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Hamiltonian dynamics of N vortices in concentric annular region

Oktay K Pashaev and Oguz Yilmaz

Izmir Institute of Technology, Department of Mathematics, Izmir, Turkey, 35430

E-mail: oguzyilmaz@iyte.edu.tr

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Abstract
The problem of N vortex dynamics in annular domain is considered. The region is canonical one and allows by conformal mapping apply results to an arbitrary position of two cylinders in the plane. Using previous solution, obtained by the authors in terms of the q-elementary functions [1] we now concentrate on the Hamiltonian formulation of the problem. The integrability of the problem of two vortices in the annular domain according to Liouville has been proved by using canonical transformations. Different motion characteristics depending on initial conditions are studied.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Point vortex motion has been an interesting topic for hydrodynamicists for over a century. The fact that equations of motion may be derived from a Hamiltonian makes the problem more interesting. This was already known to Helmholtz [2] and Kirchhoff [3]. The questions of integrability and non-integrability naturally arises when one considers the Hamiltonian systems. We know that the problem of three vortices in the plane and special symmetric arrangements of vortices are integrable [4]. This is related with existence of integrals of motion in the plane: two translations and one rotation. However, the three vortex problem becomes non-integrable in the bounded domain due to the lack of sufficient number of integrals of motion, even for special symmetric domains as the disk and the annular domain. Moreover, by the method of images, the boundary can be replaced by the couple of vortex images in the first case, and by infinite set of images in the second one. This also requires additional number of integrals of motion.

The previous study of the authors [1] deals with the problem of finding complex velocity potential arising from the arbitrary number of vortices placed in an annular region. The problem was solved by using the method of images for the annular domain, first discussed by Poincaré [5], and exactly formulated in terms of q-calculus [1]. Solution in terms of elliptic functions for one vortex problem was given by Johnson and McDonald [6] and extension to multiply connected domains by Crowdy and Marshall (see [7] and [8]). If we compare these solutions with q-elementary functions approach, then the latter shows very fast convergence and is suitable for numerical calculations.

The problem of two vortices inside and outside of circular domains were studied in [9]. In this paper we solve the two-vortex problem in the annular domain. First we briefly remind the main steps of constructing solution in terms of q-elementary functions, for the one vortex problem, and for the special N-vortex polygon problem. In particular, we emphasize the divergency problem for infinite sums of vortex images arising in this approach. Then we study the Hamiltonian structure and the integrability for $N = 2$ vortices in the annular domain. In this paper, the problem is exactly solved by canonical transformations and different motion characteristics of vortices depending on initial conditions are studied.

2. *N* vortices in the annular domain

We briefly repeat main steps constructing the solution [1] for the convenience of the reader. In derivation, here we will emphasize especially the divergency problem for infinite sum of vortex images, which we did not discuss sufficiently in [1] .

Before we discuss the details of vortices in concentric annular domains, we shall point out that the case of eccentric circles can be reduced to the present case of concentric circles by a linear transformation. Let us consider two eccentric cylinders with the larger circle containing the smaller one in its interior. Without loss of generality assume that the radius of the larger circle is unity and that its center is at the origin. The underlying idea behind the linear fractional transformation is to find a point, a, in the interior of the small disc such that the symmetric point of a with respect to both cylinders is the same. Hence, the mapping

$$k\frac{z-a}{z-1/\bar{a}},$$

where k is a complex constant, will map the unique point a to the origin, its symmetric point to infinity and eccentric circles to concentric circles. The transformation is unique except for a multiplicative constant, but the ratio of radii of the transformed circles is always unique. This mapping was given as an exercise in the classical monographs of Ahlfors [10] and Henrici [11].

2.1. *Complex velocity and vortex images*

The problem of N point vortices with strengths $\kappa_1, \ldots, \kappa_N$ at positions $z_1, \ldots, z_N$, respectively, in the annular domain $D : \{r_1 \leqslant |z| \leqslant r_2\}$ is considered. The region is bounded by two concentric circles: $C_1 : z\bar{z} = r_1^2$ and $C_2 : z\bar{z} = r_2^2$. The complex velocity is given by the Laurent series

$$\bar{V}(z) = \sum_{k=1}^{N} \frac{\mathrm{i}\kappa_k}{z - z_k} + \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} \frac{b_{n+1}}{z^{n+1}} \tag{1}$$

which has to satisfy the boundary conditions at both cylinders,

$$[z\,\bar{V}(z) + \bar{z}\,V(\bar{z})]|_{C_k} = 0, \qquad k = 1, 2. \tag{2}$$

We obtain the general solution in the form [1]

$$\bar{V}(z) = \sum_{k=1}^{N} \mathrm{i}\kappa_k \frac{1}{z - z_k} + \frac{\mathrm{i}\kappa}{z} \sum_{k=1}^{N} \frac{\mathrm{i}\kappa_k}{z(q-1)} \left[Ln_q \left(1 - \frac{z}{z_k}\right) - Ln_q \left(1 - \frac{z\bar{z}_k}{r_1^2}\right) \right.$$
$$\left. + Ln_q \left(1 - \frac{r_2^2}{z\bar{z}_k}\right) - Ln_q \left(1 - \frac{z_k}{z}\right) \right], \tag{3}$$

where

$$Ln_q(1+z) \equiv (q-1)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^n}{q^n-1} = (q-1)\sum_{n=1}^{\infty}\frac{z}{q^n+z}, \tag{4}$$

$q \equiv r_2^2/r_1^2 > 1$, $0 < |z| < q$. Here $b_1 = \mathrm{i}\kappa$ is an arbitrary parameter, determining point vortex at the origin with strength κ which cannot be fixed by the boundary conditions. To fix arbitrariness of this solution, we have to impose an additional condition. We can chose it in the form

$$\oint_{C_1} \bar{V}(z)\,\mathrm{d}z = 0. \tag{5}$$

This condition can be justified by correctness of the limiting procedure $r_1 \to 0$, $r_2 \to \infty$ to the planar problem, so that no singularity at the origin should arise (see [12, 13]).

