SOLUTION OF MAXWELL EQUATIONS ON DEFORMED SPHERICAL DOMAINS: APPLICATIONS TO THE SCATTERING PROBLEMS

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Barış ATEŞ

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İZMİR

We approve the thesis of **Barış ATEŞ**

Examining Committee Members:

Prof. Dr. Oğuz YILMAZ
Department of Mathematics, İzmir Institute of Technology

Prof. Dr. Durmuş Ali DEMİR
Department of Physics, İzmir Institute of Technology

Prof. Dr. Oktay PASHAEV
Department of Mathematics, İzmir Institute of Technology

Prof. Dr. Hamza POLAT
Department of Physics, Dokuz Eylül University

Prof. Dr. Taner Abdullah OĞUZER
Department of Electrical and Electronics Engineering , Dokuz Eylül University

27 July 2015

Prof. Dr. Oğuz YILMAZ
Supervisor, Department of Mathematics
İzmir Institute of Technology

Prof. Dr. Oğuz YILMAZ
Head of the Department of
Mathematics

Prof. Dr. Bilge KARAÇALI
Dean of the Graduate School of
Engineering and Sciences

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ABSTRACT

SOLUTION OF MAXWELL EQUATIONS ON DEFORMED SPHERICAL DOMAINS: APPLICATIONS TO THE SCATTERING PROBLEMS

In the present work, firstly we consider analytic solution of the Maxwell's Equations in the vacuum in the presence of conducting deformed spherical body. Deformation is made in the normal direction of sphere with a small perturbation parameter β and arbitrarily chosen smooth deformation function $f(\theta, \varphi)$. The azimuthal and polar angle dependence of the function is preserved till the end. Using the Debye Potentials the solution in the exterior domain of deformed conducting spherical body is given. In addition to this, scattering of electromagnetic plane waves from non-spherical dielectric and conducting objects are considered. In order to find scattered and transmitted fields, in contrast to common use of vector wave functions and their orthogonality properties, the scalar functions and orthogonalities of Associated Legendre Polynomials are used. All the surface integrals are evaluated analytically. The corrections to the coefficients of scattered and transmitted fields up to the second order are obtained and expressed in terms of the Clebsch-Gordon coefficients.

ÖZET

MAXWELL DENKLEMLERİNİN DEFORME EDİLMİŞ KÜRESEL BÖLGELERDE ÇÖZÜMÜ: SAÇILMA PROBLEMLERİNE UYGULAMALARI

Bu çalışmada, ilk olarak, boşlukta bulunan bir iletken deforme küre için Maxwell Denklemleri'nin analitik çözümleri göz önüne alınmıştır. Deformasyon, kürenin normali doğrultusunda küçük bir deformasyon parametresi β ve keyfi seçilen düzgün bir deformasyon fonksiyonu $f(\theta,\varphi)$ ile yapılmıştır. Deformasyon fonksiyonunun azimut ve kutupsal açıya bağlılığı bütün işlemler boyunca korunmuştur. Debye Potansiyellleri kullanılarak iletken deforme küre dışındaki çözümler verilmiştir. Buna ek olarak, elektromanyetik düzlemsel dalgaların deforme edilmiş iletken ve dielektrik nesnelerden saçılması incelenmiştir. Saçılan ve nüfuz eden alanları bulmak için, yaygın olarak kullanılan vektörel dalga fonksiyonları ve bunların diklik özellikleri yerine, skaler fonksiyonların ve Asosiye Legendre polinomlarının dikliği kullanılmıştır . Tüm yüzey integralleri analitik olarak hesaplanmıştır . Saçılan ve nüfuz eden alanların katsayılarındaki düzeltmeler ikinci mertebeye kadar yapılmıştır ve Clebsch-Gordon katsayıları cinsinden ifade edilmiştir.

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CHAPTER 1

INTRODUCTION

The main motivation of the present work comes from the Casimir Effect (Casimir, 1948). The Casimir energy is quite sensitive to the boundaries such that it could result in the change of sign of the energy (Bordag, 2001) . In the work of (Ahmedov, 2009), for the scalar fields, a deformed spherical cavity is considered and it is showed that the energy change is meaningful when second order correction are included in the calculations. In order to study electromagnetic Casimir energy, one has to find fields in the cavity or has to consider scattering problem and find scattering coefficients (Bordag, 2001). The subject of the present work is the essential part of the Electromagnetic Casimir energy for deformed spherical cavity, because once the classical fields obtained, the quantized electromagnetic fields could be obtained by the standart procedures (Rahi, 2009) and then corresponding Casimir energy can be obtained.

The nonspherical particles are important objects in the theory of Electromagnetism, Atmospheric physics, Acoustic Theory, Fluid Dynamics and for other branches of science (Mischenko, 2004). Because of the fact that these objects are more realistic. The nonspherical particles are obtained by small deformation of spheres. In perturbation theory the effect of surface perturbation can be analysed up to the desired order of perturbation parameter. In general, higher order corrections means more detailed analysis. In the present work perturbation theory is used up to the second order. The required order of perturbation is related with the amount of the deviation from the perfect geometries. When deviations are big, the first order perturbation does not enough (Barton, 1999), (Xie, 2010). The relation between quality factor and the shape of scatterer is analyzed in (Lai, 1991) and (White, 2012) and it is shown that, in general, perturbation of boundary reduces the quality factors. In addition to this, in the theory of acoustic, it is shown that the second order correction is the dominant one on the resonance frequency, when the spherical boundary is deformed (Mehl, 1982), (Mehl, 2007).

The theory of scattering of electromagnetic plane waves from homogeneous isotropic sphere has a special name called Lorenz-Mie-Debye theory, date back to the initiating works of Gustav Mie (Mie, 1908), Ludvig Lorenz (Lorenz, 1890) and Peter Debye (Debye, 1909). Historical and theoretical development of the Mie theory could be found in (Lock, 2009), (Horvath, 2009). The analytical solution of the Maxwell's Equations are restricted for limited perfect geometries like sphere, infinite circular cylinder and rectangular boundaries and few others (Morse,1953). Thus, exact solution of scattering of electromagnetic waves are restricted with the scatterers whose boundaries fit with the coordinate surfaces.

A general method, which is perfectly suitable for all given shape of cavities, is not known. In order to enlarge analysis to the arbitrary shapes, several elegant methods are constructed. Some of them are; Point maching Method (Oguchi, 1960), (Morrison, 1974), Method of Moments (Harrington, 1987), Generalized Multipole Method (Ludwig, 1991), Volume integral equation Method (Lakhatakia, 1993), Finite difference time domain Method (Yee, 1966), (Taflove, 1995), Finite element Method (Volakis, 1994), T matrix Method (Waterman, 1971), (Waterman, 1979), (Barber, 1975), Sh matrix Method (Petrov, 2006) and Perturbation Method (Yeh, 1964).

The perturbation Method used in the present work was first studied in (Yeh, 1964) for dielectric bodies and in (Erma, 1968), (Erma, 1968) for the conductors up to the first order corrections. The scattered and transmitted fields in the presence of inhomogeneous deformed bodies later were considered in (Raval, 1971) and (Raval, 1971). Scattering from irregular bodies by using Dirichlet and Neumann problems was successfully discussed in (Eyges, 1976). Mie scattering from conducting ensemble of rough surfaces later considered in (Schiffer, 1989). A privileged property belongs to the Perturbation theory is that it allows one to carry out calculations analytically. In the work of (Dubertrant, 2008) and (Wiersig, 2012) by using perturbation theory, effect of rotationally symmetric deformations on the characteristics of cavity are analytically analysed. Because of its effectiveness, the perturbation method is still in use to solve important physical and engineering problems (Barton, 1991), (Xie, 2010), (Panda,2012), (Panda,2013), (Jadhao,2015), (Mezei,2015), (Jadhao,2015). Perturbation theory not only allows us to extend solution for the geometries for which analytic solution does not exist but also it can be applied for the geometries for which Helmholtz equation has solution. An example such geometries could be spheroid. The Helmholtz equation can be separated in spheroidal coordinates and solution can be expressed in terms of spheroidal functions (Li,2001). Perturbation theory allows one to find an approximate solution for the spheroid in the simplicity of using spherical coordinates instead of using spheroidal coordinates and functions, (Li,2001), (Kotsis, 2007), (Zouros, 2014), (Zouros, 2015), (Mushiake,1956), (Asano,1975).

The position vector of a point on the deformed sphere is given by the following expression;

$$\vec{r}(\theta,\varphi) = R\Big(1 + 1\beta f(\theta,\varphi)\Big)\hat{r}, \qquad (1.1)$$

where r, φ, θ standard variables of spherical coordinates, $\hat{r} = \vec{r}/|\vec{r}|$ is the unit vector in the radial direction, R is the radius of the unperturbed sphere, β is the small perturbation parameter, $f(\theta,\varphi)$ is the arbitrarily predefined, periodic smooth function

$$\begin{aligned} f(\theta + k\pi, \varphi + n2\pi) &= f(\theta,\varphi) \quad k, n \ \epsilon \mathbb{Z} \\ \left|\beta f(\theta,\varphi)\right| &<< 1 \end{aligned} \qquad (1.2)$$

The normal vector to this surface up to second order in perturbation parameter is found to be;

$$\hat{n} = \left(1 - \beta^2\Big(\frac{f_\theta^2}{2} + \frac{f_\varphi^2}{2\sin^2\theta}\Big)\right)\hat{r} + \Big(-\beta f_\theta + \beta^2 f f_\theta\Big)\hat{\theta} + \left(-\beta\frac{f_\varphi}{\sin\theta} + \beta^2\frac{f f_\varphi}{\sin\theta}\right)\hat{\varphi} \qquad (1.3)$$

where the f_θ and f_φ denotes the θ and φ derivatives of the deformation function $f(\theta,\varphi)$ respectively.

Here it is assumed that the fields are well behaved, that is to say; small changes on the boundary leads to the small changes on the solutions. Since Maxwell's Equations are linear one can search a solution in the following form (Raval, 1971);

$$\begin{aligned} E(r,\theta,\varphi) &= E^0(r,\theta,\varphi) + \beta E^1(r,\theta,\varphi) + \beta^2 E^2(r,\theta,\varphi) + O(\beta^3) \\ B(r,\theta,\varphi) &= B^0(r,\theta,\varphi) + \beta B^1(r,\theta,\varphi) + \beta^2 B^2(r,\theta,\varphi) + O(\beta^3) \end{aligned} \qquad (1.4)$$

where the fields $E(r,\theta,\varphi)$, $B(r,\theta,\varphi)$ are total electric and magnetic fields and $E^0(r,\theta,\varphi)$, $B^0(r,\theta,\varphi)$ are responsible for the fields before deformation. The fields $E^1(r,\theta,\varphi)$, $B^1(r,\theta,\varphi)$ and $E^2(r,\theta,\varphi)$, $B^2(r,\theta,\varphi)$ are the first and second corrections respectively. In principle, for more accurate solution one can add higher and higher order corrections, throughout the thesis only the terms up to the second order in deformation parameter β will be kept.

Let the fields are harmonic in time, then with the help of Debye Potentials, these fields can be expressed in spherical coordinates as;

$$
\begin{aligned}
E_r^{(0,1,2)} &= \sum_{n,m} \frac{i a_{nm}^{(0,1,2)}}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial r^2} + k^2\right)(\, kr z_n(kr)) P_n^m(\cos\theta) e^{im\varphi} \qquad (1.5)\\
E_\theta^{(0,1,2)} &= \sum_{n,m} \frac{i a_{nm}^{(0,1,2)}}{\omega\mu\epsilon r}\frac{\partial}{\partial r}(kr\, z_n(kr))\frac{\partial}{\partial\theta} P_n^m(\cos\theta) e^{im\varphi}\\
&- \sum_{n,m} \frac{im}{\epsilon r\sin\theta} b_{nm}^{(0,1,2)}\; kr\, z_n(kr) P_n^m(\cos\theta) e^{im\varphi}\\
E_\varphi^{(0,1,2)} &= -\sum_{n,m} \frac{m a_{nm}^{(0,1,2)}}{\omega\mu\epsilon r\sin\theta}\frac{\partial}{\partial r}(kr\, z_n(kr)) P_n^m(\cos\theta) e^{im\varphi}\\
&+ \sum_{n,m} \frac{b_{nm}^{(0,1,2)}}{\epsilon r} kr z_n(kr)\frac{\partial}{\partial\theta} P_n^m(\cos\theta) e^{im\varphi}\\
H_r^{(0,1,2)} &= \sum_{n,m} \frac{i b_{nm}^{(0,1,2)}}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial r^2} + k^2\right)(kr z_n(kr)) P_n^m(\cos\theta) e^{im\varphi}\\
H_\theta^{(0,1,2)} &= \sum_{n,m} \frac{im}{\mu r\sin\theta} a_{nm}^{(0,1,2)}\; kr\, z_n(kr) P_n^m(\cos\theta) e^{im\varphi}\\
&+ \sum_{n,m} \frac{i b_{nm}^{(0,1,2)}}{\omega\mu\epsilon r}\frac{\partial}{\partial r}(\, kr\, z_n(kr))\frac{\partial}{\partial\theta} P_n^m(\cos\theta) e^{im\varphi}\\
H_\varphi^{(0,1,2)} &= -\sum_{n,m} \frac{1}{\mu r} a_{nm}^{(0,1,2)}\; kr\, z_n(kr)\frac{\partial}{\partial\theta} P_n^m(\cos\theta) e^{im\varphi}\\
&- \sum_{n,m} \frac{m b_{nm}^{(0,1,2)}}{\omega\mu\epsilon r\sin\theta}\frac{\partial}{\partial r}(kr z_n(kr)) P_n^m(\cos\theta) e^{im\varphi}
\end{aligned}
$$

where $z_n(kr)$ is the appropriate spherical Bessel functions (Arfken, 2015), $P_n^m(\cos\theta)$ is the Associated Legendre polynomials and the constants $a_{nm}^{(0)}, b_{nm}^{(0)}, a_{nm}^{(1)}, b_{nm}^{(1)}, a_{nm}^{(2)}, b_{nm}^{(2)}$ appear in the solution because of Superposition principle. The upper indices (0,1,2) denotes the fields that they belong. The relation between the coefficients will be determined from the boundary conditions.

The thesis is organized as follows; we begin with the short review of the solution of the Maxwell Equations in spherical coordinates, using these well known solutions (Eom, 2004), with the help of the perturbation method , boundary conditions for deformed spherical body will be considered. Relation between components of perturbed and unperturbed fields will be given explicitly. We have solved the Maxwell equations and found the fields in the presence of deformed spherical conducting body up to the second order corrections. All angular integrals are treated analytically. In Chapter 3; we will discuss electromagnetic scattering phenomena. We start with the review of scattering of uniform electromagnetic plane waves from conducting spheres and then we discuss scattering from conducting deformed spheres. First and second order corrections are given. We continue our discussion with dielectric spheres as a preparation to study scattering from deformed dielectric spheres. Corrections to the scattered and transmitted fields up to the second order are given. After that, we give numerical results in Chapter 4. We will compare our results with the known ones in the literature. Finally we make a conclusion.

CHAPTER 2

SOLUTION OF MAXWELL EQUATIONS

In the present Chapter we start with a short review of Maxwell Equations for time harmonic electromagnetic fields (Harrington, 2001) and solution of Maxwell equations by Debye potentials (Eom, 2004). Boundary conditions for deformed spherical body will be considered. Relation between components of perturbed and unperturbed fields will be given explicitly.

