Breadth first algorithms for APP detectors over MIMO channels

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***Abstract*— For iterative decoding of multiple antenna systems concatenated with an outer error correcting code, it is important to use an a posteriori probability detector for the MIMO detection to achieve near capacity performance. To avoid full APP detection, we propose a reduced complexity detector based on breadth first algorithms. Although these algorithms are sub-optimal, we show that they can provide a good list of candidates for the APP calculation. Furthermore, by exploiting the a priori information delivered from the outer decoder, it is possible to decrease the MIMO detector complexity at each iteration. Using simulation results, we will compare the performance of the proposed detectors with the list sphere detector.**

keywords : MIMO detection, fading channels, reduced complexity algorithm, soft decoding, concatenated codes

I. Introduction

Multiple input multiple output (MIMO) channels can in theory greatly increase the capacity of wireless communication links. Among the schemes that have been designed for these channels are the Vertical Bell Labs Layered Space Time (V-BLAST) [1] and the orthogonal and quasi orthogonal space time block codes. Although the classical detector for the V-BLAST scheme is based on a nulling and cancellation (NC) algorithm and in order to achieve near capacity, we should associate this scheme with a channel code such as a convolutional code or a Turbo code. Consequently, we need an a posteriori probability (APP) MIMO detector to pass an extrinsic information to the outer decoder. It is well known that a full APP detector becomes computationally intractable when the number of antennas and the size of the constellation increase [2]. Two depth first algorithms have been proposed to reduce the complexity of the APP detector : the list version of the sphere decoder [3] [4] and the list sequential detector [5]. In this paper, instead of searching for a list of the best candidates, we propose a reduced-complexity APP detector that will search for a list of good candidates. This search will be done using two breadth first algorithms applied to the tree structure of the space time scheme : the M algorithm [6] and the T algorithm [7]. While the complexity of the first decoder is constant at each iteration, using a priori information from the outer code, we will show that the complexity of the second one can be reduced at each iteration. Using numerical simulations, we will compare the performance of these APP MIMO detectors with the list version of the sphere decoder. A modified version of the M algorithm has also been independently proposed in [8].

II. Linear model for MIMO channel

We consider the V-BLAST or sequential multiplexing system over a MIMO channel with M transmit and N receive antennas given in Fig. 1.

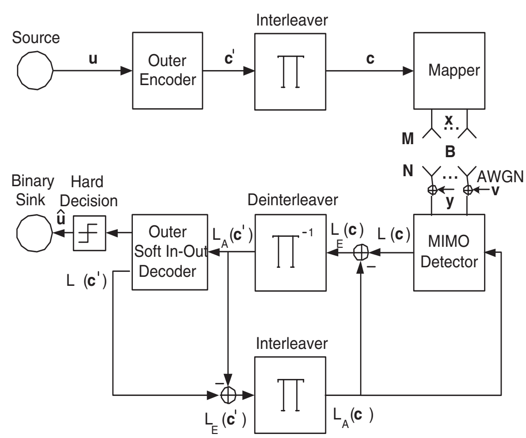

Fig. 1. MIMO transmission chain

The vector of information bits $\mathbf{u}$ is first encoded with an error correcting code and then interleaved to obtain the vector of coded bits. Then, we decompose this vector into blocks of coded bits of length $2M \times S$, $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_{2M})^T$ with $\mathbf{c}_i = (c_{i1}, \ldots, c_{iS})$.

Each vector $\mathbf{c}$ is then mapped to the vector of real symbols $\mathbf{x} = (x_1, \ldots, x_{2M})^T \in \mathcal{X}_S^{2M \times 1}$. We suppose that $\mathcal{X}_S$ is the 2^S-PAM signal set $\mathcal{X}_S = \{-2^S+1, -2^S+3, \ldots, 2^S-3, 2^S-1\}$. In other words, the complex symbols are chosen from a QAM constellation with 2^{2S} possible signal points.

Let $\mathbf{y} \in \mathbb{R}^{2N}$ the vector of received signals.

