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Relation Between Relativistic and Non-Relativistic Quantum Mechanics as Integral Transformation

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A formulation of quantum mechanics (QM) in the relativistic configurational space (RCS) is considered. A transformation connecting the non-relativistic QM and relativistic QM (RQM) has been found in an explicit form. This transformation is a direct generalization of the Kontorovich–Lebedev transformation. It is shown also that RCS gives an example of non-commutative geometry over the commutative algebra of functions.

KEY WORDS: relativistic; non-relativistic; configurational space; non-commutative differential calculus; Lorentz group representations; Fourier transformation.

1. INTRODUCTION. CONCEPT OF RELATIVISTIC CONFIGURATIONAL SPACE

The concept of relativistic configurational space (RCS)[1–5] is based on the simple observation that a free motion of a relativistic particle of mass m can be described on the basis of the Gelfand–Graev transformation,[6, 7] i.e., expansion in relativistic spherical functions. We start with the well-known fact that the equation

$$p_0^2 - \mathbf{p}^2 = m^2c^2, \qquad p_0 = \frac{E}{c} \tag{1}$$

describing the relativistic relation between energy and momentum of the particle (mass shell equation), describes at the same time the three-dimensional momentum space of constant negative curvature or the Lobachevsky

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space. The isometry group of this space is the Lorentz group. To introduce an adequate Fourier expansion, we must find the matrix elements of the unitary irreducible representations of this group. These matrix elements are the eigen-functions of the Casimir operator, or the Laplace–Beltrami operator in the Lobachevsky space (1)

$$\hat{C} = \tfrac{1}{2} M^{\mu\nu} M_{\mu\nu}, \qquad \mu, \nu = 0, 1, 2, 3 \tag{2}$$

The corresponding equation

$$\hat{C} \langle \mathbf{r} \mid \mathbf{p} \rangle = c \langle \mathbf{r} \mid \mathbf{p} \rangle \tag{3}$$

can be solved in terms of the Gelfand–Graev kernels[6, 7]

$$\langle \mathbf{r} \mid \mathbf{p} \rangle = \left(\frac{p_0 - \mathbf{p} \cdot \mathbf{n}}{mc} \right)^{-1 - i \frac{r}{\lambda}}$$

$$\mathbf{r} = r\mathbf{n}, \qquad \mathbf{n}^2 = 1, \qquad 0 \leqslant r < \infty \tag{4}$$

$$\lambda = \frac{\hbar}{mc}, \quad \text{Compton wave-length of the particle} \tag{5}$$

There are several strong arguments for considering $\mathbf{r}$ as a relativistic position vector of the particle:

1. The range of variation of r (4) coincides with that of the standard non-relativistic coordinate vector.
2. The magnitude r of $\mathbf{r}$ is Lorentz invariant in full analogy with the non-relativistic relative distance, which is Galilean invariant.
3. "Relativistic plane wave" $\langle \mathbf{r} \mid \mathbf{p} \rangle$ in the non-relativistic limit

$$|\mathbf{p}| \ll mc, \qquad r \gg \lambda, \qquad p_0 \simeq mc + \frac{\mathbf{p}^2}{2mc} \tag{6}$$

goes over into the standard non-relativistic plane wave

$$\begin{aligned} \langle \mathbf{r} \mid \mathbf{p} \rangle &= \exp\left[-\left(1 + i\frac{r}{\lambda}\right) \ln\left(\frac{p_0 - \mathbf{p} \cdot \mathbf{n}}{mc} \right) \right] \\ &\simeq \exp\left[-\left(1 + i\frac{r}{\lambda}\right) \ln\left(1 - \frac{\mathbf{p} \cdot \mathbf{n}}{mc} + \frac{\mathbf{p}^2}{2m^2c^2} + \cdots \right) \right] \\ &\simeq \exp\left(ir\frac{\mathbf{p} \cdot \mathbf{n}}{\hbar} \right) = \exp\left(i\frac{\mathbf{p} \cdot \mathbf{x}}{\hbar} \right) \end{aligned} \tag{7}$$

In the last expression **x** denotes the standard non-relativistic position vector.

4. The most important physical argument is that there exists the operator $\hat{H}_0$ of free energy in r-space

$$\hat{H}_0 = mc^2 \left\{ \cosh i\lambda \frac{\partial}{\partial r} + i \frac{\lambda}{r} \sinh i\lambda \frac{\partial}{\partial r} - \frac{\lambda^2}{r^2} \Delta_{\vartheta,\phi}\, e^{\frac{\hbar}{mc}\frac{\partial}{\partial r}} \right\} \tag{8}$$

such that

$$(\hat{H}_0 - E)\langle \mathbf{r} \,|\, \mathbf{p} \rangle = 0 \tag{9}$$

The last two equations show that the plane wave $\langle \mathbf{r} \,|\, \mathbf{p} \rangle$ is the wave function of the relativistic free particle, i.e. the state with a fixed value of the relativistic energy. It is worthwhile to stress that (9) can be considered as a solution of the problem of extracting the square root in the expression for the relativistic energy

$$E = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} \tag{10}$$

The "price" for this extraction is the presence of the exponential derivation which amounts to a finite-difference character of the Hamiltonian operator (8). The interaction can be described in terms of a potential function $V(r)$. On this basis, the quantum theory in the relativistic configurational space had been developed. This theory will be called hereafter the relativistic quantum mechanics (RQM). It proved to be an efficient approach to solving problems in a wide range: from analytic properties of relativistic wave functions and amplitudes to relativistic confinement models of composite hadrons. We refer the reader to Refs. 1–5 and further references therein where this theory is presented.