2.1. The Maxwell Equations

The Maxwell equations are given with the following system of equations;

$$\nabla \times \vec{\mathcal{E}} + \frac{\partial \vec{\mathcal{B}}}{\partial t} = 0 \tag{2.1}$$

$$\nabla \times \vec{\mathcal{H}} - \frac{\partial \vec{\mathcal{D}}}{\partial t} = \vec{\mathcal{J}} \tag{2.2}$$

$$\nabla \cdot \vec{\mathcal{B}} = 0 \tag{2.3}$$

$$\nabla \cdot \vec{\mathcal{D}} = \rho \tag{2.4}$$

where $\vec{\mathcal{E}}$ is the electric intensity, $\vec{\mathcal{H}}$ is the magnetic intensity, $\vec{\mathcal{D}}$ is the electric flux density, $\vec{\mathcal{B}}$ is the magnetic flux density, $\vec{\mathcal{J}}$ is the electric current density, ρ is the electric charge density (Harrington, 2001). The electric and magnetic fluxes are related with the electric and magnetic intensity vectors in the vacuum by the following rules;

$$\vec{\mathcal{D}} = \epsilon_0 \vec{\mathcal{E}} \tag{2.5}$$

$$\vec{\mathcal{B}} = \mu_0 \vec{\mathcal{H}} \tag{2.6}$$

where ϵ_0 is the permittivity of the vacuum and μ_0 is the permeability of the vacuum.

2.2. Time Harmonic Electromagnetic Fields

The vector fields $\vec{\mathcal{E}}, \vec{\mathcal{H}}, \vec{\mathcal{D}}, \vec{\mathcal{B}}$ depend on time. In order to separate time dependence, we suppose that the time dependence can be represented by the harmonic function (Harrington, 2001). If the time dependence is represented with cosine function then we can write

$$\vec{\mathcal{E}} = Re(\vec{E}e^{-i\omega t}). \tag{2.7}$$

In this representation the complex vector $\vec{E}$ is time independent and called the Phasor. It is also obvious that the components of the real vector $\vec{\mathcal{E}}$ are related with the components of the complex vector as

$$\mathcal{E}_\alpha = Re(E_\alpha e^{-i\omega t}). \tag{2.8}$$

The complex vector $\vec{E}$ is called Complex electric intensity. The phasor representation enables one to replace time derivatives with $(-i\omega)$. Hence the Maxwell equations takes the form

$$\nabla \times \vec{E} - i\omega\vec{B} = 0 \tag{2.9}$$
$$\nabla \times \vec{H} + i\omega\vec{D} = \vec{J} \tag{2.10}$$
$$\nabla \cdot \vec{B} = 0 \tag{2.11}$$
$$\nabla \cdot \vec{D} = \rho. \tag{2.12}$$

2.3. Solution of Maxwell Equations with Debye Potentials

The Maxwell equations, in source free region are coupled differential equations of electric and magnetic fields (Eom, 2004).

$$\nabla \times \vec{E} - i\omega\vec{B} = 0 \tag{2.13}$$

$$\nabla \times \vec{H} + i\omega\vec{D} = 0 \tag{2.14}$$

$$\nabla \cdot \vec{B} = 0 \tag{2.15}$$

$$\nabla \cdot \vec{D} = 0 \tag{2.16}$$

In order to reduce these vector equations to scalar equations we use the last two equations. Since the divergence of $\vec{E}$ and $\vec{B}$ are zero, we can express this vectors as a curl of a vector

$$\vec{E} = -\frac{1}{\epsilon}\nabla \times \vec{F} \tag{2.17}$$

$$\vec{H} = \frac{1}{\mu}\nabla \times \vec{A}. \tag{2.18}$$

First and second equations allow us to write final result;

$$\vec{E} = -\frac{1}{\epsilon}\nabla \times \vec{F} + \frac{i}{\omega\mu\epsilon}\nabla \times \nabla \times \vec{A} \tag{2.19}$$

$$\vec{H} = \frac{1}{\mu}\nabla \times \vec{A} + \frac{i}{\omega\mu\epsilon}\nabla \times \nabla \times \vec{F}. \tag{2.20}$$

By a special choice in the form of this new vectors $\vec{A}$ and $\vec{F}$, we can reduce Maxwell equations to scalar equation. To find electric and magnetic field we will consider two special choice;

1) In spherical coordinate system let us choose $\vec{F} = 0$ and $\vec{A}(r, \theta, \varphi) = \hat{r}A_r(r, \theta, \varphi)$. This choice results in; $\vec{H} = \frac{1}{\mu}\nabla \times \vec{A}$. The magnetic field can be evaluated immediately as

following;

$$
\begin{aligned}
H_r &= 0 & (2.21)\\
H_\theta &= \frac{1}{\mu r \sin\theta}\frac{\partial A_r}{\partial \varphi} & (2.22)\\
H_\varphi &= -\frac{1}{\mu r}\frac{\partial A_r}{\partial \theta}. & (2.23)
\end{aligned}
$$

To find an explicit expression for radial component of vector potential we choose a gauge (Eom, 2004)

$$\frac{\partial A_r}{\partial r} = i\omega\mu\epsilon\Phi_1 \tag{2.24}$$

which allows to show that the radial component of vector potential $\vec{A}$ satisfies following Helmhotz Equation.

$$(\nabla^2 + k^2)\frac{A_r(r,\theta,\varphi)}{r} = 0 \tag{2.25}$$

where $k^2 = \omega^2\mu\epsilon$. Solution of the above equation can be obtained by separation of variables as (Arfken, 2015);

$$A_r(r,\ \theta,\ \varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_{nm}\ kr\, z_n(kr)P_n^m(\cos\theta)e^{im\varphi}. \tag{2.26}$$

The complex Electric field can be expressed in terms of vector potential $\vec{A}$ by the following relation;

$$\vec{E} = \frac{i}{\omega\mu\epsilon}\nabla\times\nabla\times\vec{A}. \tag{2.27}$$

The components of the Electric field are explicitly given with the following relations;

$$E_r = \frac{i}{\omega\mu\epsilon}(\frac{\partial^2}{\partial r^2} + k^2)A_r \tag{2.28}$$

$$E_\theta = \frac{i}{\omega\mu\epsilon r}\frac{\partial^2 A_r}{\partial r\partial\theta} \tag{2.29}$$

$$E_\varphi = \frac{i}{\omega\mu\epsilon r\sin\theta}\frac{\partial^2 A_r}{\partial r\partial\varphi}. \tag{2.30}$$

Thus the components of Electric and Magnetic fields are expressed only in terms of scalar function $A_r(r, \theta, \varphi)$. Since the H_r is zero in this setting this decomposition is called TM wave decomposition (Eom, 2004).

2) Second way could be starting with the choice;

$$\vec{A}(r, \theta, \varphi) = 0$$
$$\vec{F}(r, \theta, \varphi) = \hat{r}F_r(r, \theta, \varphi).$$

Electric and magnetic fields in terms of vector potential $\vec{F}$ is given by;

$$\vec{E} = -\frac{1}{\epsilon}\nabla\times\vec{F}$$
$$\vec{H} = \frac{i}{\omega\mu\epsilon}\nabla\times\nabla\times\vec{F}. \tag{2.31}$$

Together with the gauge condition (Eom, 2004)

$$\Phi_2 = -\frac{i}{\omega\mu\epsilon}\frac{\partial F_r}{\partial r}, \tag{2.32}$$

we find that the scalar function $F_r(r, \theta, \varphi)$ satisfies following Helmholtz equation

$$(\nabla^2 + k^2)\frac{F_r(r, \theta, \varphi)}{r} = 0, \tag{2.33}$$

whose solution is given by

$$F_r(r,\theta,\ \varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} b_{nm}\ krz_n(kr)P_n^m(\cos\theta)e^{im\varphi} \tag{2.34}$$

where b_{nm} is constant, $P_n^m(\cos\theta)$ is the associated Legendre functions, $z_n(kr)$ is the appropriate spherical Bessel functions. If the domain, in which solution is considered, contains origin then $z_n(kr)$ is chosen to be spherical Bessel $j_n(kr)$ to have a regular solution at the origin. When r approaches to infinity $z_n(kr)$ is chosen to be spherical Hankel function of first kind $H_n^1(kr)$ in order to get outgoing waves (Arfken, 2015).

$$\begin{aligned}
E_r &= 0 && (2.35)\\
E_\theta &= -\frac{1}{\epsilon r\sin\theta}\frac{\partial F_r}{\partial\varphi} && (2.36)\\
E_\varphi &= \frac{1}{\epsilon r}\frac{\partial F_r}{\partial\theta} && (2.37)\\
H_r &= \frac{i}{\omega\mu\epsilon}(\frac{\partial^2}{\partial r^2}+k^2)F_r && (2.38)\\
H_\theta &= \frac{i}{\omega\mu\epsilon r}\frac{\partial^2 F_r}{\partial r\partial\theta} && (2.39)\\
H_\varphi &= \frac{i}{\omega\mu\epsilon r\sin\theta}\frac{\partial^2 F_r}{\partial r\partial\varphi}. && (2.40)
\end{aligned}$$

In this setting radial component of electric field E_r is zero so this representation is called TE wave decomposition.

Summation of TE and TM waves gives total Electric and Magnetic fields as;

$$E_r = \frac{i}{\omega\mu\epsilon}(\frac{\partial^2}{\partial r^2} + k^2)A_r \tag{2.41}$$

$$E_\theta = \frac{i}{\omega\mu\epsilon r}\frac{\partial^2 A_r}{\partial r\partial\theta} - \frac{1}{\epsilon r\sin\theta}\frac{\partial F_r}{\partial\varphi} \tag{2.42}$$

$$E_\varphi = \frac{i}{\omega\mu\epsilon r\sin\theta}\frac{\partial^2 A_r}{\partial r\partial\varphi} + \frac{1}{\epsilon r}\frac{\partial F_r}{\partial\theta} \tag{2.43}$$

$$H_r = \frac{i}{\omega\mu\epsilon}(\frac{\partial^2}{\partial r^2} + k^2)F_r \tag{2.44}$$

$$H_\theta = \frac{1}{\mu r\sin\theta}\frac{\partial A_r}{\partial\varphi} + \frac{i}{\omega\mu\epsilon r}\frac{\partial^2 F_r}{\partial r\partial\theta} \tag{2.45}$$

$$H_\varphi = -\frac{1}{\mu r}\frac{\partial A_r}{\partial\theta} + \frac{i}{\omega\mu\epsilon r\sin\theta}\frac{\partial^2 F_r}{\partial r\partial\varphi}. \tag{2.46}$$

Writing the explicit form of A_r and F_r gives;

$$\begin{aligned}
E_r &= \sum_{n,m}\frac{ia_{nm}}{\omega\mu\epsilon}(\frac{\partial^2}{\partial r^2} + k^2)(\,krz_n(kr))P_n^m(\cos\theta)e^{im\varphi} \qquad (2.47)\\
E_\theta &= \sum_{n,m}\frac{ia_{nm}}{\omega\mu\epsilon r}\frac{\partial}{\partial r}(kr\,z_n(kr))\frac{\partial}{\partial\theta}P_n^m(\cos\theta)e^{im\varphi}\\
&- \sum_{n,m}\frac{imb_{nm}}{\epsilon r\sin\theta}\;kr\,z_n(kr)P_n^m(\cos\theta)e^{im\varphi}\\
E_\varphi &= -\sum_{n,m}\frac{ma_{nm}}{\omega\mu\epsilon r\sin\theta}\frac{\partial}{\partial r}(kr\,z_n(kr))P_n^m(\cos\theta)e^{im\varphi}\\
&+ \sum_{n,m}\frac{b_{nm}}{\epsilon r}krz_n(kr)\frac{\partial}{\partial\theta}P_n^m(\cos\theta)e^{im\varphi}\\
H_r &= \sum_{n,m}\frac{ib_{nm}}{\omega\mu\epsilon}(\frac{\partial^2}{\partial r^2} + k^2)(krz_n(kr))P_n^m(\cos\theta)e^{im\varphi}\\
H_\theta &= \sum_{n,m}\frac{ima_{nm}}{\mu r\sin\theta}\;kr\,z_n(kr)P_n^m(\cos\theta)e^{im\varphi}\\
&+ \sum_{n,m}\frac{ib_{nm}}{\omega\mu\epsilon r}\frac{\partial}{\partial r}(\,kr\,z_n(kr))\frac{\partial}{\partial\theta}P_n^m(\cos\theta)e^{im\varphi}\\
H_\varphi &= -\sum_{n,m}\frac{a_{nm}}{\mu r}\;kr\,z_n(kr)\frac{\partial}{\partial\theta}P_n^m(\cos\theta)e^{im\varphi}\\
&- \sum_{n,m}\frac{mb_{nm}}{\omega\mu\epsilon r\sin\theta}\frac{\partial}{\partial r}(krz_n(kr))P_n^m(\cos\theta)e^{im\varphi}.
\end{aligned}$$

2.4. Solution of Maxwell's Equation in the Presence of Conducting Deformed Spherical Body

In this section, we will try to construct Electric and Magnetic fields in the presence of conducting deformed spherical body.

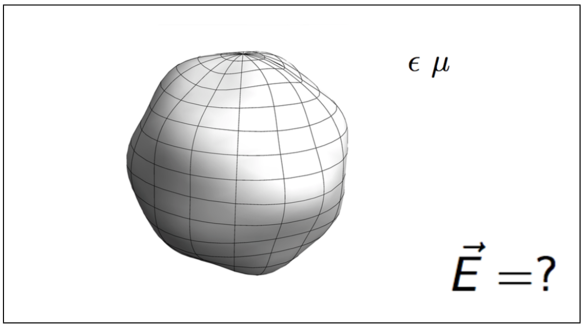

Figure 2.1. Conducting deformed body in vacuum.

The constitutive parameters ϵ and μ are arbitrary real constants. Special choice of parameters, $\epsilon = \epsilon_0$ and $\mu = \mu_0$ enables to find the solution of Maxwell Equations in vacuum. Since the shape of deformed body could be arbitrary no analytic exact form for the electric and magnetic fields is known. In order to find electric and magnetic fields perturbation method will be used. Perturbation method allows us to make calculations within the simplicity of spherical functions, namely, spherical Bessel functions and spherical harmonics.

Let us begin with the position vector of a point on deformed sphere;

$$\vec{r} = (R + R\beta f(\theta, \varphi))\hat{r} \tag{2.48}$$

where β is dimensionless small deformation parameter and $f(\theta, \varphi)$ is arbitrary smooth function. Thus the angle dependent radius of deformed sphere is;

$$\tilde{R}(\theta, \varphi) = (1 + \beta f(\theta, \varphi))R. \tag{2.49}$$

In electromagnetism, perturbation theory is used for the objects whose shapes are slightly differ from the ones for which solutions are known.

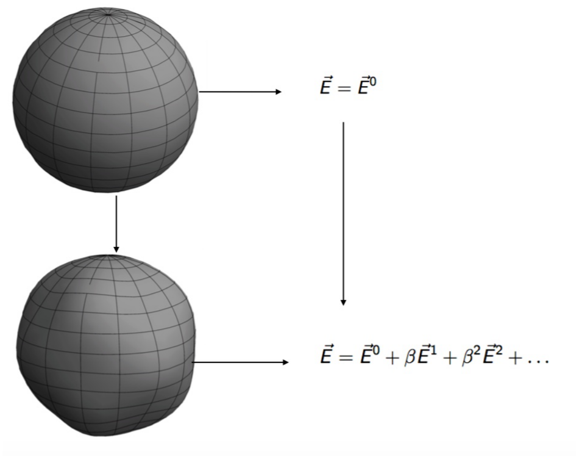

Figure 2.2. Effect of perturbation on fields.

In the present work we deal with the objects which have small deviations from the sphere. Perturbation theory assumes that the fields have also small changes due to the boundary deviations. The leading term belongs to the original shape. Other terms are correction terms originating from the boundary deviations. These additional terms are infinite but in practice one can keep the correction terms up to a definite order. In this thesis we will keep up to the second order. Second order corrections are also included in the calculations of the (Farias, 1994) but at some level they use Rayleigh approximation (Vandehulst, 1981) in order to get numeric values. But in the present thesis we keep all the second order terms till the end.

