \lambda), \Pi(t_2 + \lambda)) = \max\{\hat{f}_{2\lambda} - \phi^{*\mathrm{T}}(t_2 + \lambda)(\hat{\tau}(t_2 + \lambda) - \tau)\} \quad (88)$$

where $\hat{f}_{2\lambda}(\cdot) \in \mathbb{R}$ is an auxiliary signal defined as

$$\hat{f}_{2\lambda} \triangleq f(\hat{\tau}(t_2 + \lambda), \Pi(t_2 + \lambda)) - f(\tau, \Pi(t_2 + \lambda)) \quad (89)$$

Since $\phi^*(t_2 + \lambda)$ results in the minimum value of $a^*_+(\hat{\tau}(t_2 + \lambda), \Pi(t_2 + \lambda))$, the left-hand side of (88) can be upper bounded as follows

$$a^*_+(\hat{\tau}(t_2 + \lambda), \Pi(t_2 + \lambda)) \le \max\{\hat{f}_{2\lambda} - \phi^{*\mathrm{T}}(t_2)(\hat{\tau}(t_2 + \lambda) - \tau)\} \quad (90)$$

After adding and subtracting the terms $\hat{f}_2(\cdot)$ and $\phi^{*\mathrm{T}}(t_2)\hat{\tau}(t_2)$ to the right-hand side of (90), and then simplifying results in the following expression

$$a^*_+(\hat{\tau}(t_2 + \lambda), \Pi(t_2 + \lambda)) \le \max\{\hat{f}_{2\lambda} - \hat{f}_2 - \phi^{*\mathrm{T}}(t_2)(\hat{\tau}(t_2 + \lambda) - \hat{\tau}(t_2))\} + \max\{\hat{f}_2 - \phi^{*\mathrm{T}}(t_2)(\hat{\tau}(t_2) - \tau)\} \quad (91)$$

where the fact that $\max(a + b) \le \max(a) + \max(b)$ was utilized. After utilizing (86), the expression given in (91) can be written as follows

$$a^*_+(\hat{\tau}(t_2 + \lambda), \Pi(t_2 + \lambda)) \le \max\{\hat{f}_{2\lambda} - \hat{f}_2 - \phi^{*\mathrm{T}}(t_2)(\hat{\tau}(t_2 + \lambda) - \hat{\tau}(t_2))\} + a^*_+(\hat{\tau}(t_2), \Pi(t_2)) \quad (92)$$

where the right-hand side of the expression can be upper bounded as follows

$$a^*_+(\hat{\tau}(t_2 + \lambda), \Pi(t_2 + \lambda)) \le \max\{\hat{f}_{2\lambda} - \hat{f}_2\} + \max\{-\phi^{*\mathrm{T}}(t_2)(\hat{\tau}(t_2 + \lambda) - \hat{\tau}(t_2))\} + a^*_+(\hat{\tau}(t_2), \Pi(t_2)) \quad (93)$$

The expression given in (93) can be rewritten as follows

$$a^*_+(\hat{\tau}(t_2 + \lambda), \Pi(t_2 + \lambda)) \le L_2(2L_1 + \upsilon L_\phi)\lambda + \upsilon \min\{1, \bar{\delta}\} L_\phi^2 \lambda \quad (94)$$

where (80), (84), (85), (87), and (89) were utilized. Since $\min\{1,\bar{\delta}\} \leq 1$, (94) can be rewritten as follows

$$\begin{aligned} a^*_+(\hat{\tau}(t_2+\lambda),\Pi(t_2+\lambda)) &\leq L_2(2L_1+\upsilon L_\phi)\lambda+\upsilon L_\phi^2\lambda \\ &\leq (2L_2L_1+\upsilon L_2L_\phi+\upsilon L_\phi^2)\lambda \end{aligned} \tag{95}$$

The inequality given in (95) is rewritten as follows

$$a^*_+(\hat{\tau}(t_2+\lambda),\Pi(t_2+\lambda))sat(r)(2L_2L_1+\upsilon L_2L_\phi+\upsilon L_\phi^2)\lambda \tag{96}$$

where the fact that $sat(r) \leq 1$ was utilized. After multiplying both sides of (96) by -1, and utilizing Property 2 (see Appendix D), the lower bound on the term $-a^* sat(r)$ is obtained as follows

$$-a^*(\hat{\tau}(t_2+\lambda),\Pi(t_2+\lambda))sat(r) \geq -(2L_2L_1+\upsilon L_2L_\phi+\upsilon L_\phi^2)\lambda \tag{97}$$

