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Cable-stiffened foldable elastica for movable structures

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ABSTRACT

A structural element is proposed, made of a row of rigid voussoirs joined by a passing through cable. When the cable is tensioned, the ensemble acquires stiffness and, for appropriate contact profiles of the voussoirs, the response of the element under applied loads is governed by the same equations of Euler's elastica or, in equivalent terms, of a non-linear spline. Releasing the connecting cable, the structure is loosened and can be closely packed. With this system one can reproduce, at least in principle, any desired profile in the "stiff" configuration, and construct free form foldable surfaces of any shape.

1. Introduction

The proposed structural system belongs to the class of kinetic structures, and in particular aims at meeting the recent demands for mobile solutions capable of achieving different-in-type equilibrium configurations.

The demand for transforming spaces is pushing researchers to revise and improve the kinematic structures built in the past. The category wider constructed are line-supported structures (large roofs of stadiums and theaters [1]), that use a minor number of joints and members, but usually they are very heavy and do not allow to obtain complex surfaces. Between these Hoberman, inside the Adaptive Building Initiative, suggested rigid panels, variously connected through hinges or sliding joints [2]. The proposed solutions, however, are a juxtaposition of modules, each of which operates isolated, they are not self-supporting and must therefore be used in fixed façades or roofs. Because of this, they are actually unable to architectonically transform a whole space.

Major of research regards point-supported structures made of bars. Calatrava designed simple kinematic chain structures [3] and conceived symmetrical solids made of bars [4]. Others are modular scissor-like structures as those introduced by Pinero [5], deployable tensegrity grids researched by Motro [6] and symmetrical assemblies of over constrained mechanisms [7,8].

To stabilize the structure in the final configuration, further rods can be temporarily added. Kokawa used cables that run along specific lines of the articulated mechanism: once tensioned, they block the mechanism itself, eliminating the degrees of freedom [9]. Recently, the opportunity to use a snap-trough effect (first introduced by Zeigler [10]) has been studied: some modules have incompatibilities with the length of the structural components, so that, while moving, the bars instantaneously pass from an equilibrium configuration to another, significantly distant from the former [11]. In order to obtain a better control of the structural movement, especially for space applications, Pellegrino creatively uses or combines complaints, cables, pistons, achieving systems which can be controlled by one motor [12].

Although of great aesthetic value, all the aforementioned applications have only two configurations, typically open/closed or extended/compact. They have to be designed to achieve the specific shape of a planar surface, barrel vault, a platonic solid, or to approximate a sphere. When instead a few topical configurations are desired, more degrees of freedom have to be controlled. For example, Inoue and his colleagues have developed a flexible truss, called Variable Geometry Truss – VGT. This is a beam with some piston actuators and hinges: by controlling the lengths of extendable rods, it is possible to partially fold the beam's pieces and thus obtain curved shapes [13]. Within this class, the most studied examples are the scissors-like structures. These are formed essentially by a basic modulus composed of two cross bars connected by an intermediate pivot. Assembled together in 2D or 3D arrangements, they can form various shapes by maintaining a single degree of freedom. If one or more hinges are added, the number of

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degrees of freedom is augmented, so that the resulting mechanism can achieve different in type configurations [14].

The scissor-like structures involve just revolute joints, but their number is so large that it represents a major drawback. Moreover, they provide a discontinuous (zigzag) profile, so that any cover surface (usually formed by a flexible soft membrane) has to follow the resulting articulated shape. This aspect can penalize the use of rigid panels, which perform better than soft coatings for the organization of the interior space. Some solutions employing hard panels have been attempted [15], but the panels have to be divided into very small pieces and they lose compactness in the closed configuration.

The structure here proposed is a linear system composed of rigid voussoirs, stiffened by a tensioned cable passing through sheaths obtained by drilling the voussoirs in longitudinal direction. Simply varying the location and profile of the sheaths inside the voussoirs, it is possible, at least in principle, to approximate any curvature. Remarkably, the bending stiffness of the resulting system turns out to be proportional to the tensile force in the cable and to the curvature of the borders of the adjacent voussoirs. The range of configurations that the system can achieve is therefore much wider than the solutions presented above. Moreover, by simply varying the tensile force in the cable (pulling or releasing the cable), it is possible to change the stiffness of the system, thus obtaining diverse equilibrium configurations under the same applied loads. Moreover, for example in the case of low-rise arches, the decrease of the bending stiffness may trigger a snap-through instability, thus allowing for large movements of the system at the price of little variations of the tensile force in the cable.

With respect to the drawbacks previously mentioned for the other types of movable structures, the system is self-supporting. Moreover, the profile of the flexible beams is a continuous polygonal surface. Consequently, it easily fits to support a cover made of panels, be they a simple concertina or, where appropriate, panels hinged in both directions.

Hereinafter the characteristics and potentiality of the proposed system are detailed. The organization of the text is the following. In Section 2 the mechanical properties of the system (elastic stiffness and stress) are analyzed in the simplest case of a straight row of circular voussoirs. Analogies are evidenced in Section 3 with Euler's elastica and with spline curves. Extensions to lenticular-shaped voussoirs and/or curved assemblies are studied in Section 4. Possible applications are considered in Section 5 and discussed in the concluding chapter.

2. The proposed structural system

The load-bearing element of the proposed structural system is a flexible one-dimensional modulus, composed of a series of voussoirs connected by a tensioned cable. The cable is inserted into sheaths that run inside the voussoirs, where it is supposed to slide with negligible friction. When the cable is tensioned, the voussoirs are brought into contact one another: the higher the tensile force in the cable, the stiffer the composite system becomes. The principle is that of a Bowden cable. It is also similar to the technique used in segmental construction of un-bonded post-tensioned concrete, where the various segments are dry-connected by the force transmitted by steel tendons.

In the simplest case, one may suppose that the voussoirs are circular disks put into contact at their borders. Suppose that, in each voussoir, the cable passes through a diameter of the disk. If the cable is tensioned and fixed at the ends of the row, clearly the system assumes a straight configuration, of the type thin-line-drawn in the background of Fig. 1. If the system is deformed by an external action (applied loads), the sheaths become no longer aligned, so that the cable is strained because its length changes. At least as a first order approximation, suppose that the voussoirs are rigid, each one with the same radius R. Denoting with φ_i the angle that, after the deformation, is formed by the ith disk of the row and the next one $i+1$ (Fig. 2), then the length of the cable locally increases of the quantity δ_i, which is given by

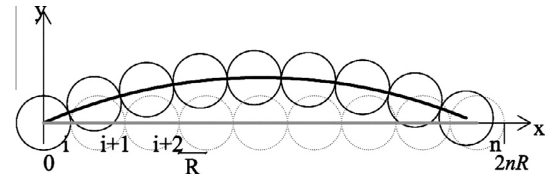

Fig. 1. Longitudinal section view of the system in the deformed state. In the background, the reference straight configuration.

$$\delta_i = 2[R - R\cos(\varphi_i/2)]. \tag{2.1}$$

Supposing that φ_i is small, using Taylor's expansion one has

$$\delta_i = 2R\left[1 - 1 + \frac{1}{2}\left(\frac{\varphi_i}{2}\right)^2\right] = R\left(\frac{\varphi_i}{2}\right)^2. \tag{2.2}$$