As it is clear from the figure 2.2, boundary deviations also lead to the corrections on surface normal vector,

$$\begin{aligned}\vec{n} \;=\; & R^2\Big([\sin\theta + 2\beta f\sin\theta + \beta^2 f^2\sin\theta]\hat{r} \\ & +[-\beta f_\theta\sin\theta - \beta^2 f f_\theta\sin\theta]\hat{\theta} + [-\beta f_\varphi - \beta^2 f f_\varphi]\hat{\varphi}\Big) \qquad (2.50)\end{aligned}$$

from which we deduce that the unit normal vector up to the β^2 term is;

$$\hat{n} = \left(1 - \beta^2\left(\frac{f_\theta^2}{2} + \frac{f_\varphi^2}{2\sin^2\theta}\right)\right)\hat{r} + \left(-\beta f_\theta + \beta^2 f f_\theta\right)\hat{\theta} + \left(-\beta\frac{f_\varphi}{\sin\theta} + \beta^2\frac{f f_\varphi}{\sin\theta}\right)\hat{\varphi} \qquad (2.51)$$

where the f_θ and f_φ denotes the θ and φ derivatives of the deformation function $f(\theta,\varphi)$ respectively.

The boundary value problem on conducting deformed spherical body requires vanishing tangential component of Electric field and vanishing normal component of Magnetic field.

$$\hat{n} \times \vec{E}|_{r=\tilde{R}} = 0, \quad \hat{n}.\vec{B}|_{r=\tilde{R}} = 0. \qquad (2.52)$$

Since the position vector of deformed sphere depends on arbitrary smooth function f, in general there isn't any coordinate system appropriate to solve the boundary value problem for the Maxwell equation in the presence of deformed sphere. In the previous section, we have constructed the electric and magnetic fields in spherical coordinate system. Here, by using these well known results (Harrington, 2001), (Eom, 2004), we will try to construct Electric and magnetic fields for the deformed sphere. On constructing the solution we have some flash lights,

1) We have to take into account that when the deformation parameter β is set to be zero, the deformed sphere reduce to the standard sphere so this must be true for the fields. Setting $\beta = 0$ in the electric and magnetic fields for deformed sphere, one will obtain electric and magnetic fields for the sphere;

$$\vec{E}(r,\theta,\varphi)\Big|_{\beta=0} = \vec{E}^0(r,\theta,\varphi) \qquad (2.53)$$

$$\vec{B}(r,\theta,\varphi)\Big|_{\beta=0} = \vec{B}^0(r,\theta,\varphi) \qquad (2.54)$$

where $\vec{E}^0$ and $\vec{B}^0$ are the electric and magnetic fields for unperturbed sphere.

2) The Maxwell equation is linear thus it admits superposition principle.

3) The fields are well behaved, that is to say; small changes on the boundary lead to the small changes on the solutions.

Thus we can write the fields for the deformed sphere as sum of unperturbed fields, first order corrections and second order corrections (Yeh, 1964), (Erma, 1968), (Erma, 1968);

$$\vec{E}(r,\theta,\varphi) = \vec{E}^0(r,\theta,\varphi) + \beta\vec{E}^1(r,\theta,\varphi) + \beta^2\vec{E}^2(r,\theta,\varphi) \tag{2.55}$$
$$\vec{B}(r,\theta,\varphi) = \vec{B}^0(r,\theta,\varphi) + \beta\vec{B}^1(r,\theta,\varphi) + \beta^2\vec{B}^2(r,\theta,\varphi). \tag{2.56}$$

As it is well known that to solve the Maxwell equations one needs to develop special techniques. One way to solve is to relate vector wave equation with scalar Helmholtz equation known as TE (Transverse Electric) and TM (Transverse Magnetic) decomposition. The TE and TM wave decomposition strongly depends on the chosen coordinate system. We choose coordinate systems according to the boundaries under the consideration. Thus there is a relation between TE, TM wave decomposition and boundaries. If the boundary for example is sphere then the spherical coordinate system enables (in some cases) to decompose fields in to the TE_r and TM_r waves which fit the boundary conditions and gives no additional difficulties.

The fact that $\vec{E}^1(r,\theta,\varphi)$, $\vec{B}^1(r,\theta,\varphi)$ and $\vec{E}^2(r,\theta,\varphi)$, $\vec{B}^2(r,\theta,\varphi)$ fields satisfy the Maxwell equations, the difference among them only occurs on the coefficients thus the components of each field must be of the following form;

$$\begin{aligned}
E_r^{(0,1,2)} &= \sum_{n,m} \frac{ia_{nm}^{(0,1,2)}}{\omega\mu\epsilon}(\frac{\partial^2}{\partial r^2} + k^2)(krz_n(kr))P_n^m(\cos\theta)e^{im\varphi} \\
E_\theta^{(0,1,2)} &= \sum_{n,m} \frac{ia_{nm}^{(0,1,2)}}{\omega\mu\epsilon r}\frac{\partial}{\partial r}(kr\, z_n(kr))\frac{\partial}{\partial\theta}P_n^m(\cos\theta)e^{im\varphi} \\
&- \sum_{n,m} \frac{im}{\epsilon r\sin\theta}b_{nm}^{(0,1,2)}\; kr\, z_n(kr)P_n^m(\cos\theta)e^{im\varphi} \\
E_\varphi^{(0,1,2)} &= -\sum_{n,m} \frac{ma_{nm}^{(0,1,2)}}{\omega\mu\epsilon r\sin\theta}\frac{\partial}{\partial r}(kr\, z_n(kr))P_n^m(\cos\theta)e^{im\varphi} \\
&+ \sum_{n,m} \frac{b_{nm}^{(0,1,2)}}{\epsilon r}krz_n(kr)\frac{\partial}{\partial\theta}P_n^m(\cos\theta)e^{im\varphi} \\
H_r^{(0,1,2)} &= \sum_{n,m} \frac{ib_{nm}^{(0,1,2)}}{\omega\mu\epsilon}(\frac{\partial^2}{\partial r^2} + k^2)(krz_n(kr))P_n^m(\cos\theta)e^{im\varphi} \\
H_\theta^{(0,1,2)} &= \sum_{n,m} \frac{im}{\mu r\sin\theta}a_{nm}^{(0,1,2)}\; kr\, z_n(kr)P_n^m(\cos\theta)e^{im\varphi} \\
&+ \sum_{n,m} \frac{ib_{nm}^{(0,1,2)}}{\omega\mu\epsilon r}\frac{\partial}{\partial r}(kr\, z_n(kr))\frac{\partial}{\partial\theta}P_n^m(\cos\theta)e^{im\varphi} \\
H_\varphi^{(0,1,2)} &= -\sum_{n,m} \frac{1}{\mu r}a_{nm}^{(0,1,2)}\; kr\, z_n(kr)\frac{\partial}{\partial\theta}P_n^m(\cos\theta)e^{im\varphi} \\
&- \sum_{n,m} \frac{mb_{nm}^{(0,1,2)}}{\omega\mu\epsilon r\sin\theta}\frac{\partial}{\partial r}(krz_n(kr))P_n^m(\cos\theta)e^{im\varphi}.
\end{aligned} \tag{2.57}$$

The relation between the coefficients $a_{nm}^0, b_{nm}^0, a_{nm}^1, b_{nm}^1, a_{nm}^2, b_{nm}^2$ will be found form the boundary conditions on deformed conducting sphere given with the following equations;

$$\hat{n}\times\vec{E}(\tilde{R},\theta,\varphi) \;=\; 0, \qquad \hat{n}.\vec{B}(\tilde{R},\theta,\varphi) = 0$$

where $\tilde{R}$ is the angle dependent radius of deformed sphere and

$$\vec{E}(r,\theta,\varphi) = \vec{E}^0(r,\theta,\varphi) + \beta\vec{E}^1(r,\theta,\varphi) + \beta^2\vec{E}^2(r,\theta,\varphi) \tag{2.58}$$

$$\vec{B}(r,\theta,\varphi) = \vec{B}^0(r,\theta,\varphi) + \beta\vec{B}^1(r,\theta,\varphi) + \beta^2\vec{B}^2(r,\theta,\varphi) \tag{2.59}$$

and finally

$$\hat{n} = \left(1 - \beta^2(\frac{f_\theta^2}{2} + \frac{f_\varphi^2}{2\sin^2\theta})\right)\hat{r} + \left(-\beta f_\theta + \beta^2 f f_\theta\right)\hat{\theta} + \left(-\beta\frac{f_\varphi}{\sin\theta} + \beta^2\frac{f f_\varphi}{\sin\theta}\right)\hat{\varphi}. \qquad (2.60)$$

Thus for the electric field, the following condition must be satisfied;

$$\Big[\left(n_\theta E_\varphi(r,\theta,\varphi) - n_\varphi E_\theta(r,\theta,\varphi)\right)\hat{r} + \left(n_\varphi E_r(r,\theta,\varphi) - n_r E_\varphi(r,\theta,\varphi)\right)\hat{\theta}$$
$$+ \left(n_r E_\theta(r,\theta,\varphi) - n_\theta E_r(r,\theta,\varphi)\right)\hat{\varphi}\Big]_{r=\tilde{R}} = 0. \qquad (2.61)$$

In order to satisfy above equation, coefficients of each unit vector must vanish.

1) The requirement of vanishing coefficient of $\hat{r}$ in (2.61) gives us;

$$\Big[n_\theta E_\varphi(r,\theta,\varphi) - n_\varphi E_\theta(r,\theta,\varphi)\Big]_{r=\tilde{R}} = 0$$
$$\Big[n_\theta(E_\varphi^0(r,\theta,\varphi) + \beta E_\varphi^1(r,\theta,\varphi) + \beta^2 E_\varphi^2(r,\theta,\varphi))$$
$$-n_\varphi(E_\theta^0(r,\theta,\varphi) + \beta E_\theta^1(r,\theta,\varphi) + \beta^2 E_\theta^2(r,\theta,\varphi))\Big]_{r=\tilde{R}} = 0. \qquad (2.62)$$

In above equality one need to expand field components in formal power series. As an example, let us consider zeroth order fields and their formal power series expansions (other fields can be expanded in a similar manner).

$$E_{r,\theta,\varphi}^0(\tilde{R},\theta,\varphi) = E_{r,\theta,\varphi}^0(R + \beta f R,\theta,\varphi) \qquad (2.63)$$
$$E_{r,\theta,\varphi}^0(\tilde{R},\theta,\varphi) = \Big[E_{r,\theta,\varphi}^0(r,\theta,\varphi) + \beta f r\frac{\partial}{\partial r}E_{r,\theta,\varphi}^0(r,\theta,\varphi)$$
$$+\frac{\beta^2 f^2 r^2}{2}\frac{\partial^2}{\partial r^2}E_{r,\theta,\varphi}^0(r,\theta,\varphi)\Big]_{r=R}.$$

Then boundary condition expressed with (2.62) takes the form;

$$\Big[n_\theta E^0_\varphi(r,\theta,\varphi) + n_\theta \beta f r \frac{\partial}{\partial r} E^0_\varphi(r,\theta,\varphi) + n_\theta \frac{\beta^2 f^2 r^2}{2} \frac{\partial^2}{\partial r^2} E^0_\varphi(r,\theta,\varphi)$$
$$+\beta n_\theta E^1_\varphi(r,\theta,\varphi) + n_\theta \beta^2 f r \frac{\partial}{\partial r} E^1_\varphi(r,\theta,\varphi)) + n_\theta \beta^2 E^2_\varphi(r,\theta,\varphi)$$
$$-n_\varphi E^0_\theta(r,\theta,\varphi) - n_\varphi \beta f r \frac{\partial}{\partial r} E^0_\theta(r,\theta,\varphi) - n_\varphi \frac{\beta^2 f^2 r^2}{2} \frac{\partial^2}{\partial r^2} E^0_\theta(r,\theta,\varphi)$$
$$-n_\varphi \beta E^1_\theta(r,\theta,\varphi) - n_\varphi \beta^2 f r \frac{\partial}{\partial r} E^1_\theta(r,\theta,\varphi) - n_\varphi \beta^2 E^2_\theta(r,\theta,\varphi)\Big]_{r=R} = 0.$$

Collecting the terms containing same power of β and using the boundary conditions for ordinary sphere we find following relations;

$$\Big[- f_\theta\Big(f r \frac{\partial}{\partial r} E^0_\varphi(r,\theta,\varphi) + E^1_\varphi(r,\theta,\varphi)\Big) + \frac{f_\varphi}{\sin\theta}\left(f r \frac{\partial}{\partial r} E^0_\theta(r,\theta,\varphi) + E^1_\theta(r,\theta,\varphi)\right)\Big]_{r=R} = 0, \tag{2.64}$$

$$\left[- f_\theta \begin{pmatrix} \frac{f^2 r^2}{2} \frac{\partial^2}{\partial r^2} E^0_\varphi(r,\theta,\varphi) + f r \frac{\partial}{\partial r} E^1_\varphi(r,\theta,\varphi) + E^2_\varphi(r,\theta,\varphi) \\ -f^2 r \frac{\partial}{\partial r} E^0_\varphi(r,\theta,\varphi) - f E^1_\varphi(r,\theta,\varphi) \end{pmatrix} + \frac{f_\varphi}{\sin\theta} \begin{pmatrix} \frac{f^2 r^2}{2} \frac{\partial^2}{\partial r^2} E^0_\theta(r,\theta,\varphi) + f r \frac{\partial}{\partial r} E^1_\theta(r,\theta,\varphi) + E^2_\theta(r,\theta,\varphi) \\ -f^2 r \frac{\partial}{\partial r} E^0_\theta(r,\theta,\varphi) - f E^1_\theta(r,\theta,\varphi) \end{pmatrix} \right]_{r=R} = 0. \tag{2.65}$$

By using equation (2.64) above equation can be written as;

$$\Big[- f_\theta(\frac{f^2 r^2}{2} \frac{\partial^2}{\partial r^2} E^0_\varphi(r,\theta,\varphi) + f r \frac{\partial}{\partial r} E^1_\varphi(r,\theta,\varphi) + E^2_\varphi(r,\theta,\varphi)) + \frac{f_\varphi}{\sin\theta}(\frac{f^2 r^2}{2} \frac{\partial^2}{\partial r^2} E^0_\theta(r,\theta,\varphi) + f r \frac{\partial}{\partial r} E^1_\theta(r,\theta,\varphi) + E^2_\theta(r,\theta,\varphi))\Big]_{r=R} = 0. \tag{2.66}$$

2) The requirement of vanishing coefficient of $\hat{\theta}$ in (2.61) gives us

$$n_\varphi E_r(r,\theta,\varphi) - n_r E_\varphi(r,\theta,\varphi)|_{r=\tilde{R}} = 0 \tag{2.67}$$

using formal power series expansion, we get following equality;

$$\begin{aligned}&\Big[n_\varphi E_r^0(r,\theta,\varphi)+\beta n_\varphi fr\frac{\partial}{\partial r}E_r^0(r,\theta,\varphi)+n_\varphi\frac{\beta^2f^2r^2}{2}\frac{\partial^2}{\partial r^2}E_r^0(r,\theta,\varphi)\\&+\beta n_\varphi E_r^1(r,\theta,\varphi)+\beta^2n_\varphi fr\frac{\partial}{\partial r}E_r^1(r,\theta,\varphi))+n_\varphi\beta^2E_r^2(r,\theta,\varphi)\\&-n_rE_\varphi^0(r,\theta,\varphi)-n_r\beta fr\frac{\partial}{\partial r}E_\varphi^0(r,\theta,\varphi)-n_r\frac{\beta^2f^2r^2}{2}\frac{\partial^2}{\partial r^2}E_\varphi^0(r,\theta,\varphi)\\&-n_r\beta E_\varphi^1(r,\theta,\varphi)-n_r\beta^2fr\frac{\partial}{\partial r}E_\varphi^1(r,\theta,\varphi)-n_r\beta^2E_\varphi^2(r,\theta,\varphi)\Big]_{r=R}=0.\end{aligned}\tag{2.68}$$

From which we get following two conditions

$$\left[\frac{f_\varphi}{\sin\theta}E_r^0(r,\theta,\varphi)+fr\frac{\partial}{\partial r}E_\varphi^0(r,\theta,\varphi)+E_\varphi^1(r,\theta,\varphi)\right]_{r=R}=0\tag{2.69}$$

$$\begin{aligned}&\Big[\frac{ff_\varphi}{\sin\theta}E_r^0(r,\theta,\varphi)-\frac{f_\varphi}{\sin\theta}(fr\frac{\partial}{\partial r}E_r^0(r,\theta,\varphi)+E_r^1(r,\theta,\varphi))\\&-\frac{f^2r^2}{2}\frac{\partial^2}{\partial r^2}E_\varphi^0(r,\theta,\varphi)-fr\frac{\partial}{\partial r}E_\varphi^1(r,\theta,\varphi)-E_\varphi^2(r,\theta,\varphi)\Big]_{r=R}=0.\end{aligned}\tag{2.70}$$