Now, the expression given in (66) can be rewritten as follows

$$\begin{aligned} \dot{\tilde{x}}_m(t_2+\lambda \geq{}& -\alpha\min\{1,\bar{\delta}\}+\bar{\varepsilon}-\upsilon L_2\min\{1,\bar{\delta}\}L_\phi T_0 \\ &-L_2(2L_1+\upsilon L_\phi)\lambda-(2L_2L_1+\upsilon L_2L_\phi+\upsilon L_\phi^2)\lambda \end{aligned} \tag{98}$$

where (82) and (97) were utilized. After substituting the definitions of c_1 and c_2 in (27) and (64), respectively, into (98), the following expression can be obtained

$$\dot{\tilde{x}}_m(t_2+\lambda) \geq \bar{\varepsilon}-c_2\min\{1,\bar{\delta}\}-c_1\lambda \tag{99}$$

Since $\min\{1,\bar{\delta}\} \leq \bar{\delta}$, the right-hand side of (99) can be lower bounded as follows

$$\dot{\tilde{x}}_m(t_2+\lambda) \geq c_3-c_1\lambda \tag{100}$$

where $c_3 \in \mathbb{R}$ is defined as

$$c_3 \triangleq \bar{\varepsilon}-c_2\bar{\delta} \tag{101}$$

Integrating both the sides of (100) over $[0, T_1]$ with T_1 being previously defined in (64), results in the following expression

$$\int_0^{T_1}\dot{\tilde{x}}_m(t_2+\lambda)\,d\lambda \geq \left(c_3\lambda-\frac{1}{2}c_1\lambda^2\right)\Big|_0^{T_1} \tag{102}$$

Simplifying the right-hand side of (102) results in the following simple expression

$$\left(c_3\lambda-\frac{1}{2}c_1\lambda^2\right)\Big|_0^{T_1} = \frac{1}{2}\frac{c_3^2}{c_1} \tag{103}$$

where (64) was utilized. After performing a change of variable $\rho = t_2+\lambda$ on the left-hand side of (102), the following expressions can be obtained

$$\begin{aligned} \int_0^{T_1}\dot{\tilde{x}}_m(t_2+\lambda)\,d\lambda &= \int_{t_2}^{t_2+T_1}\dot{\tilde{x}}_m(\rho)\,d\rho \\ &= \tilde{x}_m(\rho)\big|_{t_2}^{t_2+T_1} \\ &= \tilde{x}_m(t_2+T_1)-\tilde{x}_m(t_2) \end{aligned} \tag{104}$$

After combining (103) and (104), the expression given in (103) can be rewritten as follows

$$\tilde{x}_m(t_2+T_1)-\tilde{x}_m(t_2) \geq \frac{1}{2}\frac{c_3^2}{c_1} \tag{105}$$

Evaluating the expression in (65), with $\lambda = 0$ results

$$-\min\{1,\bar{\delta}\} < \tilde{x}_\varepsilon(t_2) < \min\{1,\bar{\delta}\} \tag{106}$$

which, after utilizing (106), can be rewritten as

$$-\varepsilon-\min\{1,\bar{\delta}\} < \tilde{x}_m(t_2) < \varepsilon+\min\{1,\bar{\delta}\} \tag{107}$$

After substituting (107) into (105), the following inequality can be written

$$\tilde{x}_m(t_2+T_1) \geq \frac{c_3^2}{2c_1}-\varepsilon-\min\{1,\bar{\delta}\} \tag{108}$$

Since $\min(a, b) \leq a$ and $\min(a, b) \leq b$, from the definition of $\bar{\delta}$ given in (63), the following inequality can be obtained

$$\bar{\delta} \leq \frac{\bar{\varepsilon}^2-4\varepsilon c_1}{2\bar{\varepsilon}c_2+4c_1} \tag{109}$$

