Journal of Physics: Conference Series **343** (2012) 012093 doi:10.1088/1742-6596/343/1/012093

Vortex Images, q-Calculus and Entangled Coherent States

Oktay K. Pashaev

Department of Mathematics, Izmir Institute of Technology, Urla-Izmir, 35430, Turkey

E-mail: oktaypashaev@iyte.edu.tr

Abstract. The two circles theorem for hydrodynamic flow in annular domain bounded by two concentric circles is derived. Complex potential and velocity of the flow are represented as q-periodic functions and rewritten in terms of the Jackson q-integral. This theorem generalizes the Milne-Thomson one circle theorem and reduces to the last on in the limit $q \to \infty$. By this theorem problem of vortex images in annular domain between coaxial cylinders is solved in terms of q-elementary functions. An infinite set of images, as symmetric points under two circles, is determined completely by poles of the q-logarithmic function, where dimensionless parameter $q = r_2^2/r_1^2$ is given by square ratio of the cylinder radii. Motivated by Möbius transformation for symmetrical points under generalized circle in complex plain, the system of symmetric spin coherent states corresponding to antipodal qubit states is introduced. By these states we construct the maximally entangled orthonormal two qubit spin coherent state basis, in the limiting case reducible to the Bell basis. Average energy of XYZ model in these states, describing finite localized structure with characteristic extremum points, appears as an energy surface in maximally entangled two qubit space. Generalizations to three and higher multiple qubits are found. We show that our entangled N qubit states are determined by set of complex Fibonacci and Lucas polynomials and corresponding Binet-Fibonacci q-calculus.

1. Introduction

One of the modern directions in which q-calculus [1] plays key role is related with quantum algebras and quantum groups as deformed versions of the usual Lie algebras with deformation parameter q. In nineteen's of twenty century, a big interest to quantum symmetries initiated large amount of work devoted to potential application of quantum symmetries to problems of quantum physics as q-harmonic oscillator, q-hydrogen atom, quantum optics, rotational and vibrational nuclear and molecular spectra, quantum integrable systems etc. Construction of representation theory of quantum groups leads to developing special part of mathematical physics as q-special functions and q-difference equations [2].

In the present paper we apply q-calculus for solution of two, at first glance not related physical problems, one from hydrodynamics and another one from quantum information theory. Both problems are connected with complex analysis notion of symmetrical points with respect to the circle [3].

In hydrodynamics it is related with method of images for bounded circular domain, where symmetrical points correspond to point vortex and its image. Application of method of images for vortex in a domain between concentric circles have been discussed by Poincare [7], who found an infinite set of images as symmetric points and indicated convergency problem for infinite sums of images. For a doubly connected domain it is well known, that any doubly connected region can be one to one and conformally mapped to the annular region bounded by two circles $r_1 < |z| < r_2$. Moreover this canonical domain is unique, up to the linear map. It means that if domain B is mapped to two different annular domains $r_1 < |z| < r_2$ and $r'_1 < |\zeta| < r'_2$, then the last ones are related by linear transformation $\zeta = e^{i\alpha}z$, thus have the same circles ratio $r_2/r_1 = r'_2/r'_1$. Then this ratio plays the role of q-parameter $q = r_2^2/r_1^2$, and solution of the problem for N point vortices can be given in terms of q-logarithmic and Jackson q-exponential functions [4].

In quantum information theory we are dealing with a qubit as a unit of quantum information. In the coherent state representation the qubit is characterized by a point in extended complex plain. Then symmetrical points in the plane determine pair of qubit states from which possible to construct entangled two qubit states [5]. It turns out that this approach can be extended to arbitrary N qubit states. By constructing entangled coherent N qubit states we find that they are determined by the set of complex Fibonacci and Lucas polynomials. The first ones can be treated as q-numbers in Binet-Fibonacci Golden calculus, which we have developed in connection with Golden quantum oscillator in [6].

The paper is organized as follows. In Section 2 we review symmetrical points in complex analysis, corresponding stereographic projections and action of Möbius transformation. In Section 3 after short review of the Milne-Thomson circle theorem for circular domain we formulate new two circle theorem for bounded flow in a region between two concentric circles. Then we find relation of complex potential and complex velocity with q-periodic functions and write solution in terms of Jackson q-integral. As an application, the problem of N point vortices in annular domain is solved by this method. In Section 4 we consider a qubit as a unit of quantum information, the coherent state representation, symmetric qubits and action of Möbius transformation on qubits. Antipodal orthogonal symmetric qubit coherent states are introduced in Section 5. In Section 6 by symmetric states, the system of two qubit maximally entangled coherent states is constructed. As an application of these states the average energy for XYZ model in these states are calculated. Extension for three qubit states and corresponding energy surface are found. Finally, in Section 7 we derive the set of N qubit entangled coherent states determined by the set of complex Fibonacci and Lucas polynomials. By Binet representation these polynomials are treated as q-numbers in the Binet-Fibonacci Golden calculus.