Expanding the q-logarithmic functions we have

$$\bar{V}(z) = \sum_{k=1}^{N}\frac{\mathrm{i}\kappa_k}{z-z_k} + \frac{\mathrm{i}\kappa}{z} + \sum_{k=1}^{N}\sum_{n=1}^{\infty}\frac{\mathrm{i}\kappa_k}{z-z_kq^n} - \sum_{k=1}^{N}\sum_{n=1}^{\infty}\frac{\mathrm{i}\kappa_k}{z-\frac{r_1^2}{\bar{z}_k}q^n}$$
$$+\sum_{k=1}^{N} i\kappa_k \sum_{n=1}^{\infty}\left[\frac{-\frac{r_2^2}{\bar{z}_k}q^{-n}}{z(z-\frac{r_2^2}{\bar{z}_k}q^{-n})}\right] - \sum_{k=1}^{N} i\kappa_k \sum_{n=1}^{\infty}\left[\frac{-z_kq^{-n}}{z(z-z_kq^{-n})}\right]. \tag{6}$$

We estimate contribution of every term in $\bar{V}(z)$ to circulation (5) around C_1. In the first sum all vortices are outside of C_1. The same is true for the second and the third sums, with images laying outside of C_1. For the fourth sum by the residues theorem we have

$$\sum_{k=1}^{N}\mathrm{i}\kappa_k\sum_{n=1}^{\infty} -\frac{r_2^2}{\bar{z}_k}q^{-n}\oint_{C_1}\frac{\mathrm{d}z}{z\big(z-\frac{r_2^2}{\bar{z}_k}q^{-n}\big)} = 0 \tag{7}$$

as well as for the last sum

$$\sum_{k=1}^{N}\mathrm{i}\kappa_k\sum_{n=1}^{\infty} z_kq^{-n}\oint_{C_1}\frac{\mathrm{d}z}{z(z-z_kq^{-n})} = 0. \tag{8}$$

It implies that in expressions (4) and (7) we can set $\kappa = 0$. However, if in the last two sums we expand terms representing vortex images by simple fractions as

$$\bar{V}(z) = \sum_{k=1}^{N}\frac{\mathrm{i}\kappa_k}{z-z_k} + \frac{\mathrm{i}\kappa}{z} + \sum_{k=1}^{N}\sum_{n=1}^{\infty}\frac{\mathrm{i}\kappa_k}{z-z_kq^n} - \sum_{k=1}^{N}\sum_{n=1}^{\infty}\frac{\mathrm{i}\kappa_k}{z-q^n\frac{r_1^2}{\bar{z}_k}}$$
$$+\sum_{k=1}^{N}\sum_{n=1}^{\infty}\left[\frac{\mathrm{i}\kappa_k}{z} - \frac{\mathrm{i}\kappa_k}{z-q^{-n}\frac{r_2^2}{\bar{z}_k}}\right] - \sum_{k=1}^{N}\sum_{n=1}^{\infty}\left[\frac{\mathrm{i}\kappa_k}{z} - \frac{\mathrm{i}\kappa_k}{z-q^{-n}z_k}\right] \tag{9}$$

we can think to cancel positive and negative terms for infinite set of images at the origin. It turns out that due to divergency of those sums we cannot skip the term with κ. Indeed if we like to rearrange terms in these series we keep term κ up to the end and then fix it to satisfy condition (5). Then we have

$$\bar{V}(z) = \sum_{k=1}^{N}\frac{\mathrm{i}\kappa_k}{z-z_k} + \frac{\mathrm{i}\kappa}{z} + \sum_{k=1}^{N}\left[\sum_{n=1}^{\infty}\frac{\mathrm{i}\kappa_k}{z-z_kq^n} + \sum_{n=1}^{\infty}\frac{\mathrm{i}\kappa_k}{z-z_kq^{-n}}\right]$$
$$-\sum_{k=1}^{N}\left[\sum_{n=0}^{\infty}\frac{\mathrm{i}\kappa_k}{z-\frac{r_1^2}{\bar{z}_k}q^{-n}} + \sum_{n=0}^{\infty}\frac{\mathrm{i}\kappa_k}{z-\frac{r_2^2}{\bar{z}_k}q^{n}}\right]. \tag{10}$$

The set of vortex images is determined completely by simple pole singularities of the q-logarithmic function. By identity $r_2^2/q^n = r_1^2/q^{n-1}$, in the above representation (10) we can combine sums so that

$$\bar{V}(z) = \sum_{k=1}^{N}\left[\sum_{n=-\infty}^{\infty}\frac{\mathrm{i}\kappa_k}{z - z_k q^n}\right] + \frac{\mathrm{i}\kappa}{z} - \sum_{k=1}^{N}\left[\sum_{n=0}^{\infty}\frac{\mathrm{i}\kappa_k}{z - \frac{r_1^2}{\bar{z}_k}q^{-n}} + \sum_{n=1}^{\infty}\frac{\mathrm{i}\kappa_k}{z - \frac{r_1^2}{\bar{z}_k}q^{n}}\right] \tag{11}$$

$$= \sum_{k=1}^{N}\mathrm{i}\kappa_k\sum_{n=-\infty}^{\infty}\left[\frac{1}{z - z_k q^n} - \frac{1}{z - \frac{r_1^2}{\bar{z}_k}q^n}\right] + \frac{\mathrm{i}\kappa}{z}. \tag{12}$$

For circulation we have

$$\oint_{C_1}\bar{V}(z)\,\mathrm{d}z = \sum_{k=1}^{N}\mathrm{i}\kappa_k\sum_{n=-\infty}^{\infty}\left[\oint\frac{\mathrm{d}z}{z - z_k q^n} - \oint\frac{\mathrm{d}z}{z - \frac{r_1^2}{\bar{z}_k}q^n}\right] + \mathrm{i}\kappa\oint\frac{\mathrm{d}z}{z} \tag{13}$$

or collecting only images inside C_1

$$\oint_{C_1}\bar{V}(z)\,\mathrm{d}z = \sum_{k=1}^{N}\mathrm{i}\kappa_k\left[\sum_{n=1}^{\infty}\oint\frac{\mathrm{d}z}{z - z_k q^{-n}} - \sum_{n=1}^{\infty}\oint\frac{\mathrm{d}z}{z - \frac{r_1^2}{\bar{z}_k}q^{-n}}\right]$$

$$- \sum_{k=1}^{N}\oint\frac{\mathrm{d}z}{z - \frac{r_1^2}{\bar{z}_k}} + \mathrm{i}\kappa\oint\frac{\mathrm{d}z}{z} = 0. \tag{14}$$

