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Golden quantum oscillator and Binet–Fibonacci calculus

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Abstract
The Binet formula for Fibonacci numbers is treated as a q-number and a q-operator with Golden ratio bases $q = \varphi$ and $Q = -1/\varphi$, and the corresponding Fibonacci or Golden calculus is developed. A quantum harmonic oscillator for this Golden calculus is derived so that its spectrum is given only by Fibonacci numbers. The ratio of successive energy levels is found to be the Golden sequence, and for asymptotic states in the limit $n \to \infty$ it appears as the Golden ratio. We call this oscillator the Golden oscillator. Using double Golden bosons, the Golden angular momentum and its representation in terms of Fibonacci numbers and the Golden ratio are derived. Relations of Fibonacci calculus with a q-deformed fermion oscillator and entangled N-qubit states are indicated.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Fibonacci numbers have been known from ancient times as 'nature's numbering system', and have applications to the growth of every living thing, from natural plants (e.g. branches of trees, the arrangement of leaves) to human proportions and architecture (the Golden section) [1]. The numbers satisfy the recursion relation

$$F_1 = F_2 = 1 \text{ (initial condition)},$$

$$F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \geqslant 2 \text{ (recursion formula)}.$$

The first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, ... For these numbers, starting from de Moivre, Lame and Binet, the next representation is known as the Binet formula [1]:

$$F_n = \frac{\varphi^n - \varphi'^n}{\varphi - \varphi'}, \tag{1}$$

where φ and φ' are the positive and negative roots of the equation

$$x^2 - x - 1 = 0.$$

These roots are explicitly

$$\varphi = \frac{1+\sqrt{5}}{2}, \qquad \varphi' = \frac{1-\sqrt{5}}{2} = -\frac{1}{\varphi}. \tag{2}$$

Number φ is known as the Golden ratio or the Golden section. There are countless works devoted to the application of the Golden ratio in many fields, from natural phenomena to architecture and music.

Fibonacci numbers can be considered as a particular realization of Fibonacci polynomials $F_n(a)$:

$$F_1(a) = 1, \qquad F_2(a) = a$$

$$F_{n+1}(a) = aF_n(a) + F_{n-1}(a), \qquad \text{for } n \geqslant 2,$$

when $a = 1$: $F_n(1) = F_n$. For $a = 2$, it gives $F_n(2)$—the Pell numbers. The Binet representation for these polynomials is easy to see as

$$F_n(a) = \frac{q^n - \left(-\frac{1}{q}\right)^n}{q - \left(-\frac{1}{q}\right)}, \tag{3}$$

where parameter $a = q - \frac{1}{q}$, so that $q = \frac{a+\sqrt{a^2+4}}{2}$ and $-\frac{1}{q} = \frac{a-\sqrt{a^2+4}}{2}$ are roots of the quadratic equation $x^2 = ax + 1$.

Here we note that the Binet formula can be considered as a special realization of the so-called q-numbers in q-calculus with two bases: q and $Q = -\frac{1}{q}$. The (Q, q) calculus generalizes Jackson's q-calculus to two parameters. In the particular case when $Q = 1$, it becomes the non-symmetrical calculus and in another case, when $Q = \frac{1}{q}$, it reduces to the symmetrical q-calculus. The (Q, q) two parametric quantum algebras have been introduced in connection with the generalized quantum q-harmonic oscillator [2, 3]. The corresponding calculus is mentioned in a convenient form for the generalization of the q-calculus in [4].

Recently, we found that the (Q, q) calculus appears naturally in the construction of the q-binomial formula for Q-commutative elements. It was inspired by non-commutative q-binomials, introduced for the description of the q-Hermite polynomial solutions of the q-heat equation in [5]. Alternatively, with the operator version of this calculus [6], we constructed the AKNS hierarchy of integrable systems, where $Q = R$ is the recursion operator of the AKNS hierarchy and q is the spectral parameter.

In this paper, we would like to explore the possibility of interpreting the Binet formula for Fibonacci polynomials and Fibonacci numbers as q-numbers, and develop the corresponding q-calculus. There are several motivations for studing this calculus.

1.1. Generalized q-deformed fermion algebra

In addition to q-bosonic quantum algebras, several attempts were made to construct q-deformed fermionic oscillators [7, 8]. These fermionic quantum algebras were applied to several problems, as the dynamic mass generation of quarks and nuclear pairing [9, 10], and as descriptive of higher order effects in many-body interactions in nuclei [11, 12].

A non-trivial q-deformation of the fermion oscillator algebra has been proposed in [7]:

$$f_q f_q^+ + \sqrt{q} f_q^+ f_q = q^{-\frac{N}{2}}, \tag{4}$$

$$[N, f_q^+] = f_q^+, \qquad [N, f_q] = -f_q; \; f_q^2 \neq 0. \tag{5}$$

In this q-deformed fermionic oscillator algebra, the Pauli exclusion principle is no longer valid. The oscillator allows more than two q-fermions in a given quantum state and admits fermion–boson transmutation. For such q-fermion algebra, the Fock space construction requires one to introduce the 'fermionic q-numbers' [7],

$$[n]_q^F = \frac{q^{-\frac{n}{2}} - (-1)^n q^{\frac{n}{2}}}{q^{-\frac{1}{2}} + q^{\frac{1}{2}}}. \tag{6}$$

For generic q, this representation is infinite dimensional. In the limit $q \to 1$, the Fock space reduces to two states: the vacuum state and one-fermion state, so that the Pauli principle gets recovered. Here, we note that this fermionic q-number (6) under substitution $q \to \frac{1}{\sqrt{q}}$ becomes the Binet formula (3) for Fibonacci polynomials $F_n(\frac{1}{\sqrt{q}} - \sqrt{q})$, and for Golden ratio base $q = \frac{1}{\varphi^2}$, it gives Fibonacci numbers (1). This relation allows us to connect Fibonacci polynomials and Fibonacci numbers, considered as q-numbers, with fermionic q-numbers of [7]. Statistical properties of these q-deformed fermions were investigated in [13] for a description of fractional statistics. Later it was shown [14] that thermodynamics of these generalized fermions should involve the q-calculus with the Jackson-type q-derivative in the form of

$$D_x f(x) = \frac{1}{x} \frac{f(q^{-1}x) - f(-qx)}{q + q^{-1}}. \tag{7}$$

Here, we find that with the substitution $q \to \frac{1}{q}$, this derivative becomes the Fibonacci derivative which we introduce in (47), and for $q \to \frac{1}{\varphi}$, it becomes the Golden derivative (48). The above consideration indicates that the Fibonacci q-calculus is a natural language for describing q-deformed fermions and their statistics.

1.2. Hecke condition for the R-matrix

Another connection is related to the quantum integrable systems approach to the theory of quantum groups via the solution of the Yang–Baxter equation for the R-matrix [15]. If one introduces the $\hat{R}$-matrix, $\hat{R} = PR$, where P is the permutation matrix, then this invertible $\hat{R}$-matrix obeys a characteristic equation. For two roots, this equation is known as the form of the Hecke condition:

$$(\hat{R} - q)\left(\hat{R} + \frac{1}{q}\right) = 0 \tag{8}$$

or

$$\hat{R}^2 = a\hat{R} + I. \tag{9}$$

By studying representations of the braid group satisfying this quadratic relation, Jones obtained a polynomial invariant in two variables for oriented links [16]. If in calculating higher powers of matrix $\hat{R}$ we repeatedly apply the Hecke condition (9), then as a result we obtain

$$\hat{R}^n = F_n(a)\hat{R} + F_{n-1}(a)I, \tag{10}$$

where $F_n(a) = aF_{n-1}(a) + F_{n-2}(a)$ are Fibonacci polynomials (3) with $a = q - \frac{1}{q}$.