3) The requirement of vanishing coefficient of $\hat{\varphi}$ in (2.61) gives us

$$n_rE_\theta(r,\theta,\varphi)-n_\theta E_r(r,\theta,\varphi)\Big|_{r=\tilde{R}}=0\tag{2.71}$$

using formal power series expansion enables us to find following equality;

$$\begin{aligned}&\Big[n_rE_\theta^0(r,\theta,\varphi)+n_r\beta fr\frac{\partial}{\partial r}E_\theta^0(r,\theta,\varphi)+n_r\frac{\beta^2f^2r^2}{2}\frac{\partial^2}{\partial r^2}E_\theta^0(r,\theta,\varphi)\\&+n_r\beta E_\theta^1(r,\theta,\varphi)+n_r\beta^2fr\frac{\partial}{\partial r}E_\theta^1(r,\theta,\varphi)+n_r\beta^2E_\theta^2(r,\theta,\varphi)\\&-n_\theta E_r^0(r,\theta,\varphi)-n_\theta\beta fr\frac{\partial}{\partial r}E_r^0(r,\theta,\varphi)-n_\theta\frac{\beta^2f^2r^2}{2}\frac{\partial^2}{\partial r^2}E_r^0(r,\theta,\varphi)\\&-n_\theta\beta E_r^1(r,\theta,\varphi)-n_\theta\beta^2fr\frac{\partial}{\partial r}E_r^1(r,\theta,\varphi)-n_\theta\beta^2E_r^2(r,\theta,\varphi)\Big]_{r=R}=0.\end{aligned}\tag{2.72}$$

From which we get following two conditions

$$\left[fr\frac{\partial}{\partial r}E^0_\theta(r,\theta,\varphi) + E^1_\theta(r,\theta,\varphi) + f_\theta E^0_r(r,\theta,\varphi)\right]_{r=R} = 0$$
$$\Big[- f f_\theta E^0_r(r,\theta,\varphi) + f_\theta\Big(fr\frac{\partial}{\partial r}E^0_r(r,\theta,\varphi) + E^1_r(r,\theta,\varphi)\Big)$$
$$+\frac{f^2r^2}{2}\frac{\partial^2}{\partial r^2}E^0_\theta(r,\theta,\varphi) + fr\frac{\partial}{\partial r}E^1_\theta(r,\theta,\varphi) + E^2_\theta(r,\theta,\varphi)\Big]_{r=R} = 0. \quad (2.73)$$

Now let us consider the boundary condition for Magnetic field

$$\hat{n}.\vec{B}(r,\theta,\varphi)|_{r=\tilde{R}} = 0 \quad (2.74)$$

which means

$$\left[n_r B_r(r,\theta,\varphi) + n_\theta B_\theta(r,\theta,\varphi) + n_\varphi B_\varphi(r,\theta,\varphi)\right]_{r=\tilde{R}} = 0 \quad (2.75)$$

formal power series of each component gives

$$\left[(fr\tfrac{\partial}{\partial r}B^0_r + B^1_r) - f_\theta B^0_\theta - \tfrac{f_\varphi}{\sin\theta}B^0_\varphi\right]_{r=R} = 0 \quad (2.76)$$
$$\Big[\frac{f^2r^2}{2}\frac{\partial^2}{\partial r^2}B^0_r(r,\theta,\varphi) + fr\frac{\partial}{\partial r}B^1_r(r,\theta,\varphi) + B^2_r(r,\theta,\varphi)$$
$$-f_\theta(fr\frac{\partial}{\partial r}B^0_\theta(r,\theta,\varphi) + B^1_\theta(r,\theta,\varphi) - fB^0_\theta(r,\theta,\varphi))$$
$$-\frac{f_\varphi}{\sin\theta}(fr\frac{\partial}{\partial r}B^0_\varphi(r,\theta,\varphi) + B^1_\varphi(r,\theta,\varphi) - fB^0_\varphi(r,\theta,\varphi))\Big]_{r=R} = 0. \quad (2.77)$$

Let us write all obtained conditions together

$$1) \quad \Big[- f_\theta\Big(fr\frac{\partial}{\partial r}E^0_\varphi(r,\theta,\varphi) + E^1_\varphi(r,\theta,\varphi)\Big)$$
$$+\frac{f_\varphi}{\sin\theta}\Big(fr\frac{\partial}{\partial r}E^0_\theta(r,\theta,\varphi) + E^1_\theta(r,\theta,\varphi)\Big)\Big]_{r=R} = 0 \quad (2.78)$$

$$2)\quad \Big[-f_\theta\bigg(\frac{f^2r^2}{2}\frac{\partial^2}{\partial r^2}E^0_\varphi(r,\theta,\varphi)+fr\frac{\partial}{\partial r}E^1_\varphi(r,\theta,\varphi)+E^2_\varphi(r,\theta,\varphi)\bigg)$$
$$+\frac{f_\varphi}{\sin\theta}\bigg(\frac{f^2r^2}{2}\frac{\partial^2}{\partial r^2}E^0_\theta(r,\theta,\varphi)+fr\frac{\partial}{\partial r}E^1_\theta(r,\theta,\varphi)+E^2_\theta(r,\theta,\varphi)\bigg)\Big]_{r=R}=0 \tag{2.79}$$

$$3)\quad \Big[\tfrac{f_\varphi}{\sin\theta}E^0_r(r,\theta,\varphi)+fr\tfrac{\partial}{\partial r}E^0_\varphi(r,\theta,\varphi)+E^1_\varphi(r,\theta,\varphi)\Big]_{r=R}=0 \tag{2.80}$$

$$4)\quad \Big[\frac{ff_\varphi}{\sin\theta}E^0_r(r,\theta,\varphi)-\frac{f_\varphi}{\sin\theta}\bigg(fr\frac{\partial}{\partial r}E^0_r(r,\theta,\varphi)+E^1_r(r,\theta,\varphi)\bigg)$$
$$-\bigg(\frac{f^2r^2}{2}\frac{\partial^2}{\partial r^2}E^0_\varphi(r,\theta,\varphi)+fr\frac{\partial}{\partial r}E^1_\varphi(r,\theta,\varphi)+E^2_\varphi(r,\theta,\varphi)\bigg)\Big]_{r=R}=0 \tag{2.81}$$

$$5)\quad \Big[fr\tfrac{\partial}{\partial r}E^0_\theta(r,\theta,\varphi)+E^1_\theta(r,\theta,\varphi)+f_\theta E^0_r(r,\theta,\varphi)\Big]_{r=R}=0 \tag{2.82}$$

$$6)\quad \Big[-ff_\theta E^0_r(r,\theta,\varphi)+f_\theta(fr\frac{\partial}{\partial r}E^0_r(r,\theta,\varphi)+E^1_r(r,\theta,\varphi))$$
$$+\frac{f^2r^2}{2}\frac{\partial^2}{\partial r^2}E^0_\theta(r,\theta,\varphi)+fr\frac{\partial}{\partial r}E^1_\theta(r,\theta,\varphi)+E^2_\theta(r,\theta,\varphi)\Big]_{r=R}=0 \tag{2.83}$$

$$7)\quad \Big[fr\tfrac{\partial}{\partial r}B^0_r(r,\theta,\varphi)+B^1_r(r,\theta,\varphi)-f_\theta B^0_\theta(r,\theta,\varphi)-\tfrac{f_\varphi}{\sin\theta}B^0_\varphi(r,\theta,\varphi)\Big]_{r=R}=0 \tag{2.84}$$

$$8)\quad \Big[\frac{f^2r^2}{2}\frac{\partial^2}{\partial r^2}B^0_r(r,\theta,\varphi)+fr\frac{\partial}{\partial r}B^1_r(r,\theta,\varphi)+B^2_r(r,\theta,\varphi)$$
$$-f_\theta\Big(fr\frac{\partial}{\partial r}B^0_\theta(r,\theta,\varphi)+B^1_\theta(r,\theta,\varphi)-fB^0_\theta(r,\theta,\varphi)\Big)$$
$$-\frac{f_\varphi}{\sin\theta}\Big(fr\frac{\partial}{\partial r}B^0_\varphi(r,\theta,\varphi)+B^1_\varphi(r,\theta,\varphi)+fB^0_\varphi(r,\theta,\varphi)\Big)\Big]_{r=R}=0. \tag{2.85}$$

2.4.1. Coefficients of First Order Perturbation

Field coefficients for the unperturbed electric and magnetic fields are already known (Kirsch,2009). Initially, in order to determine total electric and magnetic field, we have to find first order correction coefficients. Boundary condition (2.84) includes only $b^1_{lm'}$ as unknown, so it is suitable to start with it. Let us rewrite the boundary condition;

$$B^1_r(R,\theta,\varphi)=\bigg[-f(\theta,\varphi)r\frac{\partial}{\partial r}B^0_r(r,\theta,\varphi)+f_\theta(\theta,\varphi)B^0_\theta(r,\theta,\varphi)+\frac{f_\varphi(\theta,\varphi)}{\sin\theta}B^0_\varphi(r,\theta,\varphi)\bigg]_{r=R}$$

multiplying above equation with $P_l^{m'}(\cos\theta)\sin\theta e^{-im'\varphi}$ and taking angular integrals gives us;

$$
\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi} B_r^1(R,\theta,\varphi)P_l^{m'}(\cos\theta)\sin\theta e^{-im'\varphi}d\theta d\varphi \;=\\
&\int_0^{2\pi}\int_0^{\pi}\left(-f(\theta,\varphi)r\frac{\partial}{\partial r}B_r^0(r,\theta,\varphi)\right)_{r=R}P_l^{m'}(\cos\theta)e^{-im'\varphi}\sin\theta d\theta d\varphi\\
&+\int_0^{2\pi}\int_0^{\pi} f_\theta(\theta,\varphi)B_\theta^0(R,\theta,\varphi)P_l^{m'}(\cos\theta)e^{-im'\varphi}\sin\theta d\theta d\varphi\\
&+\int_0^{2\pi}\int_0^{\pi}\frac{f_\varphi(\theta,\varphi)}{\sin\theta}B_\varphi^0(R,\theta,\varphi)P_l^{m'}(\cos\theta)e^{-im'\varphi}\sin\theta d\theta d\varphi.
\end{aligned}
$$

To evaluate above expression let us first express $f(\theta,\varphi)$ in terms of spherical harmonics as in (Elwenspoek, 1982);

$$f(\theta,\varphi)=\sum_{j=0}^{\infty}\sum_{s=-j}^{j}(\tilde{f})_j^s Y_j^s(\theta,\varphi) \tag{2.86}$$

and use the relation between spherical harmonics and associated legendre polynomials (Arfken, 2015)

$$Y_j^s(\theta,\varphi)=(-1)^s\sqrt{\frac{(2j+1)(j-s)!}{4\pi(j+s)!}}P_j^s(\cos\theta)e^{is\varphi} \tag{2.87}$$

which allows us to write $f(\theta,\varphi)$ in terms of associated Legendre polynomials function as;

$$f(\theta,\varphi)=\sum_{j=0}^{\infty}\sum_{s=-j}^{j}f_j^s P_j^s(\cos\theta)e^{is\varphi}. \tag{2.88}$$

with the help of triple Legendre integrals defined in the appendix A, we find one of the first perturbation coefficients as;

$$b_{lm'}^1 \;=\; \left(\frac{\omega\mu\epsilon(2l+1)(l-m')!}{4\pi i(l+m')!\left[(\frac{\partial^2}{\partial r^2}+k^2)(krz_l(kr))\right]}\right)\times\left(\sum_{n,m}\sum_{j,s}T_1(n,m,j,s)\right)$$

where $T_1(n,m,j,s)$ is given in the appendix B. To find the other first order perturbation coefficients a^1_{nm}, let us consider boundary condition (2.80);

$$E^1_\varphi(R,\theta,\varphi) = \left(-fr\frac{\partial}{\partial r}E^0_\varphi(r,\theta,\varphi) - \frac{f_\varphi}{\sin\theta}E^0_r(r,\theta,\varphi)\right)_{r=R}.$$

After multiplication of both side with $P^{m'}_l(\cos\theta)\sin^2\theta e^{-im'\varphi}$ and taking the angular integrals, if we write the values of field from equation (2.57) together with the integral

$$\int_0^\pi \frac{\partial P^m_n(\cos\theta)}{\partial\theta}P^{m'}_l(\cos\theta)\sin^2\theta d\theta = \frac{(l-1)(l-m')}{(2l-1)}\delta_{n+1,l} - \frac{(l+1+m')(l+2)}{(2l+3)}\delta_{n-1,l}$$

we can express unknown coefficients a^1_{nm} in terms of a^0_{nm} , b^0_{nm} and previously found b^1_{nm} as;

$$\begin{aligned} a^1_{lm'} \;=\; & -\frac{\omega\mu\epsilon(2l+1)(l-m')!}{4\pi m'(l+m')!\left[\frac{1}{r}\frac{\partial}{\partial r}(krz_l(kr))\right]_{r=R}}\Bigg\{ \\ & +\frac{b^1_{l+1,m'}\delta_{m'sm'}}{\epsilon}\frac{2(l+m'+1)(l+2)(l+m')!}{(2l+3)(2l+1)(l-m')!}kz_{l+1}(kR) \\ & -\frac{b^1_{l-1,m'}\delta_{m'sm'}}{\epsilon}\frac{2(l-1)(l-m')(l+m')!}{(2l-1)(2l+1)(l-m')!}kz_{l-1}(kR) \\ & +\left(\sum_{n,m}\sum_{j,s}T_3(n,m,j,s)\right)\Bigg\}_{r=R}. \end{aligned}$$

Explicit expression of $T_3(n,m,j,s)$ is given in appendix. With this result we have evaluated the first order perturbation fields.

