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$1/N_c$ expansion and anomaly cancellation in the presence of electroweak interactions

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Abstract. We study the question of a consistent formulation of the $1/N_c$ expansion in the presence of electroweak interactions. We show that in some cases the previous formulation leads to an unrealistic picture. We improve the scheme. We derive the corresponding hypercharge and electric charge values of fermions under the requirement that the standard model in the large-N_c limit should be free of chiral gauge anomalies. We find that the resulting hypercharge and the electric charge values for quarks are the same as for the standard model.

The large-N_c approximation, having been a qualitative argument in the past [1, 2], is becoming a more quantitative tool in the study of strong interactions [3–10]. In strong intercation physics, although the effect of electroweak intercations is much smaller than the effect of the terms neglected in the large-N_c limit, in some cases these interactions give important physical contributions, such as electromagnetic self-energies [11]. Moreover, when one deals with hadronic or semi-leptonic electroweak processes the contributions from strong interactions cannot be safely neglected, such as in Z decays and rare B decays. The electroweak couplings of quarks depend on N_c through colour loops. This causes inconsistencies in the N_c counting rules [12]. Of course, one can suggest making use of the $1/N_c$ expansion only in processes where one does not face inconsistencies. However, a consistent inclusion of electroweak intercations will make the theory more elegant so that it would be successful in the whole low-energy physics domain. If one could always factorize strong and electroweak contributions in electroweak processes then one could use the $1/N_c$ expansion for the strong interaction part and the usual perturbative calculations for the electroweak part. However, we know that there can be significant non-factorizable QCD and electroweak corrections to some electroweak processes [13]. Although these contributions are calculated in perturbative QCD one can find similar non-factorizable processes in the non-perturbative region where the use of the $1/N_c$ expansion is desirable as we shall see in the following paragraphs. Therefore, the formulation of the $1/N_c$ expansion in the presence of electroweak interactions is desirable in order to improve the present scheme. In this study first we review the reasoning for the rescaling of electroweak coupling constants. We point out some shortcomings of the previous scheme for the $1/N_c$ expansion in the presence of electroweak interactions [12]. Then we improve the scheme. We find that the solutions of the anomaly cancellation equations are the same as those of the standard model for arbitrary N_c.

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We give an example where there is a non-factorizable strong interaction–electromagnetic interaction and lepton–hadron mixing, and where the best scheme for calculation is our improved $1/N_c$ expansion scheme.

Because of colour loops some Feynman graphs have more contributions than others, for example the diagrams given in figure 1 which have a combinatoric factor of N_c for N_c colours. This enables us to use an approximation scheme known as the large-N_c approximation. This scheme can most naturally be realized in a formulation where one rescales the strong coupling constant so that

$$g_3 \rightarrow g_{3c} = \frac{\sqrt{3}g_3}{\sqrt{N_c}} \qquad N_c g_{3c}^2 \rightarrow 3g_3^2 \qquad \text{as } N_c \rightarrow \infty \tag{1}$$

where N_c is the number of colours. This formulation is also, therefore, known as the $1/N_c$ expansion. In this way one neglects the graphs with less colour loops while one is effectively studying $N_c = 3$. The resulting picture suggests that large-N_c is a good approximation of the real world.

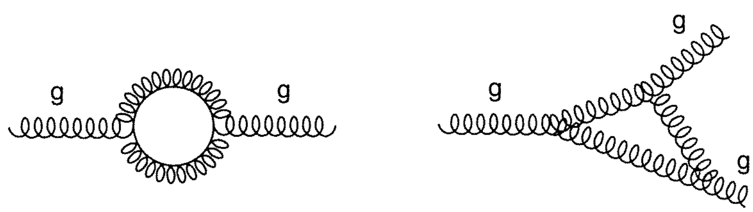

Figure 1. Feynman diagrams with colour combinatoric factors.