1.3. Entangled N-qubit spin coherent states

Another motivation comes from quantum information theory. The unit of quantum information, the qubit, in the spin coherent state representation

$$|\psi\rangle = \frac{1}{\sqrt{1 + |\psi|^2}} \begin{pmatrix} 1 \\ \psi \end{pmatrix} \tag{11}$$

is parametrized by the complex number $\psi \in C$, given by the stereographic projection $\psi = \tan\frac{\theta}{2}\, \mathrm{e}^{\mathrm{i}\phi}$, of the Bloch sphere for the qubit

$$|\theta, \phi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\phi}|1\rangle. \tag{12}$$

For an arbitrary representation j of $su(2)$, the scalar product of two coherent states is

$$\langle\phi|\psi\rangle = \frac{(1+\bar{\phi}\psi)^{2j}}{(1+|\phi|^2)^j(1+|\psi|^2)^j}, \tag{13}$$

and the orthogonality condition $\langle\phi|\psi\rangle = 0$ implies $1 + \bar{\phi}\,\psi = 0$. This constraint relates two states: one at point ϕ and another at the negative-symmetric point in the unit circle $\phi = -\frac{1}{\bar{\psi}}$ [17]. Geometrically, these points correspond to antipodal points on the Bloch sphere, $M(x, y, z)$ and $M^*(-x, -y, -z)$. In accordance with these points, we have recently constructed a maximally entangled set of orthonormal two-qubit coherent states [17],

$$|P_\pm\rangle = \frac{1}{\sqrt{2}}\left(|\psi\rangle|\psi\rangle \pm \left|-\frac{1}{\bar{\psi}}\right\rangle\left|-\frac{1}{\bar{\psi}}\right\rangle\right), \tag{14}$$

$$|G_\pm\rangle = \frac{1}{\sqrt{2}}\left(|\psi\rangle\left|-\frac{1}{\bar{\psi}}\right\rangle \pm \left|-\frac{1}{\bar{\psi}}\right\rangle|\psi\rangle\right), \tag{15}$$

with concurrence $C = 1$. These states generalize the Bell states and reduce to the last ones in the limit $\psi \to 0$ and $-\frac{1}{\bar{\psi}} \to \infty$. This construction can be extended to arbitrary N-qubit coherent states [18]. The first set of N-qubit entangled states expanded in the computational basis is

$$\frac{|\psi\rangle^N - \left|-\frac{1}{\bar{\psi}}\right\rangle^N}{\psi + \frac{1}{\bar{\psi}}} = F_1(\alpha, \beta)(|10\cdots 0\rangle + |01\cdots 0\rangle + \cdots |00\cdots 1\rangle) \tag{16}$$

$$+ F_2(\alpha, \beta)(|110\cdots 0\rangle + |101\cdots 0\rangle + \cdots |00\cdots 11\rangle) \tag{17}$$

$$\cdots + F_N(\alpha, \beta)(|111\cdots 1\rangle. \tag{18}$$

This is characterized by the set of complex Fibonacci polynomials

$$F_n(\alpha, \beta) = \frac{\psi^n - \left(-\frac{1}{\bar{\psi}}\right)^n}{\psi + \bar{\psi}^{-1}}, \tag{19}$$

with a reccurrence relation

$$F_{n+1}(\alpha, \beta) = \alpha F_n(\alpha, \beta) + \beta F_{n-1}(\alpha, \beta), \tag{20}$$

where $\alpha = \psi - \frac{1}{\bar{\psi}}$and $\beta = \frac{\psi}{\bar{\psi}}$. Another set of entangled N-qubit coherent states is

$$|\psi\rangle^N + \left|-\frac{1}{\bar{\psi}}\right\rangle^N = |00\cdots 0\rangle + L_1(\alpha, \beta)(|10\cdots 0\rangle + |01\cdots 0\rangle + \cdots |00\cdots 1\rangle) \tag{21}$$

$$+ L_2(\alpha, \beta)(|110\cdots 0\rangle + |101\cdots 0\rangle + \cdots |00\cdots 11\rangle) \tag{22}$$

$$\cdots + L_N(\alpha, \beta)(|111\cdots 1\rangle, \tag{23}$$

and it is characterized by complex Lucas polynomials $L_n(\alpha, \beta) = \psi^n + (-\frac{1}{\bar{\psi}})^n$. In the $N = 3$ qubit case, for particular $\psi \to 0$, $-\frac{1}{\bar{\psi}} \to \infty$, it gives a maximally entangled GHZ state. In the above Binet representations of complex polynomials $F_n(\alpha, \beta)$ and $L_n(\alpha, \beta)$, the negative-symmetric points ψ and $-\frac{1}{\bar{\psi}}$ are the roots of the complex quadratic equation $z^2 = \alpha z + \beta$.

From the polar representation of the complex numbers $\psi = q\,e^{i\phi}$ and $z = r\,e^{i\phi}$, we obtain $r^2 = ar + 1$, where $a = q - \frac{1}{q}$ and $r^n = rF_n(a) + F_{n-1}(a)$, with Fibonacci polynomials $F_n(a)$ (3). The complex Fibonacci polynomials are related to standard Fibonacci polynomials by the formula

$$F_n(\alpha, \beta) = F_n(a)\,e^{i\phi(n-1)} \tag{24}$$

and

$$L_n(\alpha, \beta) = L_n(a)\,e^{in\phi}, \tag{25}$$

where $\phi = \arg\psi$. The complex parameter $\alpha = \psi - \frac{1}{\bar{\psi}}$ has a simple geometrical meaning as a complex difference between symmetrical points in a unit circle. The interesting point here is that the symmetric points under the unit circle appear in the problem of vortex images in a circular domain [19], where these points correspond to the line vortex at ψ and its image in the circle at $\frac{1}{\bar{\psi}}$. Then parameter $a = |\alpha|$, $\alpha = a\,e^{i\phi}$, in Fibonacci polynomials has a geometrical meaning as the distance between the vortex and its image. In the particular case when this distance is equal to 1, $a = q - \frac{1}{q} = 1$, the position of the vortex is at the Golden ratio distance from the origin $r = \varphi = \frac{1+\sqrt{5}}{2}$ and Fibonacci polynomials turn into Fibonacci numbers. In this case, the line interval connecting the vortex and the negative-symmetric point intersects the unit circle at a point which divides this interval into two parts of length φ and $\frac{1}{\varphi}$.

The above motivations show that the Fibonacci q-calculus is interesting and rich in applications to develop. In the first part of this paper, we systematically introduce basic elements of this calculus as Fibonacci numbers, Fibonacci derivative and Fibonacci integral. Then we construct the quantum harmonic oscillator for the Golden q-calculus case, so that its spectrum is given only by Fibonacci numbers and the ratio of successive energy levels is given as the Golden sequence. For asymptotic states at $n \to \infty$, it appears as the Golden ratio. Although some results for the harmonic oscillator with generic $Q - q$ and their reductions to symmetrical and non-symmetrical cases are known [20–22], we think that the special case with negative-symmetrical and the Golden ratio bases has not been described before in the literature. Due to the importance and wide applicability of Fibonacci numbers in different fields, we think that the explicit realization of them in the form of a quantum oscillator with a Golden ratio base, which we call the Golden quantum oscillator, deserves to be studied. In particular, a realization of this type of calculus could describe the Golden ratio in non-commutative geometry and q-deformed fermions with fractional statistics.

Finally, our Golden oscillator should not be confused with the Fibonacci oscillator of [2], with the generic bases q_1 and q_2, though it is a particular case of it. In that paper, the Fibonacci calculus and its relation with the Golden ratio and Binet formula, as well as asymptotic properties of energy levels, have not been discussed.