Let then L_0 represent the unstrained length of the cable. Then, if the row is composed of n voussoirs, after the first tensioning (that puts the disks into contact) the length of the cable is $2nR$ and its initial strain is $\varepsilon_0 = (2Rn/L_0 - 1)$, with $2Rn/L_0 > 1$. Consequently, the total elongation Δ_0 of the cable is given by

$$\Delta_0 = L_0\varepsilon_0 = 2Rn - L_0. \tag{2.3}$$

Denoting with A_c and E_c the cross sectional area and the Young's modulus of the cable, respectively, then $T_0 = A_c E_c \varepsilon_0$ represents the pre-tension force, and we have equivalently

$$\Delta_0 = L_0\frac{T_0}{E_cA_c}. \tag{2.4}$$

The elastic strain energy initially stored in the cable then reads

$$U_0 = \frac{1}{2}\frac{E_cA_c}{L_0}(\Delta_0)^2. \tag{2.5}$$

On the other hand, after the deformation (Fig. 2b and c) the strain energy in the cable results

$$U = \frac{1}{2}\frac{E_cA_c}{L_0}\left(\Delta_0 + \sum_{i=1}^{n-1}\delta_i\right)^2. \tag{2.6}$$

We make the assumption that the cable is initially strongly tensioned, and that the rotations φ_i after the deformation are moderate (infinitesimal deformation), so that $\delta_i \ll \Delta_0$. Then, the increase in strain energy due to the deformation can be simplified in the form

$$\begin{aligned}\Delta U = U - U_0 &= \frac{1}{2}\frac{E_cA_c}{L_0}\left(\Delta_0 + \sum_{i=1}^{n-1}\delta_i\right)^2 - \frac{1}{2}\frac{E_cA_c}{L_0}(\Delta_0)^2 \\ &= \frac{1}{2}\frac{E_cA_c}{L_0}\left[(\Delta_0)^2 + 2\Delta_0\left(\sum_{i=1}^{n-1}\delta_i\right) + \left(\sum_{i=1}^{n-1}\delta_i\right)^2\right] - \frac{1}{2}\frac{E_cA_c}{L_0} \\ &\quad\times(\Delta_0)^2 \\ &= \frac{E_cA_c}{L_0}\Delta_0\left(\sum_{i=1}^{n-1}\delta_i\right) + o(\delta_i)^2.\end{aligned} \tag{2.7}$$

But, recalling (2.2) and (2.4), one can write

$$\Delta U = \frac{E_cA_c}{L_0}\Delta_0\left(\sum_{i=1}^{n-1}\delta_i\right) = T_0\left(\sum_{i=1}^{n-1}\delta_i\right) = \frac{1}{4}RT_0\left(\sum_{i=1}^{n-1}(\varphi_i)^2\right). \tag{2.8}$$

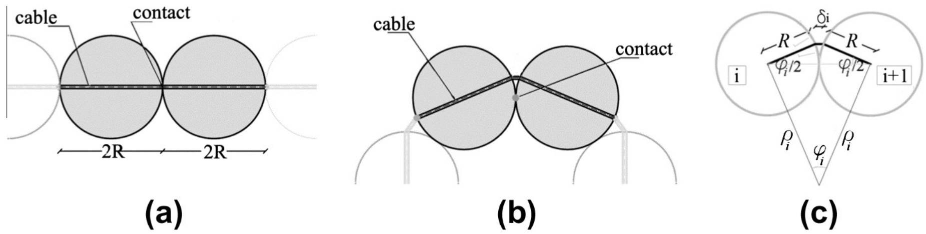

Fig. 2. Longitudinal section view of: (a) cylindrical voussoirs (disks) in the straight reference configuration; (b) voussoirs in the deformed (bent) state and (c) geometry of the deformed state.

Referring back to Fig. 2c, denote with ρ_i the "discrete" radius of curvature and with $\chi_i = 1/\rho_i$ the corresponding "discrete" curvature, such that

$$\rho_i\varphi_i = \rho_i 2\tan(\varphi_i/2) = 2R \Rightarrow \varphi_i \simeq \frac{2R}{\rho_i} = 2R\chi_i. \tag{2.9}$$

Then, using (2.9), the expression (2.8) can be conveniently rewritten in the form

$$\Delta U = \frac{1}{4}RT_0\left(\sum_{i=1}^{n-1}(2R\chi_i)^2\right) = R^3T_0\left(\sum_{i=1}^{n-1}\chi_i^2\right). \tag{2.10}$$

This latest expression can be simplified further under the hypothesis that $n \gg 1$, so that the *discrete* system can be considered in a *continuum* approximation. Introduced, as represented in Fig. 1, the reference system (x,y), such that the interval $-R \leqslant x \leqslant (2n-1)R$ denotes the reference (rectilinear) configuration of the assembly, the "discrete" set of curvatures χ_i, $i = 1,\dots,n-1$, can be approximated by a continuous function $\chi = \chi(x)$. This is the curvature of the continuous description of the polygonal line defined by the contact points of the disks, in the limit $R \to 0$. In other words, one has

$$\chi_i^2 = \frac{1}{2R}\int_{2(i-1)R}^{2i\,R}\chi^2(x)dx, i = 1,\dots,n-1. \tag{2.11}$$

Then, (2.10) reads

$$\Delta U = RT_0\left(\sum_{i=1}^{n-1}(R\chi_i)^2\right) \simeq RT_0\int_0^{2(n-1)R}[R\chi(x)]^2\frac{dx}{2R}$$
$$= \frac{1}{2}R^2T_0\int_0^{2(n-1)R}\chi^2(x)dx. \tag{2.12}$$

Indicating with Γ the undeformed rectilinear reference configuration of the assembly, i.e., the segment comprised between the centers of the first and last disks of the row, coinciding with the interval $0 \leqslant x \leqslant 2\ (n-1)R$ in Fig. 1, the increase of strain energy ΔU due to a deformation with curvature $\chi = \chi(x)$ thus results to be

$$\Delta U = \frac{1}{2}R^2T_0\int_\Gamma \chi^2(x)dx. \tag{2.13}$$

One can thus consider the minimization problem for the functional (2.13) under boundary conditions that affect in general the displacement or the rotation of the first and last disk of the row. In particular, one can either prescribe the displacement of the center of the disk and/or its rotation (geometric boundary conditions) or, as an alternative, prescribe the force acting at the disk center and/or the applied moment (natural boundary conditions). Moreover, the minimization should be performed under the constraint that the length of the deformed system is equal to the final length of the cable. However, because of the assumptions that the cable is initially strongly tensioned and that the deformation is infinitesimal, i.e., $\delta_i \ll \Delta_0$ in (2.6), the tensile force in the cable remains almost constant and the variation of its length negligible after the deformation. Consequently, in agreement with the principal hypotheses of the model, the minimization of (2.13) may be performed under the constraint that the length of the chain remains fixed (first-order approximation).

One of the most noteworthy features of the system just described is that it becomes loose when the cable is released. It is then a system that can be conveniently packaged and shipped because it would be sufficient just to tension the cable to let the assembly recover the bending stiffness associated with (2.13).

3. Analogies with elastica and splines

The form (2.13) of the strain energy functional has noteworthy similarities with the energy of Euler's elastica, and with the functional to be minimized in the variational characterization of nonlinear spline curves.

3.1. Euler's elastica

Let us denote with Γ the undistorted natural reference configuration of a thin beam. If one assumes, according to the celebrated Bernoulli's hypothesis that the effects of the normal and shearing forces are negligible with respect to bending, the curvature of the deflection curve becomes proportional to the bending moment and the elastic energy takes the form [16]

$$U = \int_\Gamma \frac{1}{2}EI\chi^2(s)ds, \tag{3.1}$$

where $\chi(s)$ is the curvature of the deflection curve at the abscissa s, defined following the length of the curve, while EI is the bending stiffness, equal to the product between the Young's modulus E of the material and the moment of inertia I of the cross sectional area of the beam. Minimization of the total energy functional under the constraint that the length of the elastica is prescribed, furnishes the variational characterization of the problem, for which existence of a solution can be proved [17].

Remarkably, comparing (2.13) with (3.1), one can see that T_0 has an effect on the stiffness of the cabled system, according to the formal equivalence

$$EI = R^2T_0. \tag{3.2}$$

Fig. 3 represents the free body diagram of the two disks showed in Fig. 2b: clearly the radius R contributes to the overall stiffness of the assembly. In fact, for the same value of the radius of curvature ρ_i, the higher is R, the greater is the arm a of the internal couple formed by the tensile force T_0 in the cable and the contact force between the disks at the contact points.

