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Merger dynamics in three-agent games

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Abstract. We present the effect of mergers, a term which we use to mean a temporary alliance, in the dynamics of the three-agent model studied by Ben-Naim, Kahng and Kim and by Rador and Mungan. Mergers are possible in three-agent games because two agents can combine forces against the third player and thus increase their probability to win a competition. We implement mergers in this three-agent model via resolving merger and no-merger units of competition in terms of a two-agent unit. This way one needs only a single parameter which we have called the competitiveness parameter. We have presented an analytical solution in the fully competitive limit. In this limit the score distribution of agents is stratified and self-similar.

1 Introduction

The use of methods inspired from physical principles has recently been of wide utility in studying various systems. Models with explanatory and predictive powers have been applied to biological, social, political, economical, animal behavioural and many other phenomena. The common property of these models is that they all involve a collection of agents interacting with a well defined set of rules. The set of rules may describe a competition for earning a game, for exchanging an attribute say money, energy or points, or for instance to fight for a position in space.

Recently such a model based on two-agent units of competition was introduced and applied to sports data [1–3]. Later that model is generalized utilizing three-agent units of competition providing qualitative understanding of emerging social structures [4]. That model being deterministic did cover the full range of possibilities. The analysis of [5], where the entire phase space of three-agent games is studied, revealed new social structures.

In the present paper we study an intriguing extension of the three-agent model; the possibility that two agents could merge forces against the third one. In [4,5] it was shown that the competitive subspace of three-agent games yields a continuous point gain rate for agents. Implementing mergers into the model we find that the agents condense at particular values of income rate. That is, as a result of mergers the society of agents becomes **stratified**. In the limit where the model becomes the most competitive the distribution for income rate becomes self-similar.

We first review the main results of [5] to introduce the three-agent model. The remaining parts of the manuscript is devoted to the study of merger dynamics.

2 Review of three agent games

In this chapter we will review the dynamics of three-agent games to have a structured manuscript. We shall omit various details as those were already discussed in depth in [5]. We shall however add emphasis on some aspects that are not discussed in the mentioned paper and also highlight points relevant for the next chapter where we will extend the model with mergers.

Let us consider a collection of N agents with a given distribution of points where $N \gg 1$. We pick three of them randomly and order their points say as $L \geq M \geq S$. We shall give one point to one and only one agent according to the following rules,

- $\{L > M > S\} \Longrightarrow \{P,\ T,\ Q\}$
- ∘ $\{L = M > S\} \Longrightarrow \left\{\frac{P+T}{2},\ \frac{P+T}{2},\ Q\right\}$
- ∘ $\{L > M = S\} \Longrightarrow \left\{P,\ \frac{T+Q}{2},\ \frac{T+Q}{2}\right\}$
- ∘ $\{L = M = S\} \Longrightarrow \left\{\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right\}$.

Here the lists on the right represent the winning probabilities of the teams with points listed on the left. In view of later chapters we refer to this model as the *single rule model* emphasizing the fact that there are no conditions on the points other than their ordering. As can be inspected, cases when some agents have equal points are resolved on the basis of *equal likelihood*. Also the fact that after every

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game one agent surely advances requires the normalization of the probabilities

$$P + T + Q = 1.$$

Now let us denote the probability to pick an agent with point x at a particular time as f_x, this should really be taken as a shorthand notation for $f(x,t)$. After a competition some teams might leave this region towards $x+1$ and some teams might enter it from $x-1$ by winning a competition in either case. This suggests the following local conservation law

$$\frac{df_x}{dt} = \sum_{y,y'} f_{x-1} f_y f_{y'}\, W(x-1,y,y') - \sum_{y,y'} f_x f_y f_{y'}\, W(x,y,y'). \qquad (1)$$

Here $W(x,y,y')$ denotes the probability that the agent with point x will win against two others with points y and y'. The microscopic rules above completely define what W is. Furthermore, the right hand side is cubic in f. This is so because N is large and in this limit the probability to pick three agents with points say x,y,z is very well approximated by the product $f_x f_y f_z$ being exact in the limit $N \to \infty$.

Since

$$\sum_x f_x = 1,$$

it is immediate that (1) also implies, as it should, the global conservation of the total number of teams, as can be checked by performing a sum over x.

The time variable in (1) has an arbitrary scaling which can be compensated by an overall factor in the definition of W since the former, in essence, represents a rate. The natural scale is such that the average points of teams is given by,

$$\bar{x}(t) \equiv \sum_{x=0}^{\infty} x\, f_x = \frac{t}{3}, \qquad (2)$$

meaning that, on average, each team participates in a single game while we increment the time variable by one unit. As only one of the participating teams in a unit of competition wins and advances its score by one, equation (2) follows. One can equivalently say that the average speed with which agents increase their points is 1/3. This normalization also means that the maximum theoretical point an agent could have acquired at time t is simply t.

The presence of sums over the discrete indices on the right hand side of (1) results in a coupled set of differential equations. For the model at hand these can be much simplified by defining

$$F_x \equiv \sum_{y=x^*}^{x-1} f_x. \qquad (3)$$

Here x^* represents the smallest point below which there are no agents. Since in our model agents do not lose points, this value does not change in time and is defined by the initial point distribution via the requirement $f_x = 0$ at $t = 0$ for $x \le x^*$. Without loss of generality on can take $x^* = 0$ and confine the points to positive values[1]. Also, from the definition of F_x the following is immediate

$$f_x = F_{x+1} - F_x. \qquad (4)$$

Summing (1) over x we obtain

$$\frac{dF_x}{dt} = -f_{x-1} \sum_{y,y'} W(x-1,y,y')\, f_y\, f_{y'}. \qquad (5)$$

We note that

$$f_{x-1} \sum_{y,x} W(x-1,y,y')\, f_y\, f_{y'} \qquad (6)$$

yields the probability that a team with score $x-1$ will win any possible choice of single competition with two other teams. Working out the sum using the rules represented by W we get the final form of the master equation,

$$\frac{dF_x}{dt} = -f_{x-1}\left[PF_{x-1}^2 + Q(1-F_x)^2 + 2TF_{x-1}(1-F_x)\right] - 2\frac{(P+T)}{2} f_{x-1}^2 F_{x-1} - 2\frac{(T+Q)}{2} f_{x-1}^2 (1-F_x) - \frac{1}{3} f_{x-1}^3. \qquad (7)$$