The purpose of this work is to establish the connection between RQM described above and the standard non-relativistic quantum mechanics (QM) in the form of an integral transformation (the Kontorovich–Lebedev transformation[8]). We also show that relativistic configurational space is an example of physical application of the ideas of non-communicative geometry (NG) and introduce this space applying the methods of NG.

The paper is organized as follows. In Sec. 2, relativistic configurational space is introduced on the basis of the NG. In Sec. 3 a short description of the one-dimensional RQM is given. Section 4 is devoted to the construction of the generalized Kontorovich–Lebedev transformation. In Sec. 5, some explicit examples of integrable problems existing in QM and RQM are considered. For the very simple case of the box potential we construct the integral transformation, connecting the QM with RQM solutions in the framework of non-commutative differential calculus. The natural result is that the relativistic energy spectrum is the immediate generalization of the non-relativistic one to which it transfers in the $c \to \infty$ limit.

2. NON-COMMUTATIVE DIFFERENTIAL GEOMETRY AND RQM

For more than a one decade the ideas of algebraic geometry and topology under the new name “non-commutative” or “quantum” differential geometry have been considered as a possible mathematical arena for comprehensive quantum field theory incorporating gravity mostly in the context of String theory (see Ref. 9 and references therein). As it often happens in the history of science, this mathematics has already been used in physics for many years.

The most known example of application of the non-commutative differential geometry in physics is quantum theory itself. The standard Heisenberg quantization procedure substitutes the phase space of classical mechanics with quantum geometry of non-commuting operators of position and momentum. We are used to say that phase space does not exist in quantum theory. However, it does not exist as “point space” of the standard differential geometry of points and curves, but still exists in the “non-commutative” sense. In accordance with ideas of algebraic geometry we study now the properties of the algebra of functions (the wave functions) on this space instead of trajectories. In the present paper, the concept of relativistic configurational space[(1–5)]) is considered as another example of the application of ideas of non-commutative geometry in physics: relativistic quantum theory is obtained from the non-relativistic one as “quantization” of configurational space.

It is sufficient for our purposes to limit ourselves to the simplest case when the algebra of functions A is commutative. The transfer to the generalized calculus[(10–15)] can be described by the following dictionary in which the first two columns are taken from the book,[(9)] and the third column corresponds to our case of commutative algebra A.

Classical	Quantum	Present consideration
Complex variable	Operator in $\mathscr{H}$	Analytic function
Real variable	Self-adjoint operator in $\mathscr{H}$	$\psi^*(z) = \psi(z^*)$
Infinitesimal	Compact operator in $\mathscr{H}$	Compact operator in $\mathscr{H}$
Differential of real or complex variable	$d\psi = [F, \psi] = F\psi - \psi F$	$d\psi = [F, \psi]$

The passage from the classical formula to the operator one is similar to the substitution of the Poisson brackets $\{\psi, \chi\}$ with commutators $[\psi, \chi]$ in the process of quantization. The standard Leibniz rule is valid in the non-commutative differential calculus

$$d(\psi\chi) = [F, \psi\chi] = [F, \psi]\,\chi + \psi[F, \chi] = d(\psi)\,\chi + \psi d(\chi) \tag{11}$$

Non-commutative differential calculus is introduced[10–15] as a deformation of the theory of differential forms. We refer the reader to these papers for a detailed description of this theory. In Ref. 15, a number of physical applications of the non-commutative differential calculus on the commutative algebra of functions could be found. Here, we outline the notion of differential calculus on an algebra A.[10–14] The differential calculus on A is a $\mathbb{Z}$-graded associative algebra over $\mathbb{C}$:

$$\Omega(A) = \bigoplus_{r \in \mathbb{Z}} \Omega^r(A), \qquad \Omega^0(A) = A, \qquad \Omega^r(A) = \{0\} \quad \forall r < 0 \tag{12}$$

The elements of $\Omega^r(A)$ are r-forms. There exists an exterior derivative ($\mathbb{C}$-linear) operator d: $\Omega^r(A) \to \Omega^{r+1}(A)$ which satisfies the following conditions:

$$d^2 = 0, \qquad d(\omega\omega') = (d\omega)\,\omega' + (-1)^r\,\omega\,d\omega' \tag{13}$$

where ω and ω' are r- and r'-forms, respectively. Here A is supposed to be the commutative algebra generated by the coordinate functions x^i, $i = 1,..., n$. In the standard differential calculus over usual manifolds, differentials commute with functions:

$$[x^i, dx^j] = 0, \qquad i, j = 1,..., n \tag{14}$$

in terms of real coordinates x^i. For us it is essential that (14) can be generalised (deformed) in different ways with (12), (13) still being true. Let us consider in more detail the deformation of (14) of the form

$$[x^i, x^j] = 0, \qquad [x^i, dx^j] = \sum_{k=1}^{n} dx^k\, C^{ij}{}_k \tag{15}$$

where $C^{ij}{}_k$ are (complex) constants which are constrained by the requirement of a consistent differential calculus.