Obviously, the stiffness is proportional to the tensile pretension force T_0. According to (3.2), one can thus tune the stiffness of the system according to the desire application, by simply varying this parameter.

The cable-stiffened system reaches and maintains a configuration that minimizes its potential energy under the action of

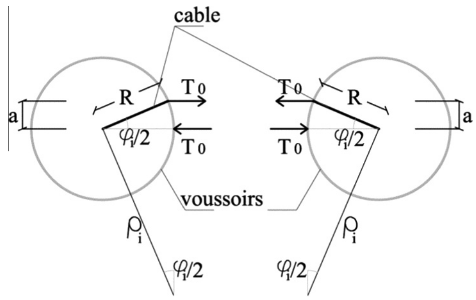

Fig. 3. Internal couple formed by the tensile force T_0 in the cable and the contact reaction force between the disks.

external forces, which depends upon the boundary conditions at the extremities. These are in terms of displacement and rotation, i.e., the usual conditions for the elastica. The mathematical characterization of the elastica through the variational approach has been the subject of extensive research. For the existence and regularity theorems under the most various boundary and load conditions the reader is referred to [18] as a comprehensive review.

Because of this analogy, the structural assembly discussed in Section 2 has been referred to as the cable-stiffened elastica.

3.2. Spline curves

Mathematical splines take their name from the Draftman's spline, a flexible lath used traditionally to enable a smooth curve to be drawn through plotted points on a drawing. This class of curves is characterized by the condition that the integral of the square of the curvature with respect to arc length should attain a minimum, under the condition that the curve passes through, or has prescribed tangent, at a certain set of points and the length of the curve is prescribed. Since the Euler–Lagrange equations associated with their variational problem are non-linear, such curves are sometimes referred to as "*non-linear splines*" [19]. This minimization is associated with the optimality condition usually referred to as "smoothest curve". From a formal point of view, the functional to be minimized coincides with (3.1) apart from the irrelevant multiplying constant *EI*. In other words, the deformed elastica geometrically coincides with the non-linear spline curve under the same boundary conditions.

This analogy is of importance for the design stage of structures employing the system, because there are computer programs that automatically can draw splines passing through a certain set of points with prescribed tangents.

More precisely, most computer programs approximate the non-linear spline with a smooth curve usually represented by a set of cubic polynomials. These are called "piecewise cubic splines". From a variational point of view, a cubic spline is derived by minimization of a functional where the curvature is approximated by the second derivative. If $y(x)$, $x \in (x_0, x_n)$ represents the equation of the spline, this is equivalent to the well-known approximation in the technical theory of elastic beams, that is

$$\chi(x) = \frac{y''(x)}{[1 + y'^2(x)]^{3/2}} \cong y''(x). \tag{3.3}$$

If no constraint about the length of the spline is added, the minimization problem

$$\min_{y(x)} \int_{x_0}^{x_n} \left(\frac{d^2y}{dx^2}\right) dx, \tag{3.4}$$

would prescribe that $y(x)$ is a cubic function. Imposing the conditions that the curve passes through a given set of points and endpoint constraints and that it is smooth, the cubic spline as pointed out by Holladay (1957) admits a unique solution [20]. For each one of the n pieces comprised between two consecutive nodes (x_{i-1}, y_{i-1}) and (x_i, y_i), $i = 1, \dots, n$, there are four unknowns to describe the cubic polynomial, and in total $2(n-1)$ continuity conditions, $2(n-1)$ smoothness conditions and four boundary conditions, i.e.,

continuity: $y(x_i^-) = y(x_i^+) = y_i$, $i = 1, \dots, n-1$;
smoothness: $y'(x_i^-) = y'(x_i^+), y''(x_i^-) = y''(x_i^+)$, $i = 1, \dots, n-1$;
boundary: $y(x_0) = y_0$, $y(x_n) = y_n$, $y'(x_0) = y'_0, y'(x_n) = y'_n$.

This procedure, which gives the curve with minimal average curvature passing through the prescribed points, may be used for a practical, preliminary, design-approach to the elastica. However, it should be noticed that here usually the length of the elastica is not prescribed, so that one has to check, *a posteriori*, that the length equals that of the cable-stiffened system.

4. More elaborated forms of the cable stiffened elastica

The cable-stiffened system represented in Fig. 1 is just one example. Many other possible forms, associated with different-in-type mechanical properties, can be obtained by simply using voussoirs of more elaborated shape, rather than simple disks. The number of possibilities is practically infinite, and hereafter a few examples are presented.

4.1. Lenticular voussoirs

Suppose now that the voussoirs are symmetric lenses, i.e., biconvex shaped figures comprised between two circular arcs with equal radii, joined at their endpoints. As showed in Fig. 4a, let 2λ represent the thickness of each lens and let R denote the radius of its profiles. Suppose then that the row, initially straight, is deformed by an external action, as represented in Fig. 4b. The relevant geometric parameters, analogously to Fig. 1c, are represented in Fig. 4c.

The treatment is similar to that of Section 2. Maintaining the same notation, the quantity δ_i is again given by (2.1) and (2.2), whereas Δ_0 takes the same form (2.4). The final expressions 2.5, 2.6, and 2.7 remain valid. One thus finds that the increase in strain energy due to the deformation, analogously to (2.8), reads

$$\Delta U = \frac{1}{4} R T_0 \left(\sum_{i=1}^{n-1} (\varphi_i)^2\right). \tag{4.1}$$

Referring to Fig. 4c, denote again with ρ_i the "discrete" radius of curvature and with $\chi_i = 1/\rho_i$ the corresponding "discrete" curvature, such that

$$\rho_i \varphi_i = \rho_i 2 \tan(\varphi_i/2) = 2\lambda \Rightarrow \varphi_i \simeq \frac{2\lambda}{\rho_i} = 2\lambda\chi_i. \tag{4.2}$$

Reasoning as before to obtain (2.10), one finds from (4.1) that

$$\Delta U = \frac{1}{4} R T_0 \left(\sum_{i=1}^{n-1} (2\lambda\chi_i)^2\right) = \lambda^2 R T_0 \left(\sum_{i=1}^{n-1} \chi_i^2\right). \tag{4.3}$$

In the continuum approximation one may write

$$\chi_i^2 = \frac{1}{2\lambda} \int_{2(i-1)\lambda}^{2i\lambda} \chi^2(x) dx, i = 1, \dots, n-1, \tag{4.4}$$

so that (4.3) can be re-written in the form

$$\Delta U = R T_0 \left(\sum_{i=1}^{n-1} (\lambda\chi_i)^2\right) \simeq R T_0 \int_0^{2(n-1)\lambda} [\lambda\chi(x)]^2 \frac{dx}{2\lambda}$$
$$= \frac{1}{2} \lambda R T_0 \int_0^{2(n-1)\lambda} \chi^2(x) dx = \frac{1}{2} \lambda R T_0 \int_\Gamma \chi^2(x) dx, \tag{4.5}$$

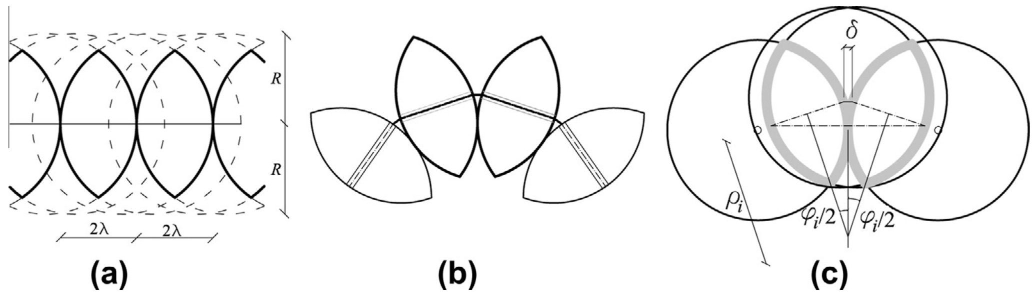

Fig. 4. Longitudinal section view of: (a) lenticular voussoirs in the straight reference configuration; (b) voussoirs in the deformed (bent) state and (c) geometry of the deformed state.

having again indicated with Γ the undeformed rectilinear reference configuration $0 \leqslant x \leqslant 2\,(n-1)\lambda$ for an assembly analogous to that of Fig. 1.