2.4.2. Coefficients of Second Order Perturbation

First order approximations could be sufficient for some cases but in general adding more and more correction terms gives more accurate. Here, we will try to find second order perturbed fields. Since unperturbed and first order perturbed fields are known in boundary condition (2.85) the only unknown term is the $b^2_{lm'}$, thus to find these coefficients, we begin with the boundary condition (2.85) ;

$$\begin{aligned}B_r^2(R,\theta,\varphi) &= \Big[-\frac{f^2r^2}{2}\frac{\partial^2}{\partial r^2}B_r^0(r,\theta,\varphi) - fr\frac{\partial}{\partial r}B_r^1(r,\theta,\varphi)\\ &\quad + f_\theta\Big(fr\frac{\partial}{\partial r}B_\theta^0(r,\theta,\varphi) + B_\theta^1(r,\theta,\varphi) - fB_\theta^0(r,\theta,\varphi)\Big)\\ &\quad + \frac{f_\varphi}{\sin\theta}\Big(fr\frac{\partial}{\partial r}B_\varphi^0(r,\theta,\varphi) + B_\varphi^1(r,\theta,\varphi) + fB_\varphi^0(r,\theta,\varphi)\Big)\Big]_{r=R}.\end{aligned}$$

In order to proceed, we can expand $f^2(\theta,\varphi)$ in terms of associated Legendre functions as we did for the function $f(\theta,\varphi)$;

$$f^2(\theta,\varphi) = \sum_{j=0}^{\infty}\sum_{s=-j}^{j} k_j^s P_j^s(\cos\theta)e^{is\varphi}. \tag{2.89}$$

In fact the expansion coefficients f_j^s in (2.88) and k_j^s are not independent but they are connected with the following formula (Miller, 1963), (Mavromatis, 1999), (Dong, 2002);

$$k_j^s = \frac{(2j+1)(j-s)!}{4\pi(j+s)!}\Big(\sum_{j_1,s_1}\sum_{j_2,s_2} f_{j_1}^{s_1} f_{j_2}^{s_2} Int_0(j_1,j_2,j,s_1,s_2,s)2\pi\delta_{s_1+s_2,s}\Big). \tag{2.90}$$

Through out the thesis we wont expand k_j^s in terms of f_j^s, we keep k_j^s in closed form because when it appears in an expression it tells us that expression comes from the second order correction. On the other hand, in the computer code, we use relation among them to determine k_j^s from the f_j^s. By the definition of the field components with equation (2.57), multiplication with the $P_l^{m'}(\cos\theta)\sin\theta e^{-im'\varphi}$ and integration gives the following final result;

$$b_{lm'}^2 = \frac{\omega\mu\epsilon(2l+1)(l-m')!}{4\pi i(kRz_l(kR))(l+m')!}\left(\sum_{n,m}\sum_{j,s}\sum_{k=5}^{18} T_k(n,m,j,s)\right). \tag{2.91}$$

Explicit expressions of T_k's are given in the appendix B. To find the coefficients a^2_{lm}, let us begin with the boundary condition (2.81);

$$
\begin{aligned}
E^2_\varphi(R,\theta,\varphi) \;=\; & \Big[\frac{f f_\varphi}{\sin\theta} E^0_r(r,\theta,\varphi) - \frac{f_\varphi}{\sin\theta}\Big(fr\frac{\partial}{\partial r}E^0_r(r,\theta,\varphi) + E^1_r(r,\theta,\varphi)\Big) \\
& -\Big(\frac{f^2 r^2}{2}\frac{\partial^2}{\partial r^2}E^0_\varphi(r,\theta,\varphi) + fr\frac{\partial}{\partial r}E^1_\varphi(r,\theta,\varphi)\Big)\Big]_{r=R} = 0.
\end{aligned}
$$

By the definition of the field components with equation (2.57), multiplication with the $P^{m'}_l(\cos\theta)\sin^2\theta e^{-im'\varphi}$ and integration gives;

$$
\begin{aligned}
a^2_{lm'} \;=\; & -\frac{\omega\mu\epsilon(2l+1)(l-m')!}{4\pi m'(l+m)![\frac{1}{r}\frac{\partial}{\partial r}(krz_l(kr))]_{r=R}}\Bigg\{ \\
& +\frac{b^2_{l+1,m'}\delta_{m'sm'}}{\epsilon}\frac{2(l+m'+1)(l+2)(l+m')!}{(2l+3)(2l+1)(l-m')!}kz_{l+1}(kR) \\
& -\frac{b^2_{l-1,m'}\delta_{m'sm'}}{\epsilon}\frac{2(l-1)(l-m')(l+m')!}{(2l-1)(2l+1)(l-m')!}kz_{l-1}(kR) \\
& +\left(\sum_{n,m}\sum_{j,s}\sum_{k=23}^{29} T_k(n,m,j,s)\right)\Bigg\}
\end{aligned}
$$

Expressions for T_k can be found in appendix B. With this result, we have completely found unknown coefficients and determined first and second order perturbed fields.

CHAPTER 3

ELECTROMAGNETIC WAVE SCATTERING

In the present Chapter, we will discuss electromagnetic scattering phenomena. We start with the review of scattering of uniform electromagnetic plane waves from conducting spheres then we discuss scattering from conducting deformed spheres. First and second order corrections are given. We continue our discussion with dielectric spheres as a preparation to study scattering from deformed dielectric spheres. Corrections to the scattered and transmitted fields upto the second order are given.

3.1. Scattering of Plane Waves From Conducting Sphere

Here we will shortly review well known solution of scattering electromagnetic plane wave from conducting sphere (Balanis, 1989). Let the incident uniform plane wave has the following form;

$$\vec{E}_i = E_0 e^{ikz} \hat{x}. \tag{3.1}$$

Let us write exponential term in spherical coordinates and expanding in terms of Legendre polynomials (Arfken, 2015)

$$e^{ik_1 r\cos\theta} = \sum_{n=0}^{\infty} a_n j_n(k_1 r) P_n(\cos\theta). \tag{3.2}$$

Using orthogonality properties of Legendre polynomials we find spherical coordinate representation of the incident electric field as (Balanis, 1989);

$$\begin{aligned}\overrightarrow{E}_i(r,\theta,\varphi) &= -iE_0\cos\varphi\sum_{n=0}^{\infty}\frac{i^n(2n+1)}{k_1 r}j_n(k_1 r)P_n^1(\cos\theta)\hat{r} \\ &+E_0\cos\theta\cos\varphi\sum_{n=0}^{\infty}i^n(2n+1)j_n(k_1 r)P_n(\cos\theta)\hat{\theta} \\ &-E_0\sin\varphi\sum_{n=0}^{\infty}i^n(2n+1)j_n(k_1 r)P_n(\cos\theta)\hat{\varphi}.\end{aligned}$$

Since electric field is known, magnetic field could be found from Maxwell equations as;

$$\begin{aligned}\overrightarrow{H}_i(r,\theta,\varphi) &= -\frac{ik_1E_0}{\omega\mu_1}\sum_{n=0}^{\infty}i^n(2n+1)\frac{j_n(k_1 r)}{k_1 r}P_n^1(\cos\theta)\sin\varphi\,\hat{r} \\ &+\frac{k_1E_0}{\omega\mu_1}\sum_{n=0}^{\infty}i^n(2n+1)j_n(k_1 r)P_n(\cos\theta)\cos\theta\sin\varphi\,\hat{\theta} \\ &+\frac{k_1E_0}{\omega\mu_1}\sum_{n=0}^{\infty}i^n(2n+1)j_n(k_1 r)P_n(\cos\theta)\cos\varphi\,\hat{\varphi}.\end{aligned}$$

The boundary conditions on conducting sphere governed by the following conditions;

$$\hat{r}\cdot\vec{B}(R,\theta,\varphi) = 0 \tag{3.3}$$

$$\hat{r}\times\vec{E}(R,\theta,\varphi) = 0, \tag{3.4}$$

where each of the fields $\vec{E}$ and $\vec{B}$ consist of two distinct fields; one of which is incident field and the other one is the scattered field, thus boundary conditions take the form;

$$\begin{aligned}B_r^{inc}+B_r^{scat} &= 0 \\ E_\theta^{inc}+E_\theta^{scat} &= 0 \\ E_\varphi^{inc}+E_\varphi^{scat} &= 0.\end{aligned}$$

In the above relations the components of the incident field are known. We have to determine scattered field components. According to field definitions given by the equation (2.57) determination of field means determination unknown coefficients a^0_{nm} and b^0_{nm}. These coefficients is found by using orthogonality relations of Legendre functions as (Harrington, 2001) ;

$$a^0_{nm} = \begin{cases} \frac{i^n(2n+1)(krj_n(kr))'}{2wn(n+1)(krh^1_n(kr))'} & : m = 1 \\ \frac{-i^n(2n+1)(krj_n(kr))}{2w(krh^1_n(kr))} & : m = -1 \end{cases}$$
$$b^0_{nm} = \begin{cases} \frac{i^n(2n+1)j_n(kr)}{2wn(n+1)h^1_n(kr)} & : m = 1 \\ \frac{-i^n(2n+1)j_n(kr)}{2wh^1_n(kr)} & : m = -1, \end{cases} \tag{3.5}$$

where primes represent the derivatives with respect to argument of the spherical Bessel functions. Since the incident and scattered fields for the conducting sphere are known, we can proceed to the fields for deformed conducting sphere.

3.1.1. Scattering from Deformed Conducting Sphere; First order Perturbed Fields

In this subsection using the previously obtain boundary conditions (2.57) we will try to find perturbed fields. Total fields outside the scatterer can be written as;

$$\overrightarrow{E}(r,\theta,\varphi) = \overrightarrow{E}^{inc}(r,\theta,\varphi) + \overrightarrow{E}^0(r,\theta,\varphi) + \beta\overrightarrow{E}^1(r,\theta,\varphi) + \beta^2\overrightarrow{E}^2(r,\theta,\varphi) \tag{3.6}$$
$$\overrightarrow{B}(r,\theta,\varphi) = \overrightarrow{B}^{inc}(r,\theta,\varphi) + \overrightarrow{B}^0(r,\theta,\varphi) + \beta\overrightarrow{B}^1(r,\theta,\varphi) + \beta^2\overrightarrow{B}^2(r,\theta,\varphi). \tag{3.7}$$

Inserting these expressions in to the boundary condition (2.84) we get

$$\begin{aligned} B^1_r(R,\theta,\varphi) = \Big[& -f(\theta,\varphi)r\frac{\partial}{\partial r}\Big(B^0_r(r,\theta,\varphi) + B^{inc}_r(r,\theta,\varphi)\Big) \\ & + f_\theta(\theta,\varphi)\Big(B^0_\theta(r,\theta,\varphi) + B^{inc}_\theta(r,\theta,\varphi)\Big) \\ & + \frac{f_\varphi(\theta,\varphi)}{\sin(\theta)}\Big(B^0_\varphi(r,\theta,\varphi) + B^{inc}_\varphi(r,\theta,\varphi)\Big)\Big]_{r=R}. \end{aligned} \tag{3.8}$$

Multiplying both side by $P_l^{m'}(\cos\theta)\sin\theta e^{-im'\varphi}$ and taking angular integrals gives us;

$$
\begin{aligned}
b^1_{lm'} &= \left(\frac{\omega\mu\epsilon(2l+1)(l-m')!}{4\pi i(l+m')!\left[(\frac{\partial^2}{\partial r^2}+k^2)(krz_l(kr))\right]}\right) \\
&\quad\times\left(\sum_{n,m}\sum_{j,s}T_1(n,m,j,s)+\sum_{n}\sum_{j,s}T_2(n,j,s)\right),
\end{aligned} \tag{3.9}
$$

where $T_1(n,m,j,s)$ and $T_2(n,j,s)$ are given in the appendix C. To find the the coefficients of a^1_{nm}, let us consider following equation obtained from the boundary condition (2.80) ;

$$
\begin{aligned}
E^1_\varphi(R,\theta,\varphi) &= -fr\frac{\partial}{\partial r}\Big(E^0_\varphi(r,\theta,\varphi)+E^{inc}_\varphi(r,\theta,\varphi)\Big)_{r=R} \\
&\quad -\frac{f_\varphi}{\sin\theta}\Big(E^0_r(r,\theta,\varphi)+E^{inc}_r(r,\theta,\varphi)\Big)_{r=R}.
\end{aligned} \tag{3.10}
$$

Multiplying both side with $P_l^{m'}(\cos\theta)\sin^2\theta e^{-im'\varphi}$ and taking the angular integrals enables us to express unknown coefficients a^1_{nm} in terms of a^0_{nm} , b^0_{nm} and previously found b^1_{nm} according to following relation;

$$
\begin{aligned}
a^1_{lm'} &= -\frac{\omega\mu\epsilon(2l+1)(l-m')!}{4\pi m'(l+m')!\left[\frac{1}{r}\frac{\partial}{\partial r}(krz_l(kr))\right]_{r=R}}\Bigg\{ \\
&\quad +\frac{b^1_{l+1,m'}\delta_{m'sm'}}{\epsilon}\frac{2(l+m'+1)(l+2)(l+m')!}{(2l+3)(2l+1)(l-m')!}kz_{l+1}(kR) \\
&\quad -\frac{b^1_{l-1,m'}\delta_{m'sm'}}{\epsilon}\frac{2(l-1)(l-m')(l+m')!}{(2l-1)(2l+1)(l-m')!}kz_{l-1}(kR) \\
&\quad +\Big(\sum_{n,m}\sum_{j,s}T_3(n,m,j,s)+\sum_{n}\sum_{j,s}T_4(n,j,s)\Big)\Bigg\}_{r=R}.
\end{aligned}
$$

Explicit expressions of $T_3(n,m,j,s)$ and $T_4(n,j,s)$ are given in appendix C. With this result, we have evaluated the first order perturbed fields.

3.1.2. Second Order Perturbed Fields

Since the fields belongs to the unperturbed and first order perturbed fields are known the second order perturbed field can be found. The boundary condition (2.85) is suitable to start with because it includes only the coefficients b^2_{nm} as unknown;

$$
\begin{aligned}
B_r^2(R,\theta,\varphi) \;=\; & \Big[-\frac{f^2r^2}{2}\frac{\partial^2}{\partial r^2}\Big(B_r^0(r,\theta,\varphi)+B_r^{inc}(r,\theta,\varphi)\Big) \\
& + f_\theta f r\frac{\partial}{\partial r}\Big(B_\theta^0(r,\theta,\varphi)+B_\theta^{inc}(r,\theta,\varphi)\Big) \\
& - f_\theta f\Big(B_\theta^0(r,\theta,\varphi)+B_\theta^{inc}(r,\theta,\varphi)\Big) \\
& + \frac{f_\varphi}{\sin\theta} f r\frac{\partial}{\partial r}\Big(B_\varphi^0(r,\theta,\varphi)+B_\varphi^{inc}(r,\theta,\varphi)\Big) \\
& - f\frac{f_\varphi}{\sin\theta}\Big(B_\varphi^0(r,\theta,\varphi)+B_\varphi^{inc}(r,\theta,\varphi)\Big) \\
& - f r\frac{\partial}{\partial r}B_r^1(r,\theta,\varphi) + f_\theta B_\theta^1(r,\theta,\varphi) + \frac{f_\varphi}{\sin\theta}B_\varphi^1(r,\theta,\varphi)\Big]_{r=R}.
\end{aligned}
$$

By the definition of the field components with equation (2.57), multiplication with the $P_l^{m'}(\cos\theta)\sin\theta e^{-im'\varphi}$ and integration gives the following final result

$$
b^2_{lm'} = \frac{\omega\mu\epsilon(2l+1)(l-m')}{4\pi i(kRz_l(kR))(l+m')!}\left(\sum_{n,m}\sum_{j,s}\sum_{k=5}^{18}T_k(n,m,j,s)+\sum_{n}\sum_{j,s}\sum_{k=19}^{22}T_k(n,j,s)\right). \quad (3.11)
$$

Explicit expressions of T_k's are given in the appendix C. To find the coefficients a^2_{lm}, let us begin with the boundary condition (2.81);

$$
\begin{aligned}
E_\varphi^2(R,\theta,\varphi) \;=\; & \Big[\frac{ff_\varphi}{\sin\theta}E_r^0(r,\theta,\varphi) - \frac{f_\varphi}{\sin\theta}\Big(fr\frac{\partial}{\partial r}E_r^0(r,\theta,\varphi)+E_r^1(r,\theta,\varphi)\Big) \\
& -\Big(\frac{f^2r^2}{2}\frac{\partial^2}{\partial r^2}E_\varphi^0(r,\theta,\varphi)+fr\frac{\partial}{\partial r}E_\varphi^1(r,\theta,\varphi)\Big)\Big]_{r=R} = 0.
\end{aligned}
$$

By the definition of the field components (2.57), multiplication with the $P_l^{m'}(\cos\theta)\sin^2\theta e^{-im'\varphi}$ and integration gives;

$$
\begin{aligned}
a^2_{lm'} = & -\frac{\omega\mu\epsilon(2l+1)(l-m')!}{4\pi m'(l+m)![\frac{1}{r}\frac{\partial}{\partial r}(krz_l(kr))]_{r=R}}\Bigg\{ \\
& + \frac{b^2_{l+1,m'}\delta_{m'sm'}}{\epsilon}\frac{2(l+m'+1)(l+2)(l+m')!}{(2l+3)(2l+1)(l-m')!}kz_{l+1}(kR) \\
& - \frac{b^2_{l-1,m'}\delta_{m'sm'}}{\epsilon}\frac{2(l-1)(l-m')(l+m')!}{(2l-1)(2l+1)(l-m')!}kz_{l-1}(kR) \\
& + \left(\sum_{n,m}\sum_{j,s}\sum_{k=23}^{29}T_k(n,m,j,s) + \sum_{n}\sum_{j,s}\sum_{k=30}^{32}T_k(n,j,s)\right)\Bigg\}.
\end{aligned}
\tag{3.12}
$$

Expressions for T_k can be found in appendix C. With this result, we have completely found unknown coefficients and determined the first and second order perturbed fields.

3.2. Scattering of Plane Waves From Deformed Dielectric Sphere

In this subsection, we shortly review scattering of plane waves from dielectric spheres (Harrington, 2001). We consider smooth deformation of a sphere whose constitutive parameters are given with real numbers ϵ_2 and μ_2 . The medium outside the deformed dielectric is characterized with real parameters are ϵ_1 and μ_1.