2. Golden *q*-calculus

In the (Q, q) calculus we have the number

$$[n]_{Q,q} = \frac{Q^n - q^n}{Q - q}. \tag{26}$$

If $Q = -\frac{1}{q}$, then this q-number gives Fibonacci polynomials

$$F_n(a) = \frac{q^n - \left(-\frac{1}{q}\right)^n}{q - \left(-\frac{1}{q}\right)} = [n]_F^q, \tag{27}$$

where $a = q - \frac{1}{q}$. In a special case with $Q = \varphi = \frac{1+\sqrt{5}}{2}$ and $q = \varphi' = \frac{1-\sqrt{5}}{2} = -\frac{1}{\varphi}$, (26) becomes Binet's formula for Fibonacci numbers as $(\varphi,\ \varphi')$ numbers:

$$F_n = \frac{\varphi^n - \varphi'^n}{\varphi - \varphi'} = [n]_{\varphi,\varphi'} \equiv [n]_F. \tag{28}$$

This definition can be extended to an arbitrary real number x,

$$[x]_{\varphi,\varphi'} \equiv [x]_F = \frac{\varphi^x - \varphi'^x}{\varphi - \varphi'} = \frac{\varphi^x - \left(-\frac{1}{\varphi}\right)^x}{\varphi + \frac{1}{\varphi}} \equiv F_x, \tag{29}$$

though due to a negative sign for the second base, it is not a real number for general x,

$$F_x = \frac{1}{\varphi + \frac{1}{\varphi}}\left(\varphi^x - \mathrm{e}^{\mathrm{i}\pi x}\frac{1}{\varphi^x}\right) = \frac{1}{\sqrt{5}}\left(\varphi^x - \mathrm{e}^{\mathrm{i}\pi x}\frac{1}{\varphi^x}\right). \tag{30}$$

Instead of the real number x, we can consider the complex numbers $z = x + \mathrm{i}y$,

Example. It is easy to see that

$$\lim_{n\to\infty}\frac{[n+1]_F}{[n]_F} = \lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \varphi.$$

The addition formula for Golden numbers is given in the form

$$[n+m]_F = F_{n+m} = \varphi^n F_m + \left(-\frac{1}{\varphi}\right)^m F_n. \tag{31}$$

Using (28) we can obtain

$$\varphi^N = \varphi F_N + F_{N-1}, \qquad \varphi'^N = \varphi' F_N + F_{N-1}, \tag{32}$$

and the above formula (31) can be rewritten as

$$\begin{aligned} F_{n+m} &= F_n F_{m-1} + F_{n+1} F_m \\ &= F_{n-1} F_m + F_n F_{m+1}. \end{aligned} \tag{33}$$

The substraction formula can be obtained from it by changing $m \to -m$ as

$$F_{n-m} = [n-m]_F = \varphi^n [-m]_F + \left(-\frac{1}{\varphi}\right)^{-m} [n]_F, \tag{34}$$

or by using the equality

$$[-n]_F = -(-1)^{-n}[n]_F$$

it can also be written as

$$\begin{aligned} [n-m]_F &= \left(-\frac{1}{\varphi}\right)^{-m} ([n]_F - \varphi^{n-m}[m]_F) \\ &= \left(-\frac{1}{\varphi}\right)^{-m} F_n - \frac{\varphi^n}{(-1)^m} F_m, \end{aligned} \tag{35}$$

or

$$F_{n-m} = \left(-\frac{1}{\varphi}\right)^{-m} F_n - \frac{\varphi^n}{(-1)^m} F_m. \tag{36}$$

Definition (higher Fibonacci numbers).

$$F_n^{(m)} \equiv \frac{(\varphi^m)^n - (\varphi'^m)^n}{\varphi^m - \varphi'^m} = [n]_{\varphi^m,\varphi'^m} \tag{37}$$

and $F_n^{(1)} \equiv F_n$.

By the definition, the multiplication rule for Golden numbers is given by the following formula

$$[nm]_{\varphi,-\frac{1}{\varphi}} = F_{nm} = [n]_{\varphi,-\frac{1}{\varphi}}[m]_{\varphi^n,(-\frac{1}{\varphi})^n} = F_n F_m^{(n)}, \tag{38}$$

and the division rule is

$$\left[\frac{m}{n}\right]_{\varphi,\varphi'} = \frac{[m]_{\varphi,\varphi'}}{[n]_{\varphi^{m/n},\varphi'^{m/n}}} = \frac{[m]_{\varphi^{1/n},\varphi'^{1/n}}}{[n]_{\varphi^{1/n},\varphi'^{1/n}}}$$

$$F_{\frac{m}{n}} = \frac{F_m}{F_n^{(\frac{m}{n})}} = \frac{F_m^{(\frac{1}{n})}}{F_n^{(\frac{1}{n})}}. \tag{39}$$

Higher Fibonacci numbers can be written as a ratio of Fibonacci numbers:

$$F_n^{(m)} = \frac{F_{mn}}{F_m}. \tag{40}$$

From definition (28), we have the following relation:

$$F_{-n} = (-1)^{n+1} F_n. \tag{41}$$

For any real x, y,

$$[x+y]_F = \varphi^x [y]_F + \left(-\frac{1}{\varphi}\right)^y [x]_F$$

$$= \varphi^y [x]_F + \left(-\frac{1}{\varphi}\right)^x [y]_F, \tag{42}$$

which are written in terms of Fibonacci numbers as follows:

$$F_{x+y} = \varphi^x F_y + \left(-\frac{1}{\varphi}\right)^y F_x$$

$$= \varphi^y F_x + \left(-\frac{1}{\varphi}\right)^x F_y. \tag{43}$$

For real x, we have the Fibonacci recurrence relation:

$$[x]_F = [x-1]_F + [x-2]_F \Rightarrow F_x = F_{x-1} + F_{x-2}. \tag{44}$$

Example. Golden π,

$$F_\pi = [\pi]_F \simeq 4.730\,68 + 0.093\,9706\mathrm{i}$$

3. Fibonacci and Golden derivative

Now we introduce the Fibonacci derivative operator

$$F^q_{x\frac{\mathrm{d}}{\mathrm{d}x}} = \frac{q^{x\frac{\mathrm{d}}{\mathrm{d}x}} - \left(-\frac{1}{q}\right)^{x\frac{\mathrm{d}}{\mathrm{d}x}}}{q+q^{-1}} = \left[x\frac{\mathrm{d}}{\mathrm{d}x}\right]^q_F \tag{45}$$

and the Golden derivative operator

$$F_{x\frac{\mathrm{d}}{\mathrm{d}x}} = \frac{\varphi^{x\frac{\mathrm{d}}{\mathrm{d}x}} - \varphi'^{x\frac{\mathrm{d}}{\mathrm{d}x}}}{\varphi - \varphi'} = \left[x\frac{\mathrm{d}}{\mathrm{d}x}\right]_F. \tag{46}$$

Then the Fibonacci derivative of the function $f(x)$ is

$$F^q_{x\frac{\mathrm{d}}{\mathrm{d}x}} f(x) = D^q_F f(x) = \frac{f(qx) - f\left(-\frac{x}{q}\right)}{\left(q+\frac{1}{q}\right)x}, \tag{47}$$

and for the Golden derivative we have

$$F_{x\frac{\mathrm{d}}{\mathrm{d}x}} f(x) = D_F f(x) = \frac{f(\varphi x) - f\left(-\frac{x}{\varphi}\right)}{\left(\varphi + \frac{1}{\varphi}\right)x} = \frac{\left(M_\varphi - M_{-\frac{1}{\varphi}}\right) f(x)}{\left(\varphi + \frac{1}{\varphi}x\right)}. \tag{48}$$

Here, arguments are scaled by the Golden ratio: $x \to \varphi x$ and $x \to -\frac{x}{\varphi}$. It can be written in terms of the Golden ratio dilatation operator

$$M_\varphi f(x) = f(\varphi x),$$

where $f(x)$ is a smooth function and its operator form can also be written as

$$M_\varphi = \varphi^{x\frac{\mathrm{d}}{\mathrm{d}x}} = \left(\frac{1+\sqrt{5}}{2}\right)^{x\frac{\mathrm{d}}{\mathrm{d}x}}.$$

We call the function $A(x)$ the Golden periodic function if

$$D_F A(x) = 0, \tag{49}$$

which implies

$$A(\varphi x) = A\left(-\frac{1}{\varphi}x\right). \tag{50}$$

As an example of the Golden periodic function, we have

$$A(x) = \sin\left(\frac{\pi}{\ln\varphi}\ln|x|\right). \tag{51}$$

Example 1. Application of the Golden derivative operator D_F on x^n generates Fibonacci numbers:

$$D_F x^n = F_n x^{n-1}$$

or

$$F_n = \frac{D_F x^n}{x^{n-1}}.$$

Example 2.

$$D_F\, \mathrm{e}^x = \sum_{n=0}^{\infty} \frac{F_n}{n!} x^n$$

or

$$D_F\, \mathrm{e}^x = \frac{\mathrm{e}^{\varphi x} - \mathrm{e}^{-\frac{x}{\varphi}}}{\varphi + \frac{1}{\varphi}} = \frac{2\,\mathrm{e}^{\frac{x}{2}} \sinh\frac{\sqrt{5}}{2}x}{\sqrt{5}x} = \sum_{n=0}^{\infty} \frac{F_n}{n!} x^n.$$