In conclusion, the form of the strain energy is perfectly analogous to that of (2.13), with the only exception that the term R^2 is substituted by λR. In general, the total length of the chain is a design *datum*, whereas the number n of voussoirs is governed by construction requirements. It follows that the length λ is the assigned length. Therefore, it can be seen from (4.5) that the higher the radius R of the profiles in contact, the stiffer the assembly results.

4.2. Curved assembly

The tensioning of the cable brings the voussoirs into contact, tangent one other, but there is only one configuration that is associated with the minimal length of the cable under the hypothesis that voussoirs are non-deformable. Such configuration depends upon the shape of the voussoirs only: if they are properly designed, in principle the reference configuration (no external load applied) can reproduce any arbitrary curve. In the following, it will be shown how to construct a cable stiffened elastica that, when the cable is tensioned, forms an arbitrary planar curve. The voussoirs are assumed to be symmetric and to have convex contact profiles, which at first may be assumed to be circular. They have to be designed so to be tangent each other in the tensioned configuration.

Suppose that the reference configuration has to reproduce a project curve, *a priori* prescribed by the designer, like the one represented in Fig. 5. At first, an assembly formed by n equal disks of radius R_i is explained. The first step is to approximate the curve with a polyline formed by segments of length $2R_i$: a series of circumferences of radius $2R_i$ are drawn, the first centered and one end of the curve, the followings each centered on the intersection point between the previous one and the curve (Fig. 5b). The points connecting all the centers of the circumferences of radius $2R_i$ are the knots of the approximating polyline. The disk-shaped voussoirs are posed in the obtained knots, and being their radius R_i, then they do not intersect instead are tangent to each other (Fig. 5c).

Of course, the smaller R_i, the closer is the approximation between the polyline and the project curve. Fig. 6a shows the approximating polyline of a target curve that is obtained with disks of radius R_i (construction circumferences of radius $2R_i$); Fig. 6b shows the same target curve, approximated by a polyline composed of disks of radius $3R_i$, constructed using circumferences of radius $6R_i$; Fig. 6c compares the project curves and the polylines so obtained. Where the curvature of the project curve is large (small), one should use small (large) disks to achieve a better approximation.

The sheaths hosting the cable have to be located in the contact points of the disk, orthogonally to the common tangent line. Although, at least in principle, any placement with the aforementioned property is acceptable, the sheath profile should reproduce,

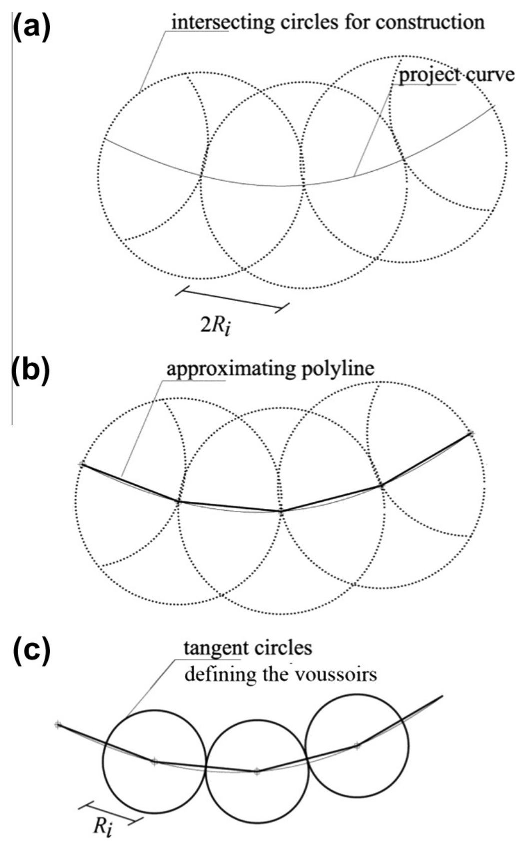

Fig. 5. Curved cable stiffened elastica. Project curve (a), approximating polyline (b), circular voussoirs that approximate the project curve (c).

as much as possible the approximating polyline of the project curve, with smooth fillets in proximity of the contact point in order to avoid cutting edges.

Once the approximating polyline has been defined, it is possible to use lenticular voussoirs, instead of circular disks. To do so, one can maintain the same contact points, but enlarge the radius of curvature of contact profiles. The geometric construction is reported in Fig. 7.

The form of the strain energy for the system becomes particularly simple provided that the initial curvature is *small*. Let the assembly be composed of n lenticular voussoirs, of the type represented in Fig. 7, and suppose that the initial configuration of the cable approximately follows the polyline mentioned in Fig. 6. Because of the initial curvature, the length of the cable passing through the voussoirs may vary from one another (Fig. 7). If the initial curvature is small such parameter is almost constant, but we shall keep the distinction and indicate, in general, with $2\lambda_i$,

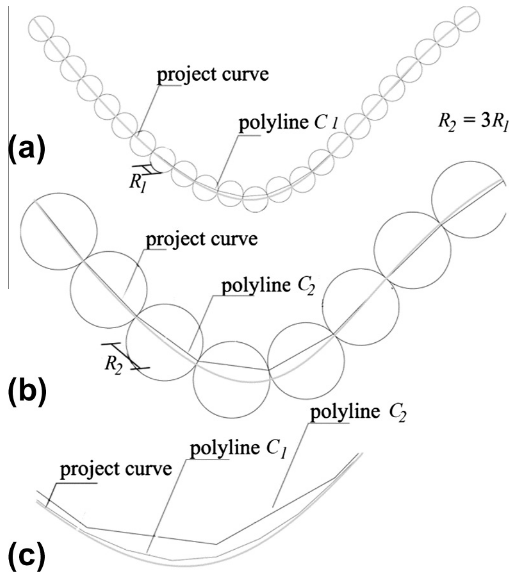

Fig. 6. The dimension and number of voussoirs influences the approximation of the target curve. The radius of the disks in (b) is three times that in (a). The project curve and the two approximating polylines are juxtaposed in (c).

$i = 1,\dots,n$, the length of the sheath inside the ith voussoir. A curvilinear abscissa s is defined, with the origin at the center of first voussoir, measuring the length of the cable up to the center of the last voussoir, so that $0 \leqslant s \leqslant \sum_{i=1}^{n-1}(\lambda_i + \lambda_{i+1})$. The contact profiles are supposed to be all arcs of circle of radius R.

The relevant geometric parameters are represented in Fig. 8. We indicate with ρ_i^0, $i = 1,\dots,n-1$, the initial radii of curvature, defined as the distance of the segments i and $i + 1$ of the cable reference-polyline from the point of intersection of the lines orthogonal to them and passing through their midpoints (that is, coinciding with the axes of voussoirs i and $i + 1$). Analogously, let φ_i^0, $i = 1,\dots,n-1$, represent the angle formed by segments i and $i + 1$ of the cable reference-polyline.