After multiplying both sides of (109) by the non-negative term $(2\bar{\varepsilon}c_2+4c_1)$, the following inequalities can be obtained

$$\begin{aligned} 2\bar{\delta}\bar{\varepsilon}c_2+4\bar{\delta}c_1 &\leq \bar{\varepsilon}^2-4\varepsilon c_1 \\ 4c_1(\bar{\delta}+\varepsilon) &\leq \bar{\varepsilon}^2-2\bar{\delta}\bar{\varepsilon}c_2 \\ 2(\bar{\delta}+\varepsilon) &\leq \frac{\bar{\varepsilon}^2-2\bar{\delta}\bar{\varepsilon}c_2}{2c_1} \end{aligned} \tag{110}$$

After adding and subtracting the term $(\bar{\delta}c_2)^2$ to the right-hand side of (110) results in

$$\begin{aligned} \frac{\bar{\varepsilon}^2-2\bar{\delta}\bar{\varepsilon}c_2}{2c_1} &= \frac{\bar{\varepsilon}^2-2\bar{\delta}\bar{\varepsilon}c_2+(\bar{\delta}c_2)^2-(\bar{\delta}c_2)^2}{2c_1} \\ &= \frac{(\bar{\varepsilon}-\bar{\delta}c_2)^2-(\bar{\delta}c_2)^2}{2c_1} \\ &= \frac{c_3^2-(\bar{\delta}c_2)^2}{2c_1} \end{aligned} \tag{111}$$

After utilizing (110) and (111), the following inequality can be obtained

$$\frac{(\bar{\delta}c_2)^2}{2c_1}+2(\bar{\delta}+\varepsilon) \leq \frac{c_3^2}{2c_1} \tag{112}$$

After utilizing (112), the inequality given in (108) can be written as follows

$$\begin{aligned}\tilde{x}_m(t_2+T_1) &\geq \frac{(\bar{\delta}c_2)^2}{2c_1}+2(\bar{\delta}+\varepsilon)-\varepsilon-\min\{1,\bar{\delta}\}\\ &\geq \frac{(\bar{\delta}c_2)^2}{2c_1}+\bar{\delta}+\varepsilon+\bar{\delta}-\min\{1,\bar{\delta}\}\\ &\geq \frac{(\bar{\delta}c_2)^2}{2c_1}+\bar{\delta}+\varepsilon\\ &\geq \bar{\delta}+\varepsilon\end{aligned} \tag{113}$$

From (5), it can be seen that the expression given in (113) implies that $\tilde{x}_\varepsilon \geq \bar{\delta}$ which contradicts (65); thus, it can be easily concluded that (62) must hold. Thus, it was shown that if $V(t_1) > \gamma$, then one of the following inequalities hold

$$|\tilde{x}_\varepsilon(t_3)| \geq \delta \min\{1,\bar{\delta}\} \quad \forall t_3 \in [t_1, t_1+T_0+T_1] \tag{114}$$

$$|\tilde{x}_\varepsilon(t_1)| > \sqrt{\gamma} \tag{115}$$

From Property 1 (see Appendix C), it follows that if (114) holds, then

$$V(t_3+T_1') \leq V(t_3) - \frac{\alpha\delta^3}{3(M+\alpha\delta)} \tag{116}$$

where $T_1' = \delta/(M+\alpha\delta)$ and M is defined in Property 1. Similarly, if (115) holds, from Property 1, it follows that

$$V(t_1+T_2') \leq V(t_1) - \frac{\alpha\sqrt{\gamma}^3}{3(M+\alpha\sqrt{\gamma})} \tag{117}$$

where $T_2' = \sqrt{\gamma}/(M+\alpha\delta)$. Since $V(t)$ is a non-increasing function, from (116) and (117), the following expression can be concluded

$$V(t_1+T_3') \leq V(t_1) - \Delta V \quad \forall V(t_1) > \gamma \tag{118}$$

where $T_3', \delta V \in \mathbb{R}$ are defined as

$$T_3' = \max\left\{T_0+T_1+T_1', T_0+T_1+T_2'\right\}$$
$$\Delta V = \min\left\{\frac{\alpha\delta^3}{3(M+\alpha\delta)}, \frac{\alpha\sqrt{\gamma}^3}{3(M+\alpha\sqrt{\gamma})}\right\}$$

Thus, it is clear from (118) that $V(t)$ decreases by a finite amount over every interval T_3' until trajectories reach Ω_ε; hence, from (37), (55), and (56), it follows that $\| \tilde{\tau}(t) \| \leq \sqrt{\gamma}$ as $t \to \infty$.

