Journal of Physics: Conference Series **343** (2012) 012020 doi:10.1088/1742-6596/343/1/012020

Damped parametric oscillator and exactly solvable complex Burgers equations

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Abstract. We obtain exact solutions of a parametric Madelung fluid model with dissipation which is linearazible in the form of Schrödinger equation with time variable coefficients. The corresponding complex Burgers equation is solved by a generalized Cole-Hopf transformation and the dynamics of the pole singularities is described explicitly. In particular, we give exact solutions for variable parametric Madelung fluid and complex Burgers equations related with the Sturm-Liouville problems for the classical Hermite, Laguerre and Legendre type orthogonal polynomials.

1. Introduction

In 1926, E. Schrödinger derived the fundamental equation of quantum mechanics and few months later E. Madelung proposed a representation of the complex wave function in terms of modulus and phase, $\Psi = \sqrt{\rho}\exp(iS/\hbar)$. It converts the linear Schrödinger equation into a system of nonlinear hydrodynamic-like equations, [1]. Originally, Madelung introduced his equations as an alternative formulation for the Schrödinger equation, and since then, the Madelung equations provide a basis for numerous classical interpretations of quantum mechanics, [2]. Later, Madelung fluid description of quantum systems became a useful tool for describing the evolution of classical/quantumlike systems, and studying the dispersionless or semiclassical limit of nonlinear partial differential equations of Schrödinger type, [3]. As known, the Madelung fluid representation is fundamental also in superconductivity theory, [4] and in the description of quantum fluids like superfluid He, [5], where the Madelung hydrodynamic variables have direct physical meaning, such as $\rho = |\Psi|^2$ plays the role of the superfluid density, and $v = \nabla(arg\Psi)$ is the superfluid velocity.

From mathematical point of view, we emphasize a remarkable property concerning integrability of the Madelung models. Since Madelung transform of the linear Schrödinger equation leads to nonlinear Madelung systems, using an inverse Madelung transform, clearly the nonlinear Madelung models are linearizable in the form of Schrödinger equation. Then, the inverse Madelung transform is a complex linearization transform similar to the Cole-Hopf transformation, and nonlinear models admitting such type of direct linearization are known as C-integrable models. From this perspective, Madelung fluid model and the complex Burgers equation are nonlinear systems belonging to the class of C-integrable models. This allows us to describe their nonlinear dynamics in terms of the corresponding linear Schrödinger equation.

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Long time ago the parametric Schrödinger model for harmonic oscillator was proposed by A. Sakharov as descriptive of quantum cosmological models, [6]. Moreover, in molecular physics, quantum chemistry, quantum optics and plasma physics many quantum-mechanical effects are treated by means of time-dependent oscillator, [7], [8]. Then, the Madelung fluid representation appears as a dual description of the same realistic systems in terms of quantum hydrodynamic variables.

Motivated by this idea, in [9], we found exact solutions for nonlinear parametric Madelung model related with the Caldirola-Kanai dissipative oscillator, [10],[11]. It was seen that the probability density $\rho_k(q,t)$ squeezes and tends to Dirac-delta distribution as $t \to \infty$, and the moving pole singularities $q_k^l(t),\ l = 1, 2, ..., k$ of the complex velocity function $V_k(q,t)$ are merging with time due to dissipation, i.e. $|q_k^l(t)| \to 0$ as $t \to \infty$.

In this work, we introduce generic parametric Madelung fluids with dissipation and complex Burgers equations linearizable in the form of parametric Schrödinger equation for harmonic oscillator related with the classical orthogonal polynomials. Indeed, quantization of the singular Sturm-Liouville problems provides a rich set of exactly solvable harmonic oscillator models with time dependent parameters, known as "quantum Sturm-Liouville problems", [12]. Then, exact solutions of the corresponding Madelung models are obtained using generalized Cole-Hopf transformation. Precisely, we discuss Madelung fluid and complex Burgers equations of Hermite, Laguerre and Legendre type. For each type, exact solutions are obtained, dynamics of the zeros of the probability density, and dynamics of the pole singularities of the complex velocity function are described explicitly. Computer plots are given to illustrate some typical behavior of the solutions.

2. Quantum Sturm-Liouville problems and the Madelung representation

In this section we outline some basic results discussed in [12] and [9]. Consider the second order differential equation

$$\frac{d}{dt}\left[\mu(t)\frac{d}{dt}\right]x(t) + [\lambda r(t)]\,x(t) = 0, \quad a < t < b, \tag{1}$$

where λ is a spectral parameter, $\mu(t) > 0$ on (a,b), $\mu(t) \in C^1(a,b)$, and $\lambda r(t) > 0$ on (a,b), $r(t) \in C(a,b)$. If singularities are allowed at the end points of the fundamental domain (a,b), and boundary conditions are imposed like $x(t)$ must be continuous or bounded or become infinite of an order less than the prescribed, one has the well known singular Sturm-Liouville problem, [13]. On the other hand, Eq.(1) can be written in the form of damped parametric oscillator $\ddot{x} + (\dot{\mu}(t)/\mu(t))\dot{x} + (\lambda r(t)/\mu(t))x = 0$, $t \in (a,b)$, where $\Gamma(t) = \dot{\mu}(t)/\mu(t)$ is the damping coefficient and $\lambda r(t)/\mu(t)$ is the frequency. Then, self-adjoint quantization leads to time-dependent quantum Hamiltonian, and Schrödinger equations for harmonic oscillator related with the classical orthogonal polynomials, also known as quantum Sturm-Liouville problems, see [12]. For generality, we shall use notation $\lambda r(t) \equiv \mu(t)\omega^2(t)$, and consider the IVP