Suppose now that, after the deformation, the initial angle φ_i^0 is varied of the quantity $\Delta\varphi_i$, $i = 1\dots n-1$, with $\Delta\varphi_i = \varphi_i^0 = 1$. Then, the length of the cable comprised between the ith and the $(i + 1)$th voussoir *locally* increases of the quantity δ_i given by

$$\delta_i = 2R[1 - \cos(\Delta\varphi_i/2)] = R\left(\frac{\Delta\varphi_i}{2}\right)^2 + o(\Delta\varphi_i)^2,$$
$$i = 1,\dots,n-1. \tag{4.6}$$

Let then L_0 represent again the unstrained length of the cable and let, as before, T_0, E_c and A_c be, respectively, the initial axial force in the cable, its elastic modulus and cross sectional area. If L is the total length of the reference polyline (Fig. 7), i.e., the length of the sheaths hosting the cable, under the hypothesis that friction is negligible and voussoirs are rigid, the total elongation Δ_0 of the cable due to prestressing is again given by (2.4). The elastic strain energy initially stored in the cable is analogous to (2.5). After the deformation, the strain energy maintains the same form (2.6). Reasoning as in (2.7) and (2.8), one eventually finds

$$\Delta U = T_0\left(\sum_{i=1}^{n-1}\delta_i\right). \tag{4.7}$$

Now, observing Fig. 8, one has, up to infinitesimals of higher order

$$\lambda_i + \lambda_{i+1} = \rho_i^0\varphi_i^0 = \rho_i(\varphi_i^0 + \Delta\varphi_i), i = 1,\dots,n-1, \tag{4.8}$$

so that

$$\Delta\varphi_i = \left(\frac{1}{\rho_i} - \frac{1}{\rho_i^0}\right)\rho_i^0\varphi_i^0 = \left(\frac{1}{\rho_i} - \frac{1}{\rho_i^0}\right)(\lambda_i + \lambda_{i+1}), i$$
$$= 0,\dots,n-1. \tag{4.9}$$

Therefore, using (4.6), one has from (4.7) that

$$\Delta U = T_0\left(\sum_{i=1}^{n-1}\delta_i\right) = \frac{1}{4}RT_0\left(\sum_{i=1}^{n-1}(\Delta\varphi_i)^2\right). \tag{4.10}$$

Introduced the "discrete" curvatures $\chi_i = 1/\rho_i$ and $\chi_i^0 = 1/\rho_i^0$, this expression becomes

$$\Delta U = \frac{1}{4}RT_0\left\{\sum_{i=1}^{n-1}[(\lambda_i + \lambda_{i+1})(\chi_i - \chi_i^0)]^2\right\}. \tag{4.11}$$

Passing to a continuum approximation, for $n \gg 1$, let $\chi_0 = \chi_0(s)$ and $\chi = \chi(s)$ represent the curvature of the continuous (smooth) description of the polygonal line (Figs. 6 and 7) before and after the deformation, respectively. Denoting with Γ_i the length of the continuous curve comprised between the centers of the ith and

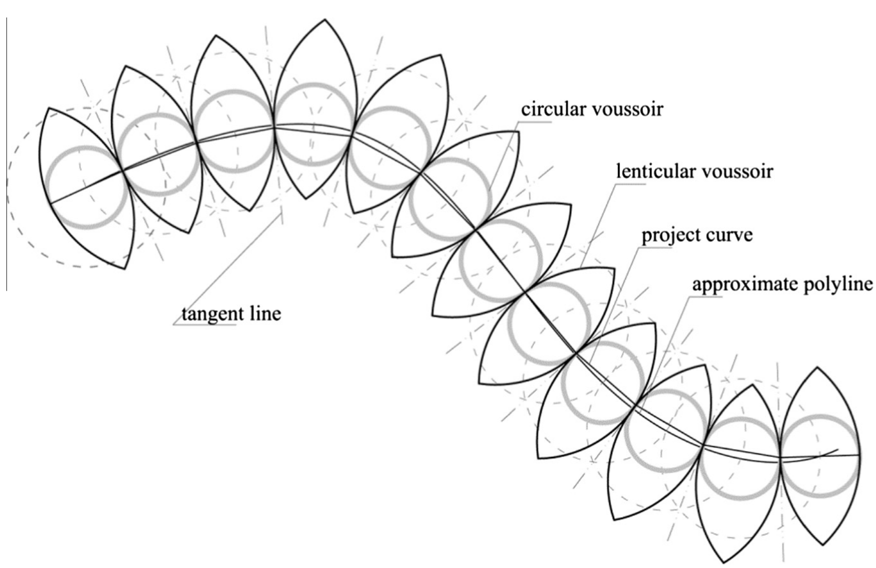

Fig. 7. Geometric construction for a curved cable-stiffened elastica with lenticular voussoirs.

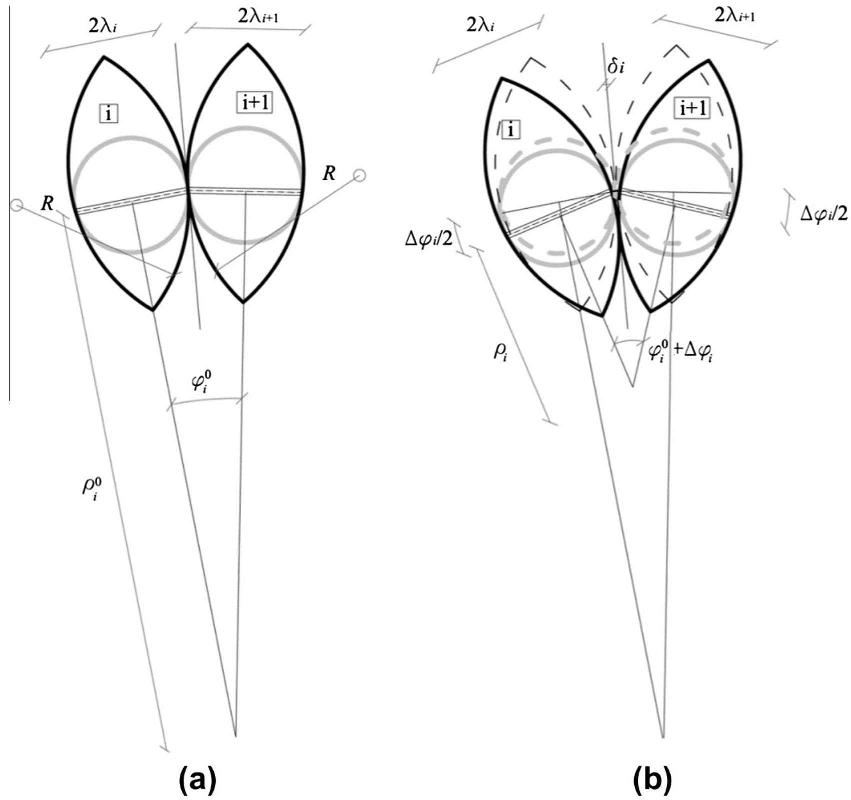

Fig. 8. Undeformed and deformed state of a curved cable-stiffened elastica with lenticular voussoirs.

$(i+1)$th voussoir, the relationship between discrete and approximate curvatures is given by

$$(\chi_i - \chi_i^0)^2 \simeq \frac{1}{\lambda_i + \lambda_{i+1}} \int_{\Gamma_i} [\chi(s) - \chi_0(s)]^2 ds, i = 1, \ldots, n-1. \quad (4.12)$$

Then, (4.11) can be re-written in the form

$$\Delta U = \frac{1}{4} R T_0 \left(\sum_{i=1}^{n-1} (\lambda_i + \lambda_{i+1}) \int_{\Gamma_i} [\chi(s) - \chi_0(s)]^2 ds \right)$$
$$= \frac{1}{4} R T_0 \left(\sum_{i=1}^{n-1} (2\bar{\lambda}) \int_{\Gamma_i} [\chi(s) - \chi_0(s)]^2 ds \right), \quad (4.13)$$

where $\bar{\lambda}$ represents a suitable weighted mean of the various λ_i, $i = 1, \ldots, n$. But if the initial curvature is small, then the parameters λ_i do not sensibly vary one-another, so that in general one might replace $\bar{\lambda}$ with the arithmetic mean of the various λ_i, $i = 1, \ldots, n$. Denoting with $\Gamma = \bigcup_{i=1}^{n-1} \Gamma_i$ the continuous reference configuration of the assembly, then (4.13) reduces to

$$\Delta U = \frac{1}{2} \bar{\lambda} R T_0 \int_{\Gamma} [\chi(s) - \chi_0(s)]^2 ds. \quad (4.14)$$

In practice, the final expression of the strain energy is analogous to that corresponding to that of an elastica with small initial curvature. Here, the bending stiffness (EI) is provided by the product $\bar{\lambda} R T_0$, similarly to the case of Section 4.1.