Residues in the first two sums cancel each other term by term. Then we fix constant κ as the total circulation

$$\kappa = \sum_{k=1}^{N}\kappa_k. \tag{15}$$

As a result we have next image representation

$$\bar{V}(z) = \sum_{k=1}^{N}\mathrm{i}\kappa_k\sum_{n=-\infty}^{\infty}\left[\frac{1}{z - z_k q^n} - \frac{1}{z - \frac{r_1^2}{\bar{z}_k}q^n}\right] + \frac{\mathrm{i}}{z}\sum_{k=1}^{N}\kappa_k. \tag{16}$$

In the limit of the vortices outside of a cylinder with radius r_1 = constant, $q \to \infty$ and $r_2 \to \infty$, it gives result (see the appendix)

$$\bar{V}(z) = \sum_{k=1}^{N}\mathrm{i}\kappa_k\left[\frac{1}{z - z_k} - \frac{1}{z - \frac{r_1^2}{\bar{z}_k}}\right] + \frac{\mathrm{i}}{z}\sum_{k=1}^{N}\kappa_k. \tag{17}$$

coinciding with the circle theorem [13].

If in (11) instead of r_1 we use expression for r_2 then following the same procedure we find that in this case we should fix $\kappa = 0$. So that we have alternative expression

$$\bar{V}(z) = \sum_{k=1}^{N}\mathrm{i}\kappa_k\sum_{n=-\infty}^{\infty}\left[\frac{1}{z - z_k q^n} - \frac{1}{z - \frac{r_2^2}{\bar{z}_k}q^n}\right]. \tag{18}$$

In the limit of the vortices inside of a cylinder with radius r_2 = constant, $q \to \infty$ and $r_1 \to 0$, it gives result

$$\bar{V}(z) = \sum_{k=1}^{N}\mathrm{i}\kappa_k\left[\frac{1}{z - z_k} - \frac{1}{z - \frac{r_2^2}{\bar{z}_k}}\right] \tag{19}$$

which also coincides with the circle theorem.

2.2. Complex potential and stream function

According to the relation

$$\bar{V}(z) = F'(z) \tag{20}$$

the complex potential of flow (16) is found in the form

$$F(z) = \sum_{k=1}^{N} \mathrm{i}\kappa_k \sum_{n=-\infty}^{\infty} \left[\ln(z - z_k q^n) - \ln\left(z - \frac{r_1^2}{\bar{z}_k} q^n\right)\right] + \mathrm{i}\sum_{k=1}^{N} \kappa_k \ln z, \tag{21}$$

or compactly

$$F(z) = \sum_{k=1}^{N} \mathrm{i}\kappa_k \sum_{n=-\infty}^{\infty} \left[\ln \frac{z - z_k q^n}{z - \frac{r_1^2}{\bar{z}_k} q^n}\right] + \mathrm{i}\sum_{k=1}^{N} \kappa_k \ln z. \tag{22}$$

and

$$F(z) = \sum_{k=1}^{N} \mathrm{i}\kappa_k \ln\left(z \prod_{n=-\infty}^{\infty} \frac{z - z_k q^n}{z - \frac{r_1^2}{\bar{z}_k} q^n}\right). \tag{23}$$

In the second form we have

$$F(z) = \sum_{k=1}^{N} \mathrm{i}\kappa_k \sum_{n=-\infty}^{\infty} \left[\ln \frac{z - z_k q^n}{z - \frac{r_2^2}{\bar{z}_k} q^n}\right]. \tag{24}$$

The stream function $\psi = (F - \bar{F})/2\mathrm{i}$ in terms of vortex images is

$$\psi(z) = \sum_{k=1}^{N} \kappa_k \sum_{n=-\infty}^{\infty} \ln\left|\frac{z - z_k q^n}{z - \frac{r_1^2}{\bar{z}_k} q^n}\right| + \sum_{k=1}^{N} \kappa_k \ln|z| \tag{25}$$

in the first form and

$$\psi(z) = \sum_{k=1}^{N} \kappa_k \sum_{n=-\infty}^{\infty} \ln\left|\frac{z - z_k q^n}{z - \frac{r_2^2}{\bar{z}_k} q^n}\right| \tag{26}$$

in the second form.

It implies the Hamiltonian [1]

$$\begin{aligned} H = &-\frac{1}{4\pi} \sum_{i,j=1 (i\neq j)}^{N} \Gamma_i \Gamma_j \log|z_i - z_j| - \frac{1}{4\pi} \sum_{i,j=1}^{N} \sum_{n=1}^{\infty} \Gamma_i \Gamma_j \log[|z_i - z_j q^n||z_i - z_j q^{-n}|] \\ &+ \frac{1}{4\pi} \sum_{i,j=1}^{N} \sum_{n=-\infty}^{\infty} \Gamma_i \Gamma_j \log|z_i \bar{z}_j - r_1^2 q^n|. \end{aligned} \tag{27}$$

2.3. Equations of motion

In order to find velocity of any vortex say, at z_k, we must subtract its own effect. From Hamiltonian function (27) we get the following system of equations of motion:

$$\dot{\bar{z}}_k = +\frac{1}{2\pi\mathrm{i}} \sum_{j=1 (j\neq k)}^{N} \frac{\Gamma_j}{z_k - z_j} + \frac{1}{2\pi\mathrm{i}} \sum_{j=1}^{N} \sum_{n=\pm 1}^{\pm\infty} \frac{\Gamma_j}{z_k - z_j q^n} - \frac{1}{2\pi\mathrm{i}} \sum_{j=1}^{N} \sum_{n=-\infty}^{\infty} \frac{\Gamma_j}{z_k - \frac{r_1^2}{\bar{z}_j} q^n}, \tag{28}$$

$(k = 1, \ldots, N)$.