Referring the reader to Refs. 10–15 (see also Ref. 2) where the general theory is developed and the consistency of differential algebra (15) is proved, we concentrate here on the main relations of the one-dimensional noncommutative differential calculus following from the hypothesis (15), which in this case takes the form

$$[x, d_+x] = i\lambda\, d_+x, \qquad \lambda = \frac{\hbar}{mc} \tag{16}$$

where λ is a Compton wave length of the particle. It follows from (16) that

$$x\, d_+x = d_+x(x+i\lambda), \qquad x^n\, d_+x = d_+x(x+i\lambda)^n \tag{17}$$

and

$$\psi(x)\, d_+x = d_+x\, \psi(x+i\lambda) \tag{18}$$

The right and left derivatives are introduced by definition as

$$d_+\psi(x) = d_+x(\vec{\partial}_+\psi(x)) = (\overleftarrow{\partial}_+\psi(x))\, d_+x \tag{19}$$

We refer the reader for additional technical details to Ref. 2, where it has been shown in particular that finite-difference differentials obey the standard Leibnitz rule:

$$\begin{aligned} d_+(\psi(x)\,\chi(x)) &= (d_+\psi(x))\,\chi(x) + \psi(x)(d_+\chi(x)) \\ &= d_+(\chi(x)\,\psi(x)) = (d_+\chi(x))\,\psi(x) + \chi(x)(d_+\psi(x)) \end{aligned} \tag{20}$$

Recall that we consider the calculus over the commutative algebra of functions so that

$$\psi(x)\,\chi(x) = \chi(x)\,\psi(x) \tag{21}$$

From (20) we find the (modified) Leibnitz rule for the derivatives

$$\begin{aligned} d_+(\psi(x)\,\chi(x)) &= d_+x(\vec{\partial}_+\psi(x))\,\chi(x) + \psi(x)\,d_+x(\vec{\partial}_+\chi(x)) \\ &= d_+x\{(\vec{\partial}_+\psi(x))\,\chi(x) + \psi(x+i\lambda)\,(\vec{\partial}_+\chi(x))\} \\ &= d_+x(\vec{\partial}_+(\psi\,\chi)) = d_+(\chi(x)\,\psi(x)) = d_+x(\vec{\partial}_+(\chi\,\psi)) \\ &= d_+x\{(\vec{\partial}_+\chi(x))\,\psi(x) + \chi(x+i\lambda)\,(\vec{\partial}_+\psi(x))\} \end{aligned} \tag{22}$$

We can also write (22) in the form

$$(\vec{\partial}_+\psi(x)\,\chi(x)) = (\vec{\partial}_+\psi(x))\,\chi(x) + \psi(x)(\vec{\partial}_+\chi(x)) + i\lambda(\vec{\partial}_+\psi(x))\,(\vec{\partial}_+\chi(x)) \tag{23}$$

from which we can clearly see what is the modification of the Leibnitz rule for the non-commutative derivatives as compared to the standard ones. The additional term $i\lambda(\vec{\partial}\psi(x))$ vanishes in the non-relativistic limit. From Eq. (22) we get

$$(\vec{\partial}_+\psi(x))(\chi(x))+\psi(x+i\lambda))(\vec{\partial}_+\chi(x))=(\vec{\partial}_+\chi(x))(\psi(x))+\chi(x+i\lambda))(\vec{\partial}_+\psi(x)) \tag{24}$$

this gives

$$\frac{(\vec{\partial}_+\psi(x))}{\psi(x+i\lambda)-\psi(x)}=\frac{(\vec{\partial}_+\chi(x))}{\chi(x+i\lambda)-\chi(x)} \tag{25}$$

To fix the *constant*, we put $\psi(x)=x$:

$$(\vec{\partial}_+x)=const(x+i\lambda-x)=i\lambda\, const \tag{26}$$

On the other hand,

$$d_+x=d_+x((\vec{\partial}_+x))=d_+x(i\lambda\, const) \tag{27}$$

and we have

$$const=\frac{1}{i\lambda} \tag{28}$$

i.e., the right derivative has the form of finite-difference operation

$$(\vec{\partial}_+\psi(x))=\frac{\psi(x+i\lambda)-\psi(x)}{i\lambda} \tag{29}$$

In the non-relativistic limit $(\vec{\partial}_+\psi(x))\to\frac{d}{dx}$. Similarly, for the left derivative we obtain the expression

$$(\overleftarrow{\partial}_+\psi(x))=\frac{\psi(x-i\lambda)-\psi(x)}{-i\lambda}=\overline{(\vec{\partial}_+\psi(x))} \tag{30}$$

To construct the relativistic energy and momentum operators in the framework of the non-commutative calculus, we need the symmetric (complex conjugated) relations

$$[x\, d_-x]=-i\lambda d_+x \tag{31}$$

$$d_-\psi(x)=d_-x(\vec{\partial}_-\psi(x))=(\overleftarrow{\partial}_-\psi(x))\, d_-x \tag{32}$$

$$\vec{\partial}_\pm=\overline{(\overleftarrow{\partial}_\pm)} \tag{33}$$

It is easy to prove that

$$[d_\pm x, \psi(x)] = \mp i d_\pm \psi(x) \tag{34}$$

Now, to obtain the non-commutative differentials corresponding to quantum operators of energy and momentum, we take the real combinations

$$d^0 x = \tfrac{1}{2}(d_+ x + d_- x), \qquad dx = \tfrac{1}{2}(d_+ x - d_- x) \tag{35}$$

The corresponding derivatives

$$\vec{\partial}^0 = \tfrac{1}{2}(\vec{\partial}_+ - \vec{\partial}_-) \qquad \text{and} \qquad \vec{\partial} = \tfrac{1}{2}(\vec{\partial}_+ + \vec{\partial}_-) \tag{36}$$

give the desired relativistic quantum operators

$$\hat{H}^0 = i\vec{\partial}^0 + 1 = mc \cosh i\lambda \frac{d}{dx}, \qquad \hat{p} = i\vec{\partial} = mc \sinh i\lambda \frac{d}{dx} \tag{37}$$