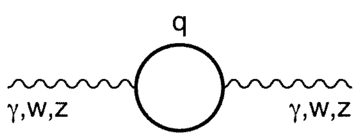

Figure 2. The diagram with the colour factor for the electroweak gauge boson self-energy.

Electroweak coupling constants are also affected by the large-N_c limit, for example through the diagram in figure 2. If we do not rescale the coupling constants, we obtain a graph of order N_c, while it should be of the same order as the bare photon propagator (i.e. of order zero). This causes an inconsistency in N_c counting rules if we do not rescale the electroweak coupling constants [12]: for example, the electromagnetic contribution to the ρ-meson self-energy tends to be of order N_c through ρ–γ mixing, while large-N_c counting rules give it to be of order N_c^0. One can find many more such examples. The triangular colour loop causes $\pi^0 \rightarrow 2\gamma$ amplitude to grow like $N_c^{1/2}$, but it should grow like $N_c^{-1/2}$. A similiar argument is true for the Δ^{++} self-energy. Another example is the well known ratio

$$R = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}$$

which is one piece of experimental evidence for $SU(3)_c$. This ratio will be proportional to N_c, which means that we are neglecting the leptonic electroweak interactions compared to hadronic electroweak interactions in the large-N_c limit, which has no validation. Moreover, the N_c factor may come in more complicated forms, such as in figure 3, where it is difficult if not impossible to tangle the strong interaction and electroweak interaction pieces of the calculation. Such graphs may give significant contributions to physical quantities [13]. As we shall see later, such effects may also arise in the non-perturbative region where the best scheme for calculation is the $1/N_c$ expansion. A similiar situation appears when we consider the $1/N_c$ baryon–baryon scattering graphs of [14] (figure 4): we notice that even the addition of one electroweak gauge boson propagator makes the scattering amplitude of order N_c^2 while it should be of order N_c according to N_c counting rules (figure 5). This is due to the fact that baryons in this scheme contain N_c quarks so the electroweak vertex gets a factor of N_c as a combinatoric. When we consider the higher orders of such graphs, one realizes that it is not easy to use the $1/N_c$ expansion effectively to choose the graphs of higher order in N_c among the ones with the same number of electromagnetic vertices. This example, and in particular, the example given in the paragraph before the last paragraph, show that the question of rescaling of the electroweak coupling constants is more than a matter of consistency of the formalism.

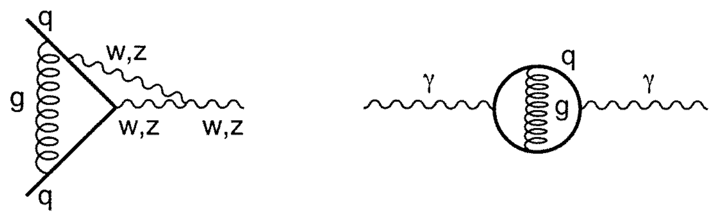

Figure 3. Diagrams with a mixing of strong and electroweak sectors.

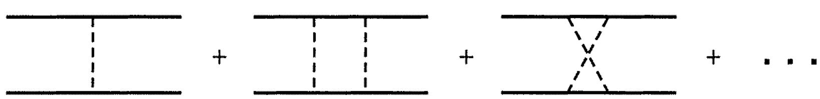

Figure 4. Baryon–baryon interaction diagrams in the $1/N_c$ expansion (full and dashed lines denote baryons and mesons, respectively).

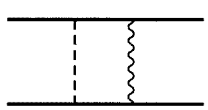

Figure 5. The leading baryon–baryon interaction diagram with addition of a photon propogator.