2.4. Motion of a point vortex in annular domain

As a first application of the above equations we consider the motion of a single vortex in the annular domain [1]:

$$\dot{z}_0 = \frac{\mathrm{i}\kappa}{\bar{z}_0(q-1)}\left[\mathrm{Ln}_q\left(1-\frac{|z_0|^2}{r_1^2}\right)-\mathrm{Ln}_q\left(1-\frac{r_2^2}{|z_0|^2}\right)\right]. \tag{29}$$

It implies that distance of the vortex from the origin is a constant of motion. Substituting $z_0(t) = |z_0|\,\mathrm{e}^{\mathrm{i}\varphi(t)}$ to (29) we get

$$\varphi(t) = \omega t + \varphi_0 \tag{30}$$

where frequency ω is a constant, depending on modulus $|z_0|$,

$$\omega = \frac{\kappa}{|z_0|^2(q-1)}\left[Ln_q\left(1-\frac{|z_0|^2}{r_1^2}\right)-Ln_q\left(1-\frac{r_2^2}{|z_0|^2}\right)\right]. \tag{31}$$

So we find that the vortex uniformly rotates around the origin,

$$z_0(t) = |z_0|\,\mathrm{e}^{\mathrm{i}\omega t+i\varphi_0} = z_0(0)\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{32}$$

with rotation frequency (31) depending on strength and initial position of the vortex, and the geometry of annular domain.

3. N vortex polygons in annular domain

As is well known the system of N vortices in the plane, in addition to the energy, admits three integrals of motion, corresponding to the pair of translations and one rotation [9, 14]. However for our problem in the annular domain only energy and the angular momentum, corresponding to rotation of the system, survives, ($\Gamma_k = -2\pi\kappa_k$),

$$I = \sum_{k=1}^{N}\Gamma_k z_k \bar{z}_k. \tag{33}$$

Conservation of angular momentum implies existence of dynamical configuration of vortices, rotating with fixed angular velocity. We restrict consideration with the simplest case when all circulations of vortices are equal each other $\Gamma_k = \Gamma$, $k = 1, \ldots, N$. We suppose that all vortices are located at the same distance r from the center, ($r_1 < r < r_2$), at vertices of the regular N polygon, described by roots of the unity. Then we have the solution

$$z_k(t) = r\,\mathrm{e}^{\mathrm{i}\omega t+\mathrm{i}\frac{2\pi}{N}k}, \qquad (k = 1, \ldots, N) \tag{34}$$

with rotation frequency

$$\omega = \frac{\Gamma}{2\pi r^2}\left\{\frac{N-1}{2}+\frac{1}{q-1}\sum_{j=1}^{N}\left[\mathrm{Ln}_q\left(1-\frac{r_2^2}{r^2}\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{N}j}\right)-\mathrm{Ln}_q\left(1-\frac{r^2}{r_1^2}\,\mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}j}\right)\right]\right\}. \tag{35}$$

For the single vortex case, when $N = 1$ frequency (35) reduces to the one obtained in section 2, (31). At the geometric mean distance the last two terms in (35) cancel each other and we obtain

$$\omega = \frac{\Gamma(N-1)}{4\pi r_1 r_2}. \tag{36}$$

So at this distance the vortex polygon is rotating with the frequency, independent of parameter q, and hence independent of the ratio r_2/r_1.

4. Two vortices in the annular region

It is clear that the system consisting of two vortices in the annular region is integrable since there are two integrals of motion, namely, the Hamiltonian and the angular momentum. But the complete integration is very difficult due to the infinite sums involved in the Kirchoff Routh function. Instead, we shall use canonical transformations to reduce the number of degrees of variables to two and then plot the constant Hamiltonian contours.

First, by using (27) we will put the motion equations in canonical form. To do that we transform the variables from (x_i, y_i) to (q_i, p_i) by

$$q_k = \sqrt{|\Gamma_k|}\mathrm{Sgn}(\Gamma_k)x_k, \qquad p_k = \sqrt{|\Gamma_k|}y_k, \quad k = 1, 2. \tag{37}$$

So that we get the motion equations in standard canonical form.

$$\dot{q}_k = \frac{\partial H'}{\partial p_k}, \qquad \dot{p}_k = -\frac{\partial H'}{\partial q_k}, \qquad k = 1, 2 \tag{38}$$

where

$$\begin{aligned}
8\pi H' = &-\Gamma_1\Gamma_2 \log\left[\frac{q_1^2+p_1^2}{|\Gamma_1|} + \frac{q_2^2+p_2^2}{|\Gamma_2|} - \frac{2}{\sqrt{|\Gamma_1\Gamma_2|}}(q_1q_2s_1s_2 + p_1p_2)\right] \\
&- \Gamma_1^2 \sum_{n=1}^{\infty} \log\left[\frac{q_1^2+p_1^2}{|\Gamma_1|}|1-q^n||1-q^{-n}|\right] \\
&- \Gamma_2^2 \sum_{n=1}^{\infty} \log\left[\frac{q_2^2+p_2^2}{|\Gamma_2|}|1-q^n||1-q^{-n}|\right] \\
&- \Gamma_1\Gamma_2 \sum_{n=\pm 1}^{\pm\infty} \log\left[\frac{q_1^2+p_1^2}{|\Gamma_1|} + \frac{q_2^2+p_2^2}{|\Gamma_2|}q^{2n} - \frac{q^n}{\sqrt{|\Gamma_1\Gamma_2|}}(q_1q_2s_1s_2 + p_1p_2)\right] \\
&+ 2\Gamma_1\Gamma_2 \sum_{n=-\infty}^{+\infty} \log\left[\frac{\left(q_1^2+p_1^2\right)\left(q_2^2+p_2^2\right)}{|\Gamma_1\Gamma_2|} - \frac{2q^n r_1^2}{\sqrt{|\Gamma_1\Gamma_2|}}(q_1q_2s_1s_2 + p_1p_2) + r_1^4 q^{2n}\right] \\
&+ 2\Gamma_1^2 \sum_{n=-\infty}^{+\infty} \left|\frac{q_1^2+p_1^2}{|\Gamma_1|} - r_1^2 q^n\right| + 2\Gamma_2^2 \sum_{n=-\infty}^{+\infty} \left|\frac{q_2^2+p_2^2}{|\Gamma_2|} - r_1^2 q^n\right|
\end{aligned} \tag{39}$$