We also have

$$\begin{aligned} d^0\psi &= i[\psi, dx^0] = d^0x(\vec{\partial}^0\psi(x)) + dx(\vec{\partial}\psi(x)) \\ d\psi &= i[\psi, dx] = d^0x(\vec{\partial}\psi(x)) + dx(\vec{\partial}^0\psi(x)) \end{aligned} \tag{38}$$

The right derivatives $\vec{\partial}^0$ and $\vec{\partial}$ obey the following Leibnitz rules:

$$\begin{aligned} \vec{\partial}^0(\psi(x)\,\chi(x)) &= (\vec{\partial}^0\psi(x))\,\chi(x) + \psi(x)\,(\vec{\partial}^0\chi(x)) \\ &\quad + i\lambda\{(\vec{\partial}^0\psi(x))(\vec{\partial}^0\chi(x)) + (\vec{\partial}\psi(x))(\vec{\partial}\chi(x))\} \\ \vec{\partial}(\psi(x)\,\chi(x)) &= (\vec{\partial}\psi(x))\,\chi(x) + \psi(x)\,(\vec{\partial}\chi(x)) \\ &\quad + i\lambda\{(\vec{\partial}^0\psi(x))(\vec{\partial}\chi(x)) + (\vec{\partial}\psi(x))(\vec{\partial}^0\chi(x))\} \end{aligned} \tag{39}$$

The transition from the non-relativistic 1-dimensional quantum theory to the relativistic one implies a transition to a two-dimensional theory (one time and one spatial dimension, or in other words the independent energy and momentum operators). We must also have a realization of the two-dimensional Poincaré Lie algebra. This problem of having the two-dimensional theory when operating with only one coordinate is solved in the framework of the 1-dimensional non-commutative differential calculus over $\mathbb{C}$ i.e., $x \to x \pm \lambda$. From the point of view of the finite-difference operators consideration of the non-commutative calculus over $\mathbb{C}$ is equivalent to consideration of the second order equations. So we achieve simultaneously two goals: Transition to the two-dimensional covariant description for quantum theory with one spatial dimension and derivation of the energy operator as the second order finite-difference operator (the necessity of the second requirement becomes in particular clear when we take the non-relativistic limit).

Now we consider the non-commutative realization of the Poincare group Lie algebra and relativistic wave functions.

The operators (37) $\hat{p}^0$ and $\hat{p}^1$ commute. Their common eigen-functions can be described as follows. Let us consider the Gelfand–Graev transformation on the hyperbola (1). In the hyper-polar coordinates

$$p^0 = mc \cosh \chi, \qquad p = mc \sinh \chi \tag{40}$$

the kernel of this transformation takes the form

$$\langle x \mid p \rangle = \left(\frac{p^0 - p}{mc} \right)^{-i \frac{x}{\lambda}} = e^{i \frac{x\chi}{\lambda}} \tag{41}$$

where

$$-\infty < x < \infty, \qquad \chi = \ln \left(\frac{p^0 + p}{mc} \right), \qquad -\infty < \chi < \infty \tag{42}$$

It is easy to see that

$$\hat{H}^0 \langle x \mid p \rangle = mc \cosh \chi \langle x \mid p \rangle, \qquad \hat{p} \langle x \mid p \rangle = mc \sinh \chi \langle x \mid p \rangle \tag{43}$$

This means that (41) can be considered as a relativistic "plane wave" describing the free motion of the particle. The variable x is the quantum coordinate.

The 2-dimensional Poincare Lie algebra

$$[\hat{M}^{10}, \hat{H}^0] = i\hat{p}, \qquad [\hat{M}^{10}, \hat{p}] = i\hat{H}^0, \qquad [\hat{H}^0, \hat{p}] = 0 \tag{44}$$

is realized in terms of the non-commutative differential calculus (as the right derivatives)

$$\hat{M}^{10} = x, \qquad \hat{p}^H = i\vec{\partial}^0 + 1, \qquad \hat{p} = -i\vec{\partial} \tag{45}$$

The plane wave (41) is the matrix element of the Lorentz group representation. This means that its transformation rule (addition theorem) reflects the transformation law of the vector (p^0, p) under the Lorentz transformation[2]

$$\begin{aligned} p'^0 &= (p(+)\,k)^0 = \cosh\chi_p \cosh\chi_k + \sinh\chi_p \sinh\chi_k = \cosh(\chi_p+\chi_k) \\ p' &= (p(+)\,k) = \sinh\chi_p \cosh\chi_k + \cosh\chi_p \sinh\chi_k = \sinh(\chi_p+\chi_k) \end{aligned} \tag{46}$$

The addition theorem for the plane waves (41) is in accordance with the one-dimensional Lorentz transformation law which is simply the shift of the rapidity χ:

$$e^{i\frac{x}{\lambda}\chi_p}\, e^{i\frac{x}{\lambda}\chi_k} = e^{i\frac{x}{\lambda}(\chi_p+\chi_k)} \tag{47}$$

Now we show that the right derivatives of the non-commutative differential calculus which obey the deformed Leibnitz rule fit the Lorentz group addition theorem (the usual derivatives evidently do not!). Using (38), we write

$$\begin{aligned} [d^0x, e^{i\frac{x}{\lambda}(\chi_p+\chi_k)}] &= d^0x(\vec{\partial}^0 e^{i\frac{x}{\lambda}(\chi_p+\chi_k)}) + dx(\vec{\partial} e^{i\frac{x}{\lambda}(\chi_p+\chi_k)}) \\ &= -i\{d^0x[\cosh(\chi_p+\chi_k)-1] - dx\sinh(\chi_p+\chi_k)\}\, e^{i\frac{x}{\lambda}(\chi_p+\chi_k)} \end{aligned} \tag{48}$$