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2\mu(t)}\frac{\partial^2\Psi}{\partial q^2} + \frac{\mu(t)\omega^2(t)}{2}q^2\Psi, \tag{2}$$

$$\Psi(q,t_0) = \psi(q), \quad -\infty < q < \infty. \tag{3}$$

It is known that, if $x(t)$ is solution of the IVP for the classical parametric oscillator

$$\ddot{x} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{x} + \omega^2(t)x = 0, \ \ x(t_0) = x_0 \neq 0, \ \ \dot{x}(t_0) = 0, \tag{4}$$

and if the initial function $\Psi(q,t_0)$ is given as

$$\varphi_k(q) = N_k e^{-\frac{\Omega_0}{2}q^2} H_k(\sqrt{\Omega_0}q), \quad k = 0,1,2,..., \tag{5}$$

then the time-evolved state is

$$\Psi_k(q,t) = N_k\sqrt{R(t)} \times \exp\left(i\left(k+\frac{1}{2}\right)\arctan(\Omega_0 g(t))\right)$$
$$\times \exp\left(i\left(\frac{\mu(t)\dot{x}(t)}{2\hbar x(t)} - \frac{\Omega_0^2}{2}g(t)R^2(t)\right)q^2\right) \times \exp\left(-\frac{\Omega_0}{2}R^2(t)q^2\right) \times H_k\left(\sqrt{\Omega_0}R(t)q\right), \tag{6}$$

where

$$g(t) = -\hbar x^2(t_0)\int^t \frac{d\xi}{\mu(\xi)x^2(\xi)}, \; g(t_0) = 0; \quad R(t) = \left(\frac{x_0^2}{x^2(t) + (\Omega_0 x(t) g(t))^2}\right)^{\frac{1}{2}}, \tag{7}$$

and the corresponding probability density $\rho_k(q,t) = |\Psi_k(q,t)|^2$ becomes

$$\rho_k(q,t) = \frac{1}{2^k k!\sqrt{\pi}} \times \sqrt{\Omega_0}R(t) \times \exp\left(-\left(\sqrt{\Omega_0}R(t)q\right)^2\right) \times H_k^2\left(\sqrt{\Omega_0}R(t)q\right), \quad k = 0,1,2,.... \tag{8}$$

Now, writing the wave function in polar form

$$\Psi(q,t) = \sqrt{\rho(q,t)}\exp\left(\frac{i}{\hbar}S(q,t)\right) = \exp\left(\frac{1}{2}\ln\rho(q,t) + \frac{i}{\hbar}S(q,t)\right), \tag{9}$$

where $\rho(q,t) \geq 0$, $S(q,t)$ is real-valued function and setting $v(q,t) = (1/\mu(t))(\partial S/\partial q)$, Schrödinger equation (2) transforms to a system of parametric Madelung fluid equations with dissipation

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{\dot{\mu}(t)}{\mu(t)}v + v\dfrac{\partial v}{\partial q} = -\dfrac{1}{\mu(t)}\dfrac{\partial}{\partial q}\left[\dfrac{-\hbar^2}{2\mu(t)}\left(\dfrac{1}{\sqrt{\rho}}\dfrac{\partial^2\sqrt{\rho}}{\partial q^2}\right) + \dfrac{\mu(t)\omega^2(t)}{2}q^2\right], \\ \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial q}\left[\rho v\right] = 0. \end{cases} \tag{10}$$

Here, $\rho(q,t)$ is the probability density, $v(q,t)$ is the velocity field and $\Gamma(t) = \dot{\mu}(t)/\mu(t)$ is the friction coefficient, which for nonconstant $\mu(t)$ reflects the dissipative nature of the system. With real-valued initial conditions $v(q,t_0) = \tilde{v}(q)$, $\rho(q,t_0) = \tilde{\rho}(q) \geq 0$, it has formal solution

$$v(q,t) = -\frac{i\hbar}{\mu(t)}\frac{\partial}{\partial q}\ln\left(\frac{\Psi(q,t)}{|\Psi(q,t)|}\right), \qquad \rho(q,t) = |\Psi(q,t)|^2, \tag{11}$$

where $\Psi(q,t)$ is solution of the Schrödinger equation (2) with initial condition

$$\Psi(q,t_0) = \sqrt{\tilde{\rho}(q)}\exp\left(\frac{i}{\hbar}\mu(t_0)\int^q \tilde{v}(\xi)d\xi\right). \tag{12}$$

Clearly, the explicit form of the solutions $v(q,t)$ and $\rho(q,t)$ depends on the properties of the initial functions $\tilde{v}(q)$ and $\tilde{\rho}(q)$. In this work, we shall consider initial functions $\tilde{v}(q)$, $\tilde{\rho}(q)$, so that in (12) one has $\Psi(q,t_0) \in L_2(R)$. Specifically, for all particular problems the initial conditions will be taken such that $\Psi(q,t_0) = \varphi_k(q)$, where $\varphi_k(q)$ is given by (5).