2. Symmetric points and Möbius transformation

In complex analysis [3], two points ψ and ψ^* are called symmetrical with respect to the circle C through ψ_1, ψ_2, ψ_3 if and only if $(\psi^*, \psi_1, \psi_2, \psi_3) = \overline{(\psi, \psi_1, \psi_2, \psi_3)}$ where the cross ratio of four points is

$$(\psi, \psi_1, \psi_2, \psi_3) = \frac{(\psi - \psi_2)(\psi_1 - \psi_3)}{(\psi - \psi_3)(\psi_1 - \psi_2)}. \tag{1}$$

The circle here is considered in the generalized form, that includes also a line, regarded as a circle with an infinite radius. On the Riemann sphere all generalized circles are coming from intersection of the sphere with a plane, so that if the plane passes the north pole, the corresponding projection would be a line. For the unit circle at the origin, we can choose $\psi_1 = -1, \psi_2 = i, \psi_3 = 1$ so that the symmetrical point of ψ is $\psi^* = 1/\bar{\psi}$. It means that points ψ and ψ^* have the same argument and are situated on the same half line from the origin, so that if one of the point is out of the circle, the second one is inside the circle, and vice versa. Hence points $\psi = 0$ and $\psi^* = \infty$ are symmetrical points with respect to the circle. The cross product (1) is invariant under the Möbius transformation, so that if a Möbius transformation carries a generalized circle C_1 into a circle C_2, then it transforms any pair of symmetrical points with respect to C_1 into a pair of symmetrical points with respect to C_2. The above symmetric points have simple meaning on the Riemann sphere:

Journal of Physics: Conference Series **343** (2012) 012093 doi:10.1088/1742-6596/343/1/012093

1. ψ and $\psi^* = \bar{\psi}$ are projections of symmetric points $M(x, y, z)$ and $M^*(x, -y, z)$
2. ψ and $\psi^* = -\bar{\psi}$ are projections of symmetric points $M(x, y, z)$ and $M^*(-x, y, z)$
3. ψ and $\psi^* = \frac{1}{\psi}$ are projections of symmetric points $M(x, y, z)$ and $M^*(x, y, -z)$
4. ψ and $\psi^* = -\frac{1}{\psi}$ are projections of symmetric points $M(x, y, z)$ and $M^*(-x, -y, -z)$

3. Vortex images in annular domain

3.1. The circle theorem

There is the Circle Theorem due to Milne-Thomson [9]. Let $f(z)$ is the complex potential of the two-dimensional irrotational flow of incompressible inviscid fluid in the z-plane ($z = x + iy$), where the singularities of $f(z)$ are all at a distance greater than r from the origin. If circular cylinder with cross section C:$|z| = r$, be introduced into the flow, then the complex potential becomes

$$F(z) = f(z) + \bar{f}\left(\frac{r^2}{z}\right). \tag{2}$$

For complex velocity of the flow $\bar{V}(z) = v_1 - iv_2 = F'(z)$ this theorem can be reformulated in the form

$$\bar{V}(z) = \bar{v}(z) - \frac{r^2}{z^2}\, v\left(\frac{r^2}{z}\right), \tag{3}$$

where $v(z)$ is a complex velocity of the flow in unbounded domain, and the second term represents correction to the complex velocity by cylinder of radius r placed at the origin. The normal velocity of the flow, proportional to $[\bar{V}(z)z + V(\bar{z})\bar{z}]$, vanishes at the surface of the cylinder $z\bar{z} = r^2$.

As an example we consider the point vortex with strength Γ at the point z_0 with complex potential

$$f(z) = \frac{\Gamma}{2\pi i}\ln(z - z_0). \tag{4}$$

Then introducing a cylinder with the center at the origin and $|z_0| > r$ gives

$$F = \frac{\Gamma}{2\pi i}\ln(z - z_0) - \frac{\Gamma}{2\pi i}\ln(\frac{r^2}{z} - \bar{z}_0) = \frac{\Gamma}{2\pi i}\ln(z - z_0) - \frac{\Gamma}{2\pi i}\ln(z - \frac{r^2}{\bar{z}_0}) + \frac{\Gamma}{2\pi i}\ln z + C. \tag{5}$$

This shows that the vortex image is located at symmetric point $z^* = r^2/\bar{z}_0$. For complex velocity we have

$$\bar{V}(z) = \frac{\Gamma}{2\pi i}\left(\frac{1}{z - z_0} - \frac{1}{z - \frac{r_1^2}{\bar{z}_0}} + \frac{1}{z}\right), \tag{6}$$

where the second term represents a vortex of strength $-\Gamma$ at the inverse point $r^2/\bar{z}_0$ with respect to the cylinder, and the last term is the positive strength vortex at the origin. Henceforth, the vortices at inverse point and at the center of cylinder, imitating the circular boundary, we shall call "vortex images" or simply "images". Therefore, in (6), there are two images; one positive image at the centre of the cylinder and another negative image at the inverse point. These two images are used to replace correctly the circular boundary in the infinite 2-D plane.

3.2. The two circles theorem

Here we formulate a new the Annular or the Two Circles Theorem.

3.2.1. The two circles theorem for complex potential Let $f(z)$ is the complex potential of the two-dimensional irrotational flow of incompressible inviscid fluid in the z-plane ($z = x + iy$), where the singularities of $f(z)$ are all at a distance greater than r_1 and less then r_2 from the origin. If two concentric circular cylinders with cross sections C_1:$|z| = r_1$ and C_2: $|z| = r_2$, $r_1 < r_2$, be introduced into the flow, then the complex potential between circles, $r_1 < |z| < r_2$, becomes

$$F(z) = f_q(z) + \bar{f}_q\left(\frac{r_1^2}{z}\right) = \sum_{n=-\infty}^{\infty} f(q^n z) + \sum_{n=-\infty}^{\infty} \bar{f}\left(q^n \frac{r_1^2}{z}\right), \tag{7}$$

where $q = \frac{r_2^2}{r_1^2}$. The proof is easy by observing that on the first circle $z\bar{z} = r_1^2$ by replacing argument in the second sum we get

$$F(z)|_{C_1} = \sum_{n=-\infty}^{\infty} (f(q^n z) + \bar{f}(q^n \bar{z})), \tag{8}$$

which means that the stream function of the flow on the first cylinder vanishes, $\Im F|_{C_1} = 0$. By substitution $r_1^2 = \frac{r_2^2}{r_1^2} r_1^2 = q\, r_1^2$ to the sum (7) and shifting summation index n to $n-1$, we have