Taking the commutator of dx^0 with the left-hand side of (47) and applying the Leibnitz rule, we obtain

$$\begin{aligned} &[d^0x, e^{i\frac{x}{\lambda}\chi_p} e^{i\frac{x}{\lambda}\chi_k}] \\ &\quad= [d^0x, e^{i\frac{x}{\lambda}\chi_p}]\, e^{i\frac{x}{\lambda}\chi_k} + e^{i\frac{x}{\lambda}\chi_p}\,[d^0x, e^{i\frac{x}{\lambda}\chi_k}] \\ &\quad= -i\{d^0x(\vec{\partial}^0 e^{i\frac{x}{\lambda}\chi_p}) + dx(\vec{\partial} e^{i\frac{x}{\lambda}\chi_p})\}\, e^{i\frac{x}{\lambda}\chi_k} - ie^{i\frac{x}{\lambda}\chi_p}\{d^0x(\vec{\partial}^0 e^{i\frac{x}{\lambda}\chi_k}) + dx(\vec{\partial} e^{i\frac{x}{\lambda}\chi_k})\} \\ &\quad= -i\{d^0x[(\vec{\partial}^0 e^{i\frac{x}{\lambda}\chi_p})\, e^{i\frac{x}{\lambda}\chi_k} + e^{i\frac{x}{\lambda}\chi_p}(\vec{\partial}^0 e^{i\frac{x}{\lambda}\chi_k}) \\ &\qquad + i(\vec{\partial}^0 e^{i\frac{x}{\lambda}\chi_p})(\vec{\partial}^0 e^{i\frac{x}{\lambda}\chi_k}) + i(\vec{\partial} e^{i\frac{x}{\lambda}\chi_p})(\vec{\partial} e^{i\frac{x}{\lambda}\chi_k})] \\ &\qquad + dx[(\vec{\partial} e^{i\frac{x}{\lambda}\chi_p})\, e^{i\frac{x}{\lambda}\chi_k} + e^{i\frac{x}{\lambda}\chi_p}(\vec{\partial} e^{i\frac{x}{\lambda}\chi_k}) + i(\vec{\partial} e^{i\frac{x}{\lambda}\chi_p})(\vec{\partial}^0 e^{i\frac{x}{\lambda}\chi_k}) \\ &\qquad + i(\vec{\partial}^0 e^{i\frac{x}{\lambda}\chi_p})(\vec{\partial} e^{i\frac{x}{\lambda}\chi_k})]\} \end{aligned} \tag{49}$$

It is easily seen that the last expression coincides with (48). Q.E.D.

The connection of the differential calculus and the addition theorem, i.e., the group of motions of the corresponding momentum space becomes essential in higher dimensions and leads to a non-local version of the gauge theory.

[2] We use Yu. A. Golfand's notation $(p(+)\,k)^\mu$ for the Lorentz transformation with the "two-velocity" $\frac{k^\mu}{m}$.

3. ONE-DIMENSIONAL RQM

We start this section with stressing again that x is a relativistic invariant, (see Ref. 3). The wave function $\psi(x)$ in the relativistic configurational space is connected with the corresponding quantity in the momentum space by the Gelfand–Graev transformation (see (58) below). In what follows, we use the unit system in which $m=\hbar=c=1$. The relativistic plane waves obey the completeness and orthogonality conditions

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \langle x \mid p \rangle \, d\Omega_p \, \langle p \mid x' \rangle = \delta(x-x')$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \langle p \mid x \rangle \, dx \, \langle x | p' \rangle = \delta(\chi-\chi') \tag{50}$$

The free Hamiltonian and momentum operators are finite-difference operators

$$\hat{H}_0 = \cosh i\lambda \frac{d}{dx}, \qquad p = -\sinh i\lambda \frac{d}{dx} \tag{51}$$

The plane wave (41) obeys the free relativistic finite-difference Schrödinger equation

$$(\hat{H}_0 - p_0)\langle x \mid p \rangle = 0 \tag{52}$$

Using $\cosh \chi = 1 + 2\sinh^2 \frac{\chi}{2}$, we can write the Hamiltonian in Eq. (52), in a “non-relativistic form” (in dimensional units)

$$\hat{h}_0 = 2mc^2 \sinh^2 \frac{i\lambda}{2}\frac{d}{dx} = \frac{\hat{k}^2}{2m} = \hat{H}_0 - mc^2 \rightarrow -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \tag{53}$$

where the corresponding momentum operator

$$\hat{k} = -2mc \sinh i \frac{\lambda}{2}\frac{d}{dx}, \qquad k = 2mc \sinh \frac{\chi}{2} \tag{54}$$

has been introduced. The relativistic Schrödinger equation takes the form

$$(\hat{h} - e)\, \psi(x) = 0 \tag{55}$$

where

$$\hat{h} = \hat{h}_0 + V(x) = \frac{\hat{k}^2}{2} + V(x), \qquad e = \frac{k^2}{2} \tag{56}$$

In a number of cases the relativistic Schrödinger equation (56) is integrable. The remarkable result is that the relativistic generalization of the harmonic oscillator case[3] is a q-oscillator[16–19] with the parameter of deformation

$$q = e^{\frac{\omega\hbar}{mc^2}} \tag{57}$$

where ω is the oscillator frequency.

