Higgs field as the gauge field corresponding to parity in the usual space-time

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Abstract

We find that the local character of field theory requires the parity degree of freedom of the fields to be considered as an additional discrete fifth dimension which is an artifact emerging due to the local description of space-time. Higgs field can be interpreted as the gauge field corresponding to this discrete dimension. Hence the noncommutative geometric derivation of the standard model follows as the manifestation of the local description of the usual space-time.

1 Introduction

Higgs field can be considered as an additional gauge field in a generalized covariant derivative. This identification is usually realized by identifying it by a connection either corresponding to an extra continuous [1] or discrete [2] dimension or corresponding to extension of the usual space-time by addition of a finite number of points either in a non-commutative differential geometry context [3] or in the usual differential geometry supplemented by some physical considerations [4]. Yet another way to obtain the generalized covariant derivative including the Higgs field is to make use of an axiomatic approach [5]. There are also some variants and improvements of these formulations [6]. The unpleasant feature of the approaches using an extended space-time is that there is no sound verification for such an enlargement of the space-time. Although the axiomatic approach is attractive in deriving the nice aspects of the scheme in the usual space-time it lacks a deeper understanding both mathematically and physically.

In this study we shall see that the noncommutative differential geometry setting directly follows from the locality of field theory in space-time. We find that the discrete space-time transformations can be incorporated into the formulation of field theory as a fifth discrete dimension corresponding to parity. We observe that in fact this discrete fifth dimension is hidden in the Dirac equation. After identification of the derivative and the gamma matrix corresponding to this dimension and after incorporating the usual gauge group structure we derive the noncommutative geometrical setting. In summary we show that the noncommutative geometrical framework is hidden in the usual space-time structure and it emerges as an artifact of field theory due to its insufficiency to include discrete space-time degrees of freedom through the usual continuous space-time coordinates .

2 Local description of space-time and the need for the fifth discrete dimension

Assume that we define a field, ϕ over a differentiable manifold M. Provided one knows $\phi(\zeta)$ at every point $\zeta \in M$ one knows every information about it. In other words in an action at a distance formulation one does not need to know $\partial_\zeta \phi$ (or more generally $D_\zeta \phi$) etc. to determine ϕ at every point. On the other hand if one uses a local formulation (i.e a field formulation) then one needs to know the variation of ϕ with ζ as well, that is,

$$\phi(\zeta) = \phi(\zeta_0) + (\zeta - \zeta_0)(\partial_\zeta \phi(\zeta))_{\zeta=\zeta_0} + ... \tag{1}$$

One can write M as a direct product of two manifolds, one expressing the internal structure, I and the other the space-time structure, S; $M = I \otimes S$. If we express S (i.e the space-time) in a local (field) formulation then we can expand ϕ in a Taylor expansion as

$$\begin{aligned} \phi(\rho,\eta) = & \quad \phi(\rho,\eta_0) + (\eta - \eta_0)(\partial_\eta \phi(\rho,\eta))_{\eta=\eta_0} + ... \\ = & \quad \phi(\rho,\eta_0) + (x - x_0)(D_x \phi(\rho,x))_{x=x_0} + ... \\ & \quad \rho \in I \ , \quad \eta \ , \eta_0 \in S \ , \quad x, x_0 \in X \end{aligned} \tag{2}$$

where X is the tangent space at η and D is the covariant derivative corresponding to the use of X as the local coordinate frame.

Physical observables are the quantities which do not change under reparametrization of ζ (i.e under symmetry transformations). In other words they are invariants under symmetry transformations. They can be constructed as

$$I = \mathcal{T}[L_\phi(\phi, D\phi) L_\zeta(\zeta, \delta\eta)] \tag{3}$$

where L_ζ represents the physical observables related to the symmetry transformations independent of ϕ and $\mathcal{T}$ denotes a generalized trace including integration. In the case of the usual 4-dimensional space-time coordinates

$$I = \int \mathcal{L}(\phi, D\phi) \sqrt{det(g_{\mu\nu})} d^4x \tag{4}$$

where $g_{\mu\nu}$ is the metric tensor.