where $s_1 = \mathrm{sgn}(\Gamma_1)$ and $s_2 = \mathrm{sgn}(\Gamma_2)$. Next we use the polar (canonical) coordinates (R_i, P_i)

$$q_1^2 + p_1^2 = 2R_1, \qquad q_2^2 + p_2^2 = 2R_2 \tag{40}$$

$$\arctan\frac{p_1}{q_1} = P_1, \qquad \arctan\frac{p_2}{q_2} = P_2 \tag{41}$$

and we get the new Hamiltonian and the motion equations

$$\begin{aligned}
8\pi H'' = &-\Gamma_1\Gamma_2 \log\left[\frac{2R_1}{|\Gamma_1|} + \frac{2R_2}{|\Gamma_2|} - \frac{4\sqrt{R_1R_2}}{\sqrt{|\Gamma_1\Gamma_2|}}\cos(P_1 - P_2s_1s_2)s_1s_2\right] \\
&- \Gamma_1^2 \sum_{n=1}^{\infty} \log\left[\frac{2R_1}{|\Gamma_1|}|1-q^n||1-q^{-n}|\right] - \Gamma_2^2 \sum_{n=1}^{\infty} \log\left[\frac{2R_2}{|\Gamma_2|}|1-q^n||1-q^{-n}|\right] \\
&- \Gamma_1\Gamma_2 \sum_{n=\pm 1}^{\pm\infty} \log\left[\frac{2R_1}{|\Gamma_1|} + \frac{2R_2}{|\Gamma_2|}q^{2n} - \frac{2q^n\sqrt{R_1R_2}}{\sqrt{|\Gamma_1\Gamma_2|}}\cos(P_1 - P_2s_1s_2)s_1s_2\right] \\
&+ 2\Gamma_1\Gamma_2 \sum_{n=-\infty}^{+\infty} \log\left[\frac{4R_1R_2}{|\Gamma_1\Gamma_2|} - \frac{4q^n r_1^2\sqrt{R_1R_2}}{\sqrt{|\Gamma_1\Gamma_2|}}\cos(P_1 - P_2s_1s_2)s_1s_2 + r_1^4 q^{2n}\right]
\end{aligned}$$

$$+2\Gamma_1^2 \sum_{n=-\infty}^{+\infty} \log |\frac{2R_1}{|\Gamma_1|} - r_1^2 q^n| + 2\Gamma_2^2 \sum_{n=-\infty}^{+\infty} \log |\frac{2R_2}{|\Gamma_2|} - r_1^2 q^n|,$$

$$\frac{\partial H''}{\partial P_k} = \dot{R}_k, \qquad \frac{\partial H''}{\partial R_k} = -\dot{P}_k, \qquad k = 1, 2. \tag{42}$$

The form of (42) suggests a new canonical transformation by which a cyclic coordinate is obtained. One possible transformation, but not the only one, is

$$Q_1 = P_1 - P_2 s_1 s_2, \qquad Q_2 = s_1 s_2 P_2 \tag{43}$$

$$G_1 = R_1, \qquad G_2 = R_1 + s_1 s_2 R_2. \tag{44}$$

We see that the new Hamiltonian (64) is cyclic in Q_2 and its conjugate G_2, which is a constant multiple of the angular momentum, and is a constant,

$$\begin{aligned}
8\pi K = &-\Gamma_1\Gamma_2 \log\left[\frac{2G_1}{|\Gamma_1|} + \frac{2(G_2 - G_1)s_1s_2}{|\Gamma_2|} - \frac{4\sqrt{G_1(G_2 - G_1)s_1s_2}}{\sqrt{|\Gamma_1\Gamma_2|}}\cos(Q_1)s_1s_2\right]\\
&- \Gamma_1^2 \sum_{n=1}^{\infty} \log\left[\frac{2G_1}{|\Gamma_1|}|1 - q^n||1 - q^{-n}|\right]\\
&- \Gamma_2^2 \sum_{n=1}^{\infty} \log\left[\frac{2(G_2 - G_1)s_1s_2}{|\Gamma_2|}|1 - q^n||1 - q^{-n}|\right]\\
&- \Gamma_1\Gamma_2 \sum_{n=\pm 1}^{\pm\infty} \log\left[\frac{2G_1}{|\Gamma_1|} + \frac{2(G_2 - G_1)s_1s_2}{|\Gamma_2|}q^{2n}\right.\\
&\left.- \frac{2q^n\sqrt{G_1(G_2 - G_1)s_1s_2}}{\sqrt{|\Gamma_1\Gamma_2|}}\cos(Q_1)s_1s_2\right]\\
&+ 2\Gamma_1\Gamma_2 \sum_{n=-\infty}^{+\infty} \log\left[\frac{4G_1(G_2 - G_1)s_1s_2}{|\Gamma_1\Gamma_2|}\right.\\
&\left.- \frac{4q^n r_1^2\sqrt{G_1(G_2 - G_1)s_1s_2}}{\sqrt{|\Gamma_1\Gamma_2|}}\cos(Q_1)s_1s_2 + r_1^4 q^{2n}\right]\\
&+ 2\Gamma_1^2 \sum_{n=-\infty}^{+\infty} \log|\frac{2G_1}{|\Gamma_1|} - r_1^2 q^n| + 2\Gamma_2^2 \sum_{n=-\infty}^{+\infty} \log|\frac{2(G_2 - G_1)s_1s_2}{|\Gamma_2|} - r_1^2 q^n|.
\end{aligned} \tag{45}$$