We have the classical real-valued equation :

$$\mathbf{y} = \mathbf{B}\mathbf{x} + \mathbf{v} \quad (1)$$

where $\mathbf{B}$ is the real channel matrix of dimension $2N \times 2M$ built from the complex channel matrix $\mathbf{H}$ by replacing each of these elements h_{ij} by $\sqrt{\frac{\rho}{M}}\begin{pmatrix}\Re(h_{ij}) & -\Im(h_{ij})\\ \Im(h_{ij}) & \Re(h_{ij})\end{pmatrix}$.

h_{ij}, the path gain between the receive antenna i to the transmit antenna j is modelled as an independent realization of complex Gaussian random variable of unit variance. $\mathbf{v}$ is a vector of independent zero mean real Gaussian noise with variance $\sigma^2 = 1/2$. We assume that the energy of x_i is $E\|x_i\|^2 = 1/2$ and consequently $E_S = M$ is the total transmit power per channel use. Then ρ is the signal to noise ratio per receive antenna.

From (1) the probability $p(\mathbf{y}|\mathbf{x})$ of a realization of the received vector given a transmitted vector is

$$p(\mathbf{y}|\mathbf{x}) = \frac{1}{(2\pi\sigma^2)^N}\exp\left(-\frac{1}{2\sigma^2}||\mathbf{y}-\mathbf{B}\mathbf{x}||^2\right) \quad (2)$$

III. APP DETECTOR FOR MIMO CHANNELS

An APP detector generates an a posteriori probability (APP) about the transmitted bits c_{ij}. The a posteriori probability of the bit c_{ij} given the received vector $\mathbf{y}$ is usually expressed as a log-likelihood ratio (L-value) as follows :

$$L(c_{ij}|\mathbf{y}) = \ln\frac{P(c_{ij}=+1|\mathbf{y})}{P(c_{ij}=-1|\mathbf{y})} \quad (3)$$

Using the Bayes theorem, (3) becomes

$$L(c_{ij}|\mathbf{y}) = \ln\frac{P(c_{ij}=+1)}{P(c_{ij}=-1)} + \ln\frac{\sum_{\mathbf{c}\in\mathbb{C}_{ij,+1}} p(\mathbf{y}|\mathbf{c})P(\mathbf{c}|c_{ij})}{\sum_{\mathbf{c}\in\mathbb{C}_{ij,-1}} p(\mathbf{y}|\mathbf{c})P(\mathbf{c}|c_{ij})} \quad (4)$$

where $i = 1,\dots,2M$ and $j = 1,\dots,S$ and $\mathbb{C}_{ij,+1}$ is the set of 2^{2MS-1} vectors $\mathbf{c}$ with $c_{ij} = +1$.

The first term is the a priori L-value $L_A(c_{ij})$ and the second term is the extrinsic L-value $L_E(c_{ij}|\mathbf{y})$ that will be passed to the next decoder.

Assuming the independence of the bits c_{ij} the extrinsic L-value becomes

$$L_E(c_{ij}|\mathbf{y}) = \ln\frac{\sum_{\mathbf{c}\in\mathbb{C}_{ij,+1}} p(\mathbf{y}|\mathbf{c})\prod_{(kl)\neq(ij)} P(c_{kl})}{\sum_{\mathbf{c}\in\mathbb{C}_{ij,-1}} p(\mathbf{y}|\mathbf{c})\prod_{(kl)\neq(ij)} P(c_{kl})} \quad (5)$$

From (2) we can derive the logarithm probability

$$\ln p(\mathbf{y}|\mathbf{c}) = -N\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}||\mathbf{y}-\mathbf{B}\mathbf{x}(\mathbf{c})||^2 \quad (6)$$

Finally using the Max-log approximation [9] $\ln\sum a_i \approx \max\ln a_i$ the extrinsic L-value can be expressed as follows

$$\begin{aligned}L_E(c_{ij}|\mathbf{y}) = &\max_{\mathbf{c}\in\mathbb{C}_{ij,+1}}\left\{-\frac{1}{2\sigma^2}||\mathbf{y}-\mathbf{B}\mathbf{x}(\mathbf{c})||^2 + \sum_{(kl)\neq(ij)}\ln P(c_{kl})\right\}\\ &-\max_{\mathbf{c}\in\mathbb{C}_{ij,-1}}\left\{-\frac{1}{2\sigma^2}||\mathbf{y}-\mathbf{B}\mathbf{x}(\mathbf{c})||^2 + \sum_{(kl)\neq(ij)}\ln P(c_{kl})\right\}\end{aligned} \quad (7)$$

where $\ln P(c_{kl})$ can be calculated from the a priori L-value $L_A(c_{kl})$.