The terms on the first line of (7) represent the bulk of interactions between three players with different scores. The second and third lines represent the cases where two players have identical scores and the last term represents the case when all the teams have the same score. The effect of terms denoting units of competition where one picks some agents with equal score are irrelevant for the late time dynamics of the system as they die out in time[2]. We will call these type of terms the *interface* terms and the rest as the *bulk* terms. So as time goes by, in a thermodynamic limit where the number of teams ranges to infinity, the majority of the contributions to the dynamics will be governed by the bulk terms. On the other hand as time goes by, almost every team will accumulate a certain number of points which, in general, will be much larger than a single point. These considerations allow one to go to a continuum limit where the terms like F_{x-1} are expanded in terms of the derivatives. A first order approximation, where one considers only the bulk terms, results in the following,

$$\frac{\partial F}{\partial \tau} = -\frac{\partial F}{\partial x}\, G'(F) = -\frac{\partial G(F)}{\partial x}, \qquad (8)$$

[1] One can also say that negative points are meaningless. However forcing positive points and mathematically being able to do so are different things. If there is a shift symmetry in the equations x^* can be chosen to be 0 without altering the asymptotic behaviour of the system. If however there is no such symmetry one simply mandates positive points at initial data. For the single rule model we have a shift symmetry but this will not be the case for the merger model. We elaborate more on this in Section 4.

[2] See for instance [1] for a detailed discussion.

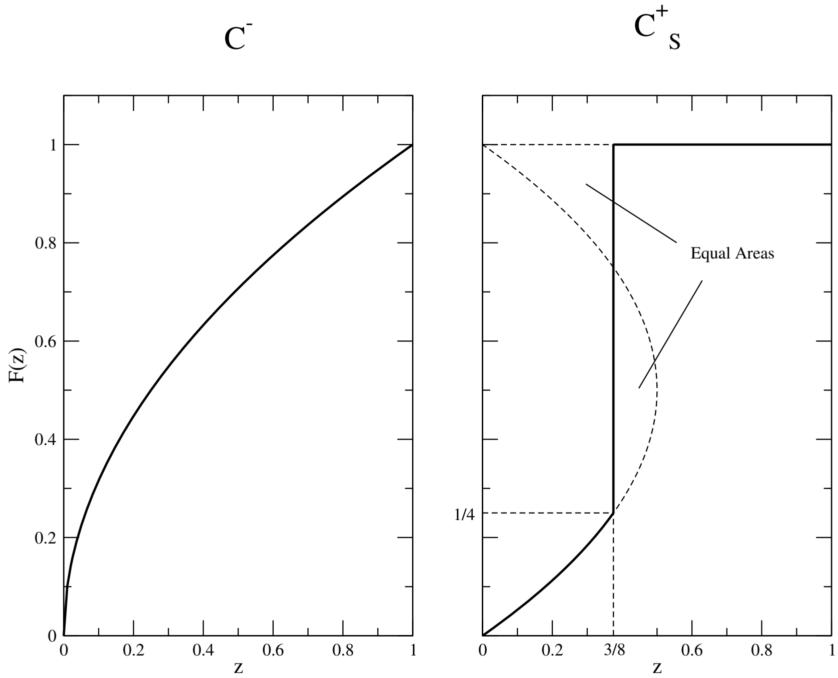

Fig. 1. Representative solutions of the single rule model relevant to our discussion. For $P = 1$ we have an example of C^- solution which in this instance is simply $F(z) = \sqrt{z}$. For $T = 1$ we have an example of C_S^+ regime; the curve is $F(z) = (1 - \sqrt{1-2z})/2$ upto the value of $z_S = 3/8$ given by Rankine-Hugoniot condition. There is a single jump here, so we have $F_l = 1/4$ and $F_r = 1$. The effect of the jump condition can be interpreted geometrically as shown in the plot to the right.

with

$$G'(F) \equiv PF^2 + 2TF(1-F) + Q(1-F)^2. \tag{9}$$

We will refer to this approximation as the *hydrodynamical limit*. We would like to mention that with this limit we still have $\bar{x} = t/3$, which corroborates the omission of the interface terms.

At this point we would like emphasize that the hydrodynamical limit for **any model** involving the struggle of three agents as a unit of competition will take the form

$$\frac{\partial F}{\partial t} = -\frac{\partial F}{\partial x} \int \int dy\, dy' \,\frac{\partial F}{\partial y}\frac{\partial F}{\partial y'} W(x, y, y'),$$

for a given function $W(x, y, y')$.

In [5] the solutions to (8) was studied via the method of characteristics, for all cases satisfying $P + T + Q = 1$, with the initial condition $F(x, 0) = \Theta(x)$ meaning that all agents starts with zero points. As is well known this is the Riemann problem. The asymptotic solutions was found to have the form $F(x, t) = F(x/t)$ as it is usually the case with hyperbolic equations coming from conservation laws. In our normalization of the time variable the maximum possible theoretical point an agent can have at time t is simply t, thus $z \equiv x/t$ satisfies $0 \le z \le 1$. Furthermore one can show that $\bar{x} = t/3$ yields

$$\int_0^1 dz F(z) = 2/3.$$

In terms of the variable $z = x/t$ one can recast (8) into

$$\frac{dF}{dz}\left[-z + PF^2 + 2TF(1-F) + Q(1-F)^2\right] = 0, \tag{10}$$

subject to the requirements, which we have discussed, that F has to satisfy. The solutions fall into the following categories

Regime C^-: for $P > T \ge Q$ and $T < 1/3$,
Regime C^0: for $T = 1/3 > Q$,
Regime C^+: for $T > 1/3$ and $P \ge T$,
Regime C_S^+: for $P < T$ and $Q \le 1/3$,
Regime S: for $Q > 1/3 > P$,
Regime C_S^-: for $Q > T$ and $P > 1/3$.

Here the superscripts $+, -$ refer to the fact that $F(z)$ has a positive or negative second derivative respectively. The superscript 0 refers to a linear curve[3]. The subscripts S mean that the solution has a shock singularity. This happens when $F(z)$ becomes double valued in z and hence the solution must be resolved via a jump. If F starts at F_l to the left of the shock and ends at F_r to the right of it the location of the jump is given by the Rankine-Hugoniot condition

$$z_S = \frac{G(F_l) - G(F_r)}{F_l - F_r} = \frac{\Delta G}{\Delta F}, \tag{11}$$

which is simply restating the fact that the number of agents leaving the shock is equal to those that enter it. In geometrical terms this means that the area between the curve $F(z)$ and $z = z_S$ to the left of the shock is equal to that area to the right of the shock.

We shall not study all types of solutions mentioned above; for our discussion on mergers we only need to draw attention to regimes C^- and C_S^+. To do so we refer the reader to Figure 1 where C^- is represented for $P = 1$ and C_S^+ for $T = 1$.

