OUT-OF-PLANE DISPLACEMENTS OF CURVED BEAMS WITH VARIABLE CURVATURE

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ABSTRACT

OUT-OF-PLANE DISPLACEMENTS OF CURVED BEAMS WITH VARIABLE CURVATURE

The differential equations of out-of-plane displacements of curved beams with variable curvature have variable coefficients. Selection of the solution method is based on the curvature function of the curved beam. In this study, Differential Quadrature Method (DQM) and Finite Element Method (FEM) are used to find the out-of-plane displacements of curved beams with variable radius of curvature. Since the parabola is very famous and known curve, it is selected as the form of curved beam. To test and validate the computer codes developed based on DQM in Matlab and based on FEM by APDL (ANSYS Parametric Design Language) in ANSYS, some typical examples are considered. As first step, convergence studies are performed to determine the number of sampling points in DQM and number of elements in FEM. After having information about aforementioned modeling parameters, comparisons between DQM and FEM results are given. The effects of variable curvature parameter of the curved beam on out-of-plane displacements are obtained. The practical application of the present model is discussed.

ÖZET

DEĞİŞKEN EĞRİLİKLİ EĞRİ ÇUBUKLARIN DÜZLEM DIŞI YERDEĞİŞTİRMELERİ

Değişken eğrilikli eğri çubukların diferansiyel denklemleri değişken katsayılıdır. Çözüm yönteminin seçimi eğri çubuğun eğrilik fonksiyonuna bağlıdır. Bu çalışmada, değişken eğrilik yarıçaplı eğri çubukların düzlem dışı yerdeğiştirmelerinin bulunması için Diferansiyel Kuadrator Yöntemi (DKY) ve Sonlu elemanlar Yöntemi (SEY) kullanılmıştır. Parabol bilinen ve meşhur bir eğri olduğundan eğri çubuğun ekseni olarak seçilmiştir. Matlab'da DKY'ye dayalı ve ANSYS deki APDL de SEY'e dayalı geliştirilmiş olan bilgisayar kodlarının testi ve doğrulanması için bazı tipik örnekler gözönüne alınmıştır. İlk adım olarak, DKY'de örnekleme noktaları ve FEM'de eleman sayılarını belirlemek için yakınsama çalışmaları yapılmıştır. Belirtilen hususda bilgi elde edildikten sonra, DKY ve SEY sonuçları arasındaki karşılaştırmalar verilmiştir. Eğri çubuğun değişken eğrilik parametresinin düzlem dışı yerdeğiştirmelere etkileri elde edilmiştir. Mevcut modelin pratik uygulamaları değerlendirilmiştir.

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LIST OF SYMBOLS

a	coefficient of parabola
a_{ij}	coefficient for derivation
b	width of the beam
C_β	coefficient for J
ds	differential length
E	modulus of elasticity
F	force
G	shear modulus
h	depth of the beam
I_{xx}	second moment of area about xx axis
J	polar moment of area
M_x	internal bending moment
M_z	internal twisting moment
n	number of sampling points
N	number of element
$P.E.$	potential energy
R	radius
s	circumferential coordinate
S	length of the beam
T_x	external bending moment
T_z	external twisting moment
V	shear force
v	transverse displacement
β	angular displacement
θ	slope
κ	curvature
ρ_0	variable radius of curvature
τ	torsion

CHAPTER 1

GENERAL INTRODUCTION

Curved beams are very interesting structural members due to both their geometries and differential equations related with solid mechanics. Curved beam can be seen sometimes as main component and sometimes used as complementary part such as stiffener. Depending on the desired functionality, they can be planar or spatial forms, and also have non-uniform cross-section and curvature. In same cases, their geometries are determined by considering the esthetical approaches.

Curved beams have two types of motions for out-of-plane displacements: (1) bending, (2) torsion. Bending and torsional motions are dependent in each other. Due to this physical reality, their analyses are not easy as independent motions.

Figure 1.1 shows a planar curved beam with fixed-free boundary conditions. It is loaded by a tip load P in vertical direction. It can be seen from Figure 1.1 that continuous line and dashed line show the undeflected and deflected shape of curved beam, respectively.

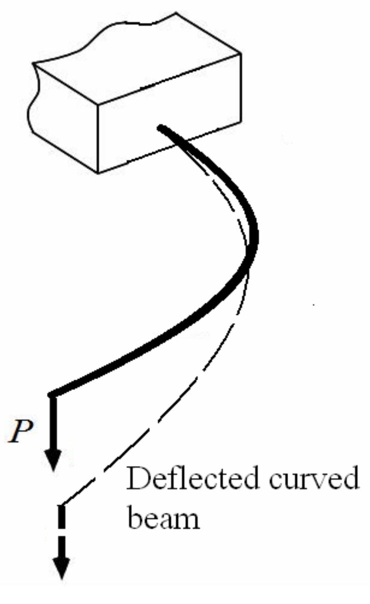

Figure 1.1. Out-of-plane deflection of a curved beam due to vertical tip load

There are many studies for out-of-plane deflection of curved beams, but only a few studies for curved beams with variable curvature. The selected ones are given in the order of publication time as follows:

Volterra and Morell (1961) studied on vibration of the curved beam in the shape of a circle, a cycloid, a catenary and a parabola by using Rayleigh-Ritz method.

Wang (1975) presented an analysis of out-of-plane vibration for a clamped elliptic arc of constant section. He used the Rayleigh-Ritz method, too.

Takahashi and Suzuki (1977) used power series to find the out-of-plane vibrations of uniform arcs in the form of ellipse.

Suzuki et al. (1978) focused on ellipse, sine catenary, hyperbola, parabola and cycloid arcs. They found the vibration characteristics by using the Rayleigh-Ritz and Lehmann-Maehly methods.

Irie et al. (1980) modeled the out-of-plane motion of a free-clamped and internal damped Timoshenko beams with circular, elliptical, catenary and parabolical neutral axes by using the transfer matrix approach.

Suzuki et al. (1983) solved exactly the problem of free vibration characteristics of a plane curved bar with an arbitrary varying cross-section such as elliptic arc bars by series solution.

Huang and Chang (1998) presented an extended methodology for analyzing the out-of-plane dynamic responses or arches. They transformed governing equations to the Laplace transform domain and then they obtained the analytical solution in the Laplace domain by using the Frobenius method.

Kim et al. (2003) derived total potential energy of non-circular curved beam for Finite Element Analysis.

If the main literature related with the Differential Transform Method is selected very carefully, the following critical papers can be found.

Bellman and Casti (1971) introduced the differential quadrature method (DQM) for the numerical solution of ordinary and partial differential equations for the first time. They used the idea behind the classical integral quadrature in their paper and focused on the solutions of linear, nonlinear and partial differential equations.

Bellman et al. (1972) presented DQM as a more efficient method than the standard finite difference method to obtain accurate numerical results by using just a few grid points. Differential quadrature approximates the derivatives as the summation of the function in the problem at the sampling points times weight coefficients. Therefore, the main task in this method is to find the proper sampling points and weight coefficients. Bellman et al. (1972) examined two methods to determine the weighting coefficients of the first order derivative. In the first method, a set of linear algebraic equations based on arbitrary distinct sampling points is considered. But, in the second one, the coordinates of grid points are chosen to be the roots of the shifted Legendre polynomial of degree N. They tested the efficacy of their approaches by carrying out a number of computational experiments.

Bert and Malik (1996) reviewed the DQM and its application to structural mechanics problems.

Shu (2000) published an excellent textbook on DQM and its application in engineering. It is the first book in this subject to describe the DQM including polynomial-based DQ (PDQ) and Fourier series expansion-based DQ (FDQ) methods. PDQ is usually used to non-periodic problems while FDQ is used to both periodic and nonperiodic problems.