Another representation of the wave function in the form

$$\Psi(q,t) = \exp\left(\frac{i}{\hbar}\mu(t)\int^q V(\xi,t)d\xi\right), \tag{13}$$

where $V(q,t)$ is a complex velocity, transforms Schrödinger equation (2) to a nonlinear complex Burgers equation with time dependent coefficients. Then, the corresponding general IVP

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{\dot{\mu}(t)}{\mu(t)}V + V\dfrac{\partial V}{\partial q} + \omega^2(t)q = \dfrac{i\hbar}{2\mu(t)}\dfrac{\partial^2 V}{\partial q^2} \ , \\ V(q,t_0) = \widetilde{V}(q) \ , \end{cases} \tag{14}$$

has formal solution given by the generalized complex Cole-Hopf transformation

$$V(q,t) = -\frac{i\hbar}{\mu(t)}\frac{\partial}{\partial q}(\ln\Psi(q,t)), \tag{15}$$

where $\Psi(q,t)$ is solution of the Schrödinger equation (2) with the initial condition

$$\Psi(q,t_0) = \exp\left(\frac{i}{\hbar}\mu(t_0)\int^q \widetilde{V}(\xi)d\xi\right).$$

3. The Hermite type Madelung fluid and complex Burgers equation

Here, we introduce nonlinear Madelung model related with the Sturm-Liouville problem for the classical Hermite polynomials. For this, we chose exponentially decreasing mass $\mu(t) = e^{-t^2}$, and constant frequency $\omega^2(t) = 2n$, $t \in (-\infty,\infty)$. Then, we obtain the system of Hermite type Madelung fluid equations

$$\begin{cases} \dfrac{\partial v}{\partial t} - 2tv + v\dfrac{\partial v}{\partial q} = -e^{t^2}\dfrac{\partial}{\partial q}\left[\dfrac{-\hbar^2}{2}e^{t^2}\left(\dfrac{1}{\sqrt{\rho}}\dfrac{\partial^2\sqrt{\rho}}{\partial q^2}\right) + ne^{-t^2}q^2\right], \\ \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial q}\left[\rho v\right] = 0. \end{cases} \tag{16}$$

Using transformation

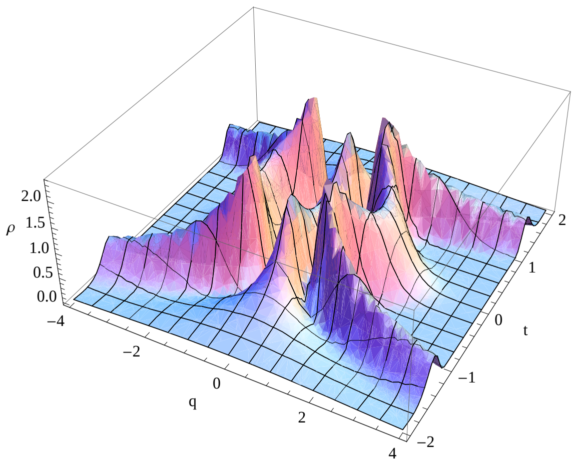
$$v(q,t) = -i\hbar e^{t^2}\frac{\partial}{\partial q}\ln\left(\frac{\Psi(q,t)}{|\Psi(q,t)|}\right), \qquad \rho(q,t) = |\Psi(q,t)|^2, \tag{17}$$

system (16) is linearizable in the form of Hermite type Schrödinger equation

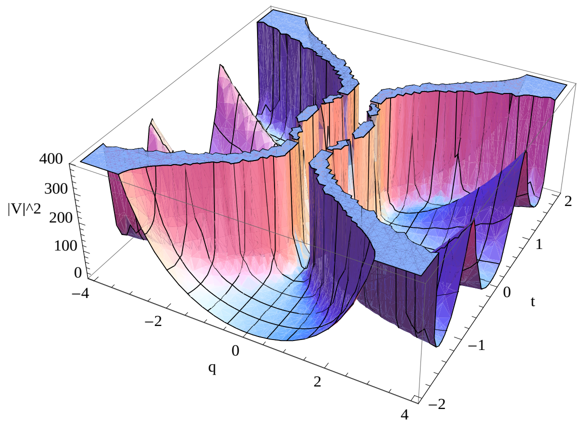
$$i\hbar\frac{\partial\Psi}{\partial t} = -\hbar^2\frac{e^{t^2}}{2}\frac{\partial^2}{\partial q^2}\Psi + ne^{-t^2}q^2\Psi, \tag{18}$$

where $\mu(t) = e^{-t^2}$ and $\omega^2(t) = 2n$, $t \in (-\infty,\infty)$, $n = 0,1,2,....$ It follows that system (16) with general initial conditions $v(q,t_0) = \tilde{v}(q)$, $\rho(q,t_0) = \tilde{\rho}(q)$, has formal solution given by (17), where $\Psi(q,t)$ is solution of the Schrödinger equation (18) with initial condition $\Psi(q,t_0) = \sqrt{\tilde{\rho}(q)}\exp\left((i/\hbar)e^{-t_0^2}\int^q\tilde{v}(\xi)d\xi\right)$. In particular, when the initial state for Eq.(18) is $\Psi(q,t_0) = \varphi_k(q)$, its solution takes the form

$$\begin{aligned} \Psi_{n,k}(q,t) &= N_k\sqrt{R_n(t)} \times \exp\left(\frac{i}{2\hbar}\left(e^{-t^2}\frac{\dot{H}_n(t)}{H_n(t)}\right)q^2\right) \times \exp\left(-\frac{i}{2}\Omega_0^2 g_n(t)R_n^2(t)q^2\right) \\ &\times \exp\left(i\left(k+\frac{1}{2}\right)\arctan(\Omega_0 g_n(t))\right) \times \exp\left(-\frac{\Omega_0}{2}R_n^2(t)q^2\right) \\ &\times H_k\left(\sqrt{\Omega_0}R_n(t)q\right), \qquad k = 0,1,2,..., \end{aligned} \tag{19}$$