$$F(z) = f_q(z) + \bar{f}_q\left(\frac{r_2^2}{z}\right) = \sum_{n=-\infty}^{\infty} f(q^n z) + \sum_{n=-\infty}^{\infty} \bar{f}\left(q^n \frac{r_2^2}{z}\right), \tag{9}$$

so that on the second circle $z\bar{z} = r_2^2$,

$$F(z)|_{C_2} = \sum_{n=-\infty}^{\infty} (f(q^n z) + \bar{f}(q^n \bar{z})) \tag{10}$$

and the stream function vanishes as well, $\Im F|_{C_2} = 0$. Here we should notice that infinite sums in the above theorem require care since could diverge when we separate or combine sums. To make concrete result convergent we can use ambiguity in definition of complex potential, which is determined up to arbitrary complex constant.

3.2.2. The two circles theorem for complex velocity The two circles theorem can be formulated for complex velocity of the flow in the next form:

$$\bar{V}(z) = \frac{(z\bar{v}(z))_q}{z} - \frac{r_1^2}{z^2} \frac{\left(\frac{r_1^2}{z} v\left(\frac{r_1^2}{z}\right)\right)_q}{\frac{r_1^2}{z}}, \tag{11}$$

where

$$f_q(z) \equiv \sum_{n=-\infty}^{\infty} f(q^n z) \tag{12}$$

so that

$$\bar{V}(z) = \sum_{n=-\infty}^{\infty} q^n \bar{v}(q^n z) - \frac{r_1^2}{z^2} \sum_{n=-\infty}^{\infty} q^n v\left(q^n \frac{r_1^2}{z}\right). \tag{13}$$

The proof is as follows. For the first circle $C_1 : \bar{z}z = r_1^2$, we have easily

$$\left[\bar{V}(z)z + V(\bar{z})\bar{z}\right]|_{C_1} = (z\bar{v}(z))_q - \left(\frac{r_1^2}{z} v\left(\frac{r_1^2}{z}\right)\right)_q + c.c. = \tag{14}$$

$$(z\bar{v}(z))_q + (\bar{z}v(\bar{z}))_q - (z\bar{v}(z))_q - (\bar{z}v(\bar{z}))_q = 0. \tag{15}$$

For the second one $C_2 : \bar{z}z = r_2^2$ we can write

$$\left[\bar{V}(z)z + V(\bar{z})\bar{z}\right]|_{C_2} = (z\bar{v}_q(z))_q - \left(\frac{r_1^2}{r_2^2}\frac{r_2^2}{z}v(\frac{r_1^2}{r_2^2}\frac{r_2^2}{z})\right)_q + c.c. \tag{16}$$

or by $q = r_2^2/r_1^2$ we get

$$\left[\bar{V}(z)z + V(\bar{z})\bar{z}\right]|_{C_2} = (z\bar{v}_q(z))_q - (q^{-1}\bar{z}\,v(q^{-1}\bar{z}))_q + (\bar{z}v_q(\bar{z}))_q - (q^{-1}z\,\bar{v}(q^{-1}z))_q. \tag{17}$$

From definition (12) by rescaling and shifting summation index we find next property of "q-function":

$$f_q(qz) = f(z), \tag{18}$$

which means that this function is the q-periodic function. For this function the q-derivative (complex) vanishes

$$D_q^z f_q(z) = \frac{f_q(qz) - f_q(z)}{(q-1)z} = 0. \tag{19}$$

This implies

$$(q^{-1}\bar{z}v(q^{-1}\bar{z}))_q = (\bar{z}v(\bar{z}))_q, \quad (q^{-1}z\bar{v}(q^{-1}z))_q = (z\bar{v}(z))_q. \tag{20}$$

Then on the second circle we have

$$\left[\bar{V}(z)z + V(\bar{z})\bar{z}\right]|_{C_2} = 0. \tag{21}$$

3.2.3. The q-Jackson integral representation The q-functions defined above (12) and the two circles theorem can be rewritten in terms of Jackson q-integral

$$\int f(x)\,d_qx \equiv (1-q)x\sum_{n=0}^{\infty} q^n f(q^n x), \tag{22}$$

so that

$$f_q(z) = \frac{1}{1-q}\int\frac{f(z)}{z}d_qz + \frac{q}{q-1}\int\frac{f(z)}{z}d_{\frac{1}{q}}z - f(z) \tag{23}$$

for complex z. For complex potential it gives

$$F(z) = \frac{1}{1-q}\int\frac{f(z)+\bar{f}(\frac{r_1^2}{z})}{z}d_qz + \frac{q}{q-1}\int\frac{f(z)+\bar{f}(\frac{r_1^2}{z})}{z}d_{\frac{1}{q}}z - f(z) - \bar{f}\left(\frac{r_1^2}{z}\right). \tag{24}$$

3.3. Point vortex between concentric circles

Now we apply the above two circles theorem to the problem of point vortex located at z_0 between two concentric circles $r_1 < |z_0| < r_2$. By using complex potential (4) and theorem (7) we get

$$F(z) = \frac{\Gamma}{2\pi i}\sum_{n=-\infty}^{\infty}\left[\ln(z - z_0q^n) - \ln(z - \frac{r_1^2}{\bar{z}_0}\,q^n)\right] = \frac{\Gamma}{2\pi i}\sum_{n=-\infty}^{\infty}\ln\frac{z - z_0q^n}{z - \frac{r_1^2}{\bar{z}_0}\,q^n}. \tag{25}$$

This gives a clear picture of the structure of vortex images. For a given vortex at z_0, the first positive sum represents contribution from the vortex and the infinite set of its even images, while the second, negative sum corresponds to odd images. The infinite set of points $..., q^{-n}z_0, ..., q^{-2}z_0, q^{-1}z_0, z_0, qz_0, q^2z_0, ..., q^nz_0, ...$ we call the q-chain. Then the set of vortex images forms two q-chains generated by vortex Γ at z_0 and its image $-\Gamma$ at $r_1^2/\bar{z}_0$.