[3] This case is identical to a two-agent model. Since the middle agent advances with the mean speed of the whole collection of agents, it is as if it is not there.

Table 1. Winning probabilities of the three agents with points $L > M > S$ for no-merger tournaments and for various possible merger cases in terms of the competitiveness parameter p defined in the text. Cases where there are equalities of points can be read from this table using the equal likelihood prescription described in Chapter II. For instance if $M = S$ the probability to win for each is $(T + Q)/2$.

	No-merger	LM merger	LS merger	MS merger	MS merger	MS merger
		$L+M>S$	$L+S>M$	$M+S<L$	$M+S>L$	$M+S=L$
P	$p^2+pq/3$	p^2	p^2	p	q	$1/2$
T	$4pq/3$	pq	q	pq	p^2	$p/2$
Q	$q^2+pq/3$	q	pq	q^2	pq	$q/2$

3 Merger dynamics

The model presented in the previous section has a single set of rules (the set $\Sigma \equiv \{P, T, Q\}$) that only depends on the ordering of points. A straightforward generalization of this model can be to extend the rules so that we have various sets of probabilities and to provide a selection rule so as to define which set should be applied for a particular choice of agents. This selection rule should of course be a condition on the points of the agents other than their ordering.

3.1 Implementing mergers

The idea of mergers is meaningful only if the game is competitive, where higher points are favoured. In this respect two agents can merge points and act as a single agent against the third one. If the allied agents lose, the third player gets the point. If on the other hand the allied agents eliminate the third player we still have to resolve a single winner. These considerations hint at the necessity to construct the rules of the three-agent step in terms of a two-agent unit of competition. If this is so achieved, after winning against the third player the agents that participated in the merger can turn against each other and play the same two-agent unit to decide which one of them will receive the point.

However, if the selection rule does not allow a merger, we have a generic three-agent step which we should take to be competitive as well to be in accord with the philosophy of the idea of mergers. Such a set of rules too can be established in terms of a two-agent unit of competition. In [5] it was shown that a natural way to achieve this is to let each three agents play a single two-agent match with each other: that is to have a tournament. All two-agent matches in this tournament are decided based only on how many whole tournaments the agents have won before: during the tournament the tournament wins of each team (which is simply the associated points L, M and S of the participants) is kept constant but they accumulate match points depending on the tournament wins. The winner of the tournament is the agent with largest number of accumulated match points. As usual ties are decided on the basis of equal likelihood.

So we have established how to implement mergers and no-mergers for each particular three-agent competition in terms of the rules of a two-agent unit of competition. Having approached the problem in this way we need to use only a single parameter; the winning probability of the higher score agent in the two-agent struggle. Let the rules for the two-agent unit be $\{l > s\} \rightarrow \{p, q\}$: the agent with point l shall get the match point with probability $p = 1-q$. This unit is competitive if $p > 1/2$. After straightforward analysis we arrive at Table 1.

We see that various cases of mergers are possible and we need a selection rule as we have argued. The simplest and most natural selection mechanism which is also in accord with the overall competitive nature of our model is to let two agents merge if and only if both their probabilities to win the unit of competition increase with respect to the no-merger case. We assume that our agents are not smart and incapable of a long term strategy; they simply respond to an increase in the probability to win the three-agent unit at hand. From the table it is evident that the agent with the highest point L will never find it profitable to merge neither with M nor with S. Also S will find it not feasible to merge with M if $M+S < L$. This leaves us with MS mergers with either $M + S > L$ or $M + S = L$. Using the selection rule stated we find that if $p > 3/4$ mergers are feasible for those agents participating in it[4]. Analysing further we can also discover that a merger unit is rejected if $L = M > S$, we remind the reader however that this is an interface term.

With the analysis above we have full knowledge of the function $W(x, y, y')$ introduced in the previous chapter. Other than this difference the mathematical set-up of the merger model is completely analogous to the single rule model of the previous chapter.

3.2 The model

The analysis of the previous section amounts to the following rules, where to be explicit at the expense of being

[4] In fact $M + S > L$ case only requires $p > 3/5$. The higher value $3/4$ quoted in the text comes from the $M + S = L$ case. This term is an interface contribution and thus will represent terms with higher derivatives or higher powers of the first derivative of F. Consequently this term is less and less important in the asymptotic future of the stem. However in a simulation it occurs in the early stages if all the agents starts out with the same points so it is honest to include it in the analysis of the selection rule.

redundant, all the details, including the surface terms, of the model are summarized.

- $L > M > S$ and $M + S < L$ no merger unit with rules $\Sigma = \{P, T, Q\}$;
- $L > M > S$ and $M + S > L$ merger unit with rules $\Sigma' = \{P', T', Q'\}$;

- $L > M > S$ and $M + S = L$ merger unit with rules $\Sigma'' = \{P'', T'', Q''\}$;
- $L = M > S$ no merger unit with rules $\left\{\frac{P+T}{2}, \frac{P+T}{2}, Q\right\}$;
- $L > M = S$ and $M + S < L$ no merger unit with rules $\left\{P, \frac{T+Q}{2}, \frac{T+Q}{2}\right\}$;
- $L > M = S$ and $M + S > L$ merger unit with rules $\left\{P', \frac{T'+Q'}{2}, \frac{T'+Q'}{2}\right\}$;
- $L > M = S$ and $M + S = L$ merger unit with rules $\left\{P'', \frac{T''+Q''}{2}, \frac{T''+Q''}{2}\right\}$;
- $L = M = S$ equally likely unit with rules $\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$.

with the definitions $P = p^2 + pq/3$, $T = 4pq/3$ and $Q = q^2 + pq/3$ for no-merger units, $P' = q$, $T' = p^2$ and $Q' = pq$ for merger units with $M + S > L$ and $P'' = 1/2$, $T'' = p/2$ and $Q'' = q/2$ for merger units with $M + S = L$.

It is important to note that if only rules Σ or Σ' are applied unconditionally during the simulation, the resulting distribution is in the C^- or C_S^+ regime respectively of the single rule model. As the reader could have already guessed, this is somewhat evident since the set Σ favours the leading agent whereas the set Σ' prefers the middle agent. Therefore when mergers are implemented the two rules are in conflict with each other. This effect is much more pronounced if we let the system approach the limit of full competitiveness by letting $p \to 1$ in which case probabilities converge to $\Sigma = \{1, 0, 0\}$ and $\Sigma' = \{0, 1, 0\}$[5]. We thus expect on general grounds that, if there is no prevalence of a single rule in the game the resulting dynamics of the system should emerge from this dialectical conflict as something that is neither a C^- nor a C_S^+ solution but a new state which bears aspects of both.