For $x = 1$, this gives the next summation formula:

$$\sum_{n=0}^{\infty} \frac{F_n}{n!} = \mathrm{e}^{\frac{1}{2}} \frac{\sinh\frac{\sqrt{5}}{2}}{\frac{\sqrt{5}}{2}}. \tag{52}$$

3.1. Golden Leibnitz rule

We derive the Golden Leibnitz rule

$$D_F(f(x)g(x)) = D_F f(x) g(\varphi x) + f\left(-\frac{x}{\varphi}\right) D_F g(x). \tag{53}$$

By symmetry, the second form of the Leibnitz rule can be derived as

$$D_F(f(x)g(x)) = D_F f(x) g\left(-\frac{x}{\varphi}\right) + f(\varphi x) D_F g(x). \tag{54}$$

These formulas can be rewritten explicitly in a symmetrical form

$$D_F(f(x)g(x)) = D_F f(x)\left(\frac{g(\varphi x) + g\left(-\frac{x}{\varphi}\right)}{2}\right) + D_F g(x)\left(\frac{f(\varphi x) + f\left(-\frac{x}{\varphi}\right)}{2}\right). \tag{55}$$

A more general form of the Golden Leibnitz formula is given with an arbitrary α,

$$\begin{aligned} D_F(f(x)g(x)) = {} & \left(\alpha f\left(-\frac{x}{\varphi}\right) + (1-\alpha) f(\varphi x)\right) D_F g(x) \\ & + \left(\alpha g(\varphi x) + (1-\alpha) g\left(-\frac{x}{\varphi}\right)\right) D_F f(x). \end{aligned}$$

Now we may compute the Golden derivative of the quotient of $f(x)$ and $g(x)$. From (53), we have

$$D_F\left(\frac{f(x)}{g(x)}\right) = \frac{D_F f(x) g(\varphi x) - D_F g(x) f(\varphi x)}{g(\varphi x) g(-\frac{x}{\varphi})}. \tag{56}$$

However, if we use (54), we obtain

$$D_F\left(\frac{f(x)}{g(x)}\right) = \frac{D_F f(x) g(-\frac{x}{\varphi}) - D_F g(x) f(-\frac{x}{\varphi})}{g(\varphi x) g(-\frac{x}{\varphi})}. \tag{57}$$

In addition to formulas (56) and (57), one may determine one more representation in a symmetrical form

$$D_F\left(\frac{f(x)}{g(x)}\right) = \frac{1}{2}\frac{D_F f(x)\left(g\left(-\frac{x}{\varphi}\right) + g(\varphi x)\right) - D_F g(x)\left(f\left(-\frac{x}{\varphi}\right) + f(\varphi x)\right)}{g(\varphi x) g\left(-\frac{x}{\varphi}\right)}. \tag{58}$$

In particular applications, one of these forms could be more useful than others.

3.2. Golden Taylor expansion

Theorem 3.2.1. *Let the Golden derivative operator D_F be a linear operator on the space of polynomials, and*

$$P_n(x) \equiv \frac{x^n}{F_n!} \equiv \frac{x^n}{F_1 F_2 \ldots F_n}$$

satisfy the following conditions:

(i) $P_0(0) = 1$ *and* $P_n(0) = 0$ *for any* $n \geqslant 1$;

(ii) *deg* $P_n = n$;

(iii) $D_F P_n(x) = P_{n-1}(x)$ *for any* $n \geqslant 1$, *and* $D_F(1) = 0$. *Then, for any polynomial* $f(x)$ *of degree N, one has the following Taylor formula:*

$$f(x) = \sum_{n=0}^{N} (D_F^n f)(0) P_n(x) = \sum_{n=0}^{N} (D_F^n f)(0) \frac{x^n}{F_n!}.$$

In the limit $N \to \infty$ (when it exists), this formula can determine some new function:

$$f_F(x) = \sum_{n=0}^{\infty} (D_F^n f)(0) \frac{x^n}{F_n!}, \tag{59}$$

which we call the Golden (or Fibonacci) function.

Example (Golden exponential)**.** The Golden exponential functions are defined as

$$e_F^x \equiv \sum_{n=0}^{\infty} \frac{x^n}{F_n!}; \qquad E_F^x \equiv \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{x^n}{F_n!}, \tag{60}$$

and for $x = 1$, we obtain the Fibonacci natural base as follows:

$$e_F^1 \equiv \sum_{n=0}^{\infty} \frac{1}{F_n!} \equiv e_F.$$

Both of these functions are entire analytic functions. For the second function, we explicitly have

$$E_F^x = 1 + \frac{x}{F_1!} - \frac{x^2}{F_2!} - \frac{x^3}{F_3!} + \frac{x^4}{F_4!} + \frac{x^5}{F_5!} - \frac{x^6}{F_6!} - \frac{x^7}{F_7!} + \frac{x^8}{F_8!} + \frac{x^9}{F_9!} - \cdots. \tag{61}$$

The Golden derivative of these exponential functions is found as

$$D_F e_F^{kx} = k e_F^{kx},$$

$$D_F E_F^{kx} = k E_F^{-kx}$$

for an arbitrary constant k (or F-periodic function). These two functions then give the general solution of the hyperbolic F-oscillator equation

$$\left(D_F^2 - k^2\right)\phi(x) = 0, \tag{62}$$

as

$$\phi(x) = A e_F^{kx} + B e_F^{-kx}, \tag{63}$$

and the elliptic F-oscillator equation

$$\left(D_F^2 + k^2\right)\phi(x) = 0, \tag{64}$$

as

$$\phi(x) = A E_F^{kx} + B E_F^{-kx}. \tag{65}$$

For an imaginary argument, we next have Euler formulas

$$e_F^{\mathrm{i}x} = \cos_F x + \mathrm{i} \sin_F x, \tag{66}$$

$$E_F^{\mathrm{i}x} = \cosh_F x + \mathrm{i} \sinh_F x, \tag{67}$$

and relations

$$\cosh_F x = \cos_F x, \tag{68}$$

$$\sinh_F x = \sin_F x, \tag{69}$$

where

$$\cosh_F x \equiv \frac{E_F^x + E_F^{-x}}{2}, \qquad \sinh_F x \equiv \frac{E_F^x - E_F^{-x}}{2}. \tag{70}$$

We note here that these relations are valid due to the alternating character of the second exponential function (61).

Example (F-oscillator)**.** For an F-oscillator with frequency ω,

$$D_F^2 x + \omega^2 x = 0, \tag{71}$$

the general solution is

$$x(t) = aE_F^{\omega t} + bE_F^{-\omega t} = a' \cosh_F \omega t + b' \sinh_F \omega t = a' \cos_F \omega t + b' \sin_F \omega t. \tag{72}$$

3.3. Golden binomial

The Golden binomial is defined as

$$(x+y)_F^n = (x + \varphi^{n-1} y)(x - \varphi^{n-3} y) \cdots (x + (-1)^{n-1} \varphi^{-n+1} y) \tag{73}$$

and it has n-zeros at the Golden ratio powers

$$\frac{x}{y} = -\varphi^{n-1}, \qquad \frac{x}{y} = -\varphi^{n-3}, \ldots, \frac{x}{y} = -\varphi^{-n+1}.$$

For the Golden binomial, the next expansion is valid

$$\begin{aligned}(x+y)_F^n \equiv (x+y)^n_{\varphi, -\frac{1}{\varphi}} &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_F (-1)^{\frac{k(k-1)}{2}} x^{n-k} y^k \\ &= \sum_{k=0}^{n} \frac{F_n!}{F_{n-k}! F_k!} (-1)^{\frac{k(k-1)}{2}} x^{n-k} y^k. \end{aligned} \tag{74}$$

The proof is easy by induction.