In the above figure $\vec{E}_{inc}$ represents incident uniform electromagnetic plane wave. $\vec{E}_{scat}$ and $\vec{E}_{trans}$ represent total scattered and transmitted fields respectively.

3.2.1. Scattering of Plane Waves From Dielectric Sphere

Let a dielectric sphere characterized with (ϵ_2, μ_2) is placed in a medium characterized with (ϵ_1, μ_1). A x polarized uniform plane wave moving in z direction with the time dependence $e^{-i\omega t}$ is given by;

$$
E_{inc} = E_0 e^{ikz}\hat{x}. \tag{3.13}
$$

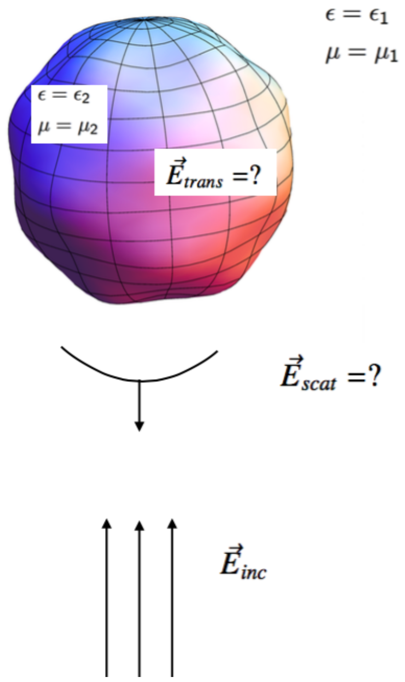

Figure 3.1. Scattering from dielectric deformed sphere.

The scattered and transmitted fields must be in the following form (Eom, 2004)

$$\begin{aligned} E_r^{(s,t)} &= \sum_{n,m} \frac{ia_{nm}^{(s,t)}}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial r^2} + k^2\right)(krz_n(kr))P_n^m(\cos\theta)e^{im\varphi} \qquad (3.14) \\ E_\theta^{(s,t)} &= \sum_{n,m} \frac{ia_{nm}^{(s,t)}}{\omega\mu\epsilon r}\frac{\partial}{\partial r}(kr\,z_n(kr))\frac{\partial}{\partial\theta}P_n^m(\cos\theta)e^{im\varphi} \\ &- \sum_{n,m} \frac{im}{\epsilon r\sin\theta}b_{nm}^{(s,t)}\;kr\,z_n(kr)P_n^m(\cos\theta)e^{im\varphi} \\ E_\varphi^{(s,t)} &= -\sum_{n,m} \frac{ma_{nm}^{(s,t)}}{\omega\mu\epsilon r\sin\theta}\frac{\partial}{\partial r}(kr\,z_n(kr))P_n^m(\cos\theta)e^{im\varphi} \\ &+ \sum_{n,m} \frac{b_{nm}^{(s,t)}}{\epsilon r}krz_n(kr)\frac{\partial}{\partial\theta}P_n^m(\cos\theta)e^{im\varphi} \end{aligned}$$

$$
\begin{aligned}
H_r^{(s,t)} &= \sum_{n,m} \frac{ib_{nm}^{(s,t)}}{\omega\mu\epsilon}(\frac{\partial^2}{\partial r^2} + k^2)(krz_n(kr))P_n^m(\cos\theta)e^{im\varphi} \\
H_\theta^{(s,t)} &= \sum_{n,m} \frac{im}{\mu r\sin\theta}a_{nm}^{(s,t)}\ kr\,z_n(kr)P_n^m(\cos\theta)e^{im\varphi} \\
&+ \sum_{n,m} \frac{ib_{nm}^{(s,t)}}{\omega\mu\epsilon r}\frac{\partial}{\partial r}(\,kr\,z_n(kr))\frac{\partial}{\partial\theta}P_n^m(\cos\theta)e^{im\varphi} \\
H_\varphi^{(s,t)} &= -\sum_{n,m} \frac{1}{\mu r}a_{nm}^{(s,t)}\ kr\,z_n(kr)\frac{\partial}{\partial\theta}P_n^m(\cos\theta)e^{im\varphi} \\
&- \sum_{n,m} \frac{mb_{nm}^{(s,t)}}{\omega\mu\epsilon r\sin\theta}\frac{\partial}{\partial r}(krz_n(kr))P_n^m(\cos\theta)e^{im\varphi}.
\end{aligned}
$$

For the dielectric case, we use the following boundary conditions (Balanis, 1989);

$$
\begin{aligned}
\hat{r}\times(\vec{E}_1 - \vec{E}_2)|_{r=R} &= 0 \\
\hat{r}\times(\vec{H}_1 - \vec{H}_2)|_{r=R} &= 0 \qquad (3.15)
\end{aligned}
$$

with

$$
\begin{aligned}
\vec{E}_1(r,\theta,\varphi) &= \vec{E}_i(r,\theta,\varphi) + \vec{E}_s^0(r,\theta,\varphi) \\
\vec{E}_2(r,\theta,\varphi) &= \vec{E}_t^0(r,\theta,\varphi), \qquad (3.16)
\end{aligned}
$$

where $\vec{E}_i$ denotes incident complex electric field, $\vec{E}_s^0$ denotes scattered complex electric field and $\vec{E}_t^0$ denotes transmitted complex electric field from sphere, writing the fields in the boundary condition, we get;

$$
\begin{aligned}
E_{s,\varphi}^0(R,\theta,\varphi) - E_{t,\varphi}^0(R,\theta,\varphi) &= -E_{i,\varphi}(R,\theta,\varphi) \qquad (3.17) \\
E_{s,\theta}^0(R,\theta,\varphi) - E_{t,\theta}^0(R,\theta,\varphi) &= -E_{i,\theta}(R,\theta,\varphi) \qquad (3.18) \\
H_{s,\varphi}^0(R,\theta,\varphi) - H_{t,\varphi}^0(R,\theta,\varphi) &= -H_{i,\varphi}(R,\theta,\varphi) \qquad (3.19) \\
H_{s,\theta}^0(R,\theta,\varphi) - H_{t,\theta}^0(R,\theta,\varphi) &= -H_{i,\theta}(R,\theta,\varphi). \qquad (3.20)
\end{aligned}
$$

In order to find the unknown coefficients $a_{nm}^{0s}, b_{nm}^{0s}, a_{nm}^{0t}, b_{nm}^{0t}$ one can not use each equation separately and use the orthogonality of Legendre polynomials because each equation contains simultaneously terms like $\frac{P_n^m}{\sin\theta}$ and $\frac{\partial P_n^m}{\partial\theta}$ which prevent to use orthogonality properties of Legendre polynomials. Multiplying (3.17) with $m' P_l^{m'} e^{-im'\varphi}$ and multiplying equation (3.18) with $i \sin\theta \frac{\partial P_l^{m'}}{\partial\theta} e^{-im'\varphi}$ summation of these two equations also multiplication of (3.19) with $\sin\theta \frac{\partial P_l^{m'}}{\partial\theta} e^{-im'\varphi}$ and multiplication of (3.20) with $im' P_l^{m'} e^{-im'\varphi}$ and adding them up with the help of following properties of Legendre Polynomials (Stratton, 1941),

$$\int_0^\pi \left(\frac{\partial P_n^m}{\partial\theta} \frac{\partial P_{n'}^m}{\partial\theta} + m^2 \frac{P_n^m}{\sin\theta} \frac{P_{n'}^m}{\sin\theta} \right) \sin\theta d\theta = \begin{cases} 0 & n \neq n' \\ \frac{2(n+m)! n(n+1)}{(2n+1)(n-m)!} & n = n' \end{cases} \tag{3.21}$$

$$\int_0^\pi \left(\frac{\partial P_n^m}{\partial\theta} \frac{P_{n'}^m}{\sin\theta} + \frac{P_n^m}{\sin\theta} \frac{\partial P_{n'}^m}{\partial\theta} \right) \sin\theta d\theta = 0, \qquad \forall n, n', m, \tag{3.22}$$

we get a linear system of equation for a_{nm}^{0s}, a_{nm}^{0t} as;

$$c_1 a_{lm'}^{0s} + c_2 a_{lm'}^{0t} = \psi_1 \tag{3.23}$$

$$c_3 a_{lm'}^{0s} + c_4 a_{lm'}^{0t} = \psi_2, \tag{3.24}$$

where

$$c_1 = -\frac{2\pi}{\omega\mu_1\epsilon_1} \frac{2l(l+1)(l+m')!}{(2l+1)(l-m')!} \left[\frac{1}{r} \frac{\partial}{\partial r} (k_1 r\, z_l^s(k_1 r)) \right]_{r=R} \tag{3.25}$$

$$c_2 = \frac{2\pi}{\omega\mu_2\epsilon_2} \frac{2l(l+1)(l+m')!}{(2l+1)(l-m')!} \left[\frac{1}{r} \frac{\partial}{\partial r} (k_2 r\, z_l^t(k_2 r)) \right]_{r=R} \tag{3.26}$$

$$c_3 = -\frac{2\pi}{\mu_1} \frac{2l(l+1)(l+m')!}{(2l+1)(l-m')!} \left(k_1\, z_l^s(k_1 r)\right)_{r=R} \tag{3.27}$$

$$c_4 = \frac{2\pi}{\mu_2} \frac{2l(l+1)(l+m')!}{(2l+1)(l-m')!} \left(k_2\, z_l^t(k_2 r)\right)_{r=R} \tag{3.28}$$

$$\psi_1 = -\int_0^{2\pi} \int_0^\pi E_{i,\varphi}(R,\theta,\varphi) \frac{m' P_l^{m'} e^{-im'\varphi}}{\sin\theta} \sin\theta d\theta d\varphi - i \int_0^{2\pi} \int_0^\pi E_{i,\theta}(R,\theta,\varphi) \frac{\partial P_l^{m'}}{\partial\theta} e^{-im'\varphi} \sin\theta d\theta d\varphi \tag{3.29}$$

$$\psi_2 = -\int_0^{2\pi} \int_0^\pi H_{i,\varphi}(R,\theta,\varphi) \frac{\partial P_l^{m'}}{\partial\theta} e^{-im'\varphi} \sin\theta d\theta d\varphi - i \int_0^{2\pi} \int_0^\pi H_{i,\theta}(R,\theta,\varphi) \frac{m' P_l^{m'} e^{-im'\varphi}}{\sin\theta} \sin\theta d\theta d\varphi. \tag{3.30}$$

Then the unknown coefficients $a^{0s}_{lm'}, a^{0t}_{lm'}$ found to be ;

$$a^{0s}_{lm'} = (c_4\psi_1(l,m') - c_2\psi_2(l,m'))/(c_1c_4 - c_2c_3) \tag{3.31}$$

$$a^{0t}_{am'} = (c_1\psi_2(l,m') - c_3\psi_1(l,m'))/(c_1c_4 - c_2c_3). \tag{3.32}$$

To find the unknown coefficients $b^{0s}_{lm'}, b^{0t}_{0m'}$ we multiply equation (3.17) with $\sin\theta\frac{\partial P^{m'}_l}{\partial\theta}e^{-im'\varphi}$ and equation (3.18) with $im'P^{m'}_l e^{-im'\varphi}$ then sum these two equations also multiply equation (3.19) with $-m'P^{m'}_l e^{-im'\varphi}$ and multiply of (3.20) with $-i\sin\theta\frac{\partial P^{m'}_l}{\partial\theta}e^{-im'\varphi}$ and adding them up, we get a simple system of equation just for $b^{0s}_{lm'}, b^{0t}_{lm'}$ as;

$$c_5 b^{0s}_{lm'} + c_6 b^{0t}_{lm'} = \psi_3 \tag{3.33}$$

$$c_7 b^{0s}_{lm'} + c_8 b^{0t}_{lm'} = \psi_4, \tag{3.34}$$

where

$$c_5 = \frac{1}{\epsilon_1}\frac{2l(l+1)(l+m')!}{(2l+1)(l-m')!}\left(k_1\, z^s_l(k_1 r)\right)_{r=R} \tag{3.35}$$

$$c_6 = -\frac{1}{\epsilon_2}\frac{2l(l+1)(l+m')!}{(2l+1)(l-m')!}\left(k_2\, z^t_l(k_2 r)\right)_{r=R} \tag{3.36}$$

$$c_7 = \frac{1}{\omega\mu_1\epsilon_1}\frac{2l(l+1)(l+m')!}{(2l+1)(l-m')!}\left[\frac{1}{r}\frac{\partial}{\partial r}(k_1 r\, z^s_l(k_1 r))\right]_{r=R} \tag{3.37}$$

$$c_8 = -\frac{1}{\omega\mu_2\epsilon_2}\frac{2l(l+1)(l+m')!}{(2l+1)(l-m')!}\left[\frac{1}{r}\frac{\partial}{\partial r}(k_2 r\, z^t_l(k_2 r))\right]_{r=R} \tag{3.38}$$

$$\psi_3 = -\int_0^{2\pi}\int_0^{\pi} E_{i,\varphi}(R,\theta,\varphi)\frac{\partial P^{m'}_l}{\partial\theta}e^{-im'\varphi}\sin\theta d\theta d\varphi$$
$$-i\int_0^{2\pi}\int_0^{\pi} E_{i,\theta}(R,\theta,\varphi)\frac{m'P^{m'}_l e^{-im'\varphi}}{\sin\theta}\sin\theta d\theta d\varphi \tag{3.39}$$

$$\psi_4 = \int_0^{2\pi}\int_0^{\pi} H_{i,\varphi}(R,\theta,\varphi)\frac{m'P^{m'}_l e^{-im'\varphi}}{\sin\theta}\sin\theta d\theta d\varphi$$
$$+i\int_0^{2\pi}\int_0^{\pi} H_{i,\theta}(R,\theta,\varphi)\frac{\partial P^{m'}_l}{\partial\theta}e^{-im'\varphi}\sin\theta d\theta d\varphi. \tag{3.40}$$

Then the unknown coefficients $b^{0s}_{lm'}, b^{0t}_{lm'}$ found to be ;

$$\begin{aligned} b^{0s}_{lm'} &= (c_8\psi_3 - c_6\psi_4)/(c_5c_8 - c_6c_7) \quad (3.41)\\ b^{0t}_{lm'} &= (c_5\psi_4 - c_7\psi_3)/(c_5c_8 - c_6c_7). \quad (3.42)\end{aligned}$$

3.2.2. First order Corrections

Having found the incident, scattered and transmitted fields for the sphere which, in our representation, are represented with the field $\vec{E}_i$, $\vec{E}^0_s$ and $\vec{E}^0_t$, we are ready to find first and second order correction fields for the deformed dielectric spherical object. The continuity of the tangential components of the electric and magnetic fields must be fulfilled

$$\begin{aligned} \hat{n}\times(\vec{E}_1 - \vec{E}_2)|_{r=\tilde{R}} &= 0\\ \hat{n}\times(\vec{H}_1 - \vec{H}_2)|_{r=\tilde{R}} &= 0, \quad (3.43)\end{aligned}$$

where

$$\begin{aligned} \vec{E}_1(r,\theta,\varphi) &= \vec{E}_i(r,\theta,\varphi) + \vec{E}^0_s(r,\theta,\varphi) + \beta\vec{E}^1_s(r,\theta,\varphi) + \beta^2\vec{E}^2_s(r,\theta,\varphi)\\ \vec{E}_2(r,\theta,\varphi) &= \vec{E}^0_t(r,\theta,\varphi) + \beta\vec{E}^1_t(r,\theta,\varphi) + \beta^2\vec{E}^2_t(r,\theta,\varphi). \quad (3.44)\end{aligned}$$