Does Eq.(4) really define the usual (Minkowski) space-time wholly? It includes ϕ and $\partial_\mu \phi$ $\mu = 0, 1, 2, 3$ so it describes the behaviour of ϕ under translations, rotations, Lorentz boosts etc. i.e the usual continuous space-time transformations hence the corresponding physical observables. In other words the generators of the continuous space-time transformations can be expressed in terms of the differetial variation of the usual 4-dimensional coordinates, x_μ. What about the discrete space-time transformations, parity and time reversal? If we do not include the spinor representaions or if we do not introduce the notion of intrinsic parity then the degrees of freedom corresponding to parity and time reversal transformations are automatically included in the Lagrangian because, in this case, once we impose Lorentz invariance the Lagrangian becomes parity and time reversal invariant. Otherwise (i.e. if we admit spinor representations of the Lorentz group with an internal group structure or admit intrinsic parity) they are not included in the above formulation unless they are included in an ad hoc way because one can not generate them from the continuous transformations. Then there are three alternatives either one uses an action at a distance formulation or one includes their effect in an ad hoc way (i.e without a differential geometric formulation) or one should extend the differential geometric formulation so that these discrete degrees of freedom are included as the local rate of change of ϕ under these transformations. In other words one must introduce an additional dimension corresponding to these degrees of freedom which disappear when one passes to an action at a distance formulation.

3 Parity as the fifth local dimension

We can summarize the previous section as follows: If one uses an action at a distance formulation it is enough to know $\phi(\zeta)$ $\zeta \in M$ for all ζ in order to have all information about the state described by ϕ but when one uses field formalism one must know $\phi(\zeta)$ and $\phi(\zeta + \delta\zeta)$ at some neigborhood of ζ or equivalently $\phi(\zeta)$ and $D\phi(\zeta)$. In the case of parity this is equivalent to saying that, in field formalism if we include the spinor reresentations of lorentz group with an internal group structure or if we admit intrinsic parity, one must know $\phi(\vec{x}, t)$ and $\phi(-\vec{x}, t)$ or equivalently $\phi(\vec{x}, t)$ and $D_p\phi(\vec{x}, t)$ at the point $(\vec{x}, t)$. One can define the sets

$$X_L = \{\vec{x}, t \mid \vec{x}, t \in S\} \quad X_R = \{-\vec{x}, t \mid \vec{x}, t \in S\} \tag{5}$$

or in a form that reflects local describtion of space-time

$$X_L = \{d\vec{x}, dt \mid \text{for all } \ d\vec{x}, dt \in O\} \quad X_R = \{-d\vec{x}, dt \mid \text{for all } \ d\vec{x}, dt \in O\} \tag{6}$$

where O is a neigborhood of $\vec{x}, t$. Parity can be considered to be an extra direction connecting X_L and X_R in addition to the usual directions x_μ $\mu = 0, 1, 2, 3$. This is due to the fact that the information about $\phi(\vec{x}, t)$ and $\phi(\vec{x} + d\vec{x}, t)$ does not tell anything about the behaviour of the field under parity because one can not generate parity through continuous transformations. When we only consider the behaviour of the field under parity transformation one can consider X_L and X_R as two discrete points in this additional direction. Below we shall formulate these observations in a more systematical way.

We consider the parity degree of freedom of the fields to correspond to a fifth discrete dimension consisting of two points

$$p_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad p_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7}$$

which are connected by parity transformation

$$P : p_{1(2)} = p_{2(1)} \ . \tag{8}$$

One notices that in fact p_1 and p_2 belong to different dimensions otherwise when a left-handed particle is created its partner right-handed would be destroyed. (But we shall see later that effectively there is only one dimension)

We shall put a two-component left-handed spinor at point p_2 and put a right-handed one at point p_1 , that is,

$$\psi(p) = \begin{pmatrix} \psi_1(p_2) \\ \psi_2(p_1) \end{pmatrix} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$$and \qquad \begin{pmatrix} \psi_2(p_1) \\ \psi_1(p_2 \end{pmatrix} = 0 \ , \quad \begin{pmatrix} \psi_1(p_1) \\ \psi_2(p_2) \end{pmatrix} = 0 \tag{9}$$

Under parity transformation

$$P : \psi(p) \to \psi'(p') = \begin{pmatrix} \psi_2(p_2) \\ \psi_1(p_1) \end{pmatrix} = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \tag{10}$$

while from Eq.(9) it follows that

$$\psi'(p) = \begin{pmatrix} \psi_2(p_1) \\ \psi_1(p_2) \end{pmatrix} = 0 \tag{11}$$