For complex velocity we have

$$\bar{V}(z) = F'(z) = \frac{\Gamma}{2\pi i} \sum_{n=-\infty}^{\infty} \left[\frac{1}{z - z_0 q^n} - \frac{1}{z - \frac{r_1^2}{\bar{z}_0} q^n} \right]. \tag{26}$$

Here we should notice that due to the multiple connected character of the domain, this expression is not unique and admits next freedom

$$\bar{V}(z) \to \bar{V}(z) + \frac{\Gamma_0}{2\pi i} \frac{1}{z} \tag{27}$$

corresponding to a vortex with arbitrary strength Γ_0 at the origin. This arbitrary parameter, determining point vortex at the origin can not be fixed by the boundary conditions. To fix arbitrariness of this solution, we have to impose an additional constraint. We can chose it in the form

$$\oint_{C_1} \bar{V}(z) dz = 0. \tag{28}$$

This condition can be justified by correctness of the limiting procedure $r_1 \to 0$, $r_2 \to \infty$ to the planar problem, so that no singularity at the origin should arise. By the residues theorem as a result we have next image representation [10]

$$\bar{V}(z) = \sum_{k=1}^{N} \frac{\Gamma_k}{2\pi i} \sum_{n=-\infty}^{\infty} \left[\frac{1}{z - z_k q^n} - \frac{1}{z - \frac{r_1^2}{\bar{z}_k} q^n} \right] + \frac{1}{2\pi i z} \sum_{k=1}^{N} \Gamma_k, \tag{29}$$

for N point vortices of strength $\Gamma_1, \Gamma_2, ..., \Gamma_N$ located at positions $z_1, z_2, ..., z_N$ correspondingly. In the limit of the vortices outside of a cylinder with radius $r_1 = constant$, $q \to \infty$ and $r_2 \to \infty$, it gives result

$$\bar{V}(z) = \sum_{k=1}^{N} \frac{\Gamma_k}{2\pi i} \left[\frac{1}{z - z_k} - \frac{1}{z - \frac{r_1^2}{\bar{z}_k}} \right] + \frac{1}{2\pi i z} \sum_{k=1}^{N} \Gamma_k, \tag{30}$$

coinciding with the circle theorem (6). If in (26) instead of r_1 we use expression for r_2 then following the same procedure we find that in this case we should fix $\Gamma_0 = 0$. So that we have alternative expression

$$\bar{V}(z) = \sum_{k=1}^{N} \frac{\Gamma_k}{2\pi i} \sum_{n=-\infty}^{\infty} \left[\frac{1}{z - z_k q^n} - \frac{1}{z - \frac{r_2^2}{\bar{z}_k} q^n} \right]. \tag{31}$$

In the limit of the vortices inside of a cylinder with radius $r_2 = constant$, $q \to \infty$ and $r_1 \to 0$, it gives result

$$\bar{V}(z) = \sum_{k=1}^{N} \frac{\Gamma_k}{2\pi i} \left[\frac{1}{z - z_k} - \frac{1}{z - \frac{r_2^2}{\bar{z}_k}} \right], \tag{32}$$

which also coincides with the circle theorem. This shows that the two circles theorem generalizes the circle theorem of Milne-Thomson and reduces to the last one in the limit $q- > \infty$.

4. Möbius transformation and qubit

In quantum computations we have a qubit as a unit of information

$$|\psi>= \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad |\psi_1|^2 + |\psi_2|^2 = 1, \tag{33}$$

then, in terms of homogeneous coordinate $\psi = \psi_2/\psi_1$ up to the global phase we have normalized qubit state

$$|\psi>= \frac{1}{\sqrt{1+|\psi|^2}} \begin{pmatrix} 1 \\ \psi \end{pmatrix} \tag{34}$$

as spin 1/2 generalized coherent state [8]. From another side, the qubit

$$|\theta, \varphi>= \cos\frac{\theta}{2}|0> + \sin\frac{\theta}{2}e^{i\varphi}|1>= \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\varphi} \end{pmatrix} \tag{35}$$

determined by point (θ, φ) on the Bloch sphere, and parameterized by the homogeneous variable $\psi = \frac{\psi_2}{\psi_1} = \tan\frac{\theta}{2}e^{i\phi}$ determines the stereographic projection of point $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ on the unit sphere to the complex plane ψ. Therefore the Bloch sphere, considered as a Riemann sphere for the extended complex plane ψ by the stereographic projection, determines the $SU(2)$ or the spin coherent state

$$|\psi>= \frac{|0> + \psi|1>}{\sqrt{1+|\psi|^2}}. \tag{36}$$

The computational basis states $|0>=|\uparrow>= \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ and $|1>=|\downarrow>= \begin{pmatrix} 0 & 1 \end{pmatrix}^T$ in this coherent state representation are just points in extended complex plane $(\Re\psi, \Im\psi) \cup \{\infty\}$, as $\psi = 0$ and $\psi = \infty$ respectively. These points are symmetrical points under the unit circle at the origin.