Now we have, as usual,

$$\frac{\partial F_x}{\partial t} = -f_{x-1} \sum_y \sum_{y'} W(x-1, y, y') f_y f_z. \qquad (12)$$

However in contrast to the single rule model the sum above is very complicated which we do not duplicate here. In the hydrodynamical limit we still have,

$$\frac{\partial F}{\partial t} = -\mathcal{G}'[F] \frac{\partial F}{\partial x}. \qquad (13)$$

[5] One may observe that in the limit where $p = 1$ the probability for the agent with the lowest point to win remains zero merger or no-merger and thus one may infer that this player has no incentive to participate in a merger with the middle agent. However letting $p = 1 - \theta$ and expanding the probabilities to first order we see that $Q = \theta/3$ and $Q' = \theta$. Thus the lowest lagger triples its chances to win in merging no matter how close to zero its winning chances are.

The particulars of it, however, are complex as expected (we have suppressed the time dependence of F's to have a readable expression):

$$\begin{aligned} \mathcal{G}'[F] = {} & P' \left[F^2(x) - F^2(x/2)\right] + P F^2(x/2) + 2(P - P') \\ & \times \int_{x/2}^{x} dy \frac{\partial F}{\partial y} F(x - y) + 2T' \left[F(2x) - F(x)\right] F(x) \\ & + 2T \left[1 - F(2x)\right] F(x) + 2(T - T') \int_x^{2x} dy \frac{\partial F}{\partial y} \\ & \times F(y - x) + Q' \left[1 + F^2(x) - 2F(x)F(2x)\right] \\ & - 2Q \left[1 - F(2x)\right] F(x) + 2(Q - Q') \int_{2x}^{\infty} dy \frac{\partial F}{\partial y} \\ & \times F(y - x). \end{aligned} \qquad (14)$$

As a check we see that letting $P' = P$ and $T' = T$ we recover the single rule model. This equation has all the complicating adjectives one can attach, the most important being non-locality, and a direct approach as that of solving for characteristics is not obvious. Nevertheless the dependence on $x/2$ and $2x$ in (14) are suggestive and have a rather simple interpretation; an agent with point x is protected against mergers of two others if the points of those are both smaller than $x/2$ and similarly an agent with point $2x$ is protected against mergers of two agents with points less than x. These games will have to be no-merger units of competition.

One can also show that the ansatz $F(x,t) = F(z \equiv x/t)$ of the single rule model is still applicable here[6]. The only concern could be the integrals but they are easily transformed accordingly. For instance,

$$\int_{x_1}^{x_2} dy \frac{\partial F}{\partial y} F(x - y) = \int_{z_1}^{z_2} d\zeta F'(\zeta) F(z - \zeta), \qquad (15)$$

where $\zeta = y/t$.

3.3 Numerical analysis

In view of the obvious difficulty of (14) we resorted to numerical simulations with the hope that they may provide us with clues. To achieve this end we simulated the microscopic system from $p = 0.750$ to $p = 1.000$ in steps of 0.001. For each of these, we take a collection of $N = 10^6$ agents and we ended the simulation when on average each agent had played about 6.5×10^6 games meaning, in our normalization, that we stopped the run when time variable t is 2×10^7. The initial condition for all the runs was $F(x, 0) = \Theta(x)$.

To start the exposition of numerical results we invite the reader to analyse Figure 2. One important aspect one can immediately realize is that upto about $p = 0.76$ the resulting distribution is exactly the same as a pure C_S^+

[6] This is a consequence of the fact that we have $W(ax, ay, ay') = W(x, y, y')$ for either the single rule or the merger model. This in turn means that the equation has $x \to ax$ and $t \to at$ symmetry.

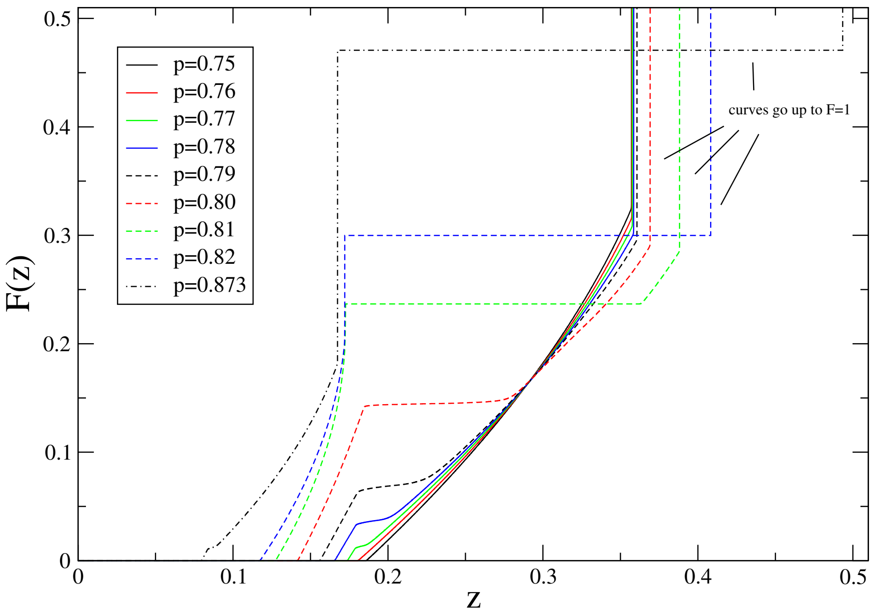

Fig. 2. (Color online) Results of the numerical simulation for various values of the competitiveness parameter p. For a detailed analysis we refer the reader to the text.

game. This can only happen if the game is dominated with mergers and no-mergers never happen; so essentially we have a single rule game. Let us call the location of the jump z_R and denote point where F vanishes as z_L. Mergers can *globally* dominate if and only if $z_R < 2z_L$ which simply implies that for any choice of three agents $M + S > L$. So for this case we can use the results of the single rule model to predict when $z_R = 2z_L$ which yields $p_0 = (\sqrt{13}+1)/6 \approx 0.76759$ in accordance with the simulations. After this point we see that the number of agents in the vicinity of z_L increase[7]. This happens because after the transition we have $z_R > 2z_L$ and hence the agents accumulated at the jump discontinuity becomes protected against mergers of the agents around z_L and some agents slide down the slope. These agents able to merge no more against the the bunch at z_R lag faster than before. Increasing p emphasizes this effect. Later on during the excursion to higher values of p, to the right of z_L a flat plateau appears. Let us call the position of the left tip of this plateau as z_P. The emergence of this plateau coincides with the condition $z_R = 2z_P$ at around $p = 0.801$. This means that the bunch condensed on the discontinuity at z_R becomes completely protected from mergers of agents to the left of z_P. Furthermore since $z_P < 2z_L$ the games played among agents to the left of z_P are all merger dominated and thus *locally*[8] of type C_S^+. Increasing p further we see that another condensation of agents occurs at z_P, which itself shifts to the left as p takes on higher values. We have turned around full circle and the solution consists of a right bunch condensed at z_R and a left bunch sitting at values less than or equal to z_P. The local structure of this left bunch is similar to the the whole structure when p was less than p_0. As the reader could have guessed a further excursion to higher values of p repeats this process and we end with a seemingly self-similar pattern. Our numerical analysis indicates that as $p \to 1$ this pattern repeats itself ad infinitum.