The application of the Golden derivative to the Golden binomial gives

$$D_F^x (x+y)_F^n = F_n (x+y)_F^{n-1},$$

$$D_F^y (x+y)_F^n = F_n (x-y)_F^{n-1},$$

which means

$$D_F^x \frac{(x+y)_F^n}{F_n!} = \frac{(x+y)_F^{n-1}}{F_{n-1}!},$$

$$D_F^y \frac{(x+y)_F^n}{F_n!} = \frac{(x-y)_F^{n-1}}{F_{n-1}!}.$$

From

$$\left(D_F^y\right)^n (x+y)_F^n,$$

for $n = 2k$, we have

$$\left(D_F^y\right)^{2k} (x+y)_F^{2k} = (-1)^k F_{2k}!$$

and for $n = 2k+1$, we obtain

$$\left(D_F^y\right)^{2k+1} (x+y)_F^{2k+1} = (-1)^k F_{2k+1}!.$$

If we introduce an F-exponential function of two arguments

$$e_F(t+x)_F \equiv \sum_{n=0}^{\infty} \frac{(t+x)_F^n}{F_n!}, \tag{75}$$

then, applying the above formulas, we have

$$D_F^t e_F(t+x)_F = e_F(t+x)_F, \tag{76}$$

$$D_F^x e_F(t+x)_F = e_F(t-x)_F, \tag{77}$$

$$\left(D_F^x\right)^2 e_F(t+x)_F = D_F^x e_F(t-x)_F = -e_F(t+x)_F. \tag{78}$$

As a result, we find the solution of the Golden heat equation is

$$\left[D_F^t + \left(D_F^x\right)^2\right] e_F(t+x)_F = 0. \tag{79}$$

In terms of the Golden binomial, we introduce the Golden polynomials

$$P_n(x) = \frac{(x-a)_F^n}{F_n!}, \tag{80}$$

where $n = 1, 2, \ldots$, and $P_0(x) = 1$ with property

$$D_F^x P_n(x) = P_{n-1}(x). \tag{81}$$

For even and odd polynomials, we have the following product representations:

$$P_{2n}(x) = \frac{1}{F_{2n}!}\prod_{k=1}^{n}(x-(-1)^{n+k}\varphi^{2k-1}a)(x+(-1)^{n+k}\varphi^{-2k+1}a), \tag{82}$$

$$P_{2n+1}(x) = \frac{(x-(-1)^n a)}{F_{2n+1}!}\prod_{k=1}^{n}(x-(-1)^{n+k}\varphi^{2k}a)(x-(-1)^{n+k}\varphi^{-2k}a). \tag{83}$$

By using (32) it is easy to find that

$$\varphi^{2k} + \frac{1}{\varphi^{2k}} = F_{2k} + 2F_{2k-1}, \tag{84}$$

$$\varphi^{2k+1} - \frac{1}{\varphi^{2k+1}} = F_{2k+1} + 2F_{2k}. \tag{85}$$

Then we can rewrite our polynomials in terms of just Fibonacci numbers:

$$P_{2n}(x) = \frac{1}{F_{2n}!}\prod_{k=1}^{n}(x^2-(-1)^{n+k}(F_{2k-1}+2F_{2k-2})xa - a^2), \tag{86}$$

$$P_{2n+1}(x) = \frac{(x-(-1)^n a)}{F_{2n+1}!}\prod_{k=1}^{n}(x^2-(-1)^{n+k}(F_{2k}+2F_{2k-1})xa + a^2). \tag{87}$$

The first few polynomials are

$$P_1(x) = (x-a) \tag{88}$$

$$P_3(x) = \tfrac{1}{2}(x+a)(x^2-3xa+a^2) \tag{89}$$

$$P_5(x) = \tfrac{1}{2\cdot3\cdot5}(x-a)(x^2+3xa+a^2)(x^2-7xa+a^2) \tag{90}$$

$$P_7(x) = \tfrac{1}{2\cdot3\cdot5\cdot8\cdot13}(x+a)(x^2-3xa+a^2)(x^2+7xa+a^2)(x^2-18xa+a^2) \tag{91}$$

...

$$P_2(x) = (x^2-xa-a^2) \tag{92}$$

$$P_4(x) = \tfrac{1}{2\cdot3}(x^2+xa-a^2)(x^2-4xa-a^2) \tag{93}$$

$$P_6(x) = \tfrac{1}{2\cdot3\cdot5\cdot8}(x^2-xa-a^2)(x^2+4xa-a^2)(x^2-11xa-a^2) \tag{94}$$

...

3.4. Noncommutative Golden ratio and Golden binomials

By choosing $q = -\frac{1}{\varphi}$ and $Q = \varphi$, in the general Q-commutative q-binomial [23], where φ is the Golden section, we obtain the Binet–Fibonacci binomial formula for the Golden non-commutative plane ($yx = \varphi xy$) (it should be compared with the Golden ratio $b = \varphi a$):

$$
\begin{aligned}
(x+y)^n_{-\frac{1}{\varphi}} &= (x+y)\left(x + \left(-\frac{1}{\varphi}\right)y\right)\left(x + \left(-\frac{1}{\varphi}\right)^2 y\right)\cdots\left(x + \left(-\frac{1}{\varphi}\right)^{n-1} y\right)\\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\varphi,-\frac{1}{\varphi}} \left(-\frac{1}{\varphi}\right)^{\frac{k(k-1)}{2}} x^{n-k}y^k\\
&= \sum_{k=0}^{n} \frac{F_n!}{F_k!F_{n-k}!}\left(-\frac{1}{\varphi}\right)^{\frac{k(k-1)}{2}} x^{n-k}y^k,
\end{aligned}
\tag{95}
$$

where F_n are the Fibonacci numbers.

3.5. Golden Pascal triangle

The Golden binomial coefficients are defined by

$$
\begin{bmatrix} n \\ k \end{bmatrix}_F = \frac{[n]_F!}{[n-k]_F![k]_F!} = \frac{F_n!}{F_{n-k}!F_k!},
\tag{96}
$$

with n and k being the non-negative integers, $n \geqslant k$ and are called the Fibonomials. Using the addition formula for Golden numbers (31), we write the following expression:

$$
F_n = F_{n-k+k} = \left(-\frac{1}{\varphi}\right)^k F_{n-k} + \varphi^{n-k}F_k,
$$

and from (32) it can be written as follows:

$$
\begin{aligned}
F_n &= F_{n-k-1}F_k + F_{n-k}F_{k+1}\\
&= F_{n-k}F_{k-1} + F_{n-k+1}F_k.
\end{aligned}
\tag{97}
$$

With the above definition (96) we have next recursion formulas

$$
\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix}_F &= \frac{\left(-\frac{1}{\varphi}\right)^k[n-1]_F!}{[k]_F![n-k-1]_F!} + \frac{\varphi^{n-k}[n-1]_F!}{[n-k]_F![k-1]_F!}\\
&= \left(-\frac{1}{\varphi}\right)^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_F + \varphi^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F
\end{aligned}
\tag{98}
$$

$$
= \varphi^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_F + \left(-\frac{1}{\varphi}\right)^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F.
\tag{99}
$$

These two rules determine the multiple Golden Pascal triangle, where $1 \leqslant k \leqslant n-1$. Then, we can construct the Golden Pascal triangle as follows:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & \swarrow & & \searrow & & & & \\
 & & & 1 & & & & 1 & & & \\
 & & \swarrow & & \searrow -\frac{1}{\varphi} & & \varphi \swarrow & & \searrow & & \\
 & 1 & & & & [2]_F & & & & 1 & \\
\swarrow & & \searrow (-\frac{1}{\varphi})^2 & & \varphi \swarrow & & \searrow -\frac{1}{\varphi} & & \varphi^2 \swarrow & & \searrow \\
 & \cdots & & & & \cdots & & & & \cdots &
\end{array}
$$

3.6. Remarkable limit

From the Golden binomial expansion (74), we have

$$(1+y)_F^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_F (-1)^{\frac{k(k-1)}{2}} y^k$$
$$= \sum_{k=0}^{n} \frac{F_n!}{F_{n-k}!F_k!} (-1)^{\frac{k(k-1)}{2}} y^k. \tag{100}$$

Then,

$$\left(1+\frac{y}{\varphi^n}\right)_F^n = \sum_{k=0}^{n} \frac{F_n!}{F_{n-k}!F_k!} (-1)^{\frac{k(k-1)}{2}} \frac{y^k}{\varphi^{nk}} \tag{101}$$

or by opening Fibonomials and taking the limit

$$\lim_{n\to\infty} \left(1+\frac{y}{\varphi^n}\right)_F^n = \sum_{k=0}^{\infty} \frac{1}{F_k!} (-1)^{\frac{k(k-1)}{2}} \frac{y^k}{\varphi^{\frac{k(k-1)}{2}} (\varphi + \frac{1}{\varphi})^k}, \tag{102}$$

$$\lim_{n\to\infty} \left(1+\frac{y}{\varphi^n}\right)_F^n = \sum_{k=0}^{\infty} \frac{1}{[k]_{-\varphi^2}!} \left(\frac{y\varphi}{\varphi^2+1}\right)^k, \tag{103}$$

where we introduced the q-number $[k]_q = 1 + q + \cdots + q^{k-1}$, with base $q = -\varphi^2$, so that

$$[k]_{-\varphi^2} = 1 + (-\varphi^2) + \cdots + (-\varphi^2)^{k-1} = \frac{(-\varphi^2)^k - 1}{(-\varphi^2) - 1}. \tag{104}$$