Chen (2006) collected his developments on numerical differential quadrature in his textbook. He demonstrated the ability for solving generic scientific and engineering problems by DQM. The book covers the generic differential quadrature, the extended differential quadrature and the related discrete element analysis methods. He emphasized that the topics in his textbook are suitable for developing solution algorithms for various computational mechanics problems with arbitrarily complex geometry. Moreover, he showed several comprehensive examples such as bars and beams, trusses, frames, general field problems, elasticity problems and bending of plates.

Zong and Zhang (2009) introduced numerous developments on DQM in their advanced level textbook. They presented complex DQ, triangular DQ, multi-scale DQ, variable order DQ, multi-domain DQ, and localized DQ. The given methods appeared due to the failing of the original direct differential quadrature (DQ) method for problems with strong nonlinearity and material discontinuity as well as for problems involving singularity, irregularity, and multiple scales.

It should be also mentioned that researchers in applied mathematics, computational mechanics, and engineering developed a range of innovative DQ-based methods given in the textbook written by Zong and Zhang (2009) to overcome the shortcomings of DQM.

In this study, in order to find the out-of-plane displacements of curved beams with variable radius of curvature, the two numerical methods, Differential Quadrature Method (DQM) and Finite Element Method (FEM), are used. The parabola is selected as the form of curved beam. The computer codes developed for DQM in Matlab and for FEM by APDL (ANSYS Parametric Design Language) in ANSYS. The developed codes are verified by comparing the results of both models. Also, parametric studies are presented.

CHAPTER 2

THEORETICAL PART OF THE STUDY

2.1. Introduction

In this chapter, theoretical backgrounds are presented. For this purpose, the parabola that is the selected curve having variable curvature is introduced geometrically in first step. In order to keep the same coordinate system with the literature for the derivation of the differential equation, a parabola is considered in z-x plane as shown in Figure 2.1.

Then, the differential equations of out-of-plane displacements of curved beams with variable curvature which have variable coefficients are derived by using vectorial approach.

Due to the variable coefficients appeared in the differential equations, its exact solution is possible only for special cases. Therefore, for this study, Differential Quadrature Method (DQM) and Finite Element Method (FEM) are considered as numerical solution methods.

After derivation of the differential equations, the numerical solution methods given above are outlined. Moreover, real constants of the selected finite element in ANSYS are given.

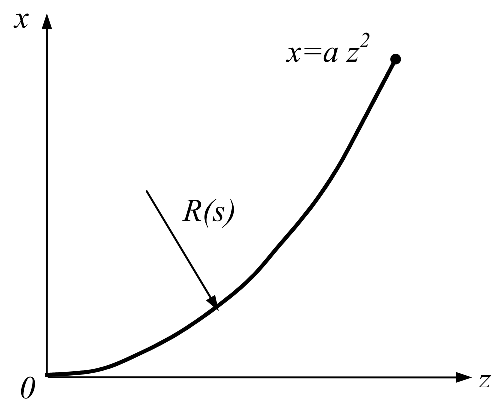

Figure 2.1. A planar curved beam with variable radius of curvature.

2.2. Geometrical Description of Curved Beam

In this section, the parabola given in Figure 2.1 is detailed mathematically. For this reason, arc length of a curve and curvature of a point on the curve are derived.

The function $f(x)$ representing a curve in x-y coordinate system shown in Figure 2.2 is considered to derive the arc length between two points on the curve. The following equations are written from Figure 2.2,

$$ds^2 = dx^2 + dy^2 \tag{2.1}$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \tag{2.2}$$

$$S = \int_{s1}^{s2} ds = \int_{x1}^{x2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \tag{2.3}$$

where s_1, s_2, x_1, and x_2 are the boundaries of the integrals. Moreover,

$$\tan\theta = dy / dx \tag{2.4}$$

$$\theta = \arctan(dy / dx) \tag{2.5}$$

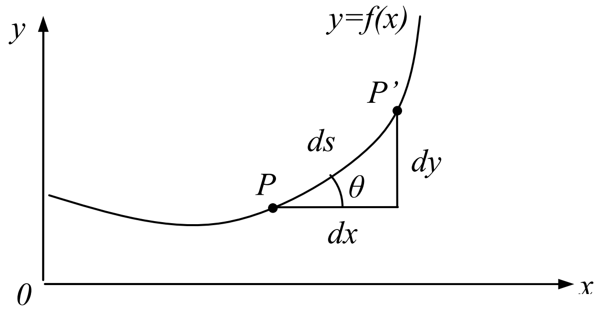

Figure 2.2. Derivation of arc length.

For derivation of the curvature of a plane curve at a point, the plot given in Figure 2.3 can be considered. Curvature is the rate of change of direction of the curve, i.e.

$$\kappa = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds} \tag{2.6}$$

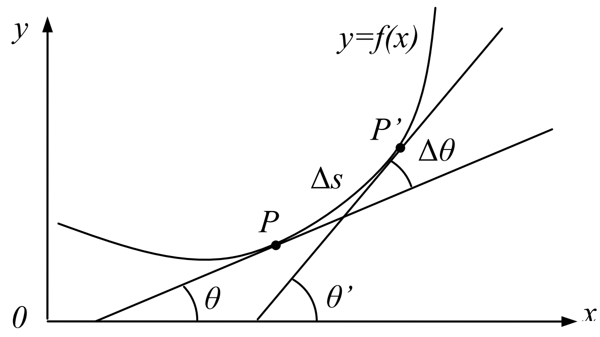

Figure 2.3. Derivation of curvature.

Equation (2.6) can be found by using chain rule as follows

$$\frac{d\theta}{ds} = \frac{d\theta}{dx}\frac{dx}{ds} \tag{2.7}$$

The first term at the right hand side of Equation (2.7) is found by using Equation (2.5) as

$$\frac{d\theta}{dx} = \frac{d}{dx}\left(\arctan\frac{dy}{dx}\right) = \frac{\frac{d}{dx}\left(\frac{dy}{dx}\right)}{1+\left(\frac{dy}{dx}\right)^2} = \frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2} \tag{2.8}$$

The second term at the right hand side of Equation (2.7) is the reciprocal of Equation (2.2),

$$\frac{dx}{ds} = \left(1+\left(\frac{dy}{dx}\right)^2\right)^{-1/2} \tag{2.9}$$

Therefore,

$$\kappa = \frac{d\theta}{ds} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} \tag{2.10}$$

2.3. Out-of-plane Displacement of Curved Beam

Equilibrium equations of out of plane displacement of a curved beam having variable radius of curvature can be obtained by using two classical ways: Vectorial Method and Hamilton's Principle. It is known that Hamilton's principle has an important advantage that is the expressing the boundary conditions without physical interpretations such as in Vectorial Method (Yardimoglu, 2012).

Vectorial Method is used here. In this method, the following force and moment equilibrium equations are used.

$$\sum_i \vec{F}_i = 0 \tag{2.11}$$

$$\sum_i \vec{M}_i = 0 \tag{2.12}$$

Figure 2.4 shows a curved beam having variable radius of curvature with internal and external forces and moments for out-of-plane displacement. It can be seen that V_y is shear force in y direction, M_x is bending moment and M_z is twisting moment. Also, F_y is external force in y direction, T_x and T_z are external bending and twisting moments, respectively. Equations (2.11) and (2.12) are given in open form by Love (1944) as