C Property 1

Property 1. The property of the proposed min–max estimator (Cao et al., 2003) states that if

$$|\tilde{x}_\varepsilon| \geq \bar{\gamma} \quad ; \qquad \bar{\gamma} \in \mathbb{R}^+ \tag{119}$$

then

$$V(t_1+T') \leq V(t_1) - \frac{\alpha\bar{\gamma}^3}{3(M+\alpha\bar{\gamma})} \tag{120}$$

where $V(\cdot)$ is the Lyapunov function defined in (37) and T', M, ψ are defined as

$$T' \triangleq \bar{\gamma}/(M+\alpha\bar{\gamma}) \tag{121}$$

$$M \triangleq \max\{|\psi(t)|\} \tag{122}$$

$$\psi \triangleq \hat{f} - f - a^* sat(r) \tag{123}$$

Proof 4. To facilitate the proof, the following lemma is stated (Cao et al., 2003).

Lemma 1. For a system of the form

$$\dot{p} = -k(t)p + s(t) \tag{124}$$

$$\dot{p}_m = -k_m p_m + s_m \tag{125}$$

where $k(t), k_m > 0$ and $|s(t)| \leq s_m \; \forall t \geq t_0$, if $p(t_0) \leq p_m(t_0) < 0$, $k(t) \leq k_m$, then $p(t) \leq p_m(t)$, $\forall t \geq t_0$ where $p_m(t) \leq 0$.

Based on (119), there are two cases that should be considered $\tilde{x}_\varepsilon \geq \bar{\gamma}$ and $\tilde{x}_\varepsilon \leq -\bar{\gamma}$. The following derivations will be made for $\tilde{x}_\varepsilon \leq -\bar{\gamma}$ case where $\tilde{x}_\varepsilon \geq \bar{\gamma}$ is very similar, thus omitted. From (1) and Remark 1, the following expression can be obtained

$$\dot{\tilde{x}}_\varepsilon = -\alpha\tilde{x}_\varepsilon + \psi(t) \tag{126}$$

Since $\hat{f}(\cdot), f(\cdot) \in \mathcal{L}_\infty$ as proved in Appendix A and $a^*(t)$ is a function of bounded signals, it follows that $\psi(t)$ can be bounded as $|\psi(t)| \leq M$. To facilitate the proof, the following differential equation is considered

$$\dot{x}_a = -\alpha x_a + M; \quad x_a(t_1) = -\bar{\gamma} \tag{127}$$

From (126)–(127), and Lemma 1, the following inequality can be obtained

$$\tilde{x}_\varepsilon(t_1+\lambda) \leq x_a(t_1+\lambda) \quad \forall\lambda \geq 0 \tag{128}$$

where $q_m(t_1+\lambda) \leq 0$. The solution of the differential equation given in (127) can be obtained as

$$x_a(t_1+\lambda) = \left(-\frac{M}{\alpha} - \bar{\gamma}\right)e^{-\alpha\lambda} + \frac{M}{\alpha} \tag{129}$$

It should be noted that, from (129), it is clear that, $\ddot{x}_a(t_1+\lambda) \leq 0 \; \forall\lambda \geq 0$; therefore $x_a(t_1+\lambda)$ is a concave function of λ, $\forall\lambda \geq 0$. After utilizing the gradient property of concave functions (Annaswamy et al., 1998), the following inequality can be written

$$x_a(t_1+\lambda) \leq x_a(t_1) + \nabla x_{a_\lambda}(t_1+\lambda-t_1) \tag{130}$$

where $\nabla x_{a_\lambda} \triangleq (\partial x_a(t_1+\lambda)/\partial\lambda)|_{\lambda=0}$. The expression given in (130) can be rewritten as follows

$$x_a(t_1+\lambda) \leq -\bar{\gamma} + (M+\alpha\bar{\gamma})\lambda \tag{131}$$

After utilizing (131), the right-hand side of (128) can be upper bounded as follows

$$\tilde{x}_\varepsilon(t_1+\lambda) \leq -\bar{\gamma} + (M+\alpha\bar{\gamma})\lambda \quad \forall\lambda \geq 0 \tag{132}$$

Substituting $\lambda = T' = \bar{\gamma}/(M+\alpha\bar{\gamma})$ in (132) results in the following inequality

$$\tilde{x}_{\varepsilon}(t) \leq 0 \qquad \forall t \in [t_1, t_1 + T'] \tag{133}$$