Quark loops in γ, W and Z self-energy diagrams and quark lines in baryons give a factor of N_c hence causing inconsistencies in N_c counting rules and causing some physical quantities to diverge. Therefore, we must impose similiar conditions on electroweak coupling constants as we do for the strong coupling constant. The diagram in figure 2 suggests that we must impose $N_c g_1^2 =$ constant, $N_c g_2^2 =$ constant for the electroweak interactions of fermions, that is,

$$g_1 \to g_{1c} = \frac{\sqrt{3} g_1}{\sqrt{N_c}} \qquad g_2 \to g_{2c} = \frac{\sqrt{3} g_2}{\sqrt{N_c}} \tag{2}$$

for a general N_c, where g_1 and g_2 are the coupling constants of $U(1)_Y$, $SU(2)_L$, respectively. This eliminates the inconsistency in N_c counting rules due to figure 2. After the rescaling given in equation (1), electroweak contributions to all physical quantities agree with N_c counting rules. For example, the baryon–electroweak gauge boson vertex becomes of the order of $\sqrt{N_c}$, which is the same N_c dependence as for the baryon–vector meson vertex as expected. Baryon–baryon scattering amplitudes through electroweak interactions become of the order N_c as they should be. One can easily check that the other examples also agree with N_c counting rules after rescaling.

The most immediate extension of the standard model corresponding to the above rescaling is to replace $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ by $SU(N_c)_c \otimes SU(2)_L \otimes U(1)_Y$ [12]. However, this extension has some shortcomings: in electroweak processes there is usually a sum over colour indices while there is no such sum for leptonic processes. For example, this scheme does not improve the situation with respect to

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \propto N_c.$$

In other words, leptonic processes will be negligible with respect to hadronic ones in the large-N_c limit which is not realistic. (In fact, one can make $R \propto O(1)$ by assuming that the coupling constants rescale only in the interaction of gauge bosons with quarks, not in their interactions with leptons, but this causes an even worse problem; the solution of the anomaly cancellation equations results in unrealistic electric charges even for baryons and mesons in the large-N_c limit.) Another even more important shortcoming of the previous formulation is that it cannot obtain the correct answer for the hadronic part of the process where quarks participate in the interaction separately (not as hadrons). The above example and $\sigma(\pi^0 \to 2\gamma)$ are such processes. This is due to the fact that the quark charges in this scheme are different from their usual values as $N_c \to \infty$, while

$$\sigma(e^+e^- \to \text{hadrons}) \propto (Q_u^2 + Q_d^2) \qquad \text{and} \qquad \sigma(\pi^0 \to 2\gamma) \propto (Q_u^2 - Q_d^2)^2 \tag{3}$$

where

$$Q_u = \frac{1}{2} + \frac{1}{2N_c} \qquad Q_d = -\frac{1}{2} + \frac{1}{2N_c} \tag{4}$$

in the scheme of [12].

Therefore, the formulation must be improved so that the above problems do not arise. In order to cure the first problem the leptonic interactions should also get a factor similiar to colour and proportional to N_c, leaving the group structure of the standard model at $N_c = 3$ untouched. We propose to extend the standard model as

$$SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \to SU(N_c)_c \otimes SU\left(\frac{N_c}{3}\right)_l \otimes SU(2)_L \otimes U(1)_Y \tag{5}$$

where $SU(N_c/3)_l$ is a global group acting only on leptons and leptons belong to the fundamental representation of this group. This condition guarantees that the lepton interactions are not suppressed with respect to quark interactions in processes where there is a sum over colours. We shall see later that the scheme given in equation (5) also solves the second problem.

Now we shall investigate the solutions of the anomaly cancellation equations for this scheme: the fermion kinetic Lagrangian for the electroweak sector reads

$$\begin{aligned}\mathcal{L}_Y = {} & \bar{Q}_{iL}[\partial\!\!\!/ + ig_{2c}(\tfrac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{B\!\!\!/}) + ig_{1c}Y_Q \phi\!\!\!/]\,Q_{iL} + \bar{l}_{iL}[\partial\!\!\!/ + ig_2(\tfrac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{B\!\!\!/}) + ig_1 Y_l \phi\!\!\!/]\,l_{iL} \\ & + \bar{u}_{iR}[\partial\!\!\!/ + ig_{1c}Y_u\phi\!\!\!/]u_{iR} + \bar{d}_{iR}[\partial\!\!\!/ + ig_{1c}Y_d\phi\!\!\!/]d_{iR} + \bar{e}_{iR}[\partial\!\!\!/ + ig_1 Y_e\phi\!\!\!/]e_{iR}\end{aligned} \tag{6}$$