Indeed, if the angular momentum is defined as

$$\begin{aligned}
I &= \Gamma_1\left(x_1^2 + y_1^2\right) + \Gamma_2\left(x_2^2 + y_2^2\right)\\
&= \Gamma_1\left(\frac{q_1^2}{|\Gamma_1|} + \frac{p_1^2}{|\Gamma_1|}\right) + \Gamma_2\left(\frac{q_2^2}{|\Gamma_2|} + \frac{p_2^2}{|\Gamma_2|}\right)\\
&= 2s_1R_1 + 2s_2R_2\\
&= 2s_1G_2
\end{aligned} \tag{46}$$

we get that G_2 is a constant multiple of angular momentum. Finally, the motion equations become

$$\frac{\partial K}{\partial Q_1} = \dot{G}_1, \qquad \frac{\partial K}{\partial G_1} = -\dot{Q}_1, \qquad \frac{\partial K}{\partial G_2} = -\dot{Q}_2, \qquad \frac{\partial K}{\partial Q_2} = 0.$$

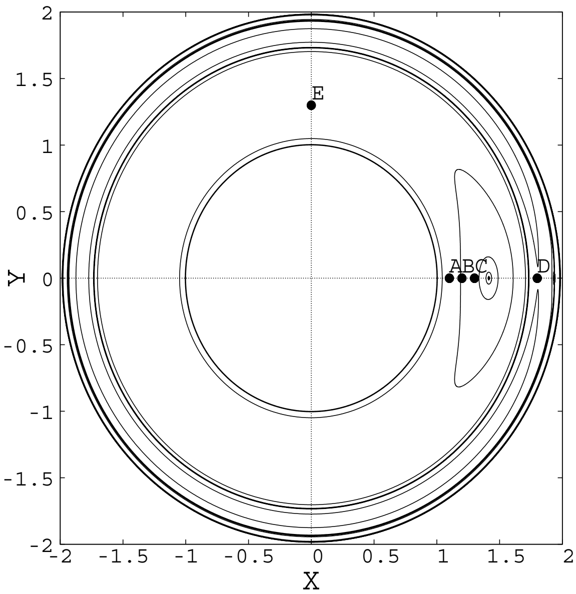

Figure 1. The Level curves for the Hamiltonian in XY coordinates. Vortices have unit strength and there is no net circulation about the cylinder; $\kappa_0 = \kappa_1 + \kappa_2$. The coordinates of the points are: $A = (1.1, 0)$, $B = (1.2, 0)$, $C = (1.3, 0)$, $D = (1.8, 0)$, and $E = (0, 1.3)$.

4.1. Vortices with equal strength

As a numerical example, we consider the case when $\Gamma_1 = \Gamma_2 = 1$ ($s_1 = s_2 = 1$) and $r_1 = 1, r_2 = 2$. Note that the only effect of the radii of the annular region is to change the ranges for the parameters and the variables of the system. We shall plot the constant H contours to demonstrate the integrability of the system and discuss the characteristics of this case. The variables are Q_1 and G_1 and the parameter is G_2. By using the geometry of the annular region, we determine the ranges for G_2, G_1 and Q_1:

$$1 < G_2 < 4, \qquad \tfrac{1}{2} < G_1 < 2, \qquad -2\pi < Q_1 < 2\pi. \tag{47}$$

In figure 1, we fix $G_2 = 2$ and observe that the system is integrable. The level curves of the Hamiltonian are shown in figure 1 in a coordinate system (X, Y), where $X = \sqrt{2R_1}\cos(P_1 - P_2)$ and $Y = \sqrt{2R_1}\sin(P_1 - P_2)$. That implies we are measuring the distance of the first vortex to the origin in the coordinate system (X, Y) that rotates with the second vortex. In fact, figure 1 corresponds to the Poincaré map in the system (p, q) where the angular momentum and the Hamiltonian are first integrals of motion. So the trajectories are plotted when the angular momentum and P_2 are fixed. Inner and outer circles in figure 1 correspond to the boundaries of the annular region. There is an elliptical singular point in figure 2 at the point $(\sqrt{2}, 0)$. We must pay attention to the fact that using figure 1 initial positions of vortices can not be chosen arbitrarily for angular momentum, G_2, is fixed ($G_2 = 2$ for figure 1). Hence, it becomes clear that when one of the vortices is placed at the point $(\sqrt{2}, 0)$, the other vortex must be placed at the same point.

Since the coordinates in figure 1 does not correspond to the physical ones we choose five different points in figure 1 and calculate the trajectories of vortices corresponding to those points. In that way, we will know what kind of vortex behavior is to be expected corresponding to different regions in figure 1. For example, if one of the vortices start off close to one of the

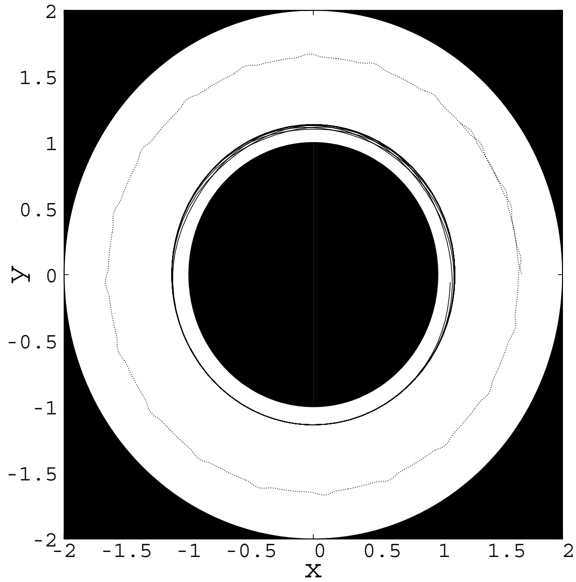

Figure 2. Motion of vortices of unit strength around the cylinder at the origin with initial vortex starting positions being (1.1, 0), (1.670 329, 0) and $\kappa_0 = \kappa_1 + \kappa_2$. Continuous and dashed lines denote the trajectories of vortices. The initial points correspond to the point A in figure 1.