Since the computing (7) is of prohibitive complexity, we will generally only use a list of sequences $\mathbf{x}$ for which the metric

$$\mathcal{M}(\mathbf{x}(\mathbf{c})) = \|\mathbf{y}-\mathbf{B}\mathbf{x}(\mathbf{c})\|^2 - 2\sigma^2\sum_{k=1}^{2M}\ln P(x_k) \quad (8)$$

is small.

It is equivalent to searching for the lattice points close to the given point $\mathbf{y}$ in the lattice Λ defined by the set $\{\mathbf{B}\mathbf{x} : \mathbf{x} \in \mathcal{X}_S^{2M\times 1}\}$ where $\mathbf{x} = (x_1, x_2, \dots, x_{2M})^T$ is the input vector the entries of which are consecutive integers.

In the following, we will assume that $2M \leq 2N$ and rank $(\mathbf{B}) = 2M$.

In [3] the authors have shown that we can rewrite $\|\mathbf{y} - \mathbf{B}\mathbf{x}(\mathbf{c})\|^2$ in the following way :

$$\|\mathbf{y}-\mathbf{B}\mathbf{x}(\mathbf{c})\|^2 = (\mathbf{x}-\hat{\mathbf{x}})^T\mathbf{B}^T\mathbf{B}(\mathbf{x}-\hat{\mathbf{x}}) + \mathbf{y}^T(\mathbf{I}-\mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T)\mathbf{y} \quad (9)$$

where $\hat{\mathbf{x}} = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{y}$ is the unconstrained least squares estimate.

Finally, the metric to be calculated for APP detection is :

$$\begin{aligned}\mathcal{M}(\mathbf{x}(\mathbf{c})) &= (\mathbf{x}-\hat{\mathbf{x}})^T\mathbf{B}^T\mathbf{B}(\mathbf{x}-\hat{\mathbf{x}}) - 2\sigma^2\sum_{k=1}^{2M}\ln P(x_k) + \mathcal{M}'\\ &= \mathbf{z}^T\mathbf{B}^T\mathbf{B}\mathbf{z} - 2\sigma^2\sum_{k=1}^{2M}\ln P(x_k) + \mathcal{M}'\\ &= \mathbf{z}^T\mathbf{R}^T\mathbf{R}\mathbf{z} - 2\sigma^2\sum_{k=1}^{2M}\ln P(x_k) + \mathcal{M}'\end{aligned} \quad (10)$$

where $\mathcal{M}' = \mathbf{y}^T(\mathbf{I} - \mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T)\mathbf{y}$, $\mathbf{x} = \hat{\mathbf{x}} + \mathbf{z}$ is a candidate vector and $\mathbf{R}$ is the upper triangular matrix with $\mathbf{B}^T\mathbf{B} = \mathbf{R}^T\mathbf{R}$ obtained using the Cholesky factorization.

Let r_{ij} an element of the matrix $\mathbf{R}$ with $i, j \leq 2M$ and $\mathbf{z} = (z_1, z_2, \dots, z_{2M})^T$. Using the exact expression we obtain

$$\mathcal{M}(\mathbf{x}(\mathbf{c})) = \sum_{i=1}^{2M}\left(q_{ii}\left(z_i + \sum_{j=i+1}^{2M} q_{ij}z_j\right)^2 - 2\sigma^2\ln P(x_i)\right) + \mathcal{M}' \quad (11)$$

where $q_{ii} = r_{ii}^2$ for $i = 1,\dots,2M$ and $q_{ij} = \frac{r_{ij}}{r_{ii}}$ for $i = 1,\dots,2M, j = i+1,\dots,2M$.

To get the list of candidates, one solution is to use a list sphere decoder [3] [4] based on the Fincke-Pohst enumeration [10] [11] or the Schnorr-Euchner refinement [12] . The list sphere decoder finds the M_{cand} best codewords to compute (7).