To recapitulate we refer to Figure 3 where we have provided representative solutions relevant to our discussion. There, the solutions we have presented are those that have a qualitative stability. That is, the location and the heights of the shocks and the form of the leftmost group will alter as we change p but the number of shocks will be constant for a while during such an excursion. We shall call these shocks *bunches* and label them as B_n with $n = 0, 1, 2\ldots$ where $n = 0$ represents the rightmost bunch. The leftmost bunch is not a pure shock but a combination of a shock and a rarefaction wave since it is locally a C_S^+ solution. The transitions between those solutions are complicated and bear the full complexity of (14) which is somewhat impenetrable. As can inferred from Figure 3, the most important aspects of these solutions are

- The games among the players of only a single bunch B_n are all mergers, since the points of these agents are localized around a particular value.
- All players in a bunch B_n are protected from mergers of two players in $\bigcup_{k>n} B_k$, since the localized points of each bunch is larger than twice that of those to the left of it.

3.4 Self similar behaviour of the model

In view of the discussion above, for values of p larger than about 0.81, we can separate the equation into two parts: one valid near the condensation of the rightmost bunch (the forerunner agents) and one for the rest of the bunches. All we need is the interaction of these two parts. From the numerical study we know that the rightmost bunch

[7] Needless to say z_R and z_L are changing as p increases. As can be expected on general grounds z_R moves to the right and z_L moves to the left.

[8] That is if we consider only the games between agents sitting in the left bunch.

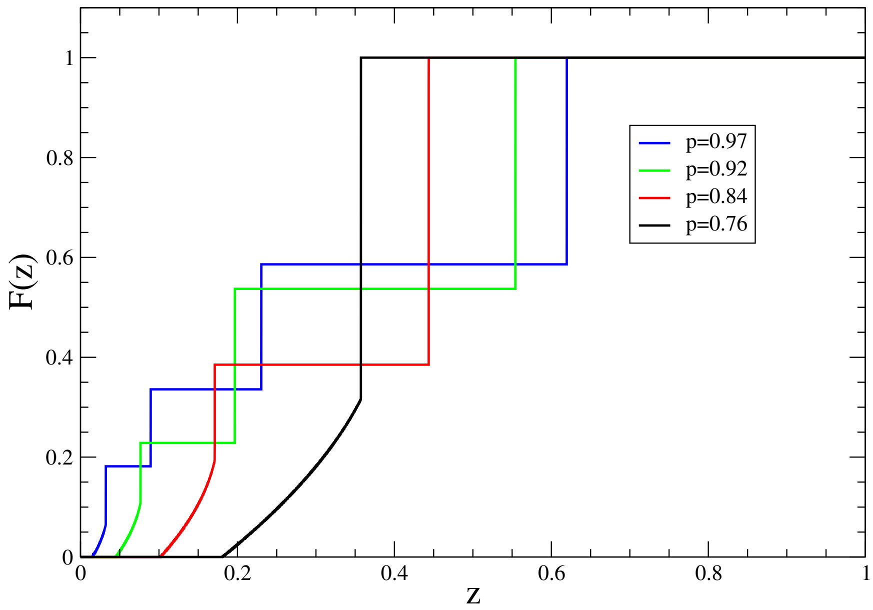

Fig. 3. (Color online) The qualitatively stable (see text) solutions of the merger model for various number of bunches.

is completely protected from mergers of two agents not in this bunch and that the games within this bunch are merger dominated. Let us start with the generic form of the equation

$$\frac{\partial F}{\partial t} = -\frac{\partial F}{\partial x} \int\int dy dy' W(x,y,y') \frac{\partial F}{\partial y}\frac{\partial F}{\partial y'}. \tag{16}$$

As we have mentioned for the merger model we can recast this in terms of $z = x/t$

$$\frac{dF}{dz}\left(-z + \mathcal{G}'[F]\right) = 0 \tag{17a}$$

$$\mathcal{G}'[F] = \int\int dy dy' W(x,y,y') \frac{\partial F}{\partial y}\frac{\partial F}{\partial y'}, \tag{17b}$$

where the integrals are over the whole domain of points.

Let us denote the rightmost bunch as B_0 and all the agents not in this bunch as B_L. The points of the agents in B_0 are coalesced near a value $x_0(t)$ which is at least twice as large as the largest point of the agents in B_L. Therefore the equation for B_0 can be written as

$$\frac{\partial F_0}{\partial t} = -\frac{\partial F_0}{\partial x} \int\int dy dy' W(x,y,y') \frac{\partial F}{\partial y}\frac{\partial F}{\partial y'} \tag{18}$$

where in evaluating W we should remember that x is near x_0. We end up in

$$\frac{\partial F_0}{\partial t} = -\frac{\partial F_0}{\partial x}\mathcal{G}'_0 \tag{19}$$

with

$$\mathcal{G}'_0 = P'(F_0-\Phi_1)^2 + 2T'(F_0-\Phi_1)(1-F_0) + Q'(1-F_0)^2 + 2P'\Phi_1(F_0-\Phi_1) + 2T'\Phi_1(1-F_0) + P\Phi_1^2. \tag{20}$$

Here Φ_1 represents the total number of agents in B_L. That is, F_0 starts at Φ_1 to the left of B_0 and ends at 1 to the right. The first line above are the self games of the rightmost bunch. The second line represents games where two agents are in B_0 and one in B_L. The third line represents games where only one agent is selected from B_0.

For values of $x < x_0/2$ we are in B_L. Denoting the cumulative function in this region as $F_L(x,t)$ we get

$$\frac{\partial F_L}{\partial t} = -\frac{\partial F_L}{\partial x}\mathcal{G}'_L \tag{21}$$

with

$$\mathcal{G}'_L = \int\int dy dy' W(x,y,y') \frac{\partial F_L}{\partial y}\frac{\partial F_L}{\partial y'} \tag{22}$$

$$+ 2T(1-\Phi_1)F_L + 2Q(1-\Phi_1)(\Phi_1 - F_L) + Q'(1-\Phi_1)^2. \tag{23}$$

Now F_L starts from 0 and ends at Φ_1 and the second (third) lines in the equation above represents picking two (one) agents from B_L. The first line represents the games where all agents are in B_L.