(a)



(b)

Figure 1: Hermite type model. a) Probability density $\rho_{2,2}(q,t)$. b) Plot of $|V_{2,2}(q,t)|^2$.

where $H_n(t)$, $n = 0,1,2,...$ are the Hermite polynomials satisfying $\ddot{H}_n - 2t\dot{H}_n + 2nH_n = 0$, $H_n(t_0) \neq 0$, $\dot{H}_n(t_0) = 0$, and the auxiliary functions are

$$g_n(t) = -\hbar H_n^2(t_0) \int^t \frac{e^{\xi^2} d\xi}{H_n^2(\xi)}, \quad g(t_0) = 0; \qquad R_n(t) = \left(\frac{H_n^2(t_0)}{{H_n}^2(t) + [\Omega_0 g_n(t) H_n(t)]^2} \right)^{\frac{1}{2}}. \tag{20}$$

Therefore, system (16) with specific initial conditions

$$v_k(q,t_0) = 0, \quad \rho_k(q,t_0) = N_k^2 \exp\left(-(\sqrt{\Omega_0} q)^2\right) H_k^2(\sqrt{\Omega_0} q), \quad k = 0,1,2,...$$

has exact solutions

$$v_n(q,t) = \left(\frac{\dot{H}_n(t)}{H_n(t)} - \hbar \Omega_0^2 e^{t^2} g_n(t) R_n^2(t) \right) q, \quad \forall k = 0,1,2,...,$$

and

$$\rho_{n,k}(q,t) = N_k^2 \times R_n(t) \times \exp\left(-\left(\sqrt{\Omega_0} R_n(t) q\right)^2 \right) \times H_k^2\left(\sqrt{\Omega_0} R_n(t) q\right), \tag{21}$$

where $g_n(t)$ and $R_n(t)$ are defined by (20).

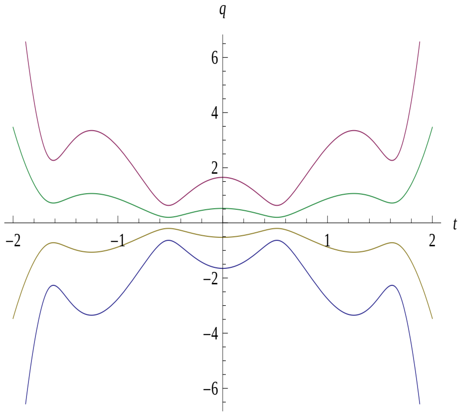
Next, we introduce the Hermite type complex Burgers equation for complex velocity field $V(q,t)$

$$\frac{\partial V}{\partial t} - 2tV + V\frac{\partial V}{\partial q} + 2nq = \frac{i\hbar}{2} e^{t^2} \frac{\partial^2 V}{\partial q^2}, \quad n = 0,1,2,.... \tag{22}$$

By the generalized complex Cole-Hopf transformation

$$V(q,t) = -i\hbar e^{t^2} \frac{\partial}{\partial q}(\ln \Psi(q,t)) \tag{23}$$

it linearizes in the form of Schrödinger equation (18). Then, the IVP for (22) with initial condition $V(q,t_0) = \widetilde{V}(q)$ has formal solution given by (23), where $\Psi(q,t)$ is solution of the IVP for the Schrödinger equation (18) with initial condition $\Psi(q,t_0) = \exp\left((i/\hbar) e^{-t_0^2} \int^q \widetilde{V}(\xi) d\xi\right)$.



(a)

Figure 2: Hermite type model. Moving poles of $V_{n,k}(q,t)$, $\ n=4, k=4$.

As an example, we consider the IVP with specific initial conditions

$$\begin{cases} \dfrac{\partial V}{\partial t} - 2tV + V\dfrac{\partial V}{\partial q} + 2nq = \dfrac{i\hbar}{2} e^{t^2} \dfrac{\partial^2 V}{\partial q^2}\ , \quad n = 0,1,2,..., \\ V_k(q,t_0) = i\hbar e^{t_0^2} \left[\Omega_0 q - \dfrac{\partial_q H_k(\sqrt{\Omega_0} q)}{H_k(\sqrt{\Omega_0} q)} \right], \quad k = 0,1,2,.... \end{cases} \tag{24}$$

It has explicit solution

$$V_{n,k}(q,t) = \left[\frac{\dot{H}_n(t)}{H_n(t)} - \hbar \Omega_0^2 e^{t^2} g_n(t) R_n^2(t) \right] q + i\hbar e^{t^2} \left[\Omega_0 R_n^2(t) q - \frac{\partial_q H_k(\sqrt{\Omega_0} R_n(t) q)}{H_k(\sqrt{\Omega_0} R_n(t) q)} \right]. \tag{25}$$