The last expression allows us to rewrite the limit in terms of the Jackson q-exponential function $e_q(x)$ with $q = -\varphi^2$,

$$\lim_{n\to\infty} \left(1+\frac{y}{\varphi^n}\right)_F^n = e_{-\varphi^2}\left(\frac{y\varphi}{\varphi^2+1}\right), \tag{105}$$

or finally we have the remarkable limit

$$\lim_{n\to\infty} \left(1+\frac{y}{\varphi^n}\right)_F^n = e_{-\varphi^2}\left(\frac{y}{\sqrt{5}}\right). \tag{106}$$

In a particular case, this gives

$$\lim_{n\to\infty} \left(1+\frac{\sqrt{5}}{\varphi^n}\right)_F^n = e_{-\varphi^2}(1). \tag{107}$$

3.7. Golden integral

3.7.1. Golden anti-derivative.

Definition 3.7.1. *The function $G(x)$ is the Golden anti-derivative of $g(x)$ if $D_F G(x) = g(x)$ and is denoted by*

$$G(x) = \int g(x)\, \mathrm{d}_F x. \tag{108}$$

Then,

$$D_F G(x) = 0 \Rightarrow G(x) = C - constant$$

or

$$D_F G(x) = 0 \Rightarrow G(\varphi x) = G\left(-\frac{x}{\varphi}\right),$$

the Golden 'periodic' function.

3.7.2. Golden–Jackson integral. By inverting equation (48) and expanding the inverse operator, we find the Jackson-type representation for the anti-derivative:

$$G(x) = \int g\left(\frac{x}{\varphi}\right) d_Q x = (1-Q)x \sum_{k=0}^{\infty} Q^k f\left(\frac{x}{\varphi} Q^k\right), \tag{109}$$

where the base $Q \equiv -\frac{1}{\varphi^2}$.

4. Golden oscillator

Now we construct the quantum oscillator with a spectrum in the form of Fibonacci numbers. Since in this oscillator the base in commutation relations is the φ-Golden ratio, we call it the Golden oscillator. The algebraic relations for the Golden Oscillator are

$$bb^+ - \varphi b^+ b = \left(-\frac{1}{\varphi}\right)^N \tag{110}$$

or

$$bb^+ + \frac{1}{\varphi} b^+ b = \varphi^N, \tag{111}$$

where N is the Hermitian number operator and φ is the deformation parameter. The bosonic Golden oscillator is defined by three operators b^+, b and N which satisfy the commutation relations:

$$[N, b^+] = b^+, \qquad [N, b] = -b. \tag{112}$$

By using the definition of the Golden number operator

$$[N]_F = \frac{\varphi^N - (-\frac{1}{\varphi})^N}{\varphi + \frac{1}{\varphi}} \equiv F_N,$$

we find the following equalities:

$$[N+1]_F - \varphi [N]_F = \left(-\frac{1}{\varphi}\right)^N \tag{113}$$

$$[N+1]_F + \frac{1}{\varphi}[N]_F = \varphi^N. \tag{114}$$

Here, the operator $(-1)^N = \mathrm{e}^{\mathrm{i}\pi N}$.

Comparing the above operator relations with the algebraic relations (110) and (111), we have

$$b^+ b = [N]_F, \qquad bb^+ = [N+1]_F.$$

Here we should note that the number operator N is not equal to b^+b as in the ordinary oscillator case. Properties of Fibonacci numbers (32) are also valid for the Fibonacci operators. By using these and the algebraic relations (110) and (111), we find the Fibonacci recurrence rule, but for operators

$$F_{N+1} = F_N + F_{N-1}. \tag{115}$$

Proposition 4.1.

$$\begin{aligned} [[N]_F, b^+] &= \{[N]_F - [N-1]_F\} b^+ \\ &= b^+\{[N+1]_F - [N]_F\}. \end{aligned} \tag{116}$$

Proposition 4.2. *We have the following equality for* $n = 0, 1, 2, \ldots$*:*

$$[[N]_F^n, b^+] = \{[N]_F^n - [N-1]_F^n\}b^+. \tag{117}$$

Proof 4.3. By using mathematical induction, showing the above equality is not difficult. □

Corollary 4.4. *For any function expandable to power series (analytic in some domain)* $F(x) = \sum_{n=0}^{\infty} c_n x^n$*, we have the following relation:*

$$\begin{aligned}[F([N]_F), b^+] &= \{F([N]_F) - F([N-1]_F)\}b^+ \\ &= b^+\{F([N+1]_F) - F([N]_F)\}\end{aligned} \tag{118}$$

and

$$b^+F([N+1]_F) = F([N]_F)b^+ \tag{119}$$

or

$$F(N)b^+ = b^+F(N+1). \tag{120}$$

By using the eigenvalue problem for the number operator

$$N|n\rangle_F = n|n\rangle_F,$$

$$[N]_F|n\rangle_F = F_N|n\rangle_F = [n]_F|n\rangle_F = F_n|n\rangle_F,$$

we obtain Fibonacci numbers as eigenvalues of $[N]_F = F_N$ operator, where we call F_N the Fibonacci operator and denote its eigenstates as $|n\rangle_{\varphi,-\frac{1}{\varphi}} \equiv |n\rangle_F$. The basis in the Fock space is defined by the repeated action of the creation operator b^+ on the vacuum state, which is annihilated by $b|0\rangle_F = 0$,

$$|n\rangle_F = \frac{(b^+)^n}{\sqrt{F_1 \cdot F_2 \cdot \cdots \cdot F_n}}|0\rangle_F, \tag{121}$$

where $[n]_F! = F_1 \cdot F_2 \cdot \cdots \cdot F_n$. Then we have

$$b^+|n\rangle_F = \sqrt{F_{n+1}}|n+1\rangle_F, \tag{122}$$

$$b|n\rangle_F = \sqrt{F_n}|n-1\rangle_F. \tag{123}$$

The number operator N in terms of F_N is written in two different forms according to even or odd eigenstates $N|n\rangle_F = n|n\rangle_F$. For $n = 2k$, we obtain

$$N = \log_{\varphi}\left(\frac{\sqrt{5}}{2}F_N + \sqrt{\frac{5}{4}F_N^2 + 1}\right), \tag{124}$$

and for $n = 2k + 1$,

$$N = \log_{\varphi}\left(\frac{\sqrt{5}}{2}F_N - \sqrt{\frac{5}{4}F_N^2 - 1}\right), \tag{125}$$

where $[N]_F$ is a Fibonacci number operator

$$[N]_F = \frac{\varphi^N - (-\frac{1}{\varphi})^N}{\varphi - (-\frac{1}{\varphi})} = F_N.$$

As a result, the Fibonacci numbers are the example of (q, Q) numbers with two bases, and one of the bases is the Golden ratio. This is why we call the corresponding q-oscillator a

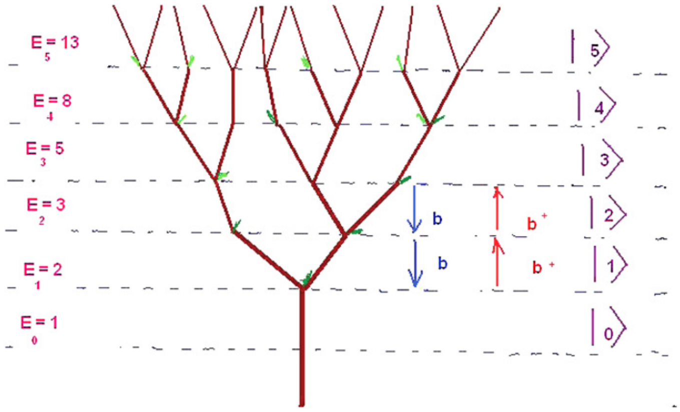

Figure 1. Quantum Fibonacci tree for the Golden oscillator.