To get this list, the square radius of the sphere should guarantee that the sphere contains at least M_{cand} points.

It is interesting to note that this algorithm can be seen as a tree search algorithm. Each path in the tree corresponds to a vector $\mathbf{x}$.

Thanks to the upper triangular form of $\mathbf{R}$, each branch of the tree can be labelled with a branch metric. For a branch at depth $2M - i + 1$ the branch metric $w(\mathbf{x}_i^{2M})$ is given by :

$$w(\mathbf{x}_i^{2M}) = \left(q_{ii} \left(z_i + \sum_{j=i+1}^{2M} q_{ij} z_j \right)^2 - 2\sigma^2 \ln P(x_i) \right) \quad (12)$$

where $\mathbf{x}_i^{2M} = (x_i, x_{i+1} \ldots, x_{2M})^T$ is the vector associated with the path from the root of the tree to the branch. Then we may write the accumulated metric corresponding to this partial path in the form

$$\begin{aligned} \mathcal{M}(\mathbf{x}_i^{2M}) &= \sum_{j=i}^{2M} w(\mathbf{x}_j^{2M}) + \mathcal{M}(\mathbf{x}_{2M+1}^{2M}) \\ &= \mathcal{M}(\mathbf{x}_{i+1}^{2M}) + w(\mathbf{x}_i^{2M}) \end{aligned} \quad (13)$$

with $\mathcal{M}(\mathbf{x}_{2M+1}^{2M}) = \mathcal{M}'$.

We will now propose to use breadth first algorithms in order to obtain the list of candidates.

A. List M APP detector

For the decoding of convolutional codes, due to the computation complexity of the Viterbi algorithm, reduced complexity algorithms have been developed. The M algorithm [6] and the T algorithm [7] are two such algorithms belonging to the breadth first decoding algorithms.

The main idea is that survivors with small weight can be omitted with a negligible probability of discarding the most likelihood path. While the M algorithm keeps a fixed number M_{cand} of paths at each step, the T algorithm keeps a variable number of survivor paths depending on a threshold parameter T.

Since it is possible to describe the problem over a tree, we can build an APP decoder for MIMO channel based on a list type version of the M algorithm.

The proposed algorithm can be described as follows :

1) initialize the memory with one path starting from the root node and with metric $\mathcal{M}(\mathbf{x}_{2M+1}^{2M}) = \mathcal{M}'$
2) extend each memorized path and update the accumulated metrics of each path
3) order the paths according to their accumulated metric then select the M_{cand} best paths among the extended paths and suppress the other paths
4) go to step 2 until we reach the leaves of the tree. Otherwise, go to step 5
5) calculate the APP

Compared to the list sphere decoder or the list sequential decoder the M_{cand} candidates obtained by the list M detector are not necessarily the closest codewords from the received point. Consequently, the expected performance of this algorithm should be lower than the two other list detectors. On the other hand, compared to the other list detectors, the breadth first structure of this algorithm has a lower computational complexity and the computational complexity doesn't depend on the channel realization. Consequently a hardware implementation will be easier to achieve.

Although the list of candidates is performed only at the first global iteration over the MIMO detector in [3] [4], we can improve the performance of the list M detector by updating the list of candidates at each global iteration using all the available a priori information. This can be done efficiently by using two separate lists (one from the channel information and one from the a priori information) [8] . This point significantly improves the performance of the detector when the number of candidates is limited.

It should be noted that a list M detector with $M_{\text{cand}} = 1$ is equivalent to the NC algorithm proposed for the hard decoding of the V-BLAST code. Compared to the NC algorithm the ordering of the column of **B** according to their Euclidian norm doesn't improve the performance of the list M algorithm.

Since we select a limited number of candidates, there is no guarantee that for each considered bit, we will keep at least one path with the complementary bit. If this problem occurs, we will consider that the corresponding bit has a high reliability and we will consequently clip the extrinsic L-value to ± 8 [3].

B. List T APP detector

In order to further reduce the complexity of the detector, it is possible to use the a priori information coming from the outer code to decrease the number of explored paths. This solution can be implemented with a list T algorithm. Step 3 of the M algorithm is then replaced as follows :

3. select the paths satisfying the metric condition 1 or 2.