When the initial curvature cannot be considered small, the treatment follows the same steps just exposed, but the resulting expressions are much more complicated and go beyond the scope of this paper.

5. Possible applications

This section collects just a few possible applications of the cable-stiffened elastica.

5.1. Beams

The simplest application consists in a straight beam. As shown in Fig. 9, the beam axis is divided into segments (polyline approximation), each one corresponding with the location of the single voussoir. At first, circular disks may be chosen. It is not necessary to maintain the whole circular shape, but the voussoirs can be cut up to a height H, sufficient to maintain into contact the circular profiles under the most severe deformation, according with the estimated deflection. The excessive material can thus be removed, and the longitudinal profile of the voussoirs will have four sides, two parallel to the axis (here rectilinear), and two arc-shaped. Once fixed the number of voussoirs and the tensile force in the cable, substantial increase of stiffness can be obtained by enlarging the radius of curvature of the contact profiles (lens-shaped voussoirs), as per (4.5). Again, these can be cut up to the height H. However, for the same overall deflection of the beam, the larger the radius, the higher is the height H that is needed to maintain the voussoirs into contact. Therefore, left aside any structural problem associated with Herzian contact pressure, which will be the subject of further work, a compromise has to be reached between the height and the radius of the constituent elements.

The response of the assembly is that of a beam whose cross-sectional inertial stiffness EI is given by (3.2).

5.2. Arches

If the design curve is a circular arch, in the reference state the curvature is constant. The approximating polyline has equally-spaced vertexes and the voussoirs can be made identical. Note however (Fig. 10c) that the sheaths do not pass through the center of the voussoirs, but have to remain slightly not-symmetric according to the construction of Fig. 7, recalling somehow the trapezoidal shape of the voussoirs in a stone arch. Following (4.14), the arch

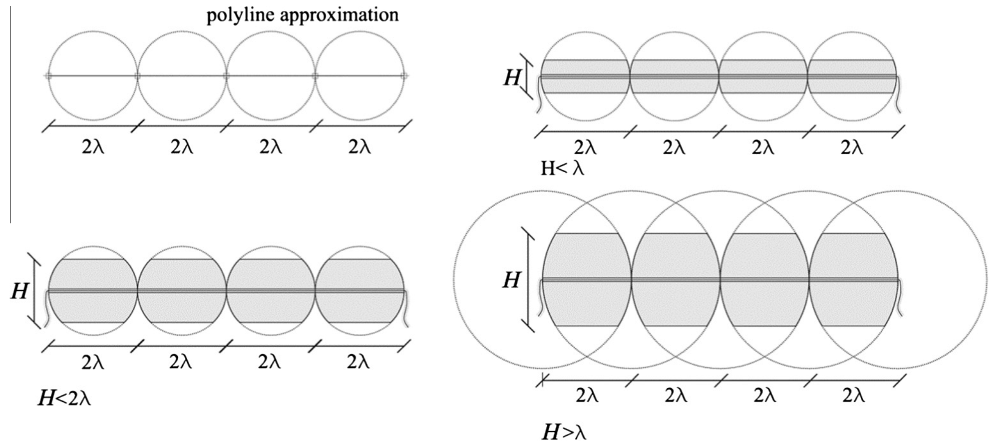

Fig. 9. In clockwise order. The generation of a polyline from the target curve, here rectilinear; the design of voussoirs of increasing height. A consistent increase of stiffness can be obtained by increasing the radius R of curvature of the contact profiles.

results stiffer by increasing the radius of the contact profile. The structural analysis of the arch is identical, in the continuum approximation, to that of a curved beam whose stiffness EI is equal to (3.2).

5.3. Free-form curve

With the same construction of Fig. 7, the cable-stiffened elastica can assume a free form curve. An example is represented in Fig. 11. The number of elements and their axial length are first decided. Then, circles delimiting the voussoir-profiles are designed as previously described. It is pointed out that their centers in general do not lay upon the curve, but move in direction opposite to the curvature. As a consequence, the drilling direction forming the sheaths is no more diametric, but is a chord whose length is inversely proportional to the local curvature. It could be observed that, due to the *a priori* fixed axial length of the elements, the final configuration of the system may not fully comply with the project one. The mismatch, negligible in size, could be partially hidden by shaping the intrados and extrados of the voussoirs according to the offset of the project curve.

5.4. Convertible forms

A primary use of this system is for *movable* structures. When there is no tension at all, the assembly is as loose as a no-tensioned cable, and can be easily rolled and transported. After tensioning the overall stiffness can be tuned up, by simply changing the tensile force T_0 in the cable. The mutual interaction and the shaping of the voussoirs let the system achieve and maintain the desired profile.

Without external loads, the cable stiffened *Elastica* achieves a configuration that depends upon the boundary conditions at the end-points only. Once external loads are applied, the system reaches the equilibrium state of a thin rod with bending stiffness EI under the same load and boundary conditions.

The ends points may act as control points to govern the movement of the whole system, but one could also think of handling more points along the curve. More elaborated movements can be obtained if the cable is fixed at an intermediate voussoir, so that the cable on the left-hand-side and the right-hand-side parts can be pulled at different tensile forces. To illustrate, assume as the starting profile that shown in Fig. 12a, obtained by properly shaping the voussoirs. Suppose that the cable had been fixed at the voussoir corresponding to point C and that, after reaching the configuration of Fig. 12a, the position and the rotation of voussoir C is held by an additional constraint. Then, one may leave unchanged the configuration on left-hand-side of C, but slack the cable on

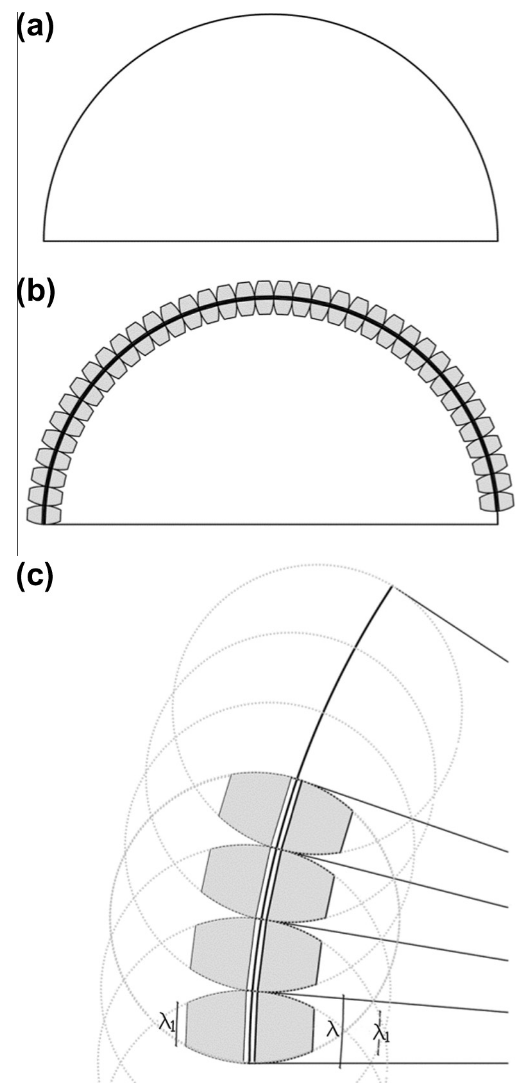

Fig. 10. An arch (a) and the corresponding cable stiffened assembly (b). Being the curvature constant in the reference state, the voussoirs can all be identical one-another (c).