Note that, since solution $\Psi_{n,k}(q,t)$ of the Hermite quantum oscillator given by (19) has zeros at points where $H_k\left(\sqrt{\Omega_0} R_n(t) q\right) = 0$, then due to Cole-Hopf transformation (23), these zeros become pole singularities of the complex Burgers solution $V_{n,k}(q,t)$ given by (25). If $\tau_k^{(l)}$, $l = 1,2,...,k$, denote the zeros of the Hermite polynomial $H_k(\xi)$, then the motion of the zeros of $\Psi_{n,k}(q,t)$ and poles of $V_{n,k}(q,t)$ for fixed $n = 0,1,...$ and $k = 1,2,...$ is described by

$$q_k^{(l)}(t) = \frac{\tau_k^{(l)}}{\sqrt{\Omega_0} R_n(t)} = \frac{\tau_k^{(l)}}{\sqrt{\Omega_0}} \left(\frac{H_n^2(t) + [\Omega_0 g_n(t) H_n(t)]^2}{H_n^2(t_0)} \right)^{\frac{1}{2}}, \quad l = 1,2,...,k. \tag{26}$$

In Fig.1 we show the behavior of the probability density $\rho_{n,k}(q,t)$ and $|V_{n,k}(q,t)|^2$ for $n = 2, k = 2$. In Fig.2, we illustrate the motion of the zeros/poles for the case $k = 4$, $n = 4$. Clearly, k is the number of the moving zeros/poles and n is the number of oscillations. At times close to $t = 0$, the singularities show oscillatory motion. Then, when $t \to \pm\infty$, one has $R_n(t) \to 0$ and $|q_k^{(l)}(t)| \to \infty$, showing that the zeros/poles go away from the origin $q = 0$. Moreover, the distance between different zeros/poles also increases with time, that is for fixed k and $i \neq j$, $|q_k^{(i)}(t) - q_k^{(j)}(t)| \to \infty$ as $t \to \pm\infty$.

4. The Laguerre type Madelung fluid and complex Burgers equation

Now, we introduce Madelung system related to the Sturm-Liouville problem for the Laguerre polynomials. We start by choosing time variable coefficients of the form $\mu(t) = te^{-t}$ and

$\omega^2(t) = n/t$, $t \in (0, \infty)$. This leads to Laguerre type parametric Madelung fluid system

$$\begin{cases} \frac{\partial v}{\partial t} + (\frac{1-t}{t})v + v\frac{\partial v}{\partial q} = -\frac{e^t}{t}\frac{\partial}{\partial q}\left[\frac{-\hbar^2 e^t}{2t}\left(\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial q^2}\right) + \frac{ne^{-t}}{2}q^2\right], \\ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q}[\rho v] = 0, \end{cases} \tag{27}$$

with variable friction $\Gamma(t) = (1-t)/t,\ \ t \in (0, \infty)$. Using transformation

$$v(q,t) = -\frac{i\hbar e^t}{t}\frac{\partial}{\partial q}\ln\left(\frac{\Psi(q,t)}{|\Psi(q,t)|}\right), \qquad \rho(q,t) = |\Psi(q,t)|^2, \tag{28}$$

it can be linearized in the form of Laguerre type Schrödinger equation

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2 e^t}{2t}\frac{\partial^2}{\partial q^2}\Psi + \frac{ne^{-t}}{2}q^2\Psi, \tag{29}$$

where $\mu(t) = te^{-t}$, and $\omega^2(t) = n/t$, $t \in (0,\infty)$, $n = 0, 1, 2, ...$ Then, system (27) with general initial conditions $v(q, t_0) = \tilde{v}(q)$, $\rho(q, t_0) = \tilde{\rho}(q)$, has formal solution given by (28), where $\Psi(q, t)$ is solution of (29) with initial condition $\Psi(q,t_0) = \sqrt{\tilde{\rho}(q)}\exp\left((i/\hbar)t_0 e^{-t_0}\int^q \tilde{v}(\xi)d\xi\right)$.

The nonstationary Schrödinger equation (29) with initial condition $\Psi(q, t_0) = \varphi_k(q)$, $k = 0, 1, 2, ...$, has exact solution

$$\begin{aligned} \Psi_{n,k}(q,t) &= N_k\sqrt{R_n(t)} \times \exp\left(\frac{i}{2\hbar}\left(te^{-t}\frac{\dot{L}_n(t)}{L_n(t)}\right)q^2\right) \times \exp\left(-\frac{i}{2}\Omega_0^2 g_n(t)R_n^2(t)q^2\right) \\ &\times \exp\left(i\left(k+\frac{1}{2}\right)\arctan(\Omega_0 g_n(t))\right) \times \exp\left(-\frac{\Omega_0}{2}R_n^2(t)q^2\right) \\ &\times H_k\left(\sqrt{\Omega_0}R_n(t)q\right), \qquad k = 0, 1, 2, ..., \end{aligned} \tag{30}$$

where $L_n(t)$, $n = 0, 1, 2, ...$ are the Laguerre polynomials satisfying

$$\ddot{L}_n + \frac{(1-t)}{t}\dot{L}_n + \frac{n}{t}L_n = 0, \quad L_n(t_0) \neq 0, \ \ \dot{L}_n(t_0) = 0,$$

and the auxiliary functions are

$$g_n(t) = -\hbar L_n^2(t_0)\int^t \frac{e^\xi d\xi}{\xi L_n^2(\xi)}, \quad g(t_0) = 0; \qquad R_n(t) = \left(\frac{L_n^2(t_0)}{{L_n}^2(t) + [\Omega_0 g_n(t)L_n(t)]^2}\right)^{\frac{1}{2}}. \tag{31}$$