4.1. Symmetric qubits

As we have seen in Section 3, symmetric points are important in the hydrodynamic theory and are related with the method of images. For point vortex in the plane bounded by the cylindrical domain or the annular domain, the symmetrical points represent vortex and its images. Now we like to introduce the coherent states corresponding to symmetric points, representing symmetrical pair of qubits with remarkable properties. Since the unit circle in the ψ plane: $|\psi|^2 = 1$, represents equator on the Bloch sphere, then any point on upper hemisphere projects to the external part of the unit circle. While the lower hemisphere is projected to internal part of the circle. It is easy to see that if point $M(x, y, z)$ is projected to ψ, then reflected in equator point $M^*(x, y, -z)$ is projected to the symmetrical point ψ^*. According to these two points, for given qubit

$$|\theta, \varphi>= \cos\frac{\theta}{2}|0> + \sin\frac{\theta}{2}e^{i\varphi}|1>= \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\varphi} \end{pmatrix} \tag{37}$$

we have "symmetric" qubit state

$$|\pi - \theta, \varphi>= \sin\frac{\theta}{2}|0> + \cos\frac{\theta}{2}e^{i\varphi}|1>= \begin{pmatrix} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\, e^{i\varphi} \end{pmatrix}. \tag{38}$$

This pair of qubit states defines the symmetric qubit coherent states. The corresponding points M and M^* on Bloch sphere are mirror images of each other in coordinate plane xy. For every complex number ψ as projection of point (θ, ϕ), we have the coherent state (34). Then every symmetric point determines the symmetric coherent state. For point $\psi^* = \frac{1}{\bar{\psi}}$. the symmetric coherent state of qubit is

$$|\psi^* >= |\frac{1}{\bar{\psi}} >= \frac{\bar{\psi}|0> + |1>}{\sqrt{1+|\psi|^2}}. \tag{39}$$

In the limiting case $\psi = 0$ and $\psi^* = \infty$ for symmetric points we get computational basis: $|\psi = 0 >= |1>$, $|\psi^* = \infty >= |0>$. Now, if one has dealing with one qubit gate represented by the linear unitary transformation, then it transforms the unit circle at origin to a generalized circle in such a way that symmetric points in the first circle transform to symmetric points with respect to the new one. It defines the transformation rule for symmetric qubit states.

5. Antipodal orthogonal symmetric coherent qubit states

According to our definition of symmetric coherent states, expansion of an arbitrary qubit state in computational basis $|\phi >= c_1|0> + c_2|1>$ can be considered as an expansion to specific symmetrical coherent states. Then we have natural generalization of this expansion to arbitrary symmetrical states

$$|\phi >= d_1|\psi> + d_2|\psi^*> \tag{40}$$

considering states $|\psi>$ and $|\psi^*>$ as a basis. However this basis is not orthonormal due to

$$<\psi^*|\psi>= \frac{2|\psi|}{1+|\psi|^2} \leq 1. \tag{41}$$

To have the orthogonal states for given state $|\psi>$ we consider the negative-symmetric state $|-\psi^*>$. This state is represented by point $-\psi^* = -1/\bar{\psi}$ which is rotation of the symmetric point ψ^* on angle π, and which belongs to the line through points ψ and ψ^*. We call this point as the negative symmetric point or negative mirror image and corresponding coherent state as the negative-symmetric coherent state. On the Bloch sphere for point $M(x, y, z)$ representing qubit state $|\theta, \varphi>$, it is given by antipodal point $M^*(-x, -y, -z)$ corresponding to state

$$|\pi - \theta, \varphi + \pi >= \sin\frac{\theta}{2}|0> - \cos\frac{\theta}{2}e^{i\varphi}|1>= \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{i\varphi} \end{pmatrix}, \tag{42}$$

which we call the antipodal qubit state. We have explicitly

$$|-\psi^* >= \frac{|0> - \psi^*|1>}{\sqrt{1+|\psi^*|^2}} = \frac{|\psi||0> - \frac{|\psi|}{\psi}|1>}{\sqrt{1+|\psi|^2}}. \tag{43}$$

Up to phase this state can be written in the form

$$|-\psi^* >= \frac{-\bar{\psi}|0> + |1>}{\sqrt{1+|\psi|^2}}. \tag{44}$$

In contrast to symmetric state (39), the negative-symmetric state (43) is orthogonal to $|\psi>$:

$$<-\psi^*|\psi>= 0 \tag{45}$$

Then, states $|\psi>$ and $|-\psi^*>$ form the orthonormal basis so that for any state

$$|\phi >= e_1|\psi> + e_2|-\psi^*> \tag{46}$$

we have

$$e_1 =< \psi|\phi >= \frac{c_1 + c_2\bar{\psi}}{\sqrt{1+|\psi|^2}}, \ \ e_2 =< -\psi^*|\phi >= \frac{-\psi c_1 + c_2}{\sqrt{1+|\psi|^2}}. \tag{47}$$

The coherent state $|\psi >$, $\psi \in C$, for arbitrary spin j is defined by

$$|\psi >= \frac{1}{(1+|\psi|^2)^j} \sum_{k=0}^{2j} \left(\frac{(2j)!}{k!(2j-k)!} \right)^{1/2} \psi^k |j, -j+k >, \tag{48}$$

and for the scalar product of two coherent states we have

$$< \phi|\psi >= \frac{(1+\bar{\phi}\,\psi)^{2j}}{(1+|\phi|^2)^j (1+|\psi|^2)^j}. \tag{49}$$

Then orthogonality condition implies $1 + \bar{\phi}\,\psi = 0$ or the negative -symmetric point in the unit circle $\phi = -\frac{1}{\bar{\psi}}$. Representation of these coherent states in terms of unit vector $\mathbf{n}$

$$< \mathbf{n_1}|\mathbf{n_2} >= e^{i\theta(\mathbf{n_1},\mathbf{n_2})} \left(\frac{1+\mathbf{n_1}\mathbf{n_2}}{2} \right)^j \tag{50}$$

shows that the above points are antipodal points on the sphere $\mathbf{n_1}\mathbf{n_2} = -1$.