Golden oscillator or a Binet–Fibonacci oscillator. The Hamiltonian for the Golden oscillator due to the operator relation (115) is written as a Fibonacci number operator:

$$H = \frac{\hbar\omega}{2}(b^+b + bb^+) = \frac{\hbar\omega}{2}\left([N+1]_F + [N]_F\right) = \frac{\hbar\omega}{2}F_{N+2},$$

where $bb^+ = [N+1]_F = F_{N+1}, \quad b^+b = [N]_F = F_N$. Here we note that our Hamiltonian is different from the q-deformed fermion Hamiltonian [7]. According to our Hamiltonian, the energy spectrum of this oscillator is written in terms of the Fibonacci number sequence:

$$E_n = \frac{\hbar\omega}{2}\left([n]_{\varphi,-\frac{1}{\varphi}} + [n+1]_{\varphi,-\frac{1}{\varphi}}\right) = \frac{\hbar\omega}{2}\left(F_n + F_{n+1}\right) = \frac{\hbar\omega}{2}F_{n+2},$$

or

$$E_n = \frac{\hbar\omega}{2}F_{n+2}.$$

The first energy eigenvalue,

$$E_0 = \frac{\hbar\omega}{2}F_2 = \frac{\hbar\omega}{2},$$

is exactly the same as the ground state energy in the ordinary case. Higher energy excited states are given by the Fibonacci sequence:

$$E_1 = \frac{\hbar\omega}{2}F_3 = \hbar\omega, \qquad E_2 = \frac{3\hbar\omega}{2}, \qquad E_3 = \frac{5\hbar\omega}{2}, \ldots .$$

In figure 1, we show the quantum Fibonacci tree for this oscillator.

The difference between the two consecutive energy levels of our oscillator is not equidistant and is found as

$$\triangle E_n = E_{n+1} - E_n = \frac{\hbar\omega}{2}F_{n+1}.$$

The ratio of two successive energy levels $\frac{E_{n+1}}{E_n} = \frac{F_{n+3}}{F_{n+2}}$ gives the Golden sequence, and for the limiting case of higher excited states $n \to \infty$, it is the Golden ratio

$$\lim_{n\to\infty}\frac{E_{n+1}}{E_n} = \lim_{n\to\infty}\frac{F_{n+3}}{F_{n+2}} = \lim_{n\to\infty}\frac{[n+3]_F}{[n+2]_F} = \frac{1+\sqrt{5}}{2} = \varphi \approx 1.618\,033\,9887.$$

This property of asymptotic energy levels to be proportional to each other by a Golden ratio leads us to call this oscillator a Golden oscillator.

We have the following relations between Golden creation and annihilation operators and the standard creation and annihilation operators:

$$b^{+} = a^{+}\sqrt{\frac{F_{N+1}}{N+1}} = \sqrt{\frac{F_N}{N}}a^{+}, \tag{126}$$

$$b = \sqrt{\frac{F_{N+1}}{N+1}}a = a\sqrt{\frac{F_N}{N}}, \tag{127}$$

which we call nonlinear unitary transformation, where $[a, a^{+}] = 1$. As a result of these relations, we can obtain the commutation relation between b^{+} and b as

$$[b, b^{+}] = bb^{+} - b^{+}b = F_{N+1} - F_N.$$

In [7] the Hamiltonian of the q-fermion oscillator was taken as

$$H_q^F = \tfrac{1}{2}\hbar\omega(b^{+}b - bb^{+}). \tag{128}$$

By using the previous formula, we have the representation of this Hamiltonian in terms of the Fibonacci operator:

$$H_q^F = \tfrac{1}{2}\hbar\omega(F_N - F_{N+1}). \tag{129}$$

Finally, using the property of Fibonacci operators, we find the q-fermion Hamiltonian as a Fibonacci operator:

$$H_q^F = -\tfrac{1}{2}\hbar\omega F_{N-1}. \tag{130}$$

To compare n-particle states for Golden and standard oscillators, we consider the following relations. From (126) and (127), we have

$$\begin{aligned}(b^{+})^n &= \left(b^{+}\sqrt{\frac{F_{N+1}}{N+1}}\right)^n \\ &= (b^{+})^n\sqrt{\frac{F_{N+n}}{N+n}\cdots\frac{F_{N+2}}{N+2}\frac{F_{N+1}}{N+1}} \\ &= (b^{+})^n\sqrt{\frac{F_{N+n}!}{F_N!}\frac{N!}{(N+n)!}}.\end{aligned} \tag{131}$$

By taking the Hermitian conjugate of this result, we obtain

$$b_q^n = \sqrt{\frac{F_{N+n}!}{F_N!}\frac{N!}{(N+n)!}}a^n. \tag{132}$$

Our next step is to show that the same set of eigenvectors $|n\rangle$ expands the whole Hilbert space both for the standard harmonic oscillator and for the Golden one. Firstly, we consider that the vacuum state $|0\rangle$ for an ordinary quantum harmonic oscillator satisfies $a|0\rangle = 0$, and the vacuum state $|0\rangle_F$ for the Golden quantum harmonic oscillator satisfies $b|0\rangle_F = 0$. From (127) we have

$$b|0\rangle_F = \sqrt{\frac{F_{N+1}}{N+1}}a|0\rangle_F = 0,$$

which gives

$$a|0\rangle_F = 0.$$

Alternatively, if $a|0\rangle_F = 0$, it implies $b|0\rangle_F = 0$. Therefore, the vacuum state $|0\rangle$ for the ordinary oscillator is exactly the same as for the Golden oscillator vacuum state $|0\rangle \equiv |0\rangle_F$. By applying $(b^+)^n$ to the vacuum state $|0\rangle_F$ and using $N|n\rangle_F = n|n\rangle_F$, we have

$$(b^+)^n|0\rangle_F = (b^+)^n\sqrt{\frac{F_{N+n}!}{F_N!}\frac{N!}{(N+n)!}}|0\rangle_F$$
$$= \sqrt{\frac{F_n!}{n!}}(b^+)^n|0\rangle, \tag{133}$$

which implies that

$$|n\rangle_F = |n\rangle.$$

As a result, we find that both the standard and the Golden harmonic oscillators have the same set of eigenstates, but with different energy eigenvalues. If for the standard oscillator the eigenstates are determined by the positive integer numbers n, $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$, then for the Golden oscillator they are given by the Fibonacci numbers F_n, $E_n = \frac{\hbar\omega}{2}F_{n+2}$.

To obtain wavefunctions in the coordinate representation, the simplest way is to represent the bosonic operators a and a^+ in terms of coordinate and momenta, so that the wavefunctions would just be standard bosonic oscillator wavefunctions in terms of Hermite polynomials. However, in this form, the representation of the operators b and b^+ is quite complicated. Another form of the wavefunction is related to the q-coordinate and q-momentum for the operators b and b^+, but with a complicated structure of the wavefunctions. One more representation of the wavefunctions is the holomorphic representation of Fock, which requires the application of the Fibonacci derivative operator. These questions would be described in our future work.

4.1. Golden angular momentum

The double Golden oscillator algebra $su_F(2)$ determines the Golden quantum angular momentum operators, defined as

$$J_+^F = b_1^+ b_2, \qquad J_-^F = b_2^+ b_1, \qquad J_z^F = \frac{N_1 - N_2}{2},$$

and satisfying the commutation relations

$$[J_+^F, J_-^F] = (-1)^{N_2} F_{2J_z} = -(-1)^{N_1} F_{-2J_z}, \tag{134}$$

$$[J_z^F, J_\pm^F] = \pm J_\pm^F, \tag{135}$$

where the Binet–Fibonacci Golden operator is

$$F_N = \frac{\varphi^N - (-\frac{1}{\varphi})^N}{\varphi + \frac{1}{\varphi}} = [N]_F.$$

The Golden quantum angular momentum operators $J_\pm^F$ may be written in terms of Fibonacci operators and standard quantum angular momentum operators $J_\pm$ as

$$J_+^F = J_+\sqrt{\frac{F_{N_1+1}}{N_1+1}}\sqrt{\frac{F_{N_2}}{N_2}} = \sqrt{\frac{F_{N_1}}{N_1}}\sqrt{\frac{F_{N_2+1}}{N_2+1}}J_+ \tag{136}$$

$$J_-^F = J_-\sqrt{\frac{F_{N_1}}{N_1}}\sqrt{\frac{F_{N_2+1}}{N_2+1}} = \sqrt{\frac{F_{N_1+1}}{N_1+1}}\sqrt{\frac{F_{N_2}}{N_2}}J_-. \tag{137}$$