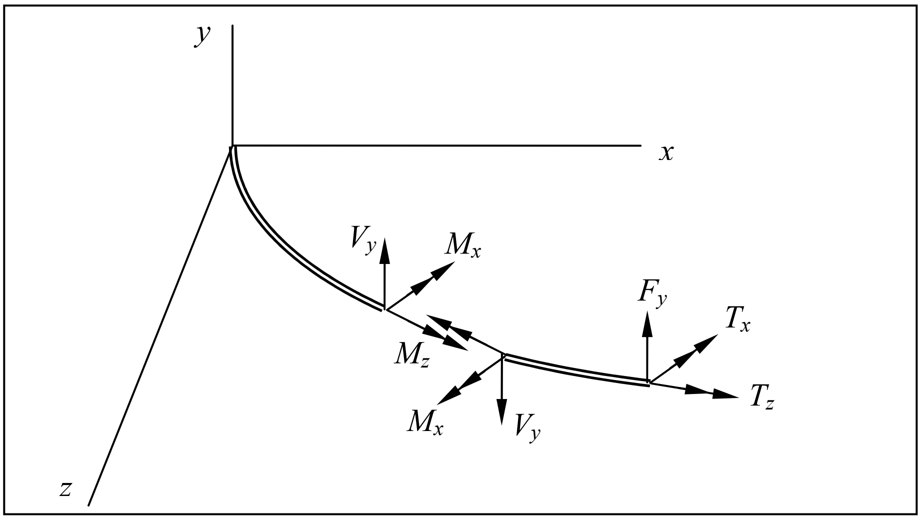

Figure 2.4. **I**nternal and external forces and moments of the curved beam.

$$\frac{dV_y(s)}{ds} + F_y(s) = 0 \tag{2.13}$$

$$\frac{dM_x(s)}{ds} + \frac{M_z(s)}{\rho_0(s)} - V_y(s) + T_x(s) = 0 \tag{2.14}$$

$$\frac{dM_z(s)}{ds} - \frac{M_x(s)}{\rho_0(s)} + T_z(s) = 0 \tag{2.15}$$

where bending moment function $M_x(s)$ and twisting moment function $M_z(s)$ are written as

$$M_x(s) = EI_{xx}\,\kappa(s) \tag{2.16}$$

$$M_z(s) = GJ\,\tau(s) \tag{2.17}$$

where $\kappa(s)$ and $\tau(s)$ are curvature and twisting functions of the curved beam, respectively, and can be found by using the equations given below

$$\kappa(s) = (\frac{\beta(s)}{\rho_0(s)} - \frac{\partial^2 v(s)}{\partial s^2}) \tag{2.18}$$

$$\tau(s) = (\frac{d\beta(s)}{ds} + \frac{1}{\rho_0(s)}\frac{\partial v(s)}{\partial s}) \tag{2.19}$$

I_{xx} in Equation (2.16) is area moment of inertia of the cross-section about *xx*-axis and determined by

$$I_{xx} = b\,h^3/12 \tag{2.20}$$

where, b and h are the width and height of the rectangular cross-section. J in Equation (2.17) is torsional constant of the cross-section and it is given for rectangular cross-section by Popov (1998) as

$$J = C_\beta \, b \, h^3 \tag{2.21}$$

where the values of parameter C_β depends on the ratio of b/h and its values are given in Table 2.1.

Table 2.1 The values of C_β for rectangular cross-section

b/h	1	1.5	2	3	6	10
C_β	0.141	0.196	0.229	0.263	0.299	0.312

Now, Equations (2.13), (2.14) and (2.15) can be written more open form by several substitutions. First of all, Equation (2.14) is substituted into Equation (2.13) to reduce the number of differential equations to two coupled equations as (Equation (2.15) is copied for the sake of completeness)

$$\frac{d^2 M_x(s)}{ds^2} + \frac{d}{ds}\left(\frac{M_z(s)}{\rho_0(s)}\right) + \frac{dT_x(s)}{ds} + F_y(s) = 0 \tag{2.22}$$

$$\frac{dM_z(s)}{ds} - \frac{M_x(s)}{\rho_0(s)} + T_z(s) = 0 \tag{2.23}$$

Secondly, Equations (2.16) and (2.17) are substituted into Equations (2.22) and (2.23) to write the following ones

$$\frac{d^2}{ds^2}(EI_{xx}\,\kappa(s)) + \frac{d}{ds}\left(\frac{GJ\,\tau(s)}{\rho_0(s)}\right) + \frac{dT_x(s)}{ds} + F_y(s) = 0 \tag{2.24}$$

$$\frac{d}{ds}(GJ\,\tau(s)) - \frac{EI_{xx}\,\kappa(s)}{\rho_0(s)} + T_z(s) = 0 \tag{2.25}$$

Then, curvature function $\kappa(s)$ and twisting function $\tau(s)$ given by Equations (2.18) and (2.19), respectively, are substituted into Equations (2.24) and (2.25) to express the differential equations in terms of linear displacement v(s) and angular displacement β(s),

$$
\begin{aligned}
&\frac{d^2}{ds^2}\left(EI_{xx}\left(\frac{\beta(s)}{\rho_0(s)}-\frac{\partial^2 v(s)}{\partial s^2}\right)\right)\\
&+\frac{d}{ds}\left(\frac{GJ}{\rho_0(s)}\left(\frac{d\beta(s)}{ds}+\frac{1}{\rho_0(s)}\frac{\partial v(s)}{\partial s}\right)\right)\\
&+\frac{dT_x(s)}{ds}+F_y(s)=0
\end{aligned}
\tag{2.26}
$$

$$
\begin{aligned}
&\frac{d}{ds}\left(GJ\left(\frac{d\beta(s)}{ds}+\frac{1}{\rho_0(s)}\frac{\partial v(s)}{\partial s}\right)\right)\\
&-\frac{EI_{xx}}{\rho_0(s)}\left(\frac{\beta(s)}{\rho_0(s)}-\frac{\partial^2 v(s)}{\partial s^2}\right)+T_z(s)=0
\end{aligned}
\tag{2.27}
$$

Finally, Equations (2.26) and (2.27) can be expanded in several steps to use numerical methods for solution as

$$
\begin{aligned}
&EI_{xx}\frac{d^2}{ds^2}\left(\frac{\beta(s)}{\rho_0(s)}\right)-EI_{xx}\frac{\partial^4 v(s)}{\partial s^4}\\
&+GJ\frac{d}{ds}\left(\frac{1}{\rho_0(s)}\frac{d\beta(s)}{ds}\right)+GJ\frac{d}{ds}\left(\frac{1}{\rho_0(s)^2}\frac{\partial v(s)}{\partial s}\right)\\
&+\frac{dT_x(s)}{ds}+F_y(s)=0
\end{aligned}
\tag{2.28}
$$

$$
\begin{aligned}
&GJ\frac{d^2\beta(s)}{ds^2}+GJ\frac{d}{ds}\left(\frac{1}{\rho_0(s)}\frac{\partial v(s)}{\partial s}\right)\\
&-\frac{EI_{xx}\,\beta(s)}{\rho_0(s)^2}-\frac{EI_{xx}}{\rho_0(s)}\left(\frac{\partial^2 v(s)}{\partial s^2}\right)+T_z(s)=0
\end{aligned}
\tag{2.29}
$$

In order to continue by using fewer notations, prime is used to represent the differentiation with respect to *s*. Thus,

$$
\begin{aligned}
& EI_{xx}\left(\frac{\beta(s)}{\rho_0(s)}\right)'' - EI_{xx}v^{iv}(s) \\
& + GJ\left(\frac{\beta'(s)}{\rho_0(s)}\right)' + GJ\left(\frac{v'(s)}{\rho_0(s)^2}\right)' \\
& + T_x'(s) + F_y(s) = 0
\end{aligned}
\tag{2.30}
$$

$$
\begin{aligned}
& GJ\beta''(s) + GJ\left(\frac{v'(s)}{\rho_0(s)}\right)' \\
& - \frac{EI_{xx}}{\rho_0(s)^2}\beta(s) + \frac{EI_{xx}}{\rho_0(s)}v''(s) \\
& + T_z(s) = 0
\end{aligned}
\tag{2.31}
$$