Therefore, system (27) with specific initial conditions

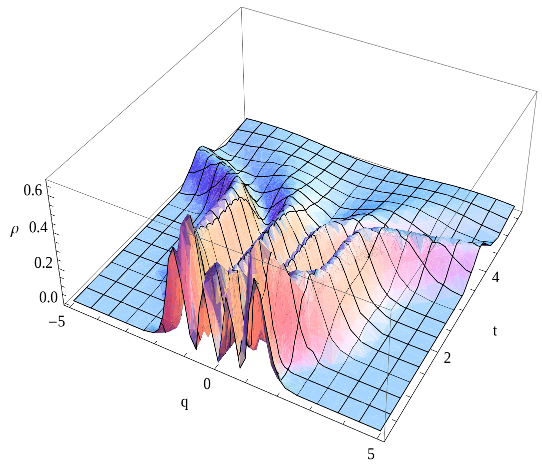
$$v_k(q,t_0) = 0, \quad \rho_k(q,t_0) = N_k^2\exp\left(-(\sqrt{\Omega_0}q)^2\right)H_k^2(\sqrt{\Omega_0}q), \quad k = 0, 1, 2, ...,$$

has exact solutions

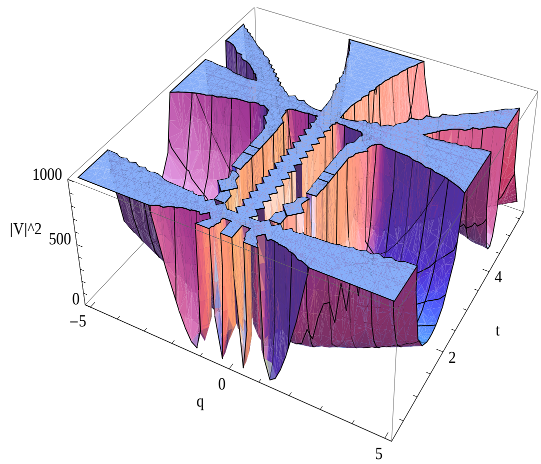
$$v_n(q,t) = \left(\frac{\dot{L}_n(t)}{L_n(t)} - \hbar\Omega_0^2\frac{e^t g_n(t)R_n^2(t)}{t}\right)q, \quad \forall k = 0, 1, 2, ...$$

and

$$\rho_{n,k}(q,t) = N_k^2 \times R_n(t) \times \exp\left(-\left(\sqrt{\Omega_0}R_n(t)q\right)^2\right) \times H_k^2\left(\sqrt{\Omega_0}R_n(t)q\right), \tag{32}$$



(a)



(b)

Figure 3: Laguerre type model. a) Probability density $\rho_{2,3}(q,t)$. b) Plot of $|V_{2,3}(q,t)|^2$.

where $g_n(t)$ and $R_n(t)$ are given by (31).

Next model which we give is the Laguerre type complex Burgers equation

$$\frac{\partial V}{\partial t} + (\frac{1-t}{t})V + V\frac{\partial V}{\partial q} + \frac{n}{t}q = i\frac{\hbar e^t}{2t}\frac{\partial^2 V}{\partial q^2}\ , \quad n = 0, 1, 2, ... \tag{33}$$

By the complex Cole-Hopf transformation

$$V(q,t) = -\frac{i\hbar e^t}{t}\frac{\partial}{\partial q}(\ln \Psi(q,t)) \tag{34}$$

it is linearizable in the form of Schrödinger equation (29). Then, Eq.(33) with general initial condition $V(q,t_0) = \widetilde{V}(q)$ has solution given by (34), where $\Psi(q,t)$ is solution of (29) with initial data $\Psi(q,t_0) = \exp\left((i/\hbar)t_0 e^{-t_0}\int^q \widetilde{V}(\xi)d\xi\right)$. The special IVP

$$\begin{cases} \dfrac{\partial V}{\partial t} + (\dfrac{1-t}{t})V + V\dfrac{\partial V}{\partial q} + \dfrac{n}{t}q = i\dfrac{\hbar e^t}{2t}\dfrac{\partial^2 V}{\partial q^2}\ , \quad n = 0,1,2,..., \\ V_k(q,t_0) = -i\dfrac{\hbar e^{t_0}}{t_0}\dfrac{d}{dq}(\ln \varphi_k(q)) = \dfrac{i\hbar e^{t_0}}{t_0}\left[\Omega_0 q - \dfrac{\partial_q H_k(\sqrt{\Omega_0}q)}{H_k(\sqrt{\Omega_0}q)}\right], \quad k = 0,1,2,... \end{cases} \tag{35}$$

has exact solutions of the form

$$V_{n,k}(q,t) = \left[\frac{\dot{L}_n(t)}{L_n(t)} - \hbar\Omega_0^2\frac{g_n(t)e^t}{t}R_n^2(t)\right]q + i\frac{\hbar e^t}{t}\left[\Omega_0 R_n^2(t)q - \frac{\partial_q H_k(\sqrt{\Omega_0}R_n(t)q)}{H_k(\sqrt{\Omega_0}R_n(t)q)}\right]. \tag{36}$$