After squaring, and then integrating both sides of (132) over $[t_1 + T', T']$, the following inequality is obtained

$$\int_{t_1}^{t_1+T'} |\tilde{x}_{\varepsilon}(\lambda)|^2 d\lambda \geq \frac{\bar{\gamma}^3}{3(M + \alpha\bar{\gamma})} \tag{134}$$

After integrating (53) over $[t_1, t_1 + T']$, the following inequality can be obtained

$$V(t_1 + T') \leq V(t_1) - \frac{\bar{\gamma}^3}{3(M + \alpha\bar{\gamma})} \tag{135}$$

where (134) was utilized. Thus, the proof of Property 1 is established.

D Property 2

Property 2. The property states the following inequality (Cao et al., 2003)

$$-a^*_-(\hat{\tau}, \Pi) \leq a^* sat\left(\frac{\tilde{x}_m}{\varepsilon}\right) \leq a^*_+(\hat{\tau}, \Pi) \tag{136}$$

where $a^*_-(\hat{\tau}, \Pi)$ denotes $a^*(t)$ when $\tilde{x}_{\varepsilon} < 0$, and $a^*_+(\hat{\tau}, \Pi)$ denotes $a^*(t)$ when $\tilde{x}_{\varepsilon} > 0$.

Proof 5. To facilitate the proof, first the left-hand side of the inequality in (136) will be proven

$$-a^*_-(\hat{\tau}, \Pi) \leq a^* sat\left(\frac{\tilde{x}_m}{\varepsilon}\right) \tag{137}$$

The solutions of the min–max optimization problem in (12)–(14) results in the following inequality (Annaswamy et al., 1998)

$$a^* \geq 0 \quad \forall \tau \in \Omega_s \tag{138}$$

From (7), it follows that $sat(\frac{\tilde{x}_m}{\varepsilon}) \geq 0$ when $\tilde{x}_m \geq 0$; thus, the following inequalities are obtained

$$a^* sat\left(\frac{\tilde{x}_m}{\varepsilon}\right) \geq 0 \tag{139}$$

$$a^* sat\left(\frac{\tilde{x}_m}{\varepsilon}\right) \geq -a^*_-(\hat{\tau}, \Pi) \tag{140}$$

where (138) was utilized. Thus, it can be concluded from (140) that if $\tilde{x}_m \geq 0$, then (137) holds.

When $\tilde{x}_m < 0$, from (5), it follows that $\tilde{x}_{\varepsilon} < 0$. Also, from (7), it follows that $-1 \leq sat(\frac{\tilde{x}_m}{\varepsilon}) < 0$. Therefore, the following inequality can be obtained

$$a^*_-(\hat{\tau}, \Pi) sat\left(\frac{\tilde{x}_m}{\varepsilon}\right) \geq -a^*_-(\hat{\tau}, \Pi) \tag{141}$$

Hence, from (141), it can be concluded that (137) holds when $\tilde{x}_m < 0$. This proves (137) for any $\tilde{x}_m(t)$. Similar analysis can be utilized to prove the right-hand side inequality of (136). Thus, the proof of Property 2 is established.

E Property 3

Property 3. The property states the following (Cao et al., 2003)

$$a^*_- = 0 \; if \; \beta = -1 \tag{142}$$

$$a^*_+ = 0 \; if \; \beta = 1 \tag{143}$$

$$\beta a^* \tilde{x}_m \leq 0 \;\; \forall \beta \tag{144}$$

Proof 6. The proof of the property follows the concept outlined in Cao et al. (2003). We included it in a detailed manner for the sake of completeness. From (23), it follows that $\beta = -1$ if q is concave; thus, the following expression can be obtained from the solutions of the min–max optimization problem given in (16)–(22)

$$a^* = 0 \quad \forall \tilde{x}_m < 0 \tag{145}$$

which proves (142). Further, when $\tilde{x}_m > 0$, the following expression can be obtained

$$\beta a^* \tilde{x}_m \leq 0 \quad \forall \tilde{x}_m > 0 \tag{146}$$

where (138) was utilized. Similarly, when $\beta = 1$, it follows that

$$a^* = 0 \quad \forall \tilde{x}_m > 0 \tag{147}$$

which proves (143). After utilizing (138), the following expression can be obtained

$$\beta a^* \tilde{x}_m \leq 0 \quad \forall \tilde{x}_m < 0 \tag{148}$$

Thus, from (146) and (148), it can be concluded that (144) holds. Hence, the proof of Property 3 is established.