There is resemblance to self-similarity in (22): the first line looks like the original equation but the interaction terms with the bunch B_0 spoils this correspondence since B_L can still gain points via these terms. However in the extreme competitiveness limit $p \to 1$ these terms are absent since T, Q and Q' all vanish. That is, agents in B_L will only gain points against themselves and will simply remain idle during any competition with the bunch B_0. Conversely bunch B_0 will use B_L as a definite source of points. In this limit (22) becomes

$$\mathcal{G}'_L = \int\int dy dy' W(x,y,y') \frac{\partial F_L}{\partial y}\frac{\partial F_L}{\partial y'}. \tag{24}$$

Let us recall however that the maximum value F_L can take is Φ_1. Defining $\tilde{F}_L \equiv F_L/\Phi_1$ and using the ansatz $z = x/t$

for a solution, the equation becomes

$$\frac{d\tilde{F}_L(z)}{dz}\left[-\frac{z}{\Phi_1^2}+\mathcal{G}_L'\right]=0. \quad (25)$$

This has exactly the same form as the original equation if we also let $z \to \Phi_1^2 z$ which one could interpret as scaling of x. However unless the initial condition can be partitioned this way, we can not say that the solution will resolve itself into a self-similar shape. The initial data we use $F(x,t) = \Theta(x)$ can be partitioned this way because $\Theta(ax) = \Theta(x)$. In conclusion if $F(x,0) = \Theta(x)$ we expect self-similar behaviour in the solution when $p = 1$ via the scaling we have described above. The procedure of extracting the rightmost bunch is somewhat similar to renormalization procedure and the decoupling mechanism in field theory where after integrating out high energy degrees of freedom we end up with a new theory. If the original theory is said to be non-renormalizable the new theory is different. If otherwise the new theory is similar in form to the original except quantities in it like fields, coupling constants etc. are scaled it is called a renormalizable theory. We see an analogy here; the extraction of the rightmost bunch yields the same form of equations for $p = 1$ and different otherwise. Thus within this sense we can say that for $p = 1$ and with $F(x,0) = \Theta(x)$ the model is renormalizable. The effect of this renormalization yields the scaling of F via $F \to \Phi_1 F$ and that of z via $z \to z\Phi_1^2$.

Given these circumstances we can repeat the *renormalization procedure* described above infinitely many times after which we shall end up with infinitely many bunches B_n localized around z_n containing $f_n \equiv \Phi_n - \Phi_{n+1}$ agents. As expected, at each step the scaling should be achieved via the same number $\Phi_1 \equiv \varphi$ if there is to be self-similarity at all. The protection of B_n against mergers of any two players in $B_{k>n}$ mandates that $z_n > 2z_{n+1}$. Reiterating this procedure we find the equation obeyed for agents in B_n to be

$$\frac{dF_n}{dz}\left[-z-2F_n(z)^2+2F_n(z)\Phi_n+\Phi_{n+1}^2\right]=0. \quad (26)$$

This can only be resolved via a shock, the location of which is found via the Rankine-Hugoniot condition

$$z_n = \frac{1}{3}(\Phi_{n+1}^2+\Phi_n\Phi_{n+1}+\Phi_n^2). \quad (27)$$

Since at each step we scale with the same number φ we have

$$\Phi_n = \varphi^n. \quad (28)$$

Which in concert with the protection mechanism mentioned means that $\varphi < 1/\sqrt{2}$ and implies the following

$$z_n = z_o\varphi^{2n}, \quad (29a)$$

$$z_0 = \frac{1}{3}(\varphi^2+\varphi+1). \quad (29b)$$

We are one equation away from a solution. A concept we may use is the stability of the solution at large times. To this end one can ask the following question; how can the bunch B_0 knows that it is the leading bunch in a self-similar pattern? To answer it let us assume the existence of a further bunch B_{-1}. Using the scaling we expect the number of agents in this bunch to be $1/\varphi - 1$. Now the location of B_0 will be stable if and only if the games lost to B_{-1} only by B_0 equals the games won against agents in $\cup_{k>0}B_k$ again only by B_0. This can only happen if the number of agents in B_{-1} equals the number of agents below B_0 meaning $1/\varphi - 1 = \varphi$. These considerations allows us to find

$$\varphi = \frac{\sqrt{5}-1}{2}, \quad (30a)$$

$$z_0 = \frac{2}{3}. \quad (30b)$$

Thus φ is the reciprocal of the Golden Ratio and z_0 is just twice the value of the mean speed of the entire system. So a player in the rightmost bunch is, *in the mean*, on the verge of being protected from mergers of *any two* randomly picked agents from the entire collection. The simulation results for which $F(x,0) = \Theta(x)$ are in very good agreement with this theoretical prediction.

It is interesting to contrast the model with mergers and the single rule model without mergers. The comparison is in Figure 4. One important aspect we realize is that the effect of implementing mergers does not affect the global behaviour of the problem; the model with mergers is like a discretization of the curve $F(z) = \sqrt{z}$ and thus overall competitiveness is still there. However the local behaviour is completely different; we have stratification of agents. Or in a different language we have the formation of distinct *social classes*. It is rather interesting to observe this behaviour when we, in effect, increased the overall competitiveness of the model. As we have mentioned before this effect is a consequence of the conflict between mergers living in the C_S^+ regime and no-mergers being in the C^- class of solutions were the simulations run unconditionally as a merger or no-merger single rule model respectively.