In the above expression, $\vec{E}_1$ is the total field in outer domain, $\vec{E}_i$ denotes incident complex electric field, $\vec{E}^0_s$ denotes scattered complex electric field from sphere, $\vec{E}^1_s$ is responsible for the first order correction to the scattered field and $\vec{E}^2_s$ is standing for second order corrections. $\vec{E}_2$ is the total field inside the scatterer, $\vec{E}^1_t$ is responsible for the first order correction to the transmitted field and $\vec{E}^2_t$ is standing for the second order correction. Now, we will try to determine the fields $\vec{E}^1_s$ and $\vec{E}^1_t$. Writing expressions for the electric and magnetic fields given with (3.44) into the (3.43) and using previously obtained boundary conditions (2.80) and (2.82) we get;

$$
\begin{aligned}
E^1_{s,\varphi}(R,\theta,\varphi) - E^1_{t,\varphi}(R,\theta,\varphi) &= \chi_1(R,\theta,\varphi) && (3.45)\\
E^1_{s,\theta}(R,\theta,\varphi) - E^1_{t,\theta}(R,\theta,\varphi) &= \chi_2(R,\theta,\varphi) && (3.46)\\
H^1_{s,\varphi}(R,\theta,\varphi) - H^1_{t,\varphi}(R,\theta,\varphi) &= \chi_3(R,\theta,\varphi) && (3.47)\\
H^1_{s,\theta}(R,\theta,\varphi) - H^1_{t,\theta}(R,\theta,\varphi) &= \chi_4(R,\theta,\varphi), && (3.48)
\end{aligned}
$$

where

$$
\begin{aligned}
\chi_1(R,\theta,\varphi) &= -\Big[\frac{f_\varphi}{\sin\theta}[E_{i,r}(r,\theta,\varphi) + E^0_{s,r}(r,\theta,\varphi) - E^0_{t,r}(r,\theta,\varphi)]\\
&\quad + fr\frac{\partial}{\partial r}[E_{i,\varphi}(r,\theta,\varphi) + E^0_{s,\varphi}(r,\theta,\varphi) - E^0_{t,\varphi}(r,\theta,\varphi)]\Big]_{r=R}\\
\chi_2(R,\theta,\varphi) &= -\Big[fr\frac{\partial}{\partial r}[E_{i,\theta}(r,\theta,\varphi) + E^0_{s,\theta}(r,\theta,\varphi) - E^0_{t,\theta}(r,\theta,\varphi)]\\
&\quad + f_\theta[E_{i,r}(r,\theta,\varphi) + E^0_{s,r}(r,\theta,\varphi) - E^0_{t,r}(r,\theta,\varphi)]\Big]_{r=R}\\
\chi_3(R,\theta,\varphi) &= -\Big[\frac{f_\varphi}{\sin\theta}[H_{i,r}(r,\theta,\varphi) + H^0_{s,r}(r,\theta,\varphi) - H^0_{t,r}(r,\theta,\varphi)]\\
&\quad + fr\frac{\partial}{\partial r}[H_{i,\varphi}(r,\theta,\varphi) + H^0_{s,\varphi}(r,\theta,\varphi) - H^0_{t,\varphi}(r,\theta,\varphi)]\Big]_{r=R}\\
\chi_4(R,\theta,\varphi) &= -\Big[fr\frac{\partial}{\partial r}[H_{i,\theta}(r,\theta,\varphi) + H^0_{s,\theta}(r,\theta,\varphi) - H^0_{t,\theta}(r,\theta,\varphi)]\\
&\quad + f_\theta[H_{i,r}(r,\theta,\varphi) + H^0_{s,r}(r,\theta,\varphi) - H^0_{t,r}(r,\theta,\varphi)]\Big]_{r=R}.
\end{aligned}
$$

The solution of linear system of equations in (3.45) will enable us find first order corrections to the scattered and transmitted fields. Writing the exact expressions of the fields from (2.57) shows that the left hand side of all equations contains infinite summations.

$$
\begin{aligned}
E^{1(s,t)}_\theta &= \sum_{n,m}\frac{ia^{1(s,t)}_{nm}}{\omega\mu\epsilon r}\frac{\partial}{\partial r}(kr\, z_n(kr))\frac{\partial}{\partial\theta}P^m_n(\cos\theta)e^{im\varphi}\\
&\quad - \sum_{n,m}\frac{im}{\epsilon r\sin\theta}b^{1(s,t)}_{nm}\; kr\, z_n(kr)P^m_n(\cos\theta)e^{im\varphi}\\
E^{1(s,t)}_\varphi &= -\sum_{n,m}\frac{ma^{1(s,t)}_{nm}}{\omega\mu\epsilon r\sin\theta}\frac{\partial}{\partial r}(kr\, z_n(kr))P^m_n(\cos\theta)e^{im\varphi}\\
&\quad + \sum_{n,m}\frac{b^{1(s,t)}_{nm}}{\epsilon r}krz_n(kr)\frac{\partial}{\partial\theta}P^m_n(\cos\theta)e^{im\varphi}
\end{aligned}
$$

$$
\begin{aligned}
H_\theta^{1(s,t)} &= \sum_{n,m} \frac{im}{\mu r \sin\theta} a_{nm}^{1(s,t)}\, kr\, z_n(kr) P_n^m(\cos\theta) e^{im\varphi} \\
&+ \sum_{n,m} \frac{ib_{nm}^{1(s,t)}}{\omega\mu\epsilon r} \frac{\partial}{\partial r}(\, kr\, z_n(kr)) \frac{\partial}{\partial\theta} P_n^m(\cos\theta) e^{im\varphi} \\
H_\varphi^{1(s,t)} &= -\sum_{n,m} \frac{1}{\mu r} a_{nm}^{1(s,t)}\, kr\, z_n(kr) \frac{\partial}{\partial\theta} P_n^m(\cos\theta) e^{im\varphi} \\
&- \sum_{n,m} \frac{mb_{nm}^{1(s,t)}}{\omega\mu\epsilon r \sin\theta} \frac{\partial}{\partial r}(krz_n(kr)) P_n^m(\cos\theta) e^{im\varphi}.
\end{aligned}
$$

In order to find the unknown coefficients $a_{nm}^{1s}, b_{nm}^{1s}, a_{nm}^{1t}, b_{nm}^{1t}$, one should notice that the constants in front of the $a_{nm}^{1(s,t)}$ in $E_\theta(H_\theta)$ and $E_\varphi(H_\varphi)$ looks similar, thus multiplying (3.45) with $m' P_l^{m'} e^{-im'\varphi}$ and multiplying equation (3.46) with $\sin\theta \frac{i\partial P_l^{m'}}{\partial\theta} e^{-im'\varphi}$ summation of these two equations also multiplication of (3.47) with $\sin\theta \frac{\partial P_l^{m'}}{\partial\theta} e^{-im'\varphi}$ and multiplication of (3.48) with $im' P_l^{m'} e^{-im'\varphi}$ and adding them up with the previously mentioned identities(3.21)-(3.22), we get a simple system of equation just for a_{nm}^{1s}, a_{nm}^{1t} as;

$$
\lambda_1 a_{lm'}^{1s} + \lambda_2 a_{lm'}^{1t} = \nu_1 \tag{3.49}
$$

$$
\lambda_3 a_{lm'}^{1s} + \lambda_4 a_{lm'}^{1t} = \nu_2, \tag{3.50}
$$

where

$$
\lambda_1 = -\frac{1}{\omega\mu_1\epsilon_1} \frac{2l(l+1)(l+m')!}{(2l+1)(l-m')!} \left[\frac{1}{r} \frac{\partial}{\partial r}(kr\, z_l^s(kr)) \right]_{r=R} \tag{3.51}
$$

$$
\lambda_2 = \frac{1}{\omega\mu_2\epsilon_2} \frac{2l(l+1)(l+m')!}{(2l+1)(l-m')!} \left[\frac{1}{r} \frac{\partial}{\partial r}(kr\, z_l^t(kr)) \right]_{r=R} \tag{3.52}
$$

$$
\lambda_3 = -\frac{1}{\mu_1} \frac{2l(l+1)(l+m')!}{(2l+1)(l-m')!} (k\, z_l^s(kr))_{r=R} \tag{3.53}
$$

$$
\lambda_4 = \frac{1}{\mu_2} \frac{2l(l+1)(l+m')!}{(2l+1)(l-m')!} (k\, z_l^t(kr))_{r=R} \tag{3.54}
$$

$$
\begin{aligned}
\nu_1 &= \int_0^{2\pi} \int_0^{\pi} \chi_1(R,\theta,\varphi) \frac{m' P_l^{m'} e^{-im'\varphi}}{\sin\theta} \sin\theta d\theta d\varphi \\
&+ i \int_0^{2\pi} \int_0^{\pi} \chi_2(R,\theta,\varphi) \frac{\partial P_l^{m'}}{\partial\theta} e^{-im'\varphi} \sin\theta d\theta d\varphi
\end{aligned} \tag{3.55}
$$

$$
\begin{aligned}
\nu_2 &= \int_0^{2\pi} \int_0^{\pi} \chi_3(R,\theta,\varphi) \frac{\partial P_l^{m'}}{\partial\theta} e^{-im'\varphi} \sin\theta d\theta d\varphi \\
&+ i \int_0^{2\pi} \int_0^{\pi} \chi_4(R,\theta,\varphi) \frac{m' P_l^{m'} e^{-im'\varphi}}{\sin\theta} \sin\theta d\theta d\varphi.
\end{aligned} \tag{3.56}
$$

Then the unknown coefficients $a^{1s}_{lm'}, a^{1t}_{lm'}$ found to be ;

$$a^{1s}_{lm'} = (\lambda_4\nu_1 - \lambda_2\nu_2)/(\lambda_1\lambda_4 - \lambda_2\lambda_3) \tag{3.57}$$

$$a^{1t}_{am'} = (\lambda_1\nu_2 - \lambda_3\nu_1)/(\lambda_1\lambda_4 - \lambda_2\lambda_3). \tag{3.58}$$

To find unknown coefficients $b^{1s}_{lm'}, b^{1t}_{lm'}$, we multiply equation (3.45) with $\sin\theta\frac{\partial P^{m'}_l}{\partial\theta}e^{-im'\varphi}$ and equation (3.46) with $im'P^{m'}_l e^{-im'\varphi}$ then sum these two equations also multiply equation (3.47) with $-m'P^{m'}_l e^{-im'\varphi}$ and multiply of (3.48) with $-i\sin\theta\frac{\partial P^{m'}_l}{\partial\theta}e^{-im'\varphi}$ and adding them up, we get a simple system of equation just for $b^{1s}_{lm'}, b^{1t}_{lm'}$ as;

$$\lambda_5 b^{1s}_{lm'} + \lambda_6 b^{1t}_{lm'} = \nu_3 \tag{3.59}$$

$$\lambda_7 b^{1s}_{lm'} + \lambda_8 b^{1t}_{lm'} = \nu_4, \tag{3.60}$$

where

$$\lambda_5 = \frac{1}{\epsilon_1}\frac{2l(l+1)(l+m')!}{(2l+1)(l-m')!}\left(k\, z^s_l(kr)\right)_{r=R} \tag{3.61}$$

$$\lambda_6 = -\frac{1}{\epsilon_2}\frac{2l(l+1)(l+m')!}{(2l+1)(l-m')!}\left(k\, z^t_l(kr)\right)_{r=R} \tag{3.62}$$

$$\lambda_7 = \frac{1}{\omega\mu_1\epsilon_1}\frac{2l(l+1)(l+m')!}{(2l+1)(l-m')!}\left[\frac{1}{r}\frac{\partial}{\partial r}(kr\, z^s_l(kr))\right]_{r=R} \tag{3.63}$$

$$\lambda_8 = -\frac{1}{\omega\mu_2\epsilon_2}\frac{2l(l+1)(l+m')!}{(2l+1)(l-m')!}\left[\frac{1}{r}\frac{\partial}{\partial r}(kr\, z^t_l(kr))\right]_{r=R} \tag{3.64}$$

$$\begin{aligned}\nu_3 &= \int_0^{2\pi}\int_0^{\pi}\chi_1(R,\theta,\varphi)\frac{\partial P^{m'}_l}{\partial\theta}e^{-im'\varphi}\sin\theta d\theta d\varphi \\ &\quad +i\int_0^{2\pi}\int_0^{\pi}\chi_2(R,\theta,\varphi)\frac{m'P^{m'}_l e^{-im'\varphi}}{\sin\theta}\sin\theta d\theta d\varphi \end{aligned} \tag{3.65}$$

$$\begin{aligned}\nu_4 &= -\int_0^{2\pi}\int_0^{\pi}\chi_3(R,\theta,\varphi)\frac{m'P^{m'}_l e^{-im'\varphi}}{\sin\theta}\sin\theta d\theta d\varphi \\ &\quad -i\int_0^{2\pi}\int_0^{\pi}\chi_4(R,\theta,\varphi)\frac{\partial P^{m'}_l}{\partial\theta}e^{-im'\varphi}\sin\theta d\theta d\varphi \end{aligned} \tag{3.66}$$

Then unknown coefficients $b^{1s}_{lm'}, b^{1t}_{lm'}$ found to be ;

$$
\begin{aligned}
b^{1s}_{lm'} &= (\lambda_8 \nu_3 - \lambda_6 \nu_4)/(\lambda_5 \lambda_8 - \lambda_6 \lambda_7) && (3.67)\\
b^{1t}_{lm'} &= (\lambda_5 \nu_4 - \lambda_7 \nu_3)/(\lambda_5 \lambda_8 - \lambda_6 \lambda_7). && (3.68)
\end{aligned}
$$

3.2.3. Second Order Corrections

Similar to the previous calculations, let us consider boundary conditions (2.81) and (2.83) which contains second order corrections. Writing the fields explicitly, we find that;

$$
\begin{aligned}
E^2_{s,\varphi}(R,\theta,\varphi) - E^2_{t,\varphi}(R,\theta,\varphi) &= \chi_5(R,\theta,\varphi) && (3.69)\\
E^2_{s,\theta}(R,\theta,\varphi) - E^2_{t,\theta}(R,\theta,\varphi) &= \chi_6(R,\theta,\varphi) && (3.70)\\
H^2_{s,\varphi}(R,\theta,\varphi) - H^2_{t,\varphi}(R,\theta,\varphi) &= \chi_7(R,\theta,\varphi) && (3.71)\\
H^2_{s,\theta}(R,\theta,\varphi) - H^2_{t,\theta}(R,\theta,\varphi) &= \chi_8(R,\theta,\varphi), && (3.72)
\end{aligned}
$$

where

$$
\begin{aligned}
\chi_5(R,\theta,\varphi) &= \Big[\frac{f f_\varphi}{\sin\theta}\left(E_{i,r} + E^0_{s,r} - E^0_{t,r}\right)\\
&\quad - \frac{f_\varphi}{\sin\theta}\left(fr\frac{\partial}{\partial r}\left(E_{i,r} + E^0_{s,r} - E^0_{t,r}\right) + E^1_{s,r} - E^1_{t,r}\right)\\
&\quad - \left(\frac{f^2 r^2}{2}\frac{\partial^2}{\partial r^2}\left(E_{i,\varphi} + E^0_{s,\varphi} - E^0_{t,\varphi}\right) + fr\frac{\partial}{\partial r}(E^1_{s,\varphi} - E^1_{t,\varphi})\right)\Big]_{r=R}\\
\chi_6(R,\theta,\varphi) &= \Big[+ f f_\theta(E_{i,r} + E^0_{s,r} - E^0_{t,r})\\
&\quad - f_\theta(fr\frac{\partial}{\partial r}(E_{i,r} + E^0_{s,r} - E^0_{t,r} + E^1_{s,r} - E^1_{t,r})\\
&\quad - \frac{f^2 r^2}{2}\frac{\partial^2}{\partial r^2}(E_{i,\theta} + E^0_{s,\theta} - E^0_{t,\theta}) - fr\frac{\partial}{\partial r}(E^1_{s,\theta} - E^1_{t,\theta})\Big]_{r=R}\\
\chi_7(R,\theta,\varphi) &= \Big[\frac{f f_\varphi}{\sin\theta}(H_{i,r} + H^0_{s,r} - H^0_{t,r})\\
&\quad - \frac{f_\varphi}{\sin\theta}\left(fr\frac{\partial}{\partial r}\left(H_{i,r} + H^0_{s,r} - H^0_{t,r}\right) + H^1_{s,r} - H^1_{t,r}\right)\\
&\quad - \left(\frac{f^2 r^2}{2}\frac{\partial^2}{\partial r^2}\left(H_{i,\varphi} + H^0_{s,\varphi} - H^0_{t,\varphi}\right) + fr\frac{\partial}{\partial r}(H^1_{s,\varphi} - H^1_{t,\varphi})\right)\Big]_{r=R}
\end{aligned}
$$

$$
\begin{aligned}
\chi_8(R,\theta,\varphi) &= \Big[+ f f_\theta (H_{i,r} + H^0_{s,r} - H^0_{t,r}) \\
&\quad - f_\theta (f r \frac{\partial}{\partial r}(H_{i,r} + H^0_{s,r} - H^0_{t,r}) + H^1_{s,r} - H^1_{t,r}) \\
&\quad - \frac{f^2 r^2}{2}\frac{\partial^2}{\partial r^2}(H_{i,\theta} + H^0_{s,\theta} - H^0_{t,\theta}) - f r \frac{\partial}{\partial r}(H^1_{s,\theta} - H^1_{t,\theta})\Big]_{r=R}. \qquad (3.73)
\end{aligned}
$$