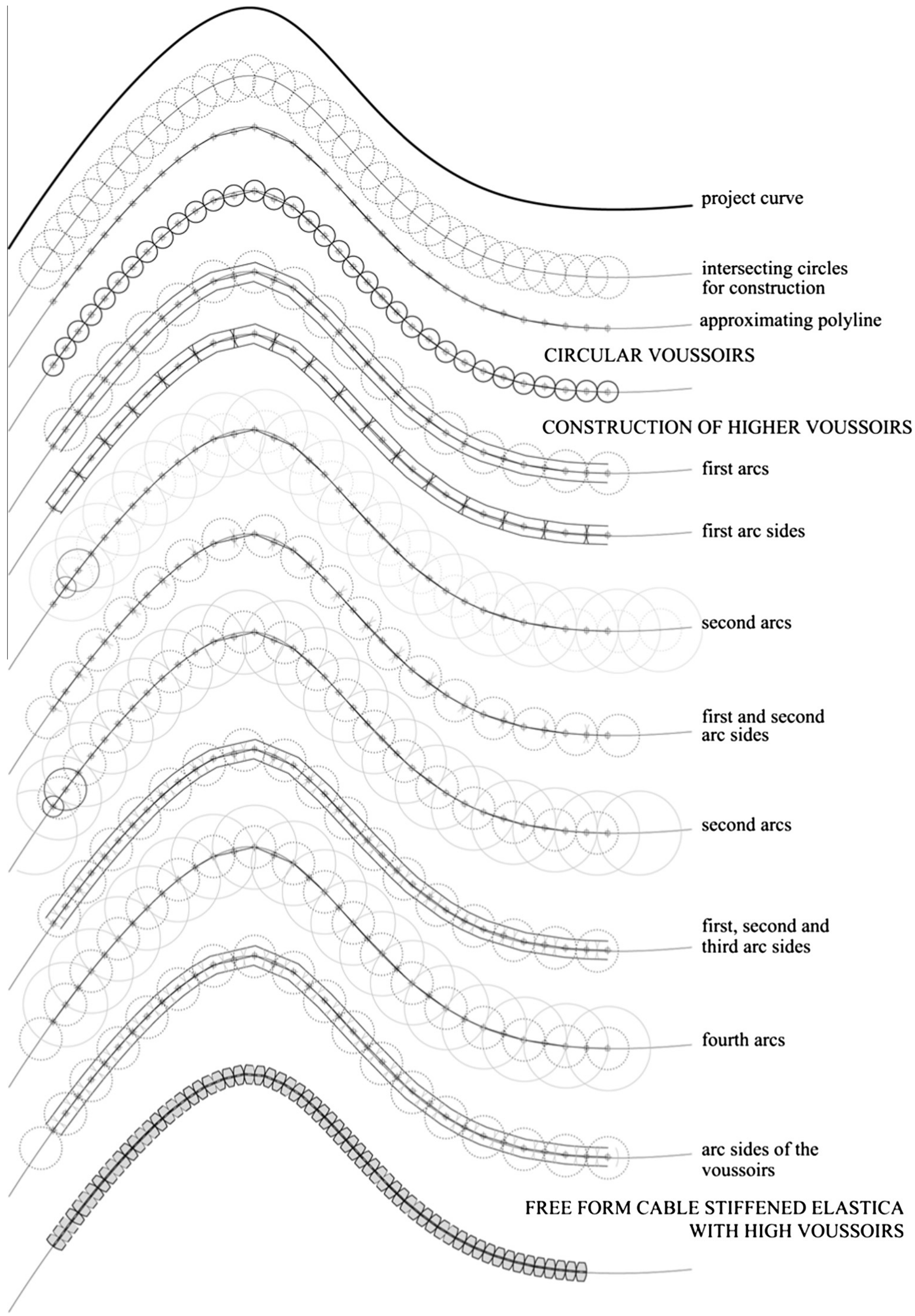

Fig. 11. In descending order. A free form curve, its approximation through a polyline of *n* segments and the corresponding cable-stiffened elastica made of circular voussoirs. The system becomes stiffer if the radius of the contact profiles is increased: they can be arcs cut from tangent circles of increased diameter. Images show the design sequence of four series of construction circles, tangent each other in every series and centered on the vertexes of the polyline. The lower image is the stiffened cable so achieved.

the other side (Fig. 12b). Depending on the length of the cable, the slack part can be constrained at another voussoir D, which acts as a new control point (Fig. 12c) and then tensioned again. One can eventually change the structural profile by moving (translate and rotate) the control points, or by varying the tensile force in each independent part of the cable.

This operation can be iterated by fixing the cable at a few intermediate voussoirs. It is not difficult to think of a device that allows tensioning the cable comprised between any two intermediate constrained voussoirs: devices of this type are already used, e.g., in the segmental construction of post-tensioned prestressed concrete beams. In this way, any desired shape can be virtually obtained and controlled.

5.5. Surfaces

Various elements can be combined to form the skeleton that sustains a covering membrane. The arrangement of the

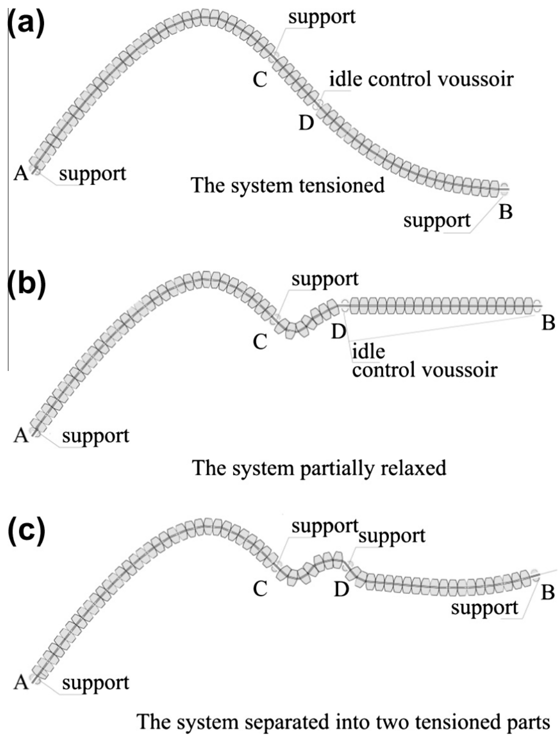

Fig. 12. Separation of the cable stiffened elastica into parts by a control points.

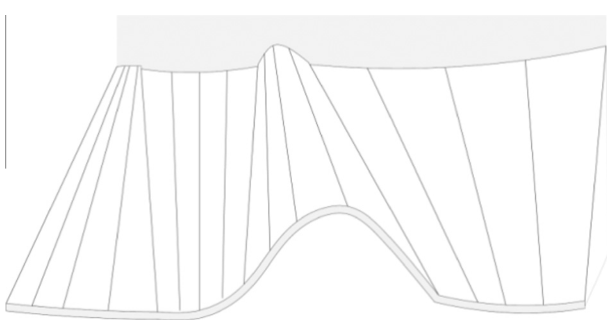

Fig. 13. A series of straight cable-stiffened-*elasticae* can form the skeleton supporting a membrane having the shape of developable surfaces.

sub-structure depends upon the target surface and the type of covering material. In the simplest case, the skeleton is formed by non-intersecting, independent cable-stiffened *elasticae*, the one adjacent to the other. This arrangement can fit, for example, the case of a fixed boundary and a developable surface (Fig. 13). Notice that the bending stiffness of each element may be controlled by simply pulling or releasing the cables, so that an active control of the cable tension can produce a "pulsing" façade or roof, adding the time-variable to the space variations.