So

$$\psi'(p') = \psi'(p) + \delta_p \psi(p) = \delta_p \psi(p) \tag{12}$$

In other words

$$\delta_p \psi = \psi'(p') = \gamma^0 \psi(p') \tag{13}$$

where it is understood that ψ_R in Eq.(9) acts as a left handed spinor while ψ_R acts as a right-handed one under SL(2,C). Now we determine the differential length in the fifth dimension, dx_4.

$$\begin{aligned} dx_4 \qquad &= dx_p = \frac{1}{im}(p_2 - p_1) = \frac{1}{im}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ &= \frac{1}{im} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{im} \gamma_5 \gamma^0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned} \tag{14}$$

Although the use of γ^0 may seem redundant in fact it is essential in order to match the differential distances in x_5 with the correct elements of ψ because 1 in $(1,-1)^T$ corresponds to the change in the coordinates of ψ_L which moved to lover position in $\delta\psi$ and vice versa for -1 and ψ_R. Therefore the derivative operator corresponding to the fifth dimension is found to be

$$\partial_4 = \frac{d_4}{dx_4} = im\gamma^0(\gamma^0\gamma_5) = im\gamma_5 \tag{15}$$

One can identify $i\gamma_5 = \gamma^4$ as the gamma matrix corresponding to the fifth dimension because

$$\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu} \qquad \mu, \nu = 0, 1, 2, 3, 4$$
$$(g^{\mu\nu}) = diag(1, -1, -1, -1, -1) \tag{16}$$

One recognizes that in fact the fifth dimension corresponding to parity transformations is hidden in Dirac equation

$$\not{D}\psi - m\psi = \not{D}\psi - \gamma^4\partial_4\psi = 0 \tag{17}$$

Although the presence of $\gamma^4\partial_4$ has no effect here we shall see in the next section that this is not the case when ψ has a local internal group structure.

One can also derive the Dirac action corresponding to Eq.(17) through this formalism

$$S_D = im \int \psi^\dagger dx_4 \gamma^\mu \partial_\mu \psi = \int \bar{\psi}(\not{p}artial - m)\psi\, d^4x \qquad \mu = 0, 1, 2, 3, 4 \tag{18}$$

where we have used the identity

$$\int_A^B \psi^\dagger\, dx_4 \gamma^\mu \partial_\mu \psi = \int_A^B \gamma^0 \gamma_5 \gamma^\mu \partial_\mu \psi\, |dx_4|$$
$$= \int_a^b \psi_L^\dagger \gamma^\mu \partial_\mu \psi_L |dx_4| - \int_b^a \psi_R^\dagger \gamma^\mu \partial_\mu \psi_R |dx_4| = \int a^b \bar{\psi} \gamma^\mu \partial_\mu \psi\, |dx_4| \tag{19}$$

This construction can be understood better through a mathematical analysis. The above gamma matrices in five dimensions correspond to spinorial representations of $SO(5)$. Because the spinorial representations of both $SO(2k)$ and $SO(2k+1)$ are 2^k dimensional the spinorial representations of both $SO(5)$ and $SO(4)$ are 4-dimensional. Hence the spinorial representations of the vectors in 5 dimensional pseudo-Euclidean space can be determined by adding another independent 2^2 dimensional matrix (i.e γ^4) to γ^μ's of 4-dimensional Minkowski space [7]. In fact this is realized another form as well: The gamma matrices of 4-dimensional Eucleadian space and the generators of $SO(4)$ together form the generators of SO(5) [8].

Another discrete space-time transformation is time reversal [9]. Fortunately time reversal does not enter as an additional discrete dimension once parity is taken to correspond to the fifth dimension because one can generate a time reversal through a rotation in plane plus an analytic extension (!) of Lorentz boosts. For example

$$\begin{aligned} &P : (x_0, x_1, x_2, x_3) \rightarrow (x_0, -x_1, -x_2, -x_3) \quad R : (x_0, -x_1, -x_2, -x_3) \rightarrow (x_0, -x_1, x_2, x_3) \\ &\Gamma : (x_0, -x_1, x_2, x_3) \rightarrow (-x_0, x_1, x_2, x_3) \end{aligned} \tag{20}$$