6. Two qubit coherent states

Here we consider two qubit coherent state

$$|\psi_1 > |\psi_2 >= \frac{1}{\sqrt{1+|\psi_1|^2}\sqrt{1+|\psi_2|^2}} \left(\begin{array}{cccc} 1 & \psi_2 & \psi_1 & \psi_1\psi_2 \end{array} \right)^T. \tag{51}$$

By proper choice of ψ_1 and ψ_2 we can construct two qubit orthonormal coherent state basis

$$\begin{aligned}
|\psi > |\psi > &= \frac{1}{1+|\psi|^2} \left(\begin{array}{cccc} 1 & \psi & \psi & \psi^2 \end{array} \right)^T, & (52)\\
|\psi > | - \psi^* > &= \frac{1}{1+|\psi|^2} \left(\begin{array}{cccc} -\bar{\psi} & 1 & -|\psi|^2 & \psi \end{array} \right)^T, & (53)\\
| - \psi^* > |\psi > &= \frac{1}{1+|\psi|^2} \left(\begin{array}{cccc} -\bar{\psi} & -|\psi|^2 & 1 & \psi \end{array} \right)^T, & (54)\\
| - \psi^* > | - \psi^* > &= \frac{1}{1+|\psi|^2} \left(\begin{array}{cccc} \bar{\psi}^2 & -\bar{\psi} & -\bar{\psi} & 1 \end{array} \right)^T. & (55)
\end{aligned}$$

It can be generated from the computational basis by operator

$$U = \frac{1}{\sqrt{1+|\psi|^2}} \left(\begin{array}{cc} 1 & -\bar{\psi} \\ \psi & 1 \end{array} \right) \tag{56}$$

applied to proper one qubit states. But due to separability, these states are not entangled as well as the computational basis. However we can generate maximally entangled Bell states from the computational basis by using a combination of Hadamard gate and a $CNOT$ gate. This allows us to introduce the new set of two qubit coherent states.

6.0.1. Maximally entangled two qubit states from coherent state basis Now we introduce the set of two qubit coherent states

$$|P_\pm> \;=\; \frac{1}{\sqrt{2}}(|\psi>|\psi>\pm|-\psi^*>|-\psi^*>), \tag{57}$$

$$|G_\pm> \;=\; \frac{1}{\sqrt{2}}(|\psi>|-\psi^*>\pm|-\psi^*>|\psi>). \tag{58}$$

These states generalize the Bell states and reduce to the last ones in the limit $\psi \to 0$ and $-\frac{1}{\psi} \to \infty$. We can show that it is maximally entangled set of orthogonal two qubit states.

Explicitly for these states we have

$$|P_+>=\frac{1}{\sqrt{2}(1+|\psi|^2)}\begin{pmatrix} 1+\bar{\psi}^2 \\ \psi-\bar{\psi} \\ \psi-\bar{\psi} \\ 1+\psi^2 \end{pmatrix}, \quad |P_->=\frac{1}{\sqrt{2}(1+|\psi|^2)}\begin{pmatrix} 1-\bar{\psi}^2 \\ \psi+\bar{\psi} \\ \psi+\bar{\psi} \\ -1+\psi^2 \end{pmatrix}, \tag{59}$$

$$|G_+>=\frac{1}{\sqrt{2}(1+|\psi|^2)}\begin{pmatrix} -2\bar{\psi} \\ 1-|\psi|^2 \\ 1-|\psi|^2 \\ 2\psi \end{pmatrix}, \quad |G_->=\frac{1}{\sqrt{2}(1+|\psi|^2)}\begin{pmatrix} 0 \\ 1+|\psi|^2 \\ -1-|\psi|^2 \\ 0 \end{pmatrix}. \tag{60}$$

6.0.2. Concurence The concurrence for pure states in the determinant form is $C_{12} = \begin{vmatrix} t_{00} & t_{01} \\ t_{10} & t_{11} \end{vmatrix}$, where $t_{ij}, (i,j=0,1)$ are coefficients of expansion for states $|\psi>$ in computational basis. Applying this definition to states (59), (60) we find that concurrence $C_{12}=1$ and these states are maximally entangled states.

The reduced density matric method and the average spin method give the same result [5] that the set of these two qubit spin coherent states is maximally entangled orthonormal set.

6.1. Maximally entangled energy surface for XYZ model
As an application here we calculate average energy for XYZ model

$$H=\frac{1}{2}[J_x\sigma_1^x\sigma_2^x+J_y\sigma_1^y\sigma_2^y+J_z\sigma_1^z\sigma_2^z] \tag{61}$$

in two qubit spin coherent states (57), (58). For the state $|P_->$ we have

$$<P_-|H|P_->=\frac{2J_+(\psi+\bar{\psi})^2-J_-[(1-\psi^2)^2+(1-\bar{\psi}^2)^2]+J_z[(1-\psi^2)(1-\bar{\psi}^2)-(\psi+\bar{\psi})^2]}{2(1+|\psi|^2)^2} \tag{62}$$

In Fig.1a we show this energy surface as function of $x=\Re\psi, y=\Im\psi$ with characteristic local minima points. For the state $|G_+>$ we have energy

$$<G_+|H|G_+>=\frac{2J_+(1-|\psi|^2)^2-4J_-[\psi^2+\bar{\psi}^2]+J_z[4|\psi|^2-(1-|\psi|^2)^2]}{2(1+|\psi|^2)^2} \tag{63}$$

and in Fig.1b we show the average energy surface for this state. It has local maxima at the origin and set of minima at the unit circle $x^2+y^2=1$.