3.5 Restricted mergers

The merger model we have presented represents a very coercive environment of competition and it is not readily susceptible to analytical study for arbitrary p. The main reason for this is its high non-locality. This non-locality is present because the two lowest laggers are allowed to merge in all cases even when $M + S = L + 1$. One could wish to contemplate other schemes of mergers where this effect is less pronounced. One way to do this is to regulate mergers. Here we present a model which is the most restricted. Let us remember that at time t the theoretical maximum point is just t. Now let us pick three agents and order their points as $L \geq M \geq S$ and *let us allow mergers only if* $S > t/2$; this makes sure that the competition is a merger unit since we always have $L < t$ and thus it is always true that $M + S \geq L$. In such an approach mergers will be represented in W via a term like $\Theta(x - t/2)$ which

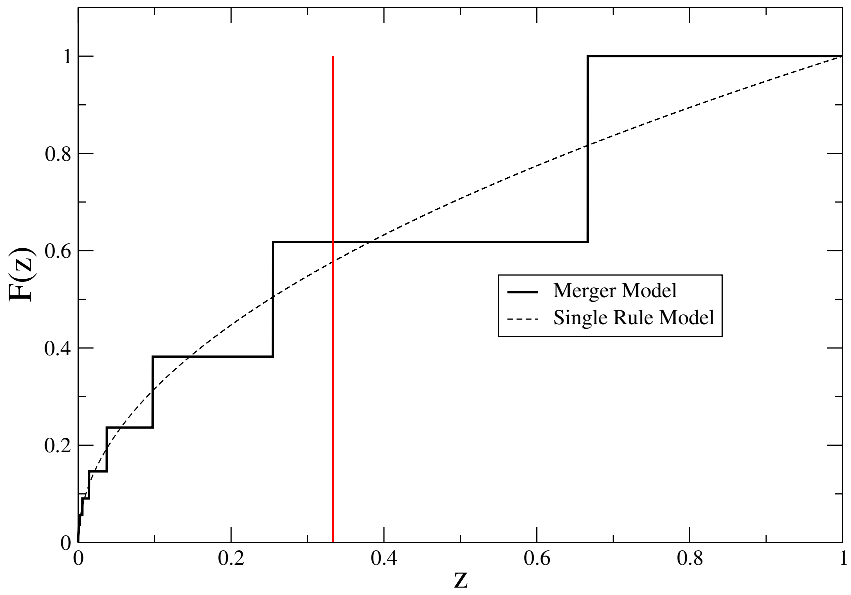

Fig. 4. (Color online) The solution to the merger model in comparison to that of the single rule model both for $p = 1$. The mean speed of agents is shown with a vertical line.

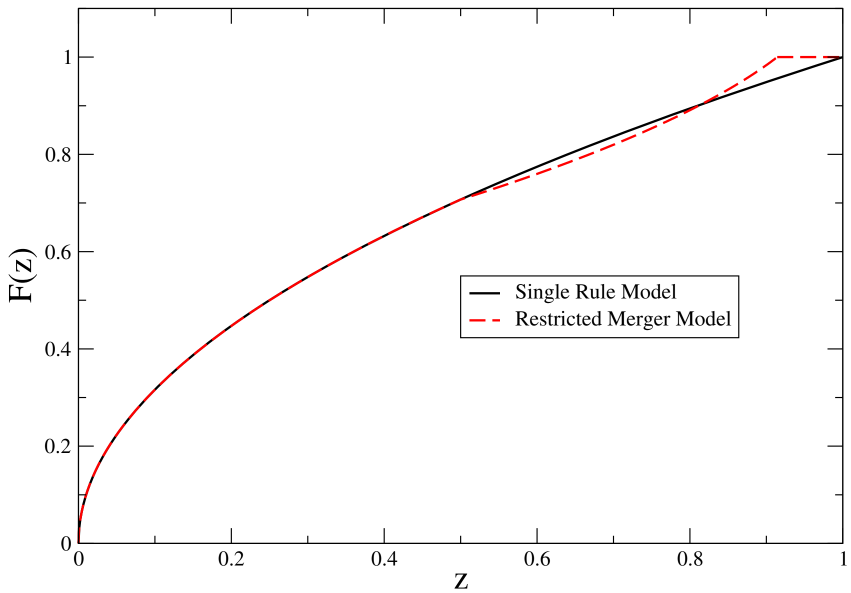

Fig. 5. (Color online) The maximally competitive limit solutions of the restricted merger model and of the single rule model.

will become $\Theta(z-1/2)$ for the asymptotic behaviour where as usual $z = x/t$. The equation becomes

$$\frac{dF}{dz}\left[-z + G'(F)\right] = 0, \qquad (31)$$

with

$$G'(F) = \begin{cases} PF^2 + 2TF(1-F) + Q(1-F)^2 \\ \qquad\qquad \text{for} \quad z < 1/2 \\ PF^2 + (P'-P)(F-\bar{F})^2 + 2T\bar{F}(1-F) \\ +2T'(1-\bar{F})(F-\bar{F}) + Q'(1-\bar{F})^2 \\ \qquad\qquad \text{for} \quad z \geq 1/2 \end{cases} \qquad (32)$$

and with $\bar{F} \equiv F(1/2)$. This equation is local and thus can be studied analytically in much the same way as the single rule model. Here we only present the solution for $p = 1$ to compare it with the unrestricted merger model.

$$F(z) = \begin{cases} \sqrt{z} & z \leq \frac{1}{2} \\ \frac{1}{2}\left(\sqrt{2} + 1 - 2\sqrt{1-z}\right) & \frac{1}{2} \leq z \leq z_r \\ 1 & z_r \leq z \end{cases} \qquad (33)$$

where we have

$$\bar{F} = \frac{1}{\sqrt{2}},$$

$$z_r = \frac{2\sqrt{2}-1}{2}.$$

Which is in very good agreement with numerical simulations. We see that the solution for $z < 1/2$ is the same as that of the single rule model. The comparison for the full range of z is given Figure 5. As the reader could have guessed the stratification effect is non-existent but the tendency of the curve to approach that regime, had mergers were unrestricted, is apparent.

4 Digression on initial conditions

Let us remember that the generic form of the equation governing the dynamics of three-agents games, in the hydrodynamical limit, is

$$\frac{\partial F}{\partial t} = -\frac{\partial F}{\partial x} G'[F]$$
$$G'[F] \equiv \int\int dy\, dy' \,\frac{\partial F}{\partial y}\frac{\partial F}{\partial y'} W(x, y, y').$$

For the single rule model of Chapter I. the integrals resolve into a simple polynomial of F. The reason for such a simplification is, for the model mentioned, the fact that W, being only a function of the ordering of points, contains only terms like $\Theta(L - M)\Theta(M - S)$ and thus obeys

$$W(ax, ay, ay') = W(x, y, y') \quad (34a)$$
$$W(x - b, y - b, y' - b) = W(x, y, y). \quad (34b)$$

The first of these equations means that the equation will be invariant under the combined transformations $x \to ax$ and $t \to at$ which makes it possible to assume an ansatz $F(x/t)$ since $F(x, 0) = \Theta(x)$ also obeys this symmetry.