Due to the variable radius of curvature $\rho_0(s)$, Equations (2.30) and (2.31) are needed one more step for the differentiation terms having divisions by radius of curvature $\rho_0(s)$. After completing that step, the differential equations are simplified, then, the following form is obtained,

$$
\begin{aligned}
& EI_{xx}v^{iv}(s) - \frac{GJ}{\rho_0(s)^2}v''(s) + \frac{2GJ\rho_0'(s)}{\rho_0(s)^3}v'(s) \\
& - \frac{EI_{xx} + GJ}{\rho_0(s)}\beta''(s) + \frac{2EI_{xx}\rho_0'(s) + GJ\rho_0'(s)}{\rho_0(s)^2}\beta'(s) \\
& - \left(\frac{2EI_{xx}\rho_0'(s)^2}{\rho_0(s)^3} - \frac{EI_{xx}\rho_0''(s)}{\rho_0(s)^2}\right)\beta(s) = T_x'(s) + F_y(s)
\end{aligned}
\tag{2.32}
$$

$$
\begin{aligned}
& - \frac{EI_{xx} + GJ}{\rho_0(s)}v''(s) + \frac{GJ\rho_0'(s)}{\rho_0(s)^2}v'(s) \\
& - GJ\beta''(s) + \frac{EI_{xx}}{\rho_0(s)^2}\beta(s) = T_z(s)
\end{aligned}
\tag{2.33}
$$

Equations (2.32) and (2.33) are coupled differential equations with variable coefficients and having two unknown functions $v(s)$ and $\beta(s)$.

Typical boundary conditions are as follows:

- Free end: shear force V_y=0, bending moment M_x=0, and twisting moment M_z=0.
- Pinned end: displacement v=0, bending moment M_x=0, and twisting moment M_z=0.
- Fixed end: displacement v=0, slope v'=0, and rotation β=0.

2.4. Discretization of Continuous Systems

2.4.1. Differential Quadrature Method

In this method, the first derivative of a function *f*(*x*) at the i^{th} sampling point is approximated by the following equation (Bellman and Casti, 1971)

$$f'(x_i) = \sum_{j=1}^{n} a_{ij} f(x_j), \quad j = 1,2,..,n \tag{2.34}$$

Open matrix form of Equation (2.32) can be written as follows:

$$\begin{Bmatrix} f'(x_1) \\ f'(x_2) \\ \vdots \\ f'(x_n) \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{Bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{Bmatrix} \tag{2.35}$$

The m^{th} order derivative can be obtained by using Equation (2.35), successively. Therefore, it can be written as

$$\begin{Bmatrix} f^m(x_1) \\ f^m(x_2) \\ \vdots \\ f^m(x_n) \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}^m \begin{Bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{Bmatrix} \tag{2.36}$$

Lagrange interpolating polynomial function can be used as solution of the differential equation (Bellman et al, 1972). The n^{th} order Lagrange interpolating polynomial function is given by

$$f_i(x) = \frac{M(x)}{(x - x_i)M_1(x_i)}, \quad i = 1,2,..,n \tag{2.37}$$

where

$$M(x)=\prod_{k=1}^{n}(x-x_k) \tag{2.38.a}$$

$$M_1(x_i)=\prod_{k=1,k\neq i}^{n}(x_i-x_k)\text{....for }\; i=1,2,..,n \tag{2.38.b}$$

Substituting Equations (2.38.a) and (2.38.b) into Equation (2.37) leads to

$$a_{ij}^{(1)}=\frac{M_1(x_i)}{(x_i-x_j)M_1(x_j)}\text{....for }\; i,j=1,2,..,n;\; i\neq j \tag{2.39}$$

$$a_{ii}^{(1)}=-\sum_{j=1,j\neq i}^{n}a_{ij}^{(1)} \tag{2.40}$$

Let the number of sampling points $n>m$, the second and third and higher order derivatives can be determined as

$$a_{ii}^{(2)}=-\sum_{k=1}^{n}a_{ik}^{(1)}a_{kj}^{(1)}\text{....for }\; i=j=1,2,..,n; \tag{2.41}$$

$$a_{ii}^{(m)}=-\sum_{k=1}^{n}a_{ik}^{(1)}a_{kj}^{(m-1)}\text{....for }\; i=j=1,2,..,n; \tag{2.42}$$

Chebyshev-Gauss-Lobatto mesh distribution given by Shu (2000) can be used for the sampling points. It is expressed as follows

$$x_i=\frac{1}{2}\left[1-\cos\left(\frac{i-1}{n-1}\pi\right)\right] \tag{2.43}$$

where

$$x_i=\frac{s_i}{S} \tag{2.44}$$

in which S is the length of the arch and s_i is the distance of the i^{th} sampling point.

The application of boundary condition can be accomplished by using Lagrange multiplier method given by Wilson (2002). In this method, the constraint equations are added to the potential energy as

$$P.E. = \frac{1}{2}\{x\}^T[K]\{x\} - \{x\}^T\{R\} + \sum_{j=1}^{J}\lambda_j[B_j]\{x\} \tag{2.45}$$

where λ_j is named as the Lagrange multiplier for the constraint j. Also, $[K]$, $\{x\}$, $\{R\}$, and $[B]$ are stiffness matrix, displacement vector, corresponding load or reaction vector, and the displacement transformation matrix, respectively.

If the potential energy given in Equation (2.45) is minimized with respect to each displacement and each Lagrange multiplier, the following set of equations is found

$$\begin{bmatrix} [K] & [B] \\ [B]^T & [0] \end{bmatrix} \begin{bmatrix} \{x\} \\ \{\lambda\} \end{bmatrix} = \begin{bmatrix} \{R\} \\ \{0\} \end{bmatrix} \tag{2.46}$$

Equation (2.46) has also boundary condition equations compared to original equilibrium equation $[K]\{x\}=\{R\}$. For n sampling points and m boundary conditions, the size of $[K]$ is $(2n)$x$(2n)$ due to the two coupled differential equations, and the size of $[B]^T$ is (m)x$(2n)$.

A curved beam axis having seven sampling points is shown in Figure 2.5.

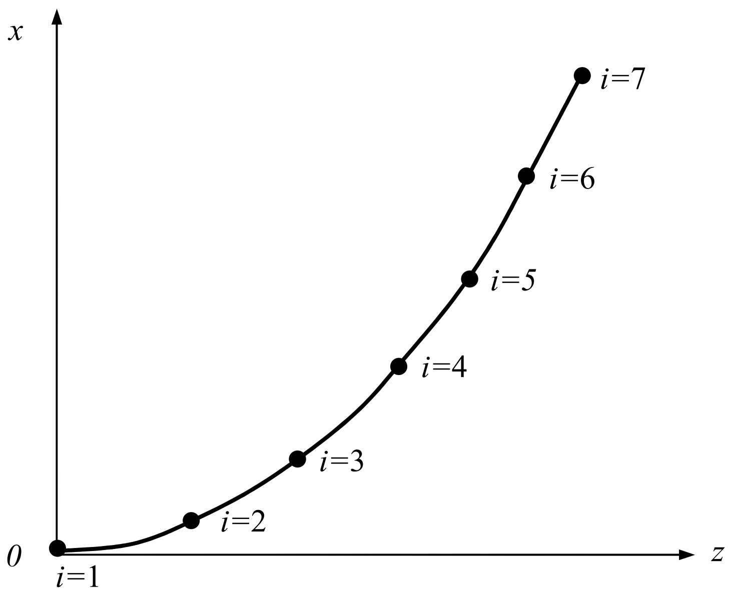

Figure 2.5. A curved beam axis having seven sampling points.