4. THE RELATION BETWEEN THE RELATIVISTIC AND NON-RELATIVISTIC CONFIGURATIONAL REPRESENTATION

Let us denote the relativistic coordinate by x and the non-relativistic coordinate by r. Correspondingly, $\psi_{rel}(x)$ is the relativistic wave function and $\psi_{nrel}(r)$ is the non-relativistic one. In what follows, we use the unit system $\hbar = c = m$, i.e., $\lambda = 1$. Our goal is to find the transformation connecting $\psi_{rel}(x)$ with $\psi_{nrel}(r)$. We have two different Fourier transformations

1. The Gelfand–Graev transformation with kernel (41)

$$\psi_{rel}(\chi) = \int_{-\infty}^{\infty} e^{-i\chi x}\psi_{rel}(x)\,dx, \qquad \psi_{rel}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\chi x}\psi_{rel}(\chi)\,d\chi \tag{58}$$

2. The usual Fourier transformation

$$\psi_{nrel}(p) = \int_{-\infty}^{\infty} e^{-ipr}\psi_{nrel}(r)\,dr, \qquad \psi_{nrel}(r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipr}\psi_{nrel}(p)\,dp \tag{59}$$

Let us suppose that (cf. Ref. 1). $\psi(\chi) = \psi(p)$. This gives

$$\int_{-\infty}^{\infty} e^{-ipr}\psi_{rel}(r)\,dr = \int_{-\infty}^{\infty} e^{-i\chi x}\psi_{nrel}(x)\,dx \tag{60}$$

and we obtain the transformation connecting the non-relativistic (r-) and the relativistic (x-) configurational representations

$$\psi_{rel}(x) = \int_{-\infty}^{\infty} B(x, r)\,\psi_{nrel}(r)\,dr \tag{61}$$

where

$$B(x, r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\chi - ipr}\,d\chi, \qquad B(x, r)^* = B(x, r) \tag{62}$$

The inverse transformation $x \to r$ is

$$\psi_{nrel}(r) = \int_{-\infty}^{\infty} A(r, x)\, \psi_{rel}(x)\, dx \tag{63}$$

where

$$A(r, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipr - i\chi x}\, dp, \qquad A(x, r)^* = A(x, r) \tag{64}$$

To calculate $B(x, r)$ in (62), we take into account (42) and consider integrals along the closed contours in a complex χ-plane. These contours for (62) must lie in the upper or lower half-plane depending on the sign of r. For $r > 0$ we close the contour in the lower half-plane by adding to (62) the integral along the straight line parallel to the real axis: $\mathrm{Im}\, \chi = -\frac{\pi}{2}$, $\mathrm{Re}\, \chi \in (+\infty, -\infty)$. For $r < 0$ we close the contour in the upper half-plane: $\mathrm{Im}\, \chi = +\frac{\pi}{2}$, $\mathrm{Re}\, \chi \in (+\infty, -\infty)$. This gives

$$B(x, r) = \frac{1}{\pi} \{\theta(r)\, e^{\frac{\pi x}{2}} K_{ix}(r) + \theta(-r)\, e^{-\frac{\pi x}{2}} K_{ix}(-r) \tag{65}$$

where $K_{ix}(r)$ is the MacDonald function. The expression for $A(r, x)$ is obtained in a similar way

$$A(r, x) = \frac{1}{\pi} \frac{x}{r} \{\theta(r)\, e^{\frac{\pi x}{2}} K_{ix}(r) + \theta(-r)\, e^{-\frac{\pi x}{2}} K_{ix}(-r)\} \tag{66}$$

Thanks to (65) and (66), the relations (61) and (63) can be considered as immediate generalization of the Kontorovich–Lebedev transformation. Using the completeness and orthogonality conditions for Kontorovich–Lebedev transformation,[8]

$$\frac{2}{\pi^2 x} \int_0^{\infty} K_{i\tau}(r) \cdot K_{i\tau}(r')\, \tau \sinh \tau\, d\tau = \delta(r - r') \tag{67}$$

$$\frac{2}{\pi^2 x}\, \tau \sinh \tau \int_0^{\infty} K_{i\tau}(r) \cdot K_{i\mu}(r)\, \frac{dr}{r} = \delta(\tau - \mu) \tag{68}$$

we obtain two orthogonality conditions for the transformation kernels (62) and (64) connecting the relativistic and non-relativistic QM

$$\int_{-\infty}^{\infty} B(x, r) \cdot A(r, x')\, dr$$

$$= \frac{x}{\pi^2} \int_{-\infty}^{\infty} \frac{dr}{r} \{\theta(r)\, e^{\frac{\pi(x+x')}{2}} K_{ix}(r)\, K_{ix'}(r) + \theta(-r)\, e^{-\frac{\pi(x+x')}{2}} K_{ix}(-r)\, K_{ix'}(-r)\}$$

$$= \frac{x}{\pi^2}\, e^{\frac{\pi(x+x')}{2}} \int_{0}^{\infty} \frac{dr}{r} K_{ix}(r)\, K_{ix'}(r) + \frac{x}{\pi^2}\, e^{-\frac{\pi(x+x')}{2}} \int_{-\infty}^{0} \frac{dr}{r} K_{ix}(r)\, K_{ix'}(r)$$

$$= xe^{\frac{\pi(x+x')}{2}} \frac{\delta(x-x')}{2x \sinh \pi x} - xe^{-\frac{\pi(x+x')}{2}} \frac{\delta(x-x')}{2x \sinh \pi x} = \delta(x-x') \tag{69}$$