Now let us shift F by a constant ϕ such that $F = \tilde{F} + \phi$ with $\phi = -(T - Q)/(1 - 3T)$. Since the single rule model obeys (34b) one can also perform a Gallilean transformation[9] on the independent variables of the form $\tilde{t} = t$ and $x = \tilde{x} - z_0\tilde{t}$. Choosing $z_0 = Q-(T-Q)^2/(1-3T)$ the equation becomes

$$\frac{\partial \tilde{F}}{\partial \tilde{t}} = -(1 - 3T)\tilde{F}^2 \frac{\partial \tilde{F}}{\partial \tilde{x}},$$

which can be recast as

$$\frac{\partial \tilde{F}}{\partial t} = -\frac{\partial \tilde{G}(\tilde{F})}{\partial x} \quad (35a)$$
$$\tilde{G}(\tilde{F}) = \frac{(1 - 3T)}{3}\tilde{F}^3, \quad (35b)$$

where $\tilde{G}(\tilde{F})$ is strictly concave for $(1 - 3t) > 0$. If on the other hand $(1 - 3t) < 0$ one can make $\tilde{G}(\tilde{F})$ strictly concave by doing $\tilde{x} \to -\tilde{x}$.

After these transformations the initial condition becomes $\tilde{F}(\tilde{x}, 0) = \Theta(\tilde{x})+\phi$ which still represents a Riemann problem. It is a known fact that the solutions to equations of type (35) converge in the infinite time limit to the solutions of the Riemann problem[10] if the initial condition obeys $\tilde{F}(\tilde{x}, 0) = F_L$ for some $x < x_L$ and $\tilde{F}(\tilde{x}, 0) = F_R$ for some $x > x_R$ where F_L and F_R are constants. Now, the dependent variable F is the normalized cumulative of a globally conserved quantity; the number of agents. Therefore for a generic initial distribution of points we have $F(x, 0) = 0$ for $x < 0$ and $F(x, 0) = 1$ for some $x > x_R$. We thus infer that for the single rule model the time asymptotics of F is independent of the initial conditions[11]. These considerations also apply to the restricted merger model we have studied since in principle it has the same general form as the single rule model.

For the merger model without restrictions we still have the symmetry in (34a) which allows us to make the $F(x/t)$ ansatz if the initial condition obeys $F(ax, 0) = F(x, 0)$. Unfortunately the shift symmetry in (34b) is absent because mergers are implemented via terms of the type $\Theta(L-M)\Theta(M-S)\Theta(M+S-L)$. This makes the equation highly non-local and in particular the Gallilean transformations will take it to an entirely different form. None of the theorems presented in the mentioned papers above hold and one would expect a strong dependence of the time asymptotics on the initial data; an observation which we have substantiated with numerical simulations.

5 Discussion

The unrestricted merger model we have presented has interesting properties. The most important being the stratification of the entire society of agents. The bunch that has the largest number of agents (this number is $(1-\varphi) \approx \%32$) is also the bunch with the largest rate of point gain. However this point gain is only two thirds of the maximum possible rate. On the other hand all the agents other than the first bunch are earning slower than the mean rate. The number of agents living below this mean is slightly lower in the single rule model where mergers are not implemented. Furthermore from our solution it is clear that the number of agents in a bunch f_n satisfies $f_n = f_{n-1} - f_{n+1}$, that is the number of agents in a bunch is like a derivative in the sense of the bunches.

It is tempting to speculate that the merger model we have presented could have applications to natural or social phenomena. The stratification phenomenon being present in various systems. For instance one could argue that a bunch, in essence, represents a single entity, the number of agents in it simply meaning that it has more activity in taking part in games. From this perspective one may interpret the merger model as one of explaining monopoly formation after a period of competition between companies. Stratification is also present in natural systems. Another tempting interpretation could be the stratification of the collection of entire living species in terms of their genetic

[9] Under the Gallilean transformation alone the equation is transformed into $\frac{dF}{dz}\left[-z + z_0 + G'(F)\right]$. That is, the z variable which in principle represents the speed of agents is shifted by a constant.

[10] See for instance [6–8].

[11] For generic initial data $F(x, 0)$ of the type mentioned in the text the simulations converge to that of $F(x, 0) = \Theta(x)$ after a comparatively larger number of Monte-Carlo cycles. An estimate of this time is presented for instance in [9].

material. If there is a competition mechanism complexified with mergers like the one described in this work one could hope to gain qualitative understanding of the formation of different strata of living organisms. Agents could be units of genetic material and the competition could be for taking part in the genetics of a (new) species.

The emergence of the stratification mechanism can be interpreted in the following way. The rules of mergers yield a complicated and non-local system. The equations are so complicated that they can not be resolved via smooth functions and the only possible escape is the formation of various shocks; there must be a solution since we are simply simulating a Monte-Carlo system with a well defined microscopic model. We believe this to be true for other systems of conservation laws, coming from well defined microscopics, where the equations become non-local.

Note added in proof

We would like to thank an anonymous referee for drawing our attention to *truels*, a three person generalization of duels. An overview of the mathematics of truels can be found in [10]. To discuss possible parallelisms between truels and the three-agent units of competition in our model we would first like to emphasize obvious differences. To do so we first would like to contrast a two-agent unit with a duel. In models we have studied in this work the outcome of a two-agent unit is determined by a single probability p representing the probability for the agent with the largest point to win. Even if there is only one unit of point difference the agent with the largest point acquires the same higher probability. It is therefore the difference between the agents that determine their fate not any intrinsic quality of the agent[12]. Another difference is that in truels, depending on the rules of engagement, there may be more than one survivor which is in contrast with our model since we allow only one agent to receive the point. This difference is somewhat minor and may be cured by demanding that the truel must end with a single survivor[13]. Yet another problem in assessing an analogy is that in most flavours of truels the worst shooter is the likely winner, in that sense truels also provide an example to survivor of the least fit. So what we understand as competitive in our work is not the same in truels. Aside mergers in our three agent games the agent with the largest point always has the largest probability to win. Even in mergers the mergees are likely to win because their points are added and they become momentarily better than the agent with the largest point in the triplet.

To reiterate on mergers, there seems to be examples of truels where the best strategy for the worst two shooters is to aim at the most skilled agent until it is eliminated. After that they may turn towards each other. In this sense we seem to have an analogy. To keep this addendum short we would like to remind the reader that we have resolved all our three-agent units of competition, merger or no-merger in terms of two-agent units. If truels are also resolved in terms of duels one can draw an analogy but this takes us away from the generic authenticity of truels. Truels are intriguing because strategical thinking yields results that are qualitatively different than duels, so the wisdom of going from two players to three lies here. In our approach the main emphasis of considering three agents is that they allow mergers in the simplest sense of the word and give a further meaning to *competitiveness*.

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[12] One can contemplate schemes where the probabilities are calculated from the points of agents but the equations that result from these models are first of all too complicated and second they do not shed further light on the general aspects of models of the type discussed in this paper.

[13] No one has to die really, the truel may be performed with paint-ball guns.