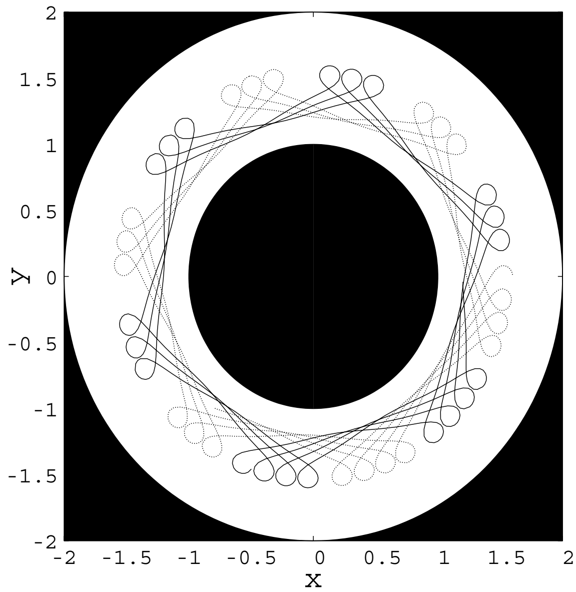

Figure 3. Motion of vortices of unit strength around the cylinder at the origin with initial vortex starting positions being (1.2, 0), (1.6, 0) and $\kappa_0 = \kappa_1 + \kappa_2$. Continuous and dashed lines denote the trajectories of vortices. The initial points correspond to the point B in figure 1.

cylinders corresponding to the points A and D in figure 1, then vortices rotate in the region without affecting each other (figures 2 and 5). If we are close to the elliptic point of figure 1, at point C, then vortices' orbit is the same; they rotate around the cylinder and around each

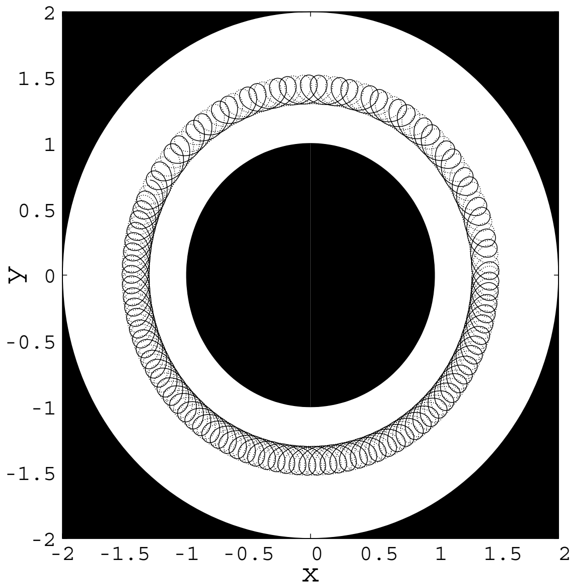

Figure 4. Motion of vortices of unit strength around the cylinder at the origin with initial vortex starting positions being (1.3, 0), (1.519 868, 0) and $\kappa_0 = \kappa_1 + \kappa_2$. Continuous and dashed lines denote the trajectories of vortices. The initial points correspond to the point C in figure 1.

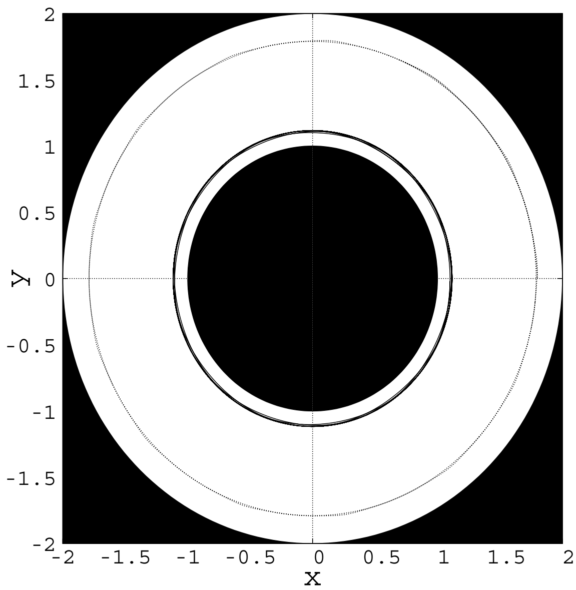

Figure 5. Motion of vortices of unit strength around the cylinder at the origin with initial vortex starting positions being (1.8, 0), (1.186 592, 0) and $\kappa_0 = \kappa_1 + \kappa_2$. Continuous and dashed lines denote the trajectories of vortices. The initial points correspond to the point D in figure 1.

other (figure 4). In the boundary of elliptic region, at point B, vortices' behavior is depicted in figure 3, which is somewhat reminiscent of the behavior in figure 4. At other points, such

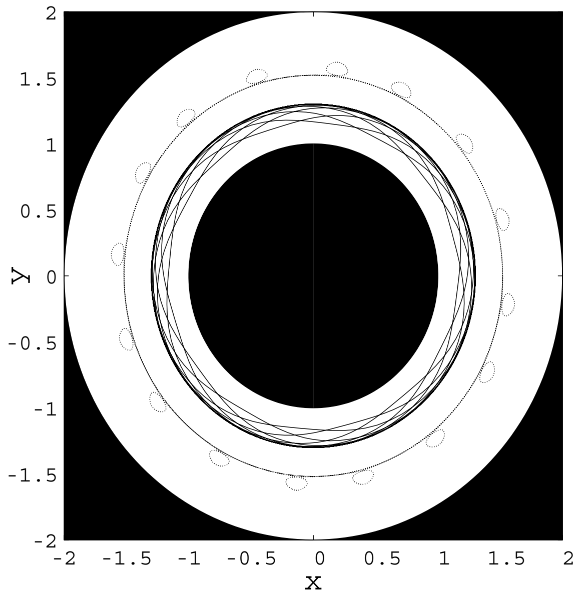

Figure 6. Motion of vortices of unit strength around the cylinder at the origin with initial vortex starting positions being (0, 1.3), (1.519 868, 0) and $\kappa_0 = \kappa_1 + \kappa_2$. Continuous and dashed lines denote the trajectories of vortices. The initial points correspond to the point E in figure 1.

as the point E where the lines passing through the origin and the vortices initial positions are orthogonal to each other, vortices do not interact with each other (figure 6).