2.4.2. Finite Element Method

Finite element displacement method is the generalization of Rayleigh-Ritz Method (Reddy 1993, Petyt 2010). The method has been developed in the second half of the last century (Cook 1989). In this method, geometrically complex shape is divided into simple geometrical shapes, such as bar, beam, plate, shell, tetrahedral solid, hexahedral solid. Each simple shape is called as finite element. Finite elements have nodes to satisfy the continuity condition with the neighbors. Therefore, nodal freedoms play critical roles in the correct modeling of the present problems geometry. In solid mechanics, nodal freedoms are displacements in x, y and z directions and rotations about x, y and z axis in general. Dividing the whole geometry into the elements is known as meshing. All elements are combined by using the continuity rules and this step is called as assembling. Mathematically, assembled system is described by systems of algebraic equations which can be written in matrix form. The most critical step, in this method, is the determination and application of the boundary conditions. Moreover, usage of reduction the geometry at the beginning of the finite element modeling by using symmetry etc reduces the size of the characteristic matrix. In solid mechanic applications, characteristics matrix of the whole system is the form of $\{F\}=[K]\{x\}$, where $[K]$ is stiffness matrix, $\{F\}$ and $\{x\}$ are external force and nodal displacements vectors, respectively.

A sample of finite element mesh for curved beam is shown in Figure 2.6.

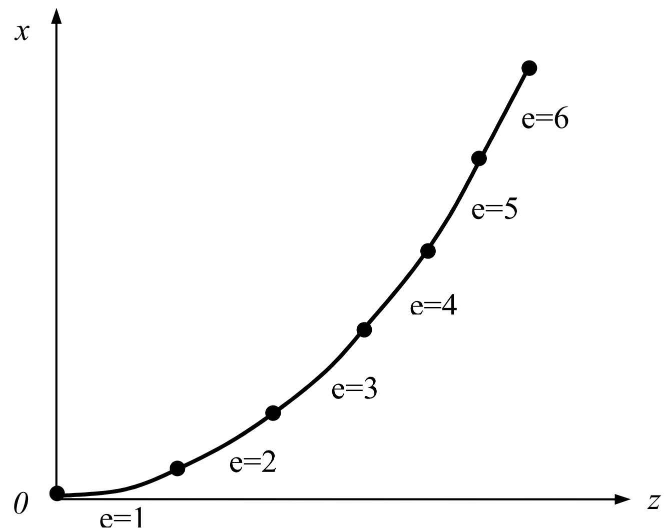

Figure 2.6. Finite element discretization of curved beam (e≡element).

Several commercial finite element analysis softwares are available. To model the curved beam for finding the out-of-plane displacements, BEAM4 (ANSYS, 2007) can be used. BEAM4 is used for tension, compression, torsion, and bending due to the nodal freedoms which are three translations and three rotations. In addition to elastic stiffness, geometric stiffness included. Nodes of BEAM4 are shown in Figure 2.7.

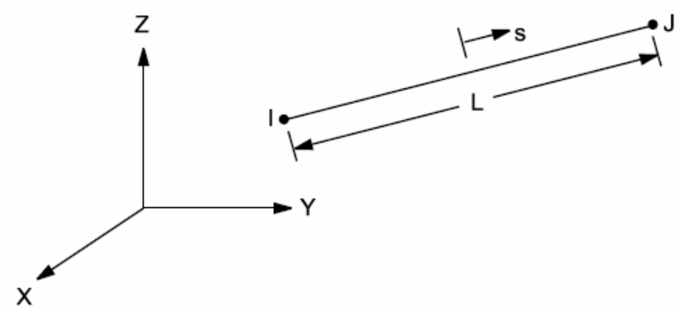

Figure 2.7. Nodes of BEAM4.
(Source: Kohnke, 2004)

The shape functions of BEAM4 are given as follows (Kohnke, 2004)

$$u=\frac{1}{2}[u_I(1-s)+u_J(1+s)] \tag{2.47}$$

$$\begin{aligned} v=&\frac{1}{2}[v_I(1-\frac{s}{2}(3-s^2))+v_J(1+\frac{s}{2}(3-s^2))] \\ &+\frac{L}{8}[\theta_{zI}(1-s^2)(1-s)-\theta_{zJ}(1-s^2)(1+s)] \end{aligned} \tag{2.48}$$

$$\begin{aligned} w=&\frac{1}{2}[w_I(1-\frac{s}{2}(3-s^2))+w_J(1+\frac{s}{2}(3-s^2))] \\ &-\frac{L}{8}[\theta_{yI}(1-s^2)(1-s)-\theta_{yJ}(1-s^2)(1+s)] \end{aligned} \tag{2.49}$$

$$\theta_x=\frac{1}{2}[\theta_{xI}(1-s)+\theta_{xJ}(1+s)] \tag{2.50}$$

The geometry of the BEAM4 is shown in Figure 2.8 from the original document. BEAM4 is based on two or three nodes. Third node of this element is needed for orientation of the element. Real constants (ANSYS, 2005) are

AREA, IZZ, IYY, TKZ, TKY, THETA

ISTRN, IXX, SHEARZ, SHEARY, SPIN, ADDMAS

where

AREA	: Cross-sectional area
IZZ and IYY	: Area moment of inertia about z and y axis, respectively
TKZ, TKY	: Thickness in z and y directions
THETA	: Orientation angle about x axis
ISTRN	: Initial strain
IXX	: Torsional moment of inertia
SHEARZ, SHEARY	: Shear deflection constant
SPIN	: Rotational frequency
ADDMAS	: Added mass/unit length

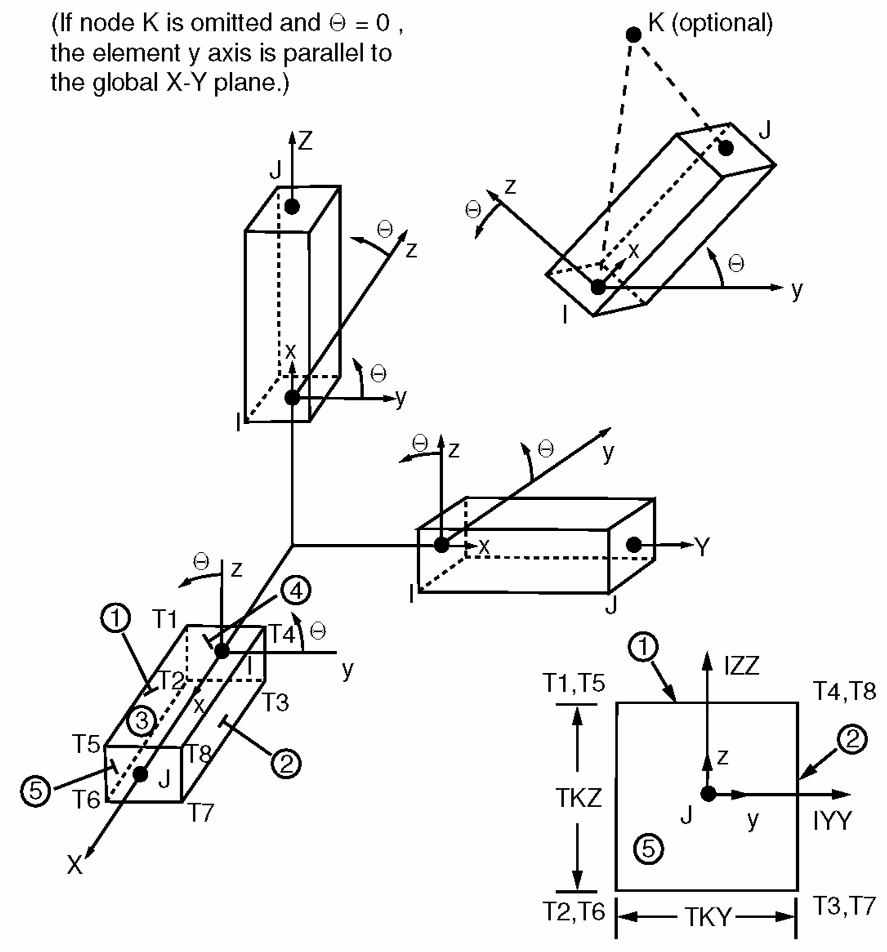

Figure 2.8. Geometry of BEAM4.
(Source: ANSYS, 2007)

CHAPTER 3

NUMERICAL APPLICATIONS AND DISCUSSION

3.1. Introduction

In this chapter, out-of-plane displacements of a curved beam with variable curvature are studied by the following numerical methods:

(a) DQM (Differential Quadrature Method): A computer program is developed in Matlab for the DQM based on Section 2.4.1.