Since solution of the Schrödinger equation, $\Psi_{n,k}(q,t)$ given by (30) has zeros at points where $H_k\left(\sqrt{\Omega_0}R_n(t)q\right) = 0$, then these zeros are pole singularities of the solution $V_{n,k}(q,t)$ given by (36). Explicitly, the motion of the zeros/poles for fixed $n = 0,1,2,...$ and $k = 1,2,...$ is described by

$$q_k^{(l)}(t) = \frac{\tau_k^{(l)}}{\sqrt{\Omega_0}R_n(t)} = \frac{\tau_k^{(l)}}{\sqrt{\Omega_0}}\left(\frac{L_n^2(t) + [\Omega_0 g_n(t)L_n(t)]^2}{L_n^2(t_0)}\right)^{\frac{1}{2}}, \quad l = 1,2,...,k. \tag{37}$$

In Fig.3, we illustrate the behavior of $\rho_{n,k}(q,t)$ and $|V_{n,k}(q,t)|^2$ for $n = 2, k = 3$, $(\hbar = \Omega_0 = 1)$. We observe that, at zeros of $L_2(t)$ there are finite time singularities of $V_{n,k}(q,t)$, and also since $k = 3$, there are three moving zeros of $\rho_{2,3}(q,t)$, which are also the moving poles of $V_{2,3}(q,t)$.

5. Legendre type Madelung fluid and complex Burgers equation

Last models which we introduce are related with the Legendre polynomials. For this, we consider time variable coefficients $\mu(t) = (1-t^2)$ and $\omega^2(t) = n(n+1)/\left(1-t^2\right)$, $t \in (-1,1)$. That gives a system of parametric Legendre type Madelung hydrodynamic equations

$$\left\{ \begin{array}{l} \dfrac{\partial v}{\partial t} - \dfrac{2t}{(1-t^2)}v + v\dfrac{\partial v}{\partial q} = -\dfrac{1}{(1-t^2)}\dfrac{\partial}{\partial q}\left[\dfrac{-\hbar^2}{2(1-t^2)}\left(\dfrac{1}{\sqrt{\rho}}\dfrac{\partial^2\sqrt{\rho}}{\partial q^2}\right) + \dfrac{n(n+1)}{2}q^2\right], \\ \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial q}\left[\rho v\right] = 0, \end{array}\right. \tag{38}$$

with friction coefficient $\Gamma(t) = -2t/\left(1-t^2\right)$, $t \in (-1,1)$. Under transformation

$$v(q,t) = -\frac{i\hbar}{(1-t^2)}\frac{\partial}{\partial q}\ln\left(\frac{\Psi(q,t)}{|\Psi(q,t)|}\right), \qquad \rho(q,t) = |\Psi(q,t)|^2, \tag{39}$$

it converts to the Legendre type Schrödinger equation

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2(1-t^2)}\frac{\partial^2}{\partial q^2}\Psi + \frac{n(n+1)}{2}q^2\Psi, \tag{40}$$

where $t \in (-1,1)$, $\mu(t) = \left(1-t^2\right)$, and $\omega^2(t) = n(n+1)/\left(1-t^2\right)$, $n = 0,1,2,...$ Then, system (38) with general initial conditions $v(q,t_0) = \tilde{v}(q)$, $\rho(q,t_0) = \tilde{\rho}(q)$, has formal solution given by (39), where $\Psi(q,t)$ is solution of (40) with initial condition $\Psi(q,t_0) = \sqrt{\tilde{\rho}(q)}\exp\left((i/\hbar)(1-t_0^2)\int^q \tilde{v}(\xi)d\xi\right)$.

Eq. (40) with initial conditions $\Psi(q,t_0) = \varphi_k(q)$ has exact solutions

$$\begin{aligned} \Psi_{n,k}(q,t) &= N_k\sqrt{R_n(t)} \times \exp\left(\frac{i}{2\hbar}\left(\left(1-t^2\right)\frac{\dot{P}_n(t)}{P_n(t)}\right)q^2\right) \times \exp\left(-\frac{i}{2}\Omega_0^2 g_n(t)R_n^2(t)q^2\right) \\ &\times \exp\left(i\left(k+\frac{1}{2}\right)\arctan(\Omega_0 g_n(t))\right) \times \exp\left(-\frac{\Omega_0}{2}R_n^2(t)q^2\right) \\ &\times H_k\left(\sqrt{\Omega_0}R_n(t)q\right), \qquad k = 0,1,2,..., \end{aligned} \tag{41}$$

where $P_n(t)$, $n = 0,1,2,...$ are the classical Legendre polynomials satisfying

$$\ddot{P}_n - \frac{2t}{(1-t^2)}\dot{P}_n + \frac{n(n+1)}{(1-t^2)}P_n = 0, \;\; P_n(t_0) \neq 0, \;\; \dot{P}_n(t_0) = 0$$