Metric condition 1:

$$\mathcal{M}(\mathbf{x}_i^{2M}) - \mathcal{M}(\mathbf{x}_i^{2M})_{min} \leq C_1 \quad (14)$$

$\mathcal{M}(\mathbf{x}_i^{2M})_{min}$ corresponds to the path obtained using the NC algorithm. Then, the NC point is the reference lattice point for the search.

Metric condition 2:

$$\mathcal{M}(\mathbf{x}_i^{2M}) \leq C_2 \quad (15)$$

When using the condition 2, if there is no remaining path, we will keep the NC solution. Here, $\hat{\mathbf{x}}$ the unconstrained least squares estimate is the reference point.

Finally we will also describe another metric condition recently proposed in [4].

Metric condition 3:

$$\mathcal{M}(\mathbf{x}_i^{2M}) - \mathcal{M}(\mathbf{x}_i^{2M})_{ML} \leq C_3 \quad (16)$$

Here, the reference point is the ML solution. Clearly, metric condition 3 gives better performances than the two previous ones. In [4] a first sphere decoder was used to get the ML point and then a list sphere APP detection was performed. However, since there is no sphere reduction in the list sphere APP detection, it is equivalent to performing a T APP detection. The only difference is the structure of the algorithms (sequential or deep first versus parallel or breadth first algorithm).

In figure 2, we have compared the histograms of the number of points visited using condition 1 and 2. For both algorithms, the average number of points was fixed to 50.

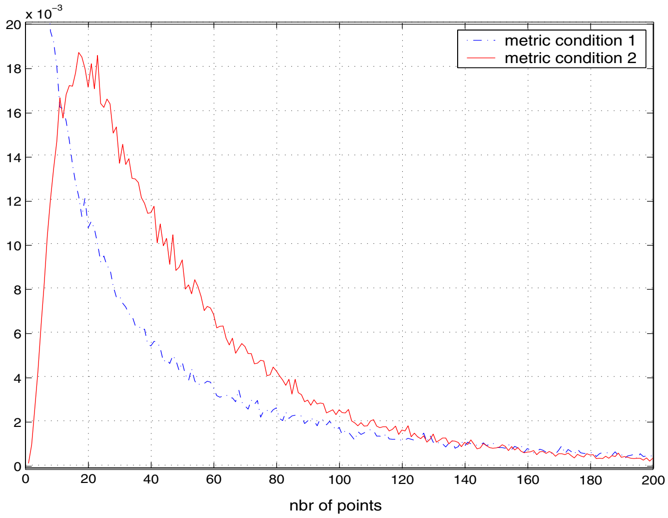

Fig. 2. Histograms of the number of points visited using metric condition 1 and 2.

The advantage of choosing the NC point as the center of the search instead of the received point y are obvious since the number of points are more concentrated around the average number of points using the metric condition 1. The same conclusion has been drawn in [4] considering the ML point as the center of the search.

C. List TM APP detector

In order to limit the memory size and the computational complexity it can be necessary to keep only a limited number of paths at each level. Consequently, we propose a TM version of the T algorithms where Step 3 is replaced as follows :

3. select the paths satisfying the metric condition 1 or 2. If the number of selected paths is greater than N_{max}, order the paths according to their accumulated metric, select the M_{cand} best paths and suppress the other paths.

IV. Simulation Results

We will first provide the definition of E_b/N_0 that is used in our performance curves. Since the average signal energy per transmitted complex symbol is E_S/M and because the fading coefficients are independent with variance 1/2 in each real dimension, the average signal energy per receive antenna is E_S. Then, the N receive antennas collect a total power of NE_S carrying $2RMS$ information bits where R is the rate of the channel code. We can consequently define the signal energy per transmitted information bit at the receiver as :

$$\left.\frac{E_b}{N_0}\right|_{dB} = \left.\frac{E_S}{N_0}\right|_{dB} + 10\log_{10}\frac{N}{2RMS} \qquad (17)$$