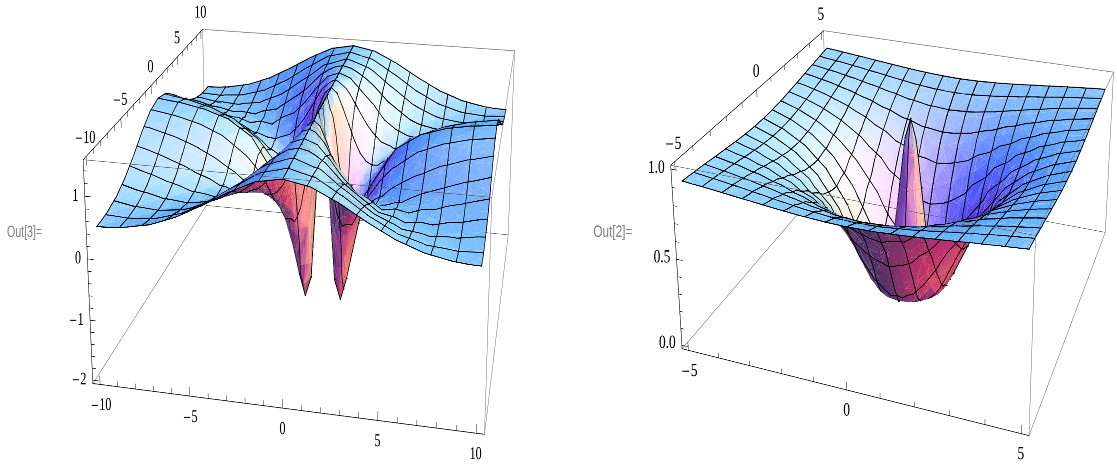

Figure 1. XYZ average energy in maximally entangled state a) $|P_- >$ state for $J_+ = 1, J_- = -0.5, J_z = 2$ b) $|G_+ >$ state for $J_+ = 1, J_- = 0, J_z = 0$

6.2. Three qubit case XYZ model

Now we consider three qubit coherent state

$$|PG_+ >= \frac{1}{\sqrt{2}}(|\psi > |\psi > |\psi > + | - \psi^* > | - \psi^* > | - \psi^* >). \tag{64}$$

This state can be obtained from maximally entangled GHZ state

$$|GHZ >= \frac{1}{\sqrt{2}}(|000 > + |111 >) \tag{65}$$

by unitary transformation $U = U \otimes U \otimes U$. This state is also maximally entangled and in the special case $\psi \to 0$ and $\psi^* \to \infty$ reduces to the GHZ state. Then we have energy

$$< PG_+|H|PG_+ >= \frac{4J_+|\psi|^2(1+|\psi|^2) + 2J_-(1+|\psi|^2)(\psi^2+\bar{\psi}^2) + J_z(1-|\psi|^2-|\psi|^4+|\psi|^6)}{(1+|\psi|^2)^3} \tag{66}$$

It is shown in Fig.2a and has four local extremum points with two maxima and two minima.

Another three qubit coherent state

$$|PG_- >= \frac{1}{\sqrt{3}}(|\psi > |\psi > | - \psi^* > + |\psi > | - \psi^* > |\psi > + | - \psi^* > |\psi > |\psi >) \tag{67}$$

is related with maximally entangled $|W >$ state

$$|W >= \frac{1}{\sqrt{3}}(|0 > |0 > |1 > + |0 > |1 > |0 > + |1 > |0 > |0 >). \tag{68}$$

For energy in this state we have

$$< PG_-|H|PG_- >= \frac{4J_+(1+|\psi|^6) - 6J_-(1+|\psi|^2)(\psi^2+\bar{\psi}^2) - J_z(1-9|\psi|^2-9|\psi|^4+|\psi|^6)}{3(1+|\psi|^2)^3} \tag{69}$$

It is shown in Fig.2b.

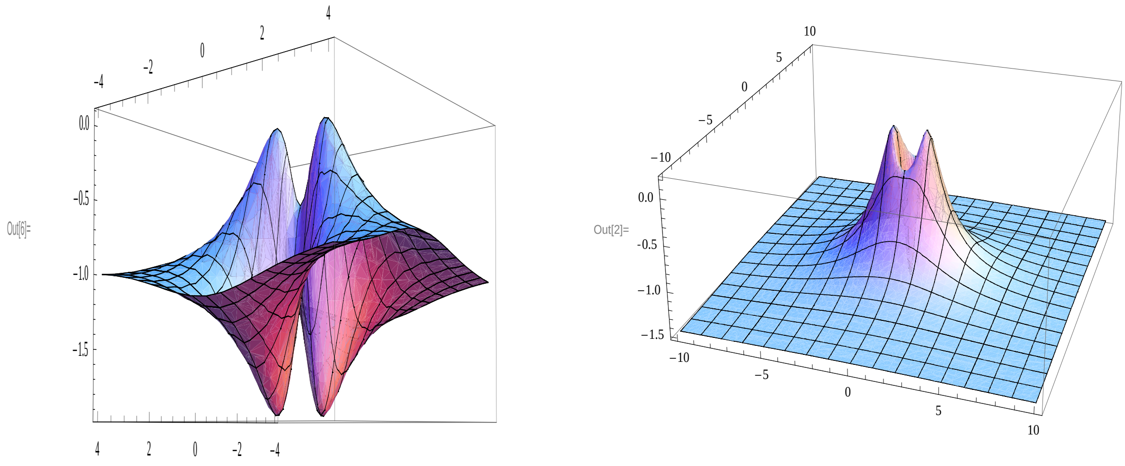

Figure 2. XYZ average energy in maximally entangled state a) $|PG_+>$ state for $J_+ = -1, J_- = -1, J_z = -1$ b) $|PG_->$ state for $J_+ = -1, J_- = -0.2, J_z = 0.5$