The skeleton for surfaces with double curvature can be obtained with intersecting elements, as shown in Fig. 14a and b. If the whole system is foldable, other problems would be related to the folding process, especially the folding directions and the behavior of the covering with respect to the skeleton, but this goes beyond the scope of this work. In any case, the key point is that elements should intersect one another. This is simply achieved using special voussoirs at the intersection points, with crossing holes where the cables can smoothly pass-through with no mutual interaction, of the type shown in Fig. 14c.

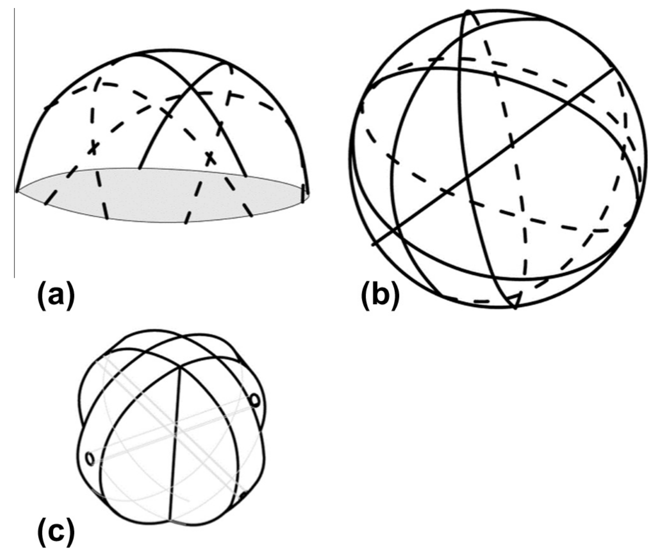

Fig. 14. To obtain complex surfaces, like a geodesic vault (a) or a sphere (b), the cable stiffened *elasticae* should intersect one another. Special voussoirs (c) need to be used at the intersection points.

6. Discussion and conclusions

The mechanical properties of the cable stiffened elastica here presented are governed, as the name suggests, by the same equations of Euler's *elastica*, interpreting the curve assumed by a thin beam under large bending deformation. Remarkably, the bending stiffness of the system is directly proportional to the tensile force in the passing through rope, in a way similar to a Bowden cable. By controlling such force, one can obtain a structure that can change its equilibrium configurations according to design purposes.

The concept can thus be used in a wide range of applications for the design of kinetic structures. From the simple profiles of a beam or of an arch, to complex profiles, the most various curves can be reproduced. Moreover, if the disposition of more elements on a spatial grid is studied, then foldable free-form surfaces can be obtained, for roofs or façades that can change their shape. Since the unique properties of the system allow achieving a proper form of expression, we may refer to it as the *Elastica for Transformable Architecture* (ETA).

The ETA is self-supporting and its stiffness can be tuned by simply pulling or releasing a cable, which is invisible because it runs inside the voussoirs. Moreover, different from other types of mechanisms, the profile of the ETA can also be controlled just by its end points: once the dead and live loads are assigned, the structure can achieve different profiles changing its boundary conditions. The ETA concept thus allows describing a pulsing interior volume, amenable of achieving completely different configurations.

From a construction point of view, perhaps the more interesting feature of the ETA is that it is a mechanism composed of rigid elements, but with no internal joints. This favors the durability and the construction. The voussoirs can be made all of the same shape and differ only in the drilling direction of the sheaths, so that manufacturing is simplified.

How to tension and release the cable has not been explicitly considered here, but could be done with servo-hydraulic control systems, with electric engines or using materials, such as shape-memory alloys, that can vary their length according to the environmental temperature. Using these, one could also conceive microscopic devices for bio-medical applications, where opening is triggered by the body temperature.

However, it cannot be forgotten that the system can be easily dismissed at occurrence. Its packaging is simple and fast because the size of the voussoirs is extremely small and to loosen the structure it is sufficient to release one cable. Therefore, temporary shields for first-aid recovery of people could be conceived, as they could be shipped and readily mounted *in situ* in the case of an exceptional event, such as an earthquake or a flooding. Military applications as well as aerospace engineering applications (antennas, space cabins or containers) can also be considered due to simplicity in unfolding.

Many aspects, however, still need to be defined. To describe an architectural volume, a cover surface is necessary. Although here not investigated, the relating characteristics of the structure can be pointed out. The ETA forms a profile almost continuous and the position of the points connecting the cover can be chosen according to the desired curvature. Both the lower and upper profile of the rigid voussoirs need not to be straight, but may be properly carved to obtain a smooth intrados and extrados. Their relatively little size with respect to the size of the structure allows an easy cover with a membrane, and could facilitate the use of rigid panels.

The proper definition of the state of stress inside the voussoirs due to Hertzian contact, as well as the stress concentration in the cable due to contact with the sheaths, are important issues that necessitate of an accurate mechanical design before bringing the theory into practice. In any case, construction of this device is elementary because it is based upon elementary concepts.

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References

[1] Ishii K. Structural design of retractable roof structures. Southampton: WIT Press; 2000.
[2] Hoberman C. Climate control. Intelligent façades. Leaf Rev 2011;9:52–4.
[3] Calatrava S. Zur Faltbarkeit von Fachwerken. PhD thesis, Zurich: ETH; 1981.
[4] Tzonis A. Santiago Calatrava. The poetics of movement. New York: Universe Publishing; 1999.
[5] Pinero EP. Project for a mobile theatre. Architect Des 1961;12:570.
[6] Motro R. Tensegrity: structural systems for the future. Oxford: Butterworth-Heinemann; 2006.
[7] Chen Y, You Z. On mobile assemblies of Bennett linkages. Proc R Soc A 2008;464(2093):1275–93.
[8] Gan WW, Pellegrino S. Closed-loop deployable structures. In: Proc 44th AIAA/ASME/ASCE/AHS/ASC structures, structural dynamics and materials conference, Norfolk, AIAA, No. 2003-1450; 2003.
[9] Kokawa T. A trial expandable arch. In: Giuliani GC, editor. Proceedings of the IASS international symposium on spatial structures: heritage, present and future, vol. 2, Milano: IASS; 1995. p. 501–10.
[10] Zeigler TR. Collapsible self-supporting structure. US Patent No. 3,968,808; 1976.
[11] Gantes CJ. Deployable structures: analysis and design. Southampton: Wit Press; 2001.
[12] Pellegrino S. Deployable structures. New York: Springer-Verlag; 2001.
[13] Inoue F. A study of movable arch structure and its external panel mechanism by variable geometry truss. In: Siviero E et al., editors. Toward the future looking to the past. Proceedings of international, conference on shell and spatial structures. Venice: IASS; 2007. p. 69–70.
[14] Akgun Y, Gantes CJ, Sobek W, Kalochairetis KE, Korkmaz K. A novel adaptive spatial scissor-hinge structural mechanism for convertible roofs. Eng Struct 2011;33(4):1365–76.
[15] Valcárcel J, Escrig F, Martín E, Vázquez JA. Analysis of expandable domes of squared modulus with self-folding roofing plates. In: Giuliani GC, editor. Proceedings of the IASS international symposium on spatial structures: heritage, present and future, vol. 2. Milano: IASS; 1995. p. 551–8.
[16] Love AEH. A treatise on the mathematical theory of elasticity. Cambridge: The University Press; 1906.
[17] Jerome JW. Smooth interpolating curves of prescribed length and minimum curvature. Proc Am Math Soc 1975;51(1):62–6.
[18] Bigoni D. Nonlinear solid mechanics. Bifurcation theory and material instability. Cambridge: Cambridge University Press; 2012.
[19] Edwards J. Exact equations of the nonlinear spline. ACM Trans Math Softw 1992;18:174–92.
[20] Holladay J. A smoothest curve approximation. Math Tables Aids Comput 1957;11:233–43.