Fig. 3 compares the BER performance of the proposed detector with the BER performance of the list sphere detector considering a 8×8 MIMO channel and the V-BLAST scheme. We will suppose that the channel is perfectly known at the receiver and changes independently at each channel use. Following [3] the transmission is organized in blocks of 9216 information bits. The channel code is a parallel concatenated convolutional (PCC) code composed of two (7,5) convolutional codes. The code rate 1/2 is obtained by puncturing. The complex symbols are taken from a QPSK constellation. For each block, we apply 4 global iterations over the MIMO detector and 8 iterations are performed for the iterative decoding of the PCC code. These parameters allow us to compare the performance of the proposed list MIMO detector with the performance given in [3]. Since the number of codewords is $2^{16} = 65536$, it is not possible to perform the full APP detector. The dashed lines correspond to the BER performance of the list sphere detector given in [3] for respectively $M_{\text{cand}} = 32, 16, 8, 4$ and 2 and the solid lines correspond to the list M detector for $M_{\text{cand}} = 32, 16, 8, 4, 2$ and 1. We can observe at BER=10^{-5} a performance degradation of 0.3 dB for $M_{\text{cand}} = 32$ and of 0.8 dB for $M_{\text{cand}} = 16$. Since the list of candidates is not as good as the list obtained using the sphere detector, the performance degradation increases when we reduce the number of candidates. When we use only one candidate for the list M detector (equivalent to the NC detector), the performance degradation is 4.8 dB at BER=10^{-5}. However to guaranty the obtaining of the M_{cand} best candidates in [3], the diameter of the sphere was much higher than using the proposed list M detector.

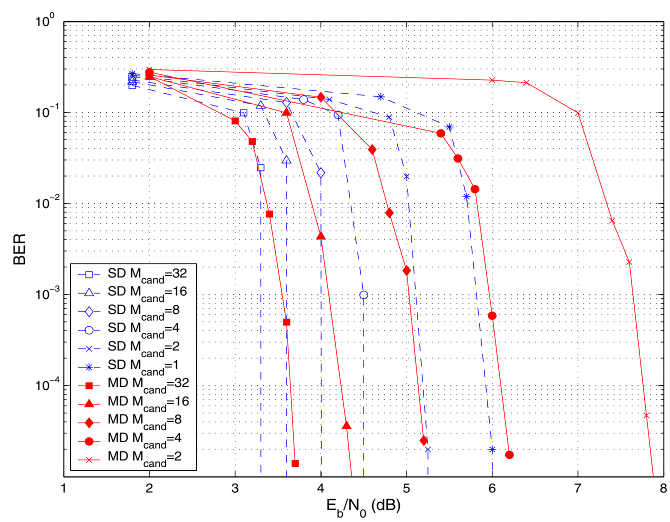

Fig. 3. BER performance over 8x8 MIMO channels, QPSK modulation, rate 1/2 memory 2 turbo codes. APP decoder : list sphere decoder $M_{\text{cand}} = 32, 16, 8, 4, 2, 1$ and list M decoder $M_{\text{cand}} = 32, 16, 8, 4, 2$.

Fig. 4 gives the BER performance using different list TM detectors. The best result is obtained when the ML point is the reference point ($C_3 = 15$ and $M_{\text{cand}} = 128$). However using the NC point as the center of the search ($C_1 = 15$ and $M_{\text{cand}} = 128$) we have a performance degradation of only 0.4 dB at $BER = 10^{-5}$. When using the Metric condition 2, we need to analyze more points to get the same performance. Fig. 5 and 6 compare respectively the performance and the complexity of the list M decoder ($M_{\text{cand}} = 16$) and the list

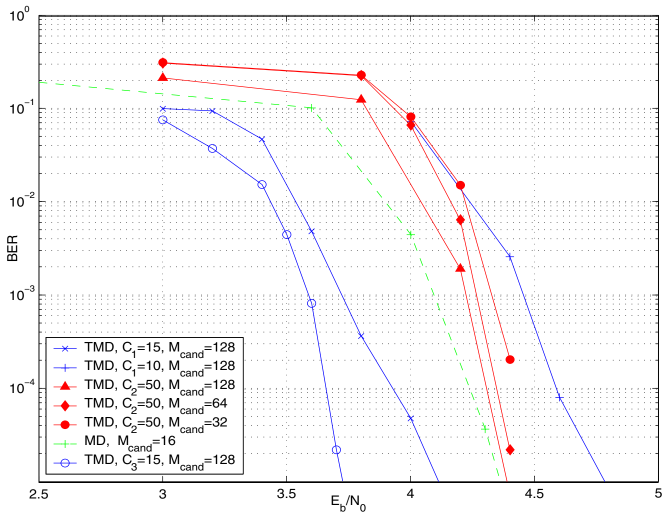

Fig. 4. BER performance of different list TM detectors using Metric conditions 1, 2 and 3.