and the auxiliary functions are

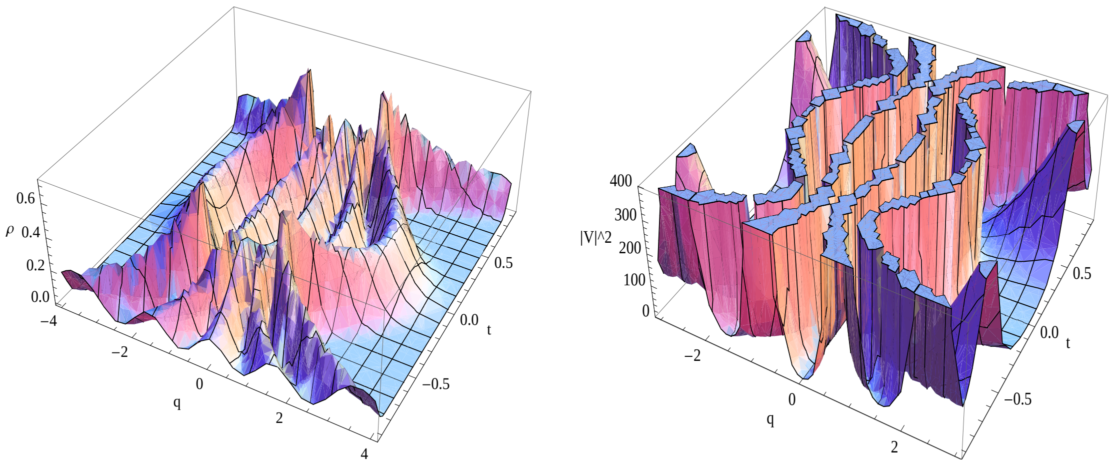
$$g_n(t) = -\hbar P_n^2(t_0)\int^t \frac{d\xi}{(1-\xi^2)P_n^2(\xi)}, \;\; g(t_0) = 0; \;\; R_n(t) = \left(\frac{P_n^2(t_0)}{{P_n}^2(t) + [\Omega_0 g_n(t)P_n(t)]^2}\right)^{\frac{1}{2}}. \tag{42}$$

Therefore, system (38) with specific initial conditions $v_k(q,t_0) = 0$, $\rho_k(q,t_0) = \varphi_k^2(q)$, $k = 0,1,2,...$, has exact solutions

$$v_n(q,t) = \left(\frac{\dot{P}_n(t)}{P_n(t)} - \hbar\Omega_0^2\frac{g_n(t)R_n^2(t)}{(1-t^2)}\right)q, \;\; \forall k = 0,1,2,... \tag{43}$$

and

$$\rho_{n,k}(q,t) = N_k^2 \times R_n(t) \times \exp\left(-\left(\sqrt{\Omega_0}R_n(t)q\right)^2\right) \times H_k^2\left(\sqrt{\Omega_0}R_n(t)q\right). \tag{44}$$



(a) (b)

Figure 4: Legendre type model. a) Probability density $\rho_{2,4}(q,t)$. b)Plot of $|V_{2,4}(q,t)|^2$.

Finally, we write the Legendre type complex Burgers equation

$$\frac{\partial V}{\partial t}-\frac{2t}{(1-t^2)}V+V\frac{\partial V}{\partial q}+\frac{n(n+1)}{(1-t^2)}q=i\frac{\hbar}{2\,(1-t^2)}\frac{\partial^2 V}{\partial q^2},\quad n=0,1,2,.... \tag{45}$$

By the generalized Cole-Hopf transform

$$V(q,t)=-\frac{i\hbar}{1-t^2}\frac{\partial}{\partial q}(\ln\Psi(q,t)), \tag{46}$$

it is linearizable in the form of Schrödinger equation (40). Then, the complex Burgers equation (45) with general initial condition $V(q,t_0)=\widetilde{V}(q)$, has formal solution given by (46), where $\Psi(q,t)$ is solution of (40) with initial condition $\Psi(q,t_0)=\exp\left((i/\hbar)(1-t_0^2)\int^q\widetilde{V}(\xi)d\xi\right)$.

In particular, the complex Burgers equation (45) with specific initial data

$$V_k(q,t_0)=\frac{i\hbar}{(1-{t_0}^2)}\left[\Omega_0 q-\frac{\partial_q H_k(\sqrt{\Omega_0}q)}{H_k(\sqrt{\Omega_0}q)}\right],\quad k=0,1,2,...,$$

has exact solution

$$V_{n,k}(q,t)=\left[\frac{\dot{P}_n(t)}{P_n(t)}-\hbar\Omega_0^2\frac{g_n(t)}{(1-t^2)}R_n^2(t)\right]q+i\frac{\hbar}{(1-t^2)}\left[\Omega_0 R_n^2(t)q-\frac{\partial_q H_k(\sqrt{\Omega_0}R_n(t)q)}{H_k(\sqrt{\Omega_0}R_n(t)q)}\right]. \tag{47}$$

The motion of the zeros of $\Psi_{n,k}(q,t)$ given by (41), and poles of solution $V_{n,k}(q,t)$ given by (47), for fixed $n=0,1,2,...$ and $k=1,2,...$ is described by

$$q_k^{(l)}(t)=\frac{\tau_k^{(l)}}{\sqrt{\Omega_0}R_n(t)}=\frac{\tau_k^{(l)}}{\sqrt{\Omega_0}}\left(\frac{P_n^2(t)+[\Omega_0 g_n(t)P_n(t)]^2}{P_n^2(t_0)}\right)^{\frac{1}{2}},\quad l=1,2,...,k. \tag{48}$$

In Fig.4, we give the plot of $\rho_{n,k}(q,t)$, and $|V_{n,k}(q,t)|^2$ for $n=2,k=4$. One can observe also, the four zeros and poles respectively, which show oscillatory motion on the time interval (-1,1).

Acknowledgments
This work is supported by TÜBİTAK, Project No: 110T679.

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