The Casimir operator for the Binet–Fibonacci case is

$$\begin{aligned} C^F &= (-1)^{-J_z}\left(F_{J_z}F_{J_z+1} + (-1)^{-N_2}J_-^F J_+^F\right) \\ &= (-1)^{-J_z}\left(-F_{J_z}F_{J_z-1} + (-1)^{-N_2}J_+^F J_-^F\right). \end{aligned} \tag{138}$$

The angular momentum operators $J_\pm^F$ and J_z^F act on the state $|j, m\rangle_F$ as

$$J_+^F|j, m\rangle_F = \sqrt{F_{j-m}F_{j+m+1}}|j, m+1\rangle_F, \tag{139}$$

$$J_-^F|j, m\rangle_F = \sqrt{F_{j+m}F_{j-m+1}}|j, m-1\rangle_F, \tag{140}$$

$$J_z^F|j, m\rangle_F = m|j, m\rangle_F. \tag{141}$$

If the eigenvalues of the Casimir operator C_j^F are determined by the product of two successive Fibonacci numbers

$$C_j^F = (-1)^{-j}F_jF_{j+1},$$

then the asymptotic ratio of two successive eigenvalues of the Casimir operator gives the Golden ratio square:

$$\lim_{j\to\infty}\frac{(-1)^{-j}F_jF_{j+1}}{(-1)^{-j+1}F_{j-1}F_j} = -\varphi^2.$$

We can also construct the representation of our F-deformed angular momentum algebra in terms of the double Golden boson representation b_1, b_2. The actions of F-deformed angular momentum operators to the state $|n_1, n_2\rangle_F$ are given as follows:

$$J_+^F|n_1, n_2\rangle_F = b_1^+b_2|n_1, n_2\rangle_F = \sqrt{F_{n_1+1}F_{n_2}}|n_1+1, n_2-1\rangle_F, \tag{142}$$

$$J_-^F|n_1, n_2\rangle_F = b_2^+b_1|n_1, n_2\rangle_F = \sqrt{F_{n_1}F_{n_2+1}}|n_1-1, n_2+1\rangle_F, \tag{143}$$

$$J_z^F|n_1, n_2\rangle_F = \tfrac{1}{2}(N_1 - N_2)|n_1, n_2\rangle_F = \tfrac{1}{2}(n_1 - n_2)|n_1, n_2\rangle_F. \tag{144}$$

The above expressions reduce to the familiar ones (139)–(141) provided we define

$$j \equiv \frac{n_1 + n_2}{2}, \qquad m \equiv \frac{n_1 - n_2}{2}$$

$$|n_1, n_2\rangle_F \equiv |j, m\rangle_F,$$

and substitute

$$n_1 \to j + m, \qquad n_2 \to j - m.$$

4.2. Symmetrical $su_{i\varphi}(2)$ quantum algebra

As an example of the symmetrical q-deformed $su_q(2)$ algebra, we choose the base as $q_i = \mathrm{i}\varphi$ and $q_j = \mathrm{i}\frac{1}{\varphi}$; then our complex equation for the base becomes

$$(\mathrm{i}\varphi)^2 = \mathrm{i}(\mathrm{i}\varphi) - 1.$$

The φ-deformed symmetrical angular momentum operators remain the same as $J_\pm^{(s)}, J_z^{(s)}$. The symmetrical quantum algebra with base $(\mathrm{i}\varphi, \frac{i}{\varphi})$ becomes

$$[J_+^\varphi, J_-^\varphi] = [2J_z]_{\frac{i}{\varphi}} = [2J_z]_{i\varphi,\frac{i}{\varphi}}(-1)^{(\frac{1}{2}-J_z)}, \tag{145}$$

where

$$[2J_z]_{\frac{i}{\varphi}} = \frac{\varphi^{2J_z} - \varphi^{-2J_z}}{\varphi - \varphi^{-1}}$$

and

$$[J_z^{(s)}, J_\pm^{(s)}] = \pm J_\pm^{(s)}. \tag{146}$$

4.3. $\tilde{su}_F(2)$ algebra

One of the special cases of the symmetrical $\tilde{su}_{(q,Q)}(2)$ algebra is constructed by choosing the Binet–Fibonacci case ($q = \varphi,\ Q = -\frac{1}{\varphi}$). The generators of the $\tilde{su}_F(2)$ algebra $\tilde{J}_\pm^\varphi, \tilde{J}_z^\varphi$ are given in terms of double bosons $b_1,\ b_2$ as follows:

$$\tilde{J}_+^F = (-1)^{-\frac{N_2}{2}} b_1^+ b_2, \tag{147}$$

$$\tilde{J}_-^F = b_2^+ b_1 (-1)^{-\frac{N_2}{2}}, \tag{148}$$

$$\tilde{J}_z^F = J_z. \tag{149}$$

satisfying the anti-commutation relation

$$\tilde{J}_+^F \tilde{J}_-^F + \tilde{J}_-^F \tilde{J}_+^F = \{\tilde{J}_+^F, \tilde{J}_-^F\} = [2J_z]_F, \tag{150}$$

and $[\tilde{J}_z^F, \tilde{J}_\pm^F] = \pm\tilde{J}_\pm^F$. The Casimir operator is written in the following forms:

$$\begin{aligned}\tilde{C}^F &= (-1)^{J_z}\{F_{J_z}F_{J_z+1} - \tilde{J}_-^F\tilde{J}_+^F\}\\ &= (-1)^{J_z}\{\tilde{J}_+^F\tilde{J}_-^F - F_{J_z}F_{J_z-1}\}.\end{aligned} \tag{151}$$

The actions of the F-deformed angular momentum operators to the states $|j,m\rangle_F$ are

$$\tilde{J}_+^F|j,m\rangle_F = (-1)^{\frac{j-m}{2}}\sqrt{F_{j-m}F_{j+m+1}}|j,m+1\rangle_F, \tag{152}$$

$$\tilde{J}_-^F|j,m\rangle_F = (-1)^{\frac{j-m}{2}}\sqrt{F_{j+m}F_{j-m+1}}|j,m-1\rangle_F, \tag{153}$$

$$\tilde{J}_z^F|j,m\rangle_F = m|j,m\rangle_F. \tag{154}$$

And the eigenvalues of Casimir operators are given by

$$\begin{aligned}\tilde{C}^F|j,m\rangle_F &= \{(-1)^m F_m F_{m+1} - (-1)^j F_{j-m}F_{j+m+1}\}|j,m\rangle_F\\ &= \{(-1)^j F_{j-m+1}F_{j+m} - (-1)^m F_m F_{m-1}\}|j,m\rangle_F.\end{aligned}$$

5. Conclusions

By interpreting the Binet formula for Fibonacci polynomials and Fibonacci numbers as q-numbers, we have developed a special version of q-calculus with negative-inverse points. From one side these points are specific for the spin coherent state description of qubits in quantum information theory, providing a unique pair of one-qubit orthogonal states. In terms of these states, maximally entangled two-, three- and arbitrary N-qubit states can be derived. The expansion of these states on the computational basis is determined by Fibonacci polynomials and Lucas polynomials. So the Fibonacci calculus developed here can have potential application in quantum information theory. In a stereographic projection picture of these states, the argument of Fibonacci polynomials has interpretation of a distance between symmetric points in a unit circle. Another potential application is related to the method of images in the problem of point vortices in circular domains, as advocated a long time ago by Poincare [24]. Here, the vortex and its image are located at symmetric points and the Kirchhoff energy of configuration depends on the distance between them. This distance is determined by the parameter a in Fibonacci polynomials and is equal to 1 for a vortex at the Golden ratio distance φ.

When this paper had already been submitted to the journal, we learned that Parthasarathy and Viswanathan in their study of the q-deformed fermion oscillator in 1991 [7] had introduced the fermion q-number, which coincides exactly with the Fibonacci polynomial as a q-number in the Binet representation. According to this, we expect that our results would be useful in describing such an oscillator, the corresponding coherent states and fractional statistics. In particular, we expect that it could be a useful tool to study the entanglement of q-fermionic states.

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