where $Q_{i\mathrm{L}}$, $l_{i\mathrm{L}}$, $u_{i\mathrm{R}}$, $d_{i\mathrm{R}}$ and $e_{i\mathrm{R}}$ are the left-handed quark, the left-handed lepton, the right-handed up quark, the right-handed down quark and the right-handed charged lepton fields, respectively. i runs over fermion generations and $SU(N_\mathrm{c})_\mathrm{c}$, $SU(N_\mathrm{c}/3)_l$ colours.

If the $1/N_\mathrm{c}$ expansion is reliable it must preserve the renormalizability of the underlying theory, $SU(3)_\mathrm{c}\otimes SU(2)_L\otimes U(1)_Y$. This means that $SU(N_\mathrm{c})\otimes SU(N_\mathrm{c}/3)\otimes SU(2)_L\otimes U(1)_Y$ must be renormalizable and hence free from chiral gauge anomalies [15, 16]. The anomaly cancellation equations [17] are the same as the standard model, that is,

$$\begin{aligned}
&2\,Y_Q - Y_u - Y_d = 0 : \mathrm{Tr}[U_Y(1)\otimes SU(N_\mathrm{c})^2]\\
&3Y_Q + Y_L = 0 : \mathrm{Tr}[U_Y(1)\otimes SU(2)_L^2]\\
&3(2Y_Q^3 - Y_u^3 - Y_d^3) + 2Y_L^3 - Y_e^3 = 0 : \mathrm{Tr}[U_Y(1)^3]\\
&3(2Y_Q - Y_u - Y_d) + 2Y_L - Y_e = 0 : \mathrm{Tr}[U_Y(1)(\mathrm{graviton})^2].
\end{aligned} \tag{7}$$

The solution is

$$(Y_Q, Y_u, Y_d, Y_L, Y_e) = (-\tfrac{1}{3}, -(1+\tfrac{1}{3}), -(1-\tfrac{1}{3}), 1, 2)\,Y_L. \tag{8}$$

The electric charges for quarks and mesons are the same as for the standard model except the electromagnetic coupling constant is rescaled as $e\to e_\mathrm{c} = 3e/\sqrt{N_\mathrm{c}}$. In order to determine the electric charge of the baryons one must determine how to generalize the baryons for a general N_c. One can take any consistent formula to generalize the baryons to arbitrary N_c [18]. One of the choices for generalizing baryons to arbitrary N_c is to impose

$$r - (N_\mathrm{c} - r) = x - y \tag{9}$$

where r, x and y are the number of up quarks for the baryon of N_c quarks, the number of up quarks for the usual baryon ($N_\mathrm{c} = 3$) and the number of down quarks for the usual baryon, respectively. Another choice is to impose

$$\frac{r}{N_\mathrm{c}} = \frac{x}{3} \qquad x + y = 3. \tag{10}$$

The electric charge of a baryon, B, composed of r up quarks and $N_\mathrm{c} - r$ down quarks is

$$Q_\mathrm{B}(r, N_\mathrm{c} - r) = rQ_\mathrm{u} + (N_\mathrm{c} - r)Q_\mathrm{d} = re_\mathrm{c} + N_\mathrm{c}e_\mathrm{c}(-\tfrac{1}{2} + \tfrac{1}{3}). \tag{11}$$