(b) FEM (Finite Element Method): A computer program is developed by using APDL (ANSYS Parametric Design Language) in ANSYS. In this program, the two dimensional geometry based on parametric inputs regarding the shape of the curved beam axis and finite element model of curved beam is formed in ANSYS. BEAM4 is selected to model the curved beam.

In first step, a typical parabolic curved beam of which axis is shown in Figure 3.1 is considered to compare the results of both methods. Boundary conditions of curved beam are fixed and free at root and tip, respectively, as shown in Figure 3.1. A tip load 20 N perpendicular to *zx* plane is applied. The convergence studies for two methods are carried out to find the proper number of sampling points and number of elements.

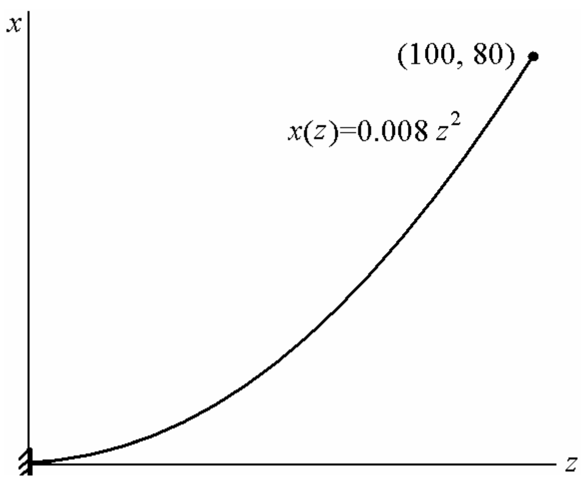

Figure 3.1. Curved beam axis for convergence test and comparisons.

In second step, in order to see the effects of parabola parameter a of the curved beam in different applications based on several geometrical restrictions on vertical displacements, various values for a are selected. The geometrical restrictions considered for parametric studies are classified as follows:

Case 1: Half parabolas with the same width,

Case 2: Half parabolas with the same depth,

Case 3: Half parabolas with the same arc length,

Case 4: Full parabolas with the same width,

Case 5: Full parabolas with the same depth,

Case 6: Full parabolas with the same arc length.

Geometrical meaning of some terms used for the cases above are given below:

- Half parabola: a part of parabola in the first quadrant as seen in Figure 2.1.
- Full parabola: a part of parabola in the first and second quadrant which obtained by adding the mirror of half parabola about vertical axis to half parabola.
- Width of the parabola: the distance across the aperture (or opening) of the parabola.
- Vertex: Origin O in the Figure 2.1.
- Depth or height of the parabola: the distance from vertex to the aperture.

The cases listed above are illustrated in Figures 3.2 to 3.7, respectively. The dimensions used to draw the parabolas are given in the co-ordinate axis.

It can be said that larger absolute value of the parabola parameter a forms narrower parabola, and smaller absolute value of the parabola parameter a forms wider parabola.

Boundary conditions of the half parabolas considered as Case 1 to Case 3 are fix-free as seen from Figure 3.1. Similar to the study done for first step, a tip load 200 N perpendicular to zx plane is applied.

Boundary conditions of the full parabolas considered as Case 4 to Case 6 are fix-fix. In this case a load 400 N perpendicular to zx plane is applied at the vertex of the parabola.

As material characteristics of curved beam, modulus of elasticity E=200000 GPa and shear modulus G=80000 GPa are taken in all studies.

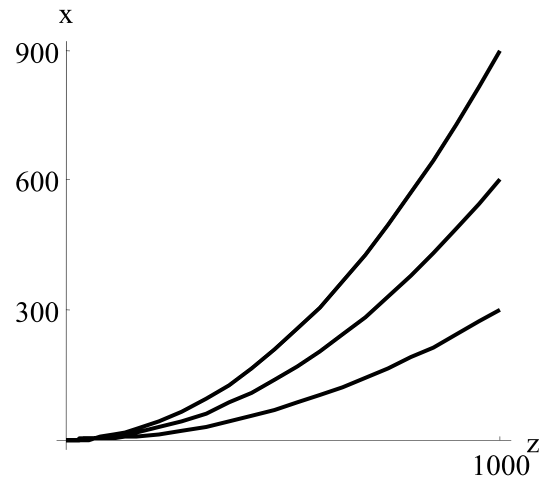

Figure 3.2. Half parabolas with the same width z_0=1000 mm.

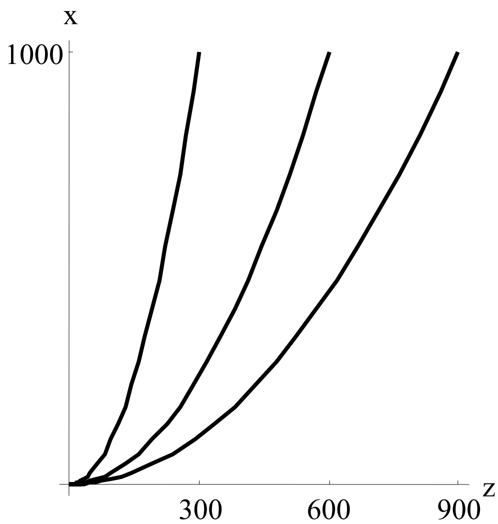

Figure 3.3. Half parabolas with the same depth x_0=1000 mm.

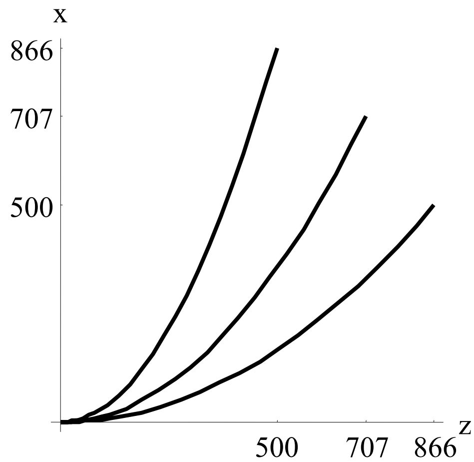

Figure 3.4. Half parabolas with the same arc length *S*=1000 mm.

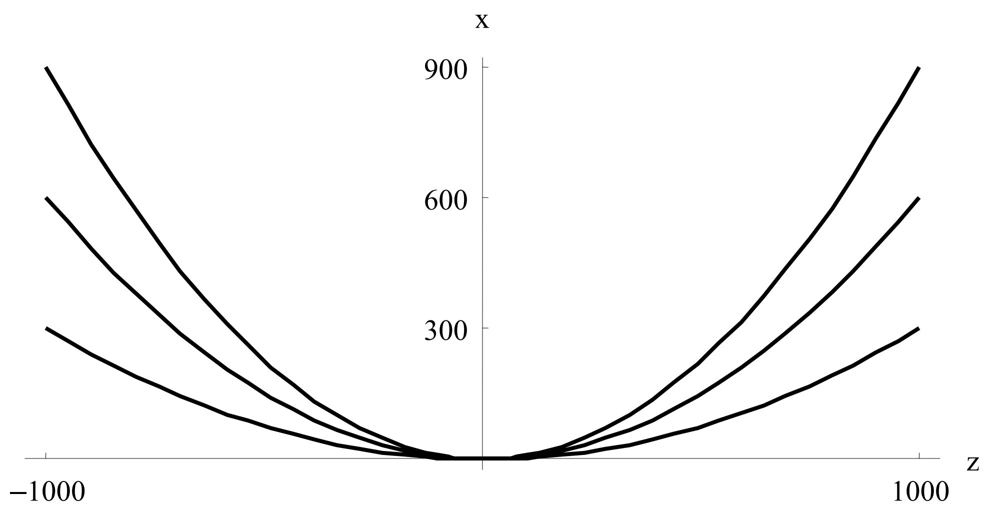

Figure 3.5. Full parabolas with the same width $2z_0$=2000 mm.