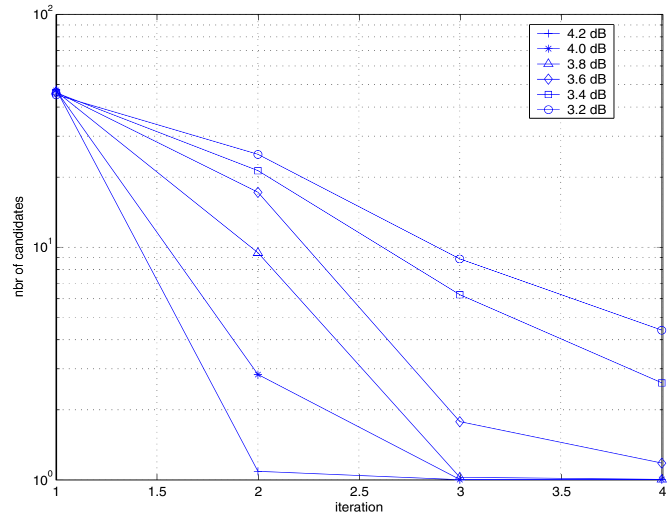

Fig. 6. Complexity vs. iteration for different E_b/N_0 of the list TM detector using Metric condition 1 with $C_1 = 15$ and $M_{\text{cand}} = 128$.

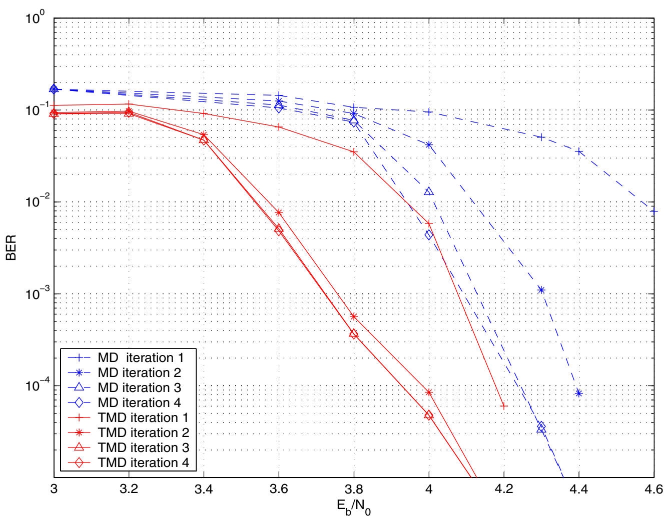

Fig. 5. BER performance using the list M detector $M_{\text{cand}} = 16$ and the list TM detector using Metric condition 1 with $C_1 = 15$ and $M_{\text{cand}} = 128$.

TM decoder ($C_1 = 15$ and $M_{\text{cand}} = 128$). By adapting the complexity of the MIMO decoder all along the iteration, we can reduce the overall complexity compare to a list M decoder.

V. Conclusion

In this paper, we have proposed to apply breadth first algorithms for the APP detection over MIMO channels. We have shown that using a list M detector we can achieve performances close to the performance of a list sphere detector but with a significantly lower complexity. Exploiting the a priori information delivered from the outer decoder, we have also introduced the list T, and TM detectors that allows to decrease the MIMO detector complexity. While the complexity is fixed for the list M algorithm, the complexity of the list T and TM algorithm decrease at each iteration. Furthermore we have shown that by choosing the NC point as the center point for the list search we can also reduce the complexity. Using simulation results, we have shown that this detector gives a better complexity performance compromise. The same conclusions hold when using space time block codes instead of the V-BLAST scheme.

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