We notice that the choice of equation (9) does not lead to correct baryon charges as $N_\mathrm{c}\to\infty$, but the choice given by equation (10) does. For example,

$$\begin{aligned}
Q_\mathrm{n} &= \frac{N_\mathrm{c}}{3}Q_\mathrm{u} + 2\frac{N_\mathrm{c}}{3}Q_\mathrm{d} = 0\\
Q_\mathrm{p} &= 2\frac{N_\mathrm{c}}{3}Q_\mathrm{u} + \frac{N_\mathrm{c}}{3}Q_\mathrm{d} = \sqrt{N_\mathrm{c}}e\\
Q_{\Delta^{++}} &= 3\sqrt{N_\mathrm{c}}Q_\mathrm{u} = \sqrt{N_\mathrm{c}}2e
\end{aligned} \tag{12}$$

for the choice corresponding to equation (10). Baryon electric charges are of order $\sqrt{N_\mathrm{c}}$ which is in agreement with the fact that the baryon self-energy is being of order N_c. All charges for arbitrary N_c apart from a possible rescaling factor are the same as their usual values. Therefore, the ambiguity in the electric charge dependence of the cross sections of $\mathrm{e}^+\mathrm{e}^-\to$ hadrons, $\pi^0\to 2\gamma$ for large N_c in the previous formulation (which can be seen by the use of equations (3) and (4)) is removed in this formulation.

At this point one may wonder if the scheme introduced here is a purely theoretical argument which is not essential for phenomenological purposes. We give a concrete example where this is not the case, in other words we give an example where the most appropriate method to be used is the $1/N_\mathrm{c}$ expansion within this scheme. First, we consider one of

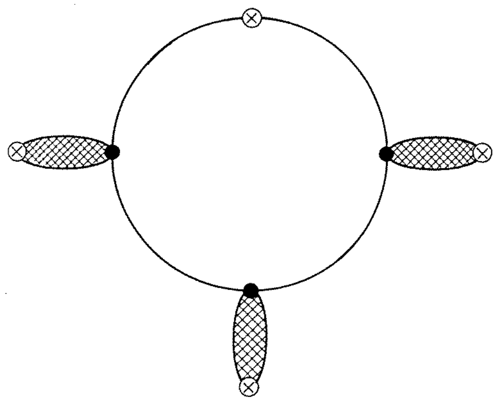

Figure 6. A diagram for the hadronic light-by-light contribution to the muon $(g-2)$ in the ENJL model at $\mathcal{O}$ leading order in the $1/N_{\mathrm{c}}$ expansion. Dots represent ENJL vertices, the circled crossed vertices are where photons connect, the cross-hatched loops are full two-point functions and the lines are constituent quark propagators.

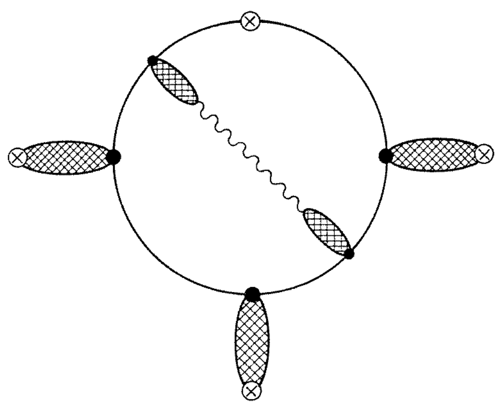

Figure 7. A diagram obtained from figure 6 by addition of a photon propagator.