F Validity of Assumptions 3 and 4

Assumptions 3 and 4 are technical assumptions that are used for the proof of convergence as given in Appendix A. In general, it is not possible to ascertain whether these assumptions are realistic. In this appendix, the validity of Assumptions 3 and 4 are discussed. To facilitate the validity argument, we add and subtract $f(\tau + \Delta\tau, \Pi)$ from the left-hand side of (3) to obtain the following expression

$$\begin{aligned} &|f(\tau + \Delta\tau, \Pi + \Delta\Pi) - f(\tau, \Pi)| \\ &\quad = |f(\tau + \Delta\tau, \Pi + \Delta\Pi) - f(\tau + \Delta\tau, \Pi) \\ &\qquad + f(\tau + \Delta\tau, \Pi) - f(\tau, \Pi)| \end{aligned} \tag{149}$$

The right-hand side of (149) can be upper bounded as follows

$$\begin{aligned} &|f(\tau + \Delta\tau, \Pi + \Delta\Pi) - f(\tau_0, \Pi)| \\ &\quad \leq |f(\tau + \Delta\tau, \Pi + \Delta\Pi) - f(\tau + \Delta\tau, \Pi)| \\ &\qquad + |f(\tau + \Delta\tau, \Pi) - f(\tau, \Pi)| \end{aligned} \tag{150}$$

where triangle inequality was utilized. After utilizing the mean value theorem (Khalil, 2002), the terms on the right-hand side of (150) can be written as follows

$$f(\tau + \Delta\tau, \Pi + \Delta\Pi) - f(\tau + \Delta\tau, \Pi) = \frac{\partial f(\tau + \Delta\tau, h_1)}{\partial h_1}\Big|_{h_1=\psi_1} (\Pi + \Delta\Pi - \Pi) \quad (151)$$

where $\psi_1 \in [\Pi, \Pi + \Delta\Pi]$ and can be chosen as $\psi_1 = \Pi + \Delta\Pi - \rho_1(\Pi + \Delta\Pi - \Pi)$ with $\rho_1 \in [0, 1]$ and

$$f(\tau + \Delta\tau, \Pi) - f(\tau, \Pi) = \frac{\partial q(h_2, \Pi)}{\partial h_2}\Big|_{h_2=\psi_2} (\tau + \Delta\tau - \tau) \quad (152)$$

where $\psi_2 \in [\tau, \tau + \Delta\tau]$ and can be chosen as $\psi_2 = \tau + \Delta\tau - \rho_2(\tau + \Delta\tau - \tau)$ with $\rho_2 \in [0, 1]$. It is assumed that $q(\cdot)$ is differentiable with respect to its arguments. Also, since the measurable signals are assumed to be bounded, we can utilize (150)–(152) to obtain the following expression

$$|f(\tau + \Delta\tau, \Pi + \Delta\Pi) - f(\tau, \Pi)| \leq L_2(\|\Delta\Pi\| + \|\Delta\tau\|) \quad (153)$$

where $L_2 \in \mathbb{R}$ is a positive constant. The expression given in (153) is same as the expression given in (3) in Assumption 4.

Similar argument can be given to show the validity of Assumption 3. To facilitate the argument, we define $t_\Delta \in \mathbb{R}$ as $t_1 \leq t_\Delta \leq t_2$. After utilizing the mean value theorem, the following expression can be obtained

$$\Pi(t_2) - \Pi(t_1) = \dot{\Pi}(t_\Delta)(t_2 - t_1) \quad (154)$$

The left-hand side of (154) can be upper bounded as follows

$$\|\Pi(t_2) - \Pi(t_1)\| \leq \|\dot{\Pi}(t_\Delta)\| \, |(t_2 - t_1)| \quad (155)$$

Since arguments of $\dot{\Pi}(\cdot)$ are assumed to be bounded, then $\dot{\Pi}(t_\Delta)$ is also bounded; hence, (155) can be written as follows

$$\|\Pi(t_1) - \Pi(t_2)\| \leq L_1|t_1 - t_2| \quad (156)$$

where $L_1 \in \mathbb{R}$ is a positive constant. It can be seen that (156) is the same expression as given in (2) in Assumption 3.