5. Conclusions

As a new application of our approach, that is q-logarithmic representation of the N vortex motion problem in the annular domain, we studied the two vortex problem.

The Hamiltonian structure and the integrability of the problem of two vortices in the annular domain, according to Liouville, has been proved using canonical transformations. We show that vortices display different motion characteristics depending on different regions of figure 1, where level curves of the Hamiltonian are plotted in a coordinate system that rotates with the second cylinder. In fact, figure 1 corresponds to the Poincaré map in the system (p, q) where the angular momentum and the Hamiltonian are first integrals of motion. To our best knowledge, these results are new.

Appendix. Limiting cases of vortex equations

Here, we show how two limiting cases of vortex equations and Hamiltonian functions arise from our problem. We consider the vortex equations

$$\dot{\bar{z}}_k = \sum_{j=1}^{N} \mathrm{i}\kappa_j \frac{1}{z_k - z_j} + \sum_{j=1}^{N} \frac{\mathrm{i}\kappa_j}{z_k(q-1)} \left[\mathrm{Ln}_q \left(1 - \frac{z_k}{z_j}\right) - \mathrm{Ln}_q \left(1 - \frac{z_k \bar{z}_j}{r_1^2}\right) + \mathrm{Ln}_q \left(1 - \frac{r_2^2}{z_k \bar{z}_j}\right) - \mathrm{Ln}_q \left(1 - \frac{z_j}{z_k}\right) \right] \quad \text{(A.1)}$$

and the Hamiltonian function in terms of q-exponential functions

$$H = -\frac{1}{4\pi}\sum_{j=1(j\neq k)}^{N}\Gamma_i\Gamma_j \ln|z_i - z_j| - \frac{1}{4\pi}\sum_{j=1(j\neq k)}^{N}\Gamma_i\Gamma_j \ln\left|\frac{E_q\left(\frac{z_i}{(1-q)z_j}\right)E_q\left(\frac{z_j}{(1-q)z_i}\right)}{E_q\left(\frac{z_i\bar{z}_j}{(1-q)r_1^2}\right)E_q\left(\frac{r_2^2}{(1-q)z_i\bar{z}_j}\right)}\right| \tag{A.2}$$

in two limits for $q \to \infty$.

(1) For $q = \frac{r_2^2}{r_1^2} \to \infty$ we consider $r_2 = \text{const}$, and $r_1 \to 0$ limit, corresponding to motion inside the circular domain with radius r_2. Using asymptotic limits

$$\lim_{q\to\infty} \mathrm{Ln}_q(1+z) = z \tag{A.3}$$

$$\lim_{q\to\infty} \frac{\mathrm{Ln}_q(1-qz)}{q-1} = \frac{z}{z-1} \tag{A.4}$$

from the above system (A.2) we get

$$\dot{\bar{z}}_k = \frac{1}{2\pi \mathrm{i}}\sum_{j=1(j\neq k)}^{N}\frac{\Gamma_j}{z_k - z_j} + \frac{1}{2\pi \mathrm{i}}\sum_{j=1}^{N}\frac{\Gamma_j\bar{z}_j}{r_2^2 - z_k\bar{z}_j} \tag{A.5}$$

which coincides with result [1]. Using asymptotic limits for q-exponential function

$$\lim_{q\to\infty} E_q\left(\frac{z}{q}\right) = 1 \tag{A.6}$$

$$\lim_{q\to\infty} E_q\left(\frac{qz}{q-1}\right) = 1 + z \tag{A.7}$$

we have the limiting case of the Hamiltonian

$$H = -\frac{1}{4\pi}\sum_{j=1(j\neq k)}^{N}\Gamma_i\Gamma_j \ln|z_i - z_j| \\ - \frac{1}{4\pi}\sum_{j=1(j\neq k)}^{N}\Gamma_i\Gamma_j \ln\left|\frac{1}{r_2 - z_k\bar{z}_j}\right| - \frac{1}{2\pi}\sum_{j=1(j\neq k)}^{N}\Gamma_i\Gamma_j \ln r_2 \tag{A.8}$$

which differs from [1] by the last constant term.

(2) For $q = \frac{r_2^2}{r_1^2} \to \infty$ we consider $r_1 = \text{const}$, and $r_2 \to \infty$ limit, corresponding to motion outside the circular domain with radius r_1. Using the above asymptotic limits we find the system

$$\dot{\bar{z}}_k = \frac{1}{2\pi \mathrm{i}}\sum_{j=1(j\neq k)}^{N}\frac{\Gamma_j}{z_k - z_j} + \frac{1}{2\pi \mathrm{i}}\sum_{j=1}^{N}\frac{\Gamma_j\bar{z}_j}{r_1^2 - z_k\bar{z}_j} + \frac{1}{2\pi \mathrm{i}}\frac{1}{z_k}\sum_{j=1}^{N}\Gamma_j. \tag{A.9}$$

These system also coincides with the one in [1]. For Hamiltonian function in the limit case we have

$$H = -\frac{1}{4\pi}\sum_{j=1(j\neq k)}^{N}\Gamma_i\Gamma_j \ln|z_i - z_j| + \frac{1}{4\pi}\sum_{j=1(j\neq k)}^{N}\Gamma_i\Gamma_j \ln\left|z_i\bar{z}_j - r_1^2\right| \\ - \frac{1}{4\pi}\sum_{j=1(j\neq k)}^{N}\Gamma_i\Gamma_j \ln|z_i\bar{z}_j| \tag{A.10}$$

which also coincides with [1].

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