the graphs (figure 6) for a light-by-light hadronic contribution to the anomalous magnetic moment $((g-2)/2)$ of the muon, where an extended Nambu–Jona–Lasinio (EJNL) model with $1/N_{\mathrm{c}}$ expansion is employed [19]. The contribution of this and similiar graphs to $(g-2)/2$ is of the order of -10×10^{-10} [19, 20] and it is larger than the precision of the forthcoming BNL-E821 experiment which is at least 4×10^{-10}. If we add a photon line as in figure 7 we obtain a non-factorizable strong interaction–electromagnetic interaction mixing graph. So one must use the $1/N_{\mathrm{c}}$ expansion in the presence of electroweak interactions. Although the previous scheme for the $1/N_{\mathrm{c}}$ expansion in the presence of electroweak intercations is not wholly consistent, it gives correct results in this case because the graph does not contain leptons. However, when we consider one of the graphs in figure 8 we notice that it does not only contain non-factorizable strong interaction and electromagnetic pieces but it also contains non-factorizable hadron–lepton contributions. In order to collect the graphs with the same order in N_{c} correctly one must count the order of N_{c}'s coming from lepton loops and photon vertices. Therefore, one must necessarily use the present scheme for evaluation of figure 8. A naive estimate shows that each such graph is suppressed by

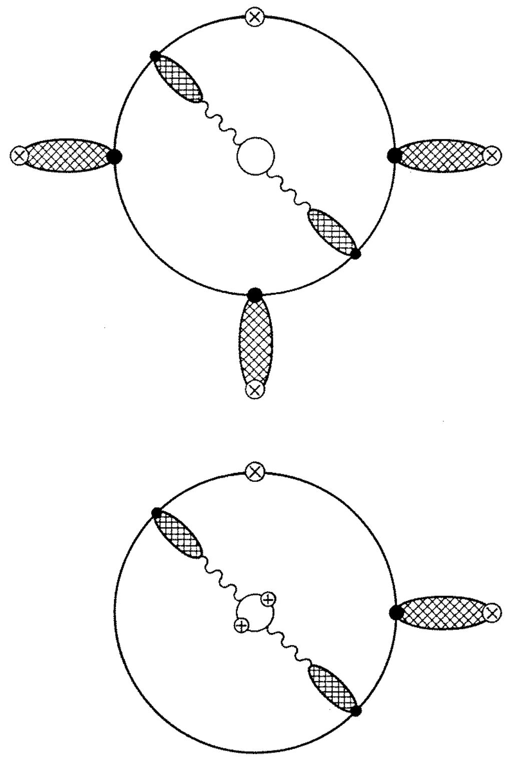

Figure 8. Some diagrams with non-factorizable strong interaction–electromagnetic interaction and hadron–lepton contributions in the ENJL model with the $1/N_\mathrm{c}$ expansion. The smaller circular loops are lepton or quark loops.

a factor of α^2 compared to figure 6 and it is multiplied by a factor of about 100 due to combinations of the internal lines. So it is one or two orders of magnitude smaller than the sensitivity of the BNL-E821 experiment and this precision can be reached in future experiments [21]. If we add the contribution from figure 7 and from the factorizable graphs the contribution becomes closer to the sensitivity of BNL-E821.

In this study we have investigated the question of determining a consistent and reliable extension of the standard model for large N_c. We have improved the analysis. Apart from being consistent with empirical data, this scheme has the virtue that it has the same electric charges and hypercharges as the standard model for an arbitrary number of colours. The only non-trivial thing in this scheme is the appearance of a $SU(N_\mathrm{c}/3)_l$ group when the number of colours is more than three. If one considers the $1/N_\mathrm{c}$ expansion just as an approximation scheme with no further deeper implications then one can take the occurrence of $SU(N_\mathrm{c}/3)_l$ just as a formal calculational tool with no physical implications. However, if one sees the $1/N_\mathrm{c}$ expansion as a sound theoretical framework for the manifestation of the real world then one should take the emergence of $SU(N_\mathrm{c}/3)_l$ as an indication of new physics beyond the standard model, for example, quark–lepton symmetric models [22] where $SU(N_\mathrm{c})_l \otimes U(1)_{Y'}$ broken to $SU(N_\mathrm{c}/3)_l \otimes U(1)_Y$ is one such framework for such an interpretation. In any case, this study serves as a basis for a consistent and realistic formulation of electroweak interactions in the $1/N_\mathrm{c}$ expansion.

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