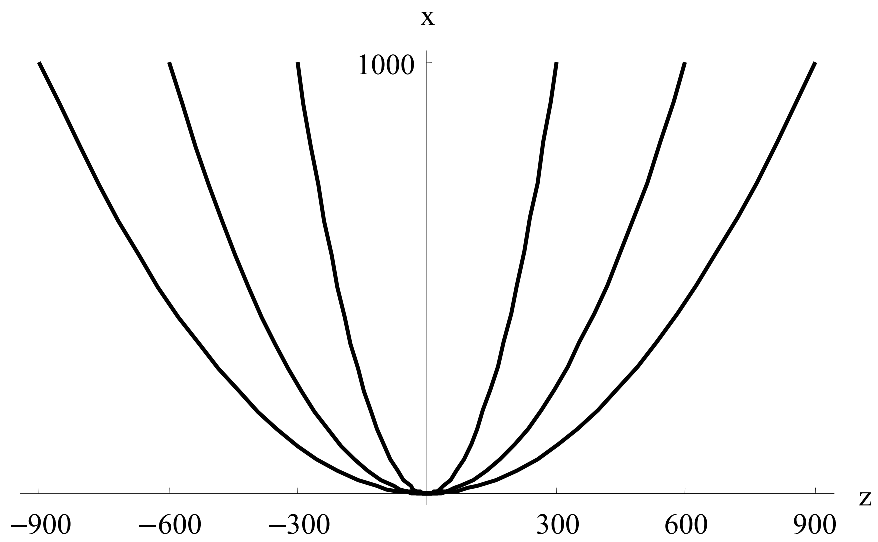

Figure 3.6. Full parabolas with the same depth $2x_0$=2000 mm.

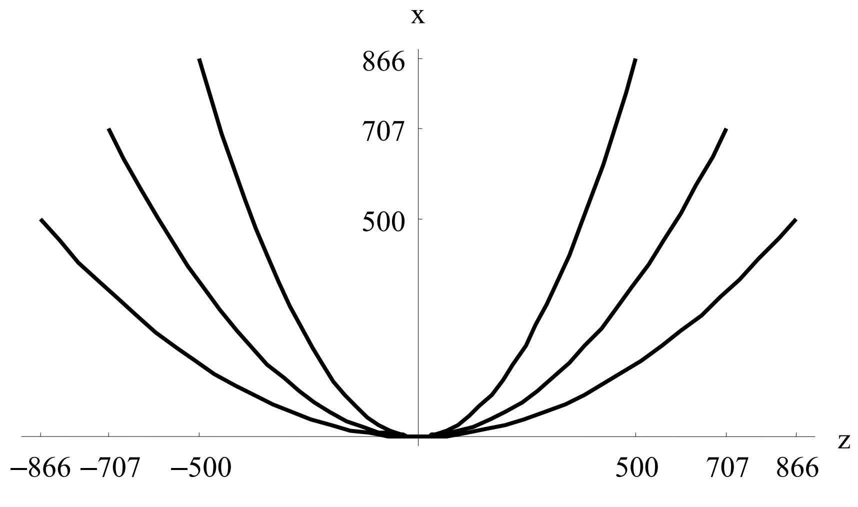

Figure 3.7. Full parabolas with the same arc length $2S$=2000 mm.

3.2. Convergence Tests for Differential Quadrature Method

For convergence study, to find the minimum number of sampling point in DQM, the displacements of curved beam with the parameters $a=a_1=0.008$, $b=5$ mm, $h=5$ mm, $S=133$ mm are calculated for different discretizations.

The results are given in Table 3.1 and Figure 3.8. It is seen from the presented results that the reasonable number of sampling point $n=24$.

Table 3.1. Convergence of displacements by using DQM

n	Displacement v (mm)
8	1.5475
12	1.5435
16	1.5413
20	1.5408
24	1.5405
28	1.5402

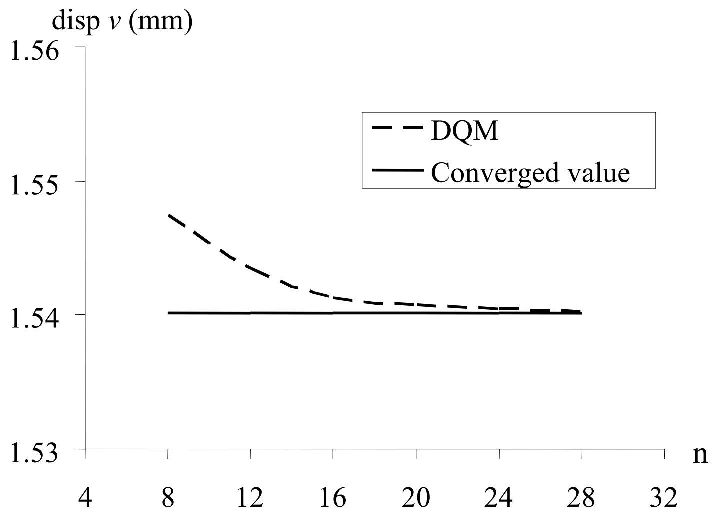

Figure 3.8. Convergence of displacements by DQM.

3.3. Convergence Tests for Finite Element Method

The results are given in Table 3.2 and Figure 3.9. It is seen from the presented results that the reasonable number of element N=26.

Table 3.2. Convergence of displacements by using FEM

N	Displacement v (mm)
6	1.5242
8	1.5306
10	1.5330
12	1.5346
14	1.5358
16	1.5368
18	1.5369
20	1.5371
22	1.5377
24	1.5376
26	1.5378
28	1.5379

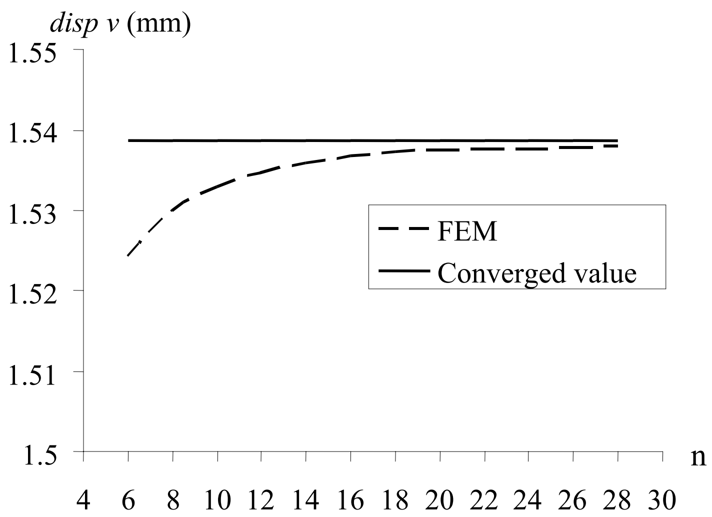

Figure 3.9. Convergence of displacements by FEM.

3.4. Parametric Studies for Out-of-Plane Displacements of Curved Beams

In all cases, 25x25 mm square cross-section is used. Solutions are found by using n=24. Displacements at the load application points are presented in Figures 3.10-3.16.

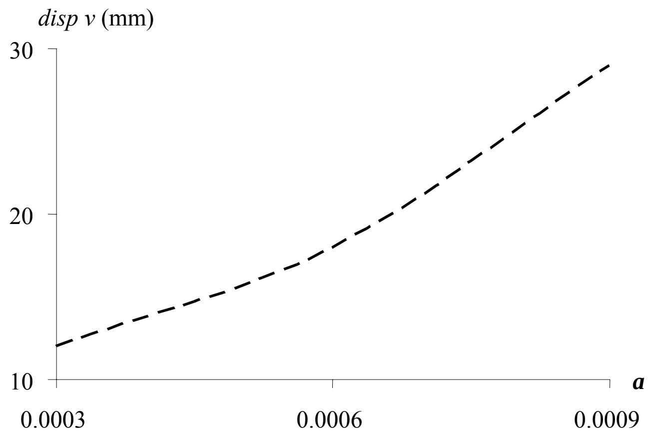

Figure 3.10. Displacements of the half parabolas with the same width.