The second formula can be obtained in a similar way

$$\int_{-\infty}^{\infty} A(r, x) \cdot B(x, r')\, dx = \delta(r-r') \tag{70}$$

In the non-relativistic limit we have $x \to r$ (i.e., x and r are of the same nature) and all relations are simplified

$$A(r, x) \to \delta(r-x) \qquad B(x, r) \to \delta(x-r)$$

$$\begin{aligned} &\int_{-\infty}^{\infty} A(r, x) \cdot B(x, r')\, dx \to \int_{-\infty}^{\infty} \delta(r-x)\, \delta(x-r')\, dx = \delta(r-r'); \\ &\int_{-\infty}^{\infty} B(x, r) \cdot A(r, x')\, dr \to \int_{-\infty}^{\infty} \delta(x-r) \cdot \delta(r-x')\, dr = \delta(x-x') \end{aligned} \tag{71}$$

The MacDonald functions $K_{ix}(r)$ obey the simple finite-difference relations

$$\sinh i \frac{\partial}{\partial x} K_{ix}(r) = -\frac{ix}{r} K_{ix}(r), \qquad \cosh i \frac{\partial}{\partial x} K_{ix}(r) = -\frac{\partial}{\partial r} K_{ix}(r) \tag{72}$$

which result in the finite-difference equations for $A(r, x)$, and $B(x, r)$

$$-i \frac{\partial}{\partial r} B(x, r) = \sinh i \frac{\partial}{\partial x} B(x, r), \quad -i \frac{\partial}{\partial r} A(r, x) = \sinh i \frac{\partial}{\partial x} A(r, x) \tag{73}$$

We also have

$$A(r, x) = \cosh i \frac{\partial}{\partial x} B(x, r) \tag{74}$$

5. SOLVING THE RELATIVISTIC SCHRÖDINGER EQUATION. THE NON-LOCAL INTEGRABLE CASES

In this section, we shall consider how the non-relativistic Schrödinger equation with a potential in the r-representation is related to the finite-difference Schrödinger equation (56) in the x-representation. We start with the Coulomb potential

$$V(r) = -g\,\frac{1}{|r|} \tag{75}$$

Applying (59) we transform to the non-relativistic momentum representation

$$V(p) = \int_{-\infty}^{\infty} e^{ipr} V(r)\, dr = g\,\ln(p^2+\mu^2) \tag{76}$$

In the one-dimensional case, in comparison with the higher dimensions the Coulomb potential is more singular. First, the singularity $\frac{1}{x}$ is not compensated by the radial part of the volume element. Second, the potential in momentum space does not decrease at infinity. For this reason, it is convenient to substitute the expressions (75) and (76) by

$$V_{reg}(r) = -g\,\frac{e^{-\mu|r|}}{|r|} = -\frac{g}{r}\,\{\vartheta(r)\,e^{-\mu r} - \vartheta(-r)\,e^{\mu r}\} \tag{77}$$

where $\vartheta(r)$ is the regularized step function subjected to the condition $\vartheta(+0) = \vartheta(-0) = \frac{1}{2}$ and

$$V_{reg}(p) = e^{-\zeta|p|}\{A + g\,\ln(p^2+\mu^2)\} \tag{78}$$

To obtain the expression for the Coulomb potential in RCS, we transform (78) with the help of the second formula in (58)

$$V(x) = \int_{-\infty}^{\infty} e^{i\chi x} V_{reg}(p)\, d\chi \tag{79}$$

After performing the partial integration we can remove the ζ-regularization and obtain

$$V(x) = \frac{ig}{x}\int_{-\infty}^{\infty} e^{i\chi x}\,\frac{2p}{p^2+\mu^2}\,\frac{dp}{d\chi}\,d\chi \tag{80}$$

Taking into account (40) we have

$$V(x)=\frac{ig}{x}\int_{-\infty}^{\infty} e^{i\chi x}\left[\frac{1}{\sinh\chi-i\mu}+\frac{1}{\sinh\chi-i\mu}\right]\cosh\chi\, d\chi \tag{81}$$

This integral can be evaluated on the basis of Jordan's lemma. We close the contours of integration in the upper or lower complex half-plane of rapidity χ depending on the sign of x. As a consequence of the periodicity of the hyperbolic functions along the imaginary axis in the χ-plane, there are infinite series of poles of the type

$$\chi_n=2\pi in \tag{82}$$

The residues at these poles must be summarized. The lower limit of summation ($n=0$ or $n=1$) is dictated by imaginary additions ($\pm\mu$) in the denominator of (81). The final expression for the Coulomb potential in the relativistic configurational x-representation is

$$V(x)=-g\,\frac{2\pi}{x}\coth\pi x \tag{83}$$

Let us analyze the last expression. $\coth\pi x$ is an even function of x, so the total expression (83) depends on $|x|$, cf. (75). At large x-es the expression $\coth\pi x$ is equal to 1 and (83) reduces to the usual Coulomb potential. At small x its behavior is different from the non-relativistic case, it has a singularity $\sim\frac{1}{x^2}$. The most important property of (83) is connected with the periodicity property of $\coth\pi x$:

$$\coth\pi(x\pm i)=\coth\pi x \tag{84}$$

which means that the factor $g'(x)=2\pi g\coth\pi x$ is a constant with respect to finite-difference differentiations in Eq. (56), and for this reason the Schrödinger equation in RCS is an integrable problem. Let us consider now the general case. The non-relativistic Schrödinger equation

$$-\frac{1}{2}\frac{d^2}{dr^2}\psi(r)+V(r)\,\psi_{nrel}(r)=E\psi_{nrel}(r) \tag{85}$$

can be transformed into the relativistic finite-difference Schrödinger equation using the modified Kontorovich–Lebedev transformation described in the previous section. We have

$$\int_{-\infty}^{\infty} B(x,r)\left(-\frac{1}{2}\frac{d^2}{dr^2}+V(r)-E\right)\psi_{nrel}(r)\,dr=0 \tag{86}$$