where

$$P = \mathrm{diag}(1, -1, -1, -1) \quad \Gamma = \mathrm{diag}(-1, -1, 1, 1)$$

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{21}$$

Here Γ is an analytic extension of the usual Lorentz transformations with

$$\gamma = \sqrt{1-\beta^2} = -1 \quad \text{for} \quad \beta = 0 \tag{22}$$

Although this choice is not physical it is a legitmate one because in the differential geometrical construction of the action functional we consider all kinds of variations of the variables. In other words this choice is already included in the construction of the action. Therefore although time reversal is an independent space-time transformation it does not introduce a new artificial dimension. (Remember that the fifth dimension itself is not true dimension. It is an artifact imposed by our insistence of a wholly differential geometric construction of action functional.) This fact can be understood in another way also: Both parity and time reversal interchange the dotted and the undotted representations of SL(2,C) so both act on the same discrete dimension. Of course when we consider an internal structure for the fields then time reversal and charge conjugation operations can be considered to correspond to some symmetries whose connection can be identified with another set of Higgs fields. But this connection is different from the one corresponding to a parity transformation because the laters correspond to the extension of the usual gauge group while the fifth dimension can be identified as gauge field without extending the gauge group.

4 Inclusion of internal structure and emergence of Higgs field

Above we have seen that the (formal) fifth dimension is hidden in the Dirac equation if we only consider space-time. When we consider an internal structure for the fields then the fifth dimension becomes more explicit. For example in the standard model (considering only the weak sector)

$$
\begin{aligned}
&\partial_4 = \frac{d_4}{dx_4} = m(\gamma^0\gamma^4)\gamma^0 \rightarrow (M \otimes \gamma^0\gamma^4)(\tau_1 \otimes I_4 \otimes I_3 \otimes \gamma^0) \\
&= -M(\tau_1 \otimes I_4 \otimes I_3) \otimes \gamma^4 \\
&\gamma^4\partial_4 = M(\tau_1 \otimes I_4 \otimes I_3) \otimes I_4
\end{aligned} \tag{23}
$$

where M =diag$(M', M'^{\dagger})$ [M'= diag$(M_u, M_d, 0, M_e)$] is the fermion mass matrix. (Note that the unusual γ_5 in the standard formulation of the noncommutative geometry [10] does not appear here because we use use Minkowski signature instead of Euclidean one.) The fermion mass matrix, M in Eq.(23) can be taken to be the generalization of the fermion masses in the usual Dirac equation. It can be better understood if we consider it to correspond to the increase of the discrete points in the fifth dimension from 2 to $2 \times 4 \times n = 8n$ while the space-time is still five dimensional. We can assume that each $\psi_{iaL(R)}$, $(a = u, d, \nu, e$ and $i = 1, 2, ...n$ for n families of fermions) has a differential variation in the fifth dimension given by m_{ia} when $\psi_{iaL(R)}$ is in the mass basis.

Then the fermion mass matrices are found from

$$\bar{\psi}_{ia}(\partial_4\psi_{ja})+(\bar{\psi}_{ia}\partial_4)\psi_{ja} = \bar{\psi}_{iaL}(x_{ija}+x^*_{jia})\psi_{jaR}+h.c. = \bar{\psi}_{aL}U_L^\dagger\, m_a\, U_R\psi_{aR}+h.c. \tag{24}$$

that is $M_{ija} = x^*_{jia} + x_{ija}$ where a non-diagonal $M_a = U_L^\dagger\, m_a\, U_R$ corresponds to the general case where ψ_{ia}'s are not in the mass basis i.e. each ψ_{ia} does not sit at one point in the fifth dimension but rather they are at general points in flavor space which can be expressed as admixtures of $1/m_{ia}$'s.

5 Conclusion

In this study we have seen that parity can be incorporated into the differential geometric construction of field theory as a fifth discrete dimension otherwise while the continuous part of space-time transformations are included in an elegant differentail geometric formulation the discrete transformations can be incorporated in an ad hoc way into the local formulation of the field theory . We have shown that the differentiation along the fifth discrete direction is nontrivial especially if we include an internal gauge structure for the fields. The corresponding connection can be identified as the Higgs field of the standard model if take the gauge group as $SU(3)_c \otimes SU(2)_L \otimes U(1)$. In other words Higgs field can be considered as a gauge field without extending the (Minkowski) space-time and the $(SU(2)_L \otimes U(1))$ gauge group. We hope that the studies in this direction supplemented with grand unification schemes of multiple intermediate symmetry breaking scales will contribute to a more unified, more predictive and renormalization group invariant account of particle physics [5,11].

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