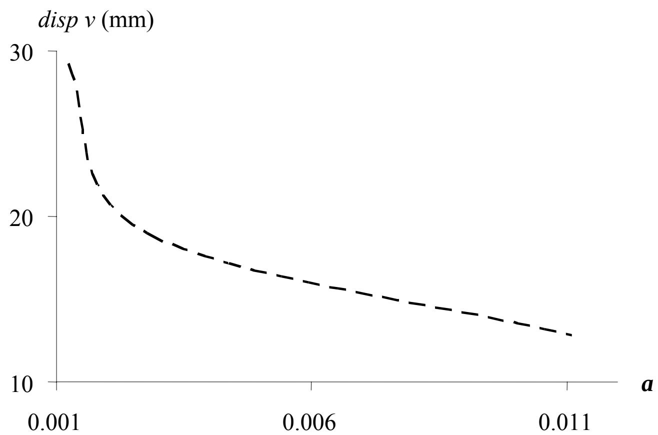

Figure 3.11. Displacements of the half parabolas with the same depth.

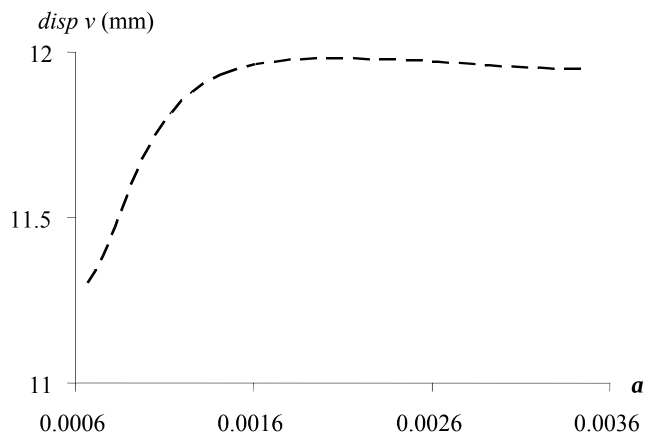

Figure 3.12. Displacements of the half parabolas with the same length.

It can be said from Figure 3.10 that displacements increase when *a* increases, since length of curved beam increases. Figure 3.11 shows opposite tendency according to Figure 3.10. However, Figure 3.12 shows very close values due to the same curved length.

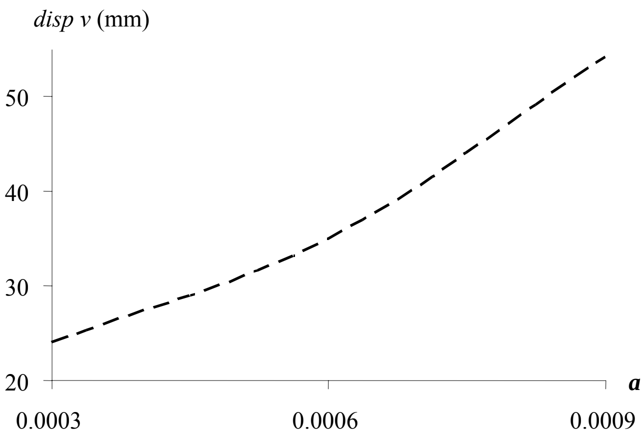

Figure 3.13. Displacements of the full parabolas with the same width.

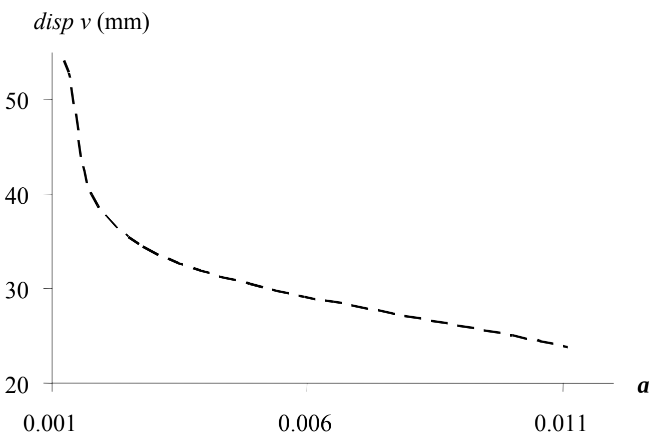

Figure 3.14. Displacements of the full parabolas with the same depth.

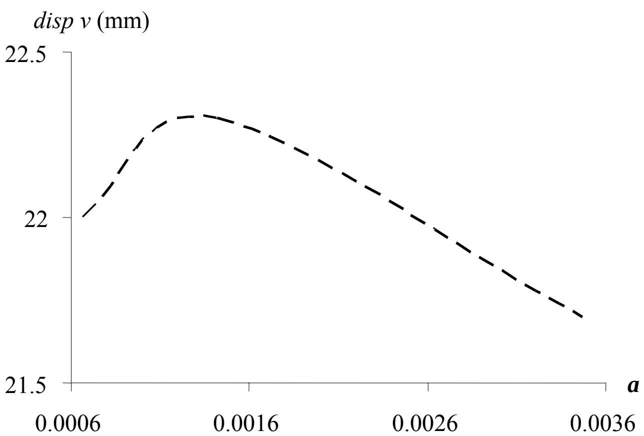

Figure 3.15. Displacements of the full parabolas with the same length.

Similar to Figures 3.10 and 3.11, Figures 3.13 and 3.14 presents the similar tendencies, respectively.

Figure 3.10 and Figure 3.13 exhibit almost linear relationship between vertical displacement v and parabola parameter a. The physical situation can be stated for these two figures as follows: when the depth of parabola increases for the parabola having the same width, vertical displacement v increases due to the increasing of the length of the parabola. It should be pointed out that when parabola parameter a increases, depth and length of the parabola increases.

But, Figure 3.11 and Figure 3.14 show that the relation between vertical displacement v and parabola parameter a is nonlinear. Now, the physical situation can be stated for these two figures as follows: when the width of parabola decreases for the parabola having the same depth, vertical displacement v decreases due to the decreasing of the length of the parabola. It should be pointed out that when parabola parameter a increases, width and length of the parabola decreases.

On the other hand, Figure 3.15 shows very interesting behavior due to the width and depth of the parabolas shown in Figure 3.7. The parabola with depth of 707 mm and with width of 2x707=1414 mm has more vertical displacement than others. The reason behind this can be explained as follows: It is known that the stiffness of a straight cantilever beam and the stiffness of a beam fixed at both ends are based on the length of the beam. Using these simple ideas, it can be said that the parabola with the depth of 866 mm is stiffer than the parabola with the depth of 707 mm due to its width. Also, the parabola with the depth of 500 mm is stiffer than the parabola with the depth of 707 mm due to the depth of it.

CHAPTER 4

CONCLUSIONS

When the differential equation has variable coefficient, its exact solution can not be obtained in all times. For these cases, numerical solutions are used. Due to the computer technologies, Power Series Method, Finite Difference Method, Finite Element Method, Differential Transform Method, and Differential Quadrature Method are developed. The Differential Quadrature Method (DQM) is a kind of the generalization of Finite Difference Method.

The out-of-plane displacements of curved beams with variable curvature are modeled mathematically by two coupled differential equations with variable coefficients. Differential Quadrature Method (DQM) is used for the current problem and tested by the results Finite Element Method (FEM). Although, FEM is numerical method and gives approximate results, when the exact results are not available for a problem, it can be used for verifications.

The effects of variable curvature parameter of the curved beam on out-of-plane displacements are studied as the practical application of DQM in structural mechanics.

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