The first term goes over to the corresponding relativistic kinetic energy term:

$$\int_{-\infty}^{\infty} B(x, r)\left(-\frac{1}{2}\frac{d^2}{dr^2}\psi_{nrel}(r)\right)dr = -\frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{d^2}{dr^2}B(x, r)\right)\psi_{nrel}(r)\,dr$$

$$= \frac{1}{2}\left(\sinh i\frac{\partial}{\partial x}\right)^2\int_{-\infty}^{\infty} B(x, r)\,\psi_{nrel}(r)\,dr$$

$$= \frac{1}{2}\left(\sinh i\frac{\partial}{\partial x}\right)^2\psi_{rel}(x) \tag{87}$$

where after performing a double partial integration over r we have used (61) and (73) to obtain the finite-difference energy operator acting on the relativistic wave function. The potential term with the local non-relativistic potential leads in general to non-local relativistic potentials. We have

$$\int_{-\infty}^{\infty} B(x, r)\,V(r)\,\psi_{nrel}(r)\,dr = \int_{-\infty}^{\infty} K(x, x_1, x_2)\,V(x_1)\,\psi_{rel}(x_2)\,dx_1\,dx_2 \tag{88}$$

where

$$K(x, x_1, x_2) = \int_{-\infty}^{\infty} B(x, r)\,A(r, x_1)\,A(r, x_2)\,dr$$

$$= \cosh i\frac{d}{dx_1}\cosh i\frac{d}{dx_2}\int_{-\infty}^{\infty} B(x, r)\,B(r, x_1)\,B(r, x_2)\,dr$$

$$= \frac{2}{(2\pi)^3}\cosh i\frac{d}{dx_1}\cosh i\frac{d}{dx_2}\cosh\frac{\pi}{2}(x+x_1+x_2)\,J(x, x_1, x_2) \tag{89}$$

The kernel $J(x, x_1, x_2)$ can be evaluated in a closed form and is expressed in terms of the 7 – parameter hypergeometric function

$$J = \delta(x, x_1, x_2)\times$$

$$\times\,{}_4F_3\left(\begin{matrix}\frac{1+ix_1+ix_2-ix}{2}, \frac{1-ix+ix_1-ix_2}{2}, \frac{1-ix-ix_1+ix_2}{2}, \frac{1-ix-ix_1-ix_2}{2}; \\ 1-ix, \frac{1-ix}{2}, \frac{2-ix}{2}\end{matrix}\;;\frac{1}{4}\right)+h.c. \tag{90}$$

where

$$\delta(x, x_1, x_2) = \frac{\Gamma(ix)\, \Gamma(\frac{1+ix_1+ix_2-ix}{2})\, \Gamma(\frac{1-ix+ix_1-ix_2}{2})\, \Gamma(\frac{1-ix-ix_1+ix_2}{2})\, \Gamma(\frac{1-ix-ix_1-ix_2}{2})}{\Gamma(1-ix)} \tag{91}$$

As an illustration, let us consider the box potential the for usual Schrödinger equation

$$V(r) = \begin{cases} 0, & -L \leqslant r \leqslant L \\ \infty, & \text{for } |r| > L \end{cases} \tag{92}$$

The well-known solution of this problem is

- the wave function

$$\psi(r) = \begin{cases} \cos \dfrac{\pi n}{2L} r, & n \text{ odd} \\ \sin \dfrac{\pi n}{2L} r, & n \text{ even} \end{cases} \tag{93}$$

- the energy spectrum (in the dimensional units)

$$E_n = \frac{\pi^2 \hbar^2 n^2}{8mL^2} \tag{94}$$

The transformation (63), (61) leads us to complicated non-local expressions for the potential and wave function in the relativistic x-representation. We emphasize that this problem is an example of the integrable case of finite-difference Schrödinger equation with the non-local potential. In fact, we can get a series of integrable non-local cases simply considering the exactly soluble problems for the one-dimensional non-relativistic Schrödinger equation. At the same time, the local potential in the relativistic x-space exists, which is an analogue of (92), and corresponds to the local integrable relativistic finite-difference Schrödinger equation. We write

$$V(x) = \begin{cases} 0, & -L \leqslant x \leqslant L \\ \infty, & \text{for } |x| > L \end{cases} \tag{95}$$

The solution is

- the wave function

$$\psi(x) = \begin{cases} \cos\dfrac{\pi n}{2L}x, & n \text{ odd} \\ \sin\dfrac{\pi n}{2L}x, & n \text{ even} \end{cases} \tag{96}$$

- the rapidity spectrum (dimensions have been restored)

$$\chi_n = \frac{\pi n \hbar}{2L} \tag{97}$$

- the spectrum of the "kinetic energy"

$$e_n = 2mc^2 \sinh^2 \frac{\pi n \hbar}{4mcL} \tag{98}$$

- the spectrum of the relativistic energy

$$E_n = mc^2 \cosh \frac{\pi n \hbar}{2mcL} \tag{99}$$

In the non-relativistic limit, it follows from (99) that

$$E_n \longrightarrow mc^2 + \frac{1}{8m}\left(\frac{\pi n \hbar}{L}\right)^2 + \frac{1}{384m^3c^2}\left(\frac{\pi n \hbar}{L}\right)^4 + \cdots \tag{100}$$

cf. (94).

The second term in (100) gives the non-relativistic energy spectrum (94) and the next term is the first relativistic correction for the energy spectrum which agrees with the expansion of (10). This example also shows that (at least in some limited, low energy region) the relativistic phenomenology, i.e., introducing the interaction as a potential, can be considered at the level of the finite-difference relativistic Schrödinger equation similarly to the non-relativistic QM case.

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