7. Entangled N qubit spin coherent states

This construction can be extended to arbitrary N-qubit coherent states. The first set of N-qubit entangled states expanded in computational basis is

$$\frac{|\psi>^N - |-\frac{1}{\bar{\psi}}>^N}{\psi + \bar{\psi}^{-1}} = F_1(\alpha,\beta)(|10...0> + |01...0> + ...|00...1>) \tag{70}$$

$$+F_2(\alpha,\beta)(|110...0> + |101...0> + ...|00...11>) \tag{71}$$

$$... + F_N(\alpha,\beta)(|111...1> \tag{72}$$

and is characterized by the set of complex Fibonacci polynomials $F_n(\alpha,\beta)$, with recurrence relation

$$F_{n+1}(\alpha,\beta) = \alpha F_n(\alpha,\beta) + \beta F_{n-1}(\alpha,\beta), \tag{73}$$

where $\alpha = \psi - \frac{1}{\bar{\psi}}$, $\beta = \frac{\psi}{\bar{\psi}}$.

Another set of entangled N-qubit coherent states is

$$|\psi>^N + |-\frac{1}{\bar{\psi}}>^N = |00...0> + L_1(\alpha,\beta)(|10...0> + |01...0> + ...|00...1>) \tag{74}$$

$$+L_2(\alpha,\beta)(|110...0> + |101...0> + ...|00...11>) \tag{75}$$

$$... + L_N(\alpha,\beta)(|111...1> \tag{76}$$

and is characterized by complex Lucas polynomials $L_n(\alpha,\beta) = \psi^n + (-\frac{1}{\bar{\psi}})^n$. In the above Binet representations of complex polynomials $F_n(\alpha,\beta)$ and $L_n(\alpha,\beta)$, the negative-symmetric points ψ and $-\frac{1}{\bar{\psi}}$ are roots of complex quadratic equation $z^2 = \alpha z + \beta$, where $\alpha = \psi - \frac{1}{\bar{\psi}}$ and $\beta = \frac{\psi}{\bar{\psi}}$. From polar representation of complex numbers $\psi = qe^{i\phi}$ and $z = re^{i\phi}$ we get $r^2 = ar + 1$, where $a = q - \frac{1}{q}$, and $r^n = rF_n(a) + F_{n-1}(a)$ with Fibonacci polynomials $F_n(a)$ [11]:

$$F_1(a) = 1, \quad F_2(a) = a,$$

$$F_{n+1}(a) = aF_n(a) + F_{n-1}(a), \quad \text{for } n \geq 2,$$

when $a = 1$: $F_n(1) = F_n$ are Fibonacci numbers. The Binet representation for these polynomials is

$$F_n(a) = \frac{q^n - (-\frac{1}{q})^n}{q - (-\frac{1}{q})}, \tag{77}$$

where parameter $a = q - \frac{1}{q}$, so that $q = \frac{a+\sqrt{a^2+4}}{2}$ and $-\frac{1}{q} = \frac{a-\sqrt{a^2+4}}{2}$ are roots of quadratic equation $x^2 = ax + 1$.

Then complex Fibonacci polynomials are related with standard Fibonacci polynomials by formula

$$F_n(\alpha, \beta) = F_n(a)\, e^{i\phi(n-1)} \tag{78}$$

and

$$L_n(\alpha, \beta) = L_n(a)\, e^{in\phi}, \tag{79}$$

where $\phi = arg\, \psi$. Complex parameter $\alpha = \psi - \frac{1}{\bar{\psi}}$ has simple geometrical meaning as a complex difference between symmetrical points in unit circle.

The interesting point to note here is that as we have seen the symmetric points under the unit circle appear in the problem of vortex images in circular domain [4], where these points correspond to the line vortex at ψ and its image in the circle at $\frac{1}{\bar{\psi}}$. Then parameter a, $\alpha = ae^{i\phi}$ in Fibonacci polynomials has simple geometrical meaning as the distance between vortex and its image. In particular case when this distance is equal one, $a = q - \frac{1}{q} = 1$, position of the vortex is at the Golden Ratio distance from origin $r = \varphi = \frac{1+\sqrt{5}}{2}$ and Fibonacci polynomials turn to Fibonacci numbers. In this case the line interval connecting vortex and the negative-symmetric point, intersects the unit circle at a point which divide this interval in two parts of length φ and $\frac{1}{\varphi}$.

7.1. Acknowledgments

This work was support by TUBITAK, TBAG Project 110T679 and Izmir Institute of Technology .

References

[1] Kac V and Cheung P 2002 Quantum Calculus, (Springer)
[2] Ismail M 2005 Classical and Quantum Orthogonal Polynomials in One Variable (London:Cambridge University Press)
[3] Ahlfors L 1979 Complex Analysis, (McGraw-Hill)
[4] Pashaev O K and Yilmaz O 2008 *J. Phys A: Math Theor* **41** 135207
[5] Pashaev O K and Gurkan Z N 2011 Solitons in maximally entangled two qubit phase space *Preprint* quant-ph/1107.1397
[6] Pashaev O K and Nalci S 2011 Golden quantum oscillator and Binet-Fibonacci calculus *Preprint* mathQA/1107.4389
[7] Poincare H 1983. Theorie des Tourbillions, (Paris: Georges Carre)
[8] Perelomov A 1986 Generalized Coherent States and Their Applications, (Berlin: Springer)
[9] Milne-Thomson L M 1968 Theoretical Hydrodynamics, (London: Macmillan)
[10] Pashaev O K and Yilmaz O 2011 *J. Phys A: Math Theor* **44** 185501
[11] Koshy T 2001 Fibonacci and Lucas Numbers with Applications, (John Wiley and Sons)