In order to find unknown coefficients $a^{2s}_{lm'}, b^{2s}_{lm'}, a^{2t}_{lm'}, b^{2t}_{lm'}$, we multiply (3.69) with $m' P^{m'}_l e^{-im'\varphi}$ and multiplying equation (3.70) with $i \sin\theta \frac{\partial P^{m'}_l}{\partial\theta} e^{-im'\varphi}$ then sum these two equations, also we multiply (3.71) with $\sin\theta \frac{\partial P^{m'}_l}{\partial\theta} e^{-im'\varphi}$ and multiply (3.72) with $im' P^{m'}_l e^{-im'\varphi}$ and adding them up, we get system of equation just for $a^{2s}_{lm'}, a^{2t}_{lm'}$ as;

$$
\begin{aligned}
\lambda_1 a^{2s}_{lm'} + \lambda_2 a^{2t}_{lm'} &= \nu_5 \qquad (3.74)\\
\lambda_3 a^{2s}_{lm'} + \lambda_4 a^{2t}_{lm'} &= \nu_6, \qquad (3.75)
\end{aligned}
$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are evaluated in previous subsection and

$$
\begin{aligned}
\nu_5 &= \int_0^{2\pi}\int_0^{\pi} \chi_5(R,\theta,\varphi)\frac{m' P^{m'}_l e^{-im'\varphi}}{\sin\theta}\sin\theta d\theta d\varphi \\
&\quad + i\int_0^{2\pi}\int_0^{\pi} \chi_6(R,\theta,\varphi)\frac{\partial P^{m'}_l}{\partial\theta} e^{-im'\varphi}\sin\theta d\theta d\varphi \qquad (3.76)\\
\nu_6 &= \int_0^{2\pi}\int_0^{\pi} \chi_7(R,\theta,\varphi)\frac{\partial P^{m'}_l}{\partial\theta} e^{-im'\varphi}\sin\theta d\theta d\varphi \\
&\quad + i\int_0^{2\pi}\int_0^{\pi} \chi_8(R,\theta,\varphi)\frac{m' P^{m'}_l e^{-im'\varphi}}{\sin\theta}\sin\theta d\theta d\varphi. \qquad (3.77)
\end{aligned}
$$

Then unknown coefficients $a^{2s}_{lm'}, a^{2t}_{lm'}$ found to be ;

$$
\begin{aligned}
a^{2s}_{lm'} &= (\lambda_4\nu_5 - \lambda_2\nu_6)/(\lambda_1\lambda_4 - \lambda_2\lambda_3) \qquad (3.78)\\
a^{2t}_{lm'} &= (\lambda_1\nu_6 - \lambda_3\nu_5)/(\lambda_1\lambda_4 - \lambda_2\lambda_3). \qquad (3.79)
\end{aligned}
$$

To find unknown coefficients b^{2s}_{nm}, b^{2t}_{nm}, we multiply equation (3.69) with $\sin\theta \frac{\partial P^{m'}_l}{\partial\theta} e^{-im'\varphi}$ and equation (3.70) with $im' P^{m'}_l e^{-im'\varphi}$ then sum these two equations, also multiply equation (3.71) with $-m' P^{m'}_l e^{-im'\varphi}$ and multiply (3.72) with $-i\sin\theta \frac{\partial P^{m'}_l}{\partial\theta} e^{-im'\varphi}$ and adding them up;

we get a simple system of equation just for b^{2s}_{nm}, b^{2t}_{nm} as;

$$\lambda_5 b^{2s}_{lm'} + \lambda_6 b^{2t}_{lm'} = \nu_7 \tag{3.80}$$

$$\lambda_7 b^{2s}_{lm'} + \lambda_8 b^{2t}_{lm'} = \nu_8, \tag{3.81}$$

where $\lambda_5, \lambda_6, \lambda_7, \lambda_8$ are previously evaluated and

$$\begin{aligned} \nu_7 &= \int_0^{2\pi}\int_0^{\pi} \chi_5(R,\theta,\varphi)\frac{\partial P_l^{m'}}{\partial\theta} e^{-im'\varphi}\sin\theta d\theta d\varphi \\ &\quad +i\int_0^{2\pi}\int_0^{\pi} \chi_6(R,\theta,\varphi)\frac{m' P_l^{m'} e^{-im'\varphi}}{\sin\theta}\sin\theta d\theta d\varphi \end{aligned} \tag{3.82}$$

$$\begin{aligned} \nu_8 &= -\int_0^{2\pi}\int_0^{\pi} \chi_7(R,\theta,\varphi)\frac{m' P_l^{m'} e^{-im'\varphi}}{\sin\theta}\sin\theta d\theta d\varphi \\ &\quad -i\int_0^{2\pi}\int_0^{\pi} \chi_8(R,\theta,\varphi)\frac{\partial P_l^{m'}}{\partial\theta} e^{-im'\varphi}\sin\theta d\theta d\varphi. \end{aligned} \tag{3.83}$$

Then unknown coefficients $b^{2s}_{lm'}, b^{2t}_{lm'}$ found to be ;

$$b^{2s}_{lm'} = (\lambda_8\nu_7 - \lambda_6\nu_8)/(\lambda_5\lambda_8 - \lambda_6\lambda_7) \tag{3.84}$$

$$b^{2t}_{lm'} = (\lambda_5\nu_8 - \lambda_7\nu_7)/(\lambda_5\lambda_8 - \lambda_6\lambda_7). \tag{3.85}$$

With these results, we have completely determined the first and second order corrections to the scattered and transmitted fields.

CHAPTER 4

VALIDATION OF CALCULATIONS AND NUMERICAL RESULTS

In this chapter, we will compare our results with the known results in literature. Since the analytic expressions obtained in the previous chapters are too complicated, we make comparisons with the help of graphics which are obtained from data files produced by the Fortran code.

The scattering cross section is defined by the following formula (Skolnik, 1981)

$$\sigma = \lim_{r \to \infty} \left(4\pi r^2 \frac{|\vec{E}^s|^2}{|\vec{E}_{inc}|^2} \right). \tag{4.1}$$

The y axes of the graphs are the back scattering cross sections (Kotsis, 2007) if not stated else. The x axes of the graphs are the size parameters given by kR where k is the wave number and R is the radius of unperturbed sphere. Using the relation between wave number and wavelength λ, we can express size parameter as $2\pi R/\lambda$, which allows one to compare wavelength of the incident wave with radius of the sphere (Bohren, 1983).

1) We start to control our calculation with a choice of constant deformation function $f(\theta, \varphi) = C$. This deformation function is the easiest and the most essential function to check the agreement of the results within itself.As illustrated in Figure 4.1, starting with the sphere with radius R and making deformation with deformation parameter β and constant deformation function $f(\theta, \varphi) = C$ results in a new sphere with the radius $R(1 + \beta C)$.

Thus the scattering cross section obtained from the first and second order deformation of the sphere with a parameter β and deformation function C must be in agreement with the scattering cross section of the sphere with radius $R(1 + \beta C)$.

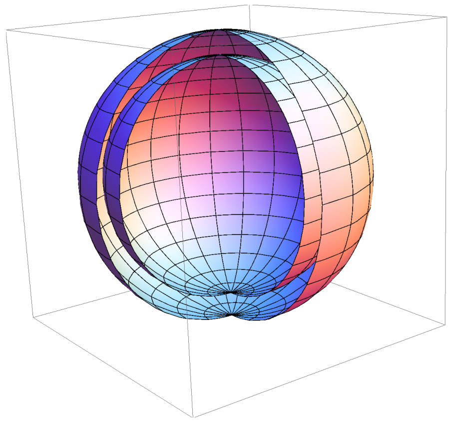

Figure 4.1. Deformation of the sphere with constant function.

The Figure 4.2 graphic is obtained from deformation of a dielectric sphere with radius 10 by the deformation function 1 and parameter $\beta = 0.05$.The medium parameters are chosen such as $\epsilon_{in}/\epsilon_{out} = (1.33)^2$ and $\mu_{in} = \mu_{out} = 1$.

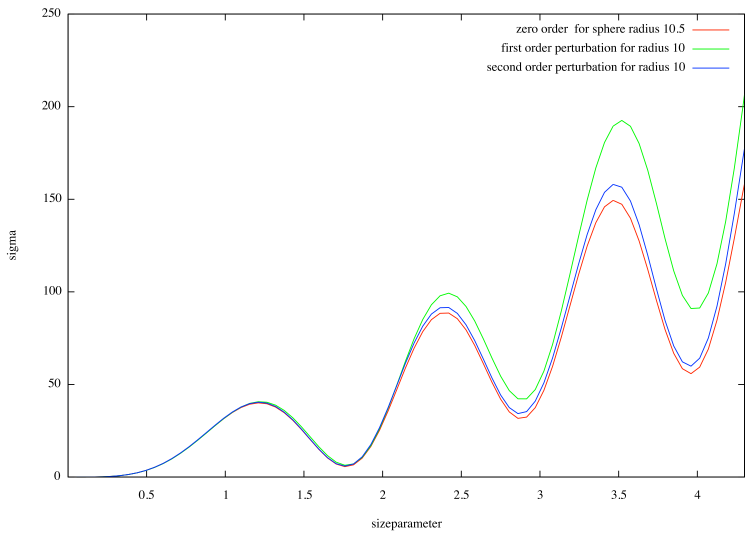

Figure 4.2. Effect of deformation of sphere with constant function.

Figure 4.2 emphasises the importance of the second order perturbation; it says that when size parameter is around 2, first order perturbation is a good approximation but when size parameters is greater then 2, we need second order perturbation in order to get more close to the exact solution.

2) After showing that our calculation is consistent within itself, we choose a deformation function independent of azimuthal angle (rotational symmetric) as;
$f(\theta, \varphi) = \sin^2(\theta)$. This deformation enables to obtain a spheroid from a sphere.

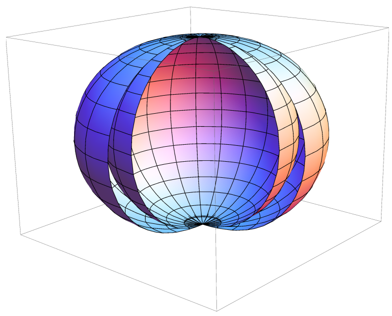

Figure 4.3. Obtaining spheroid from a sphere.

We compare our first order results with the paper of Yeh (Yeh, 1964). The parameters are chosen such as; $\beta = 0.0929705, \beta = 0.173554$ and $\epsilon_{in}/\epsilon_{out} = (1.33)^2$ and $\mu_{in} = \mu_{out} = 1$. For this choice, we found normalized cross section for the spheroid as in Figure 4.4. This figure is quite similar to the one in (Yeh, 1964) except for the values of size parameter between 2.5 and 3.0. As stated in (Erma, 1968) this difference may be caused by the sign error in the paper of Yeh.

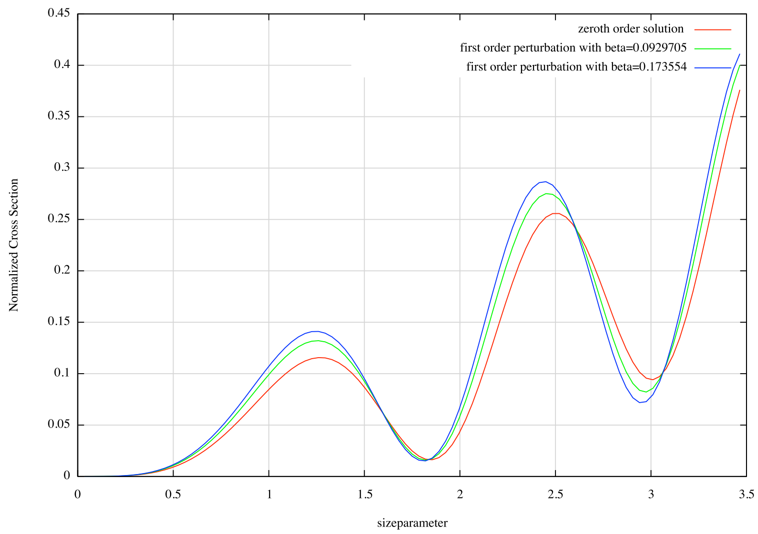

Figure 4.4. Normalized Cross Section for the spheroid.

3) We continue with the check of limiting cases of our results. One can obtain perfect conductor solution form the dielectric solution by taking the following limiting cases;

$$\frac{\mu_{in}}{\mu_{out}} \to 0, \qquad \frac{\epsilon_{out}}{\epsilon_{in}} \to 0 \tag{4.2}$$

to take above limits numerically, we take the parameters of the mediums as follows; $\mu_{in} = 10^{-15}, \mu_{out} = 10^{15}, \epsilon_{out} = 10^{-15}, \epsilon_{in} = 10^{15}$. As discussed in item 1) we take a sphere and make a constant deformation and compare obtained results with the analytical solution for the obtained new sphere. We obtain normalized cross section given with Figure 4.5.In this figure again second order perturbation is very close to the analytical results reported in page 295 of (Harrington, 2001).

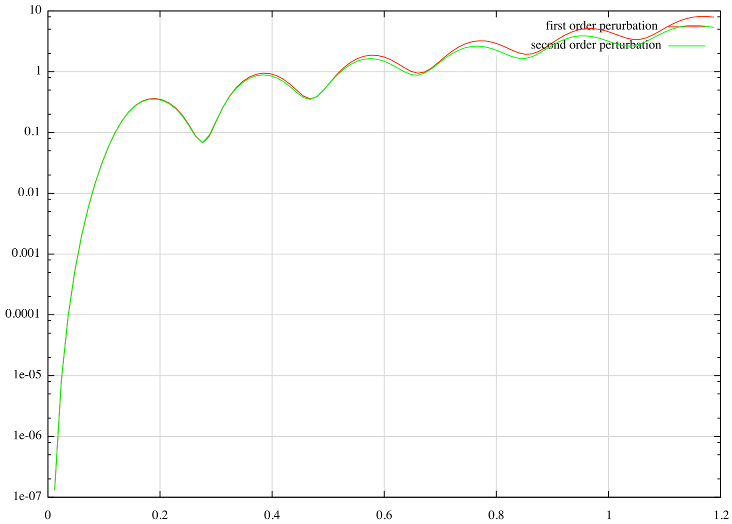

Figure 4.5. Conducting limit in logaritmic scale.

4) Up to now we have checked our results by using back scattering cross section. Now, we will make comparison by using forward scattering cross section. To do this, we choose deformation function given in (Kotsis, 2007) with;

$$\vec{r} = R\left(1 - \frac{h^2}{2}\sin^2\theta - \frac{h^4}{2}(\sin^2\theta - \frac{3}{4}\sin^4\theta)\right)\hat{r}, \tag{4.3}$$

where $h = 0.4$ (prolate spheroid). We obtained Figure 4.6 for these choices. If we compare our first order results with the Figure 6 of (Kotsis, 2007), it can be seen that they are in a great agreement.

After comparing our results with the known ones, as promised in the abstract, we can proceed to the discussion of deformed spheres which do not have rotational symmetry. In another words, we will discuss deformed spheres which are obtained by the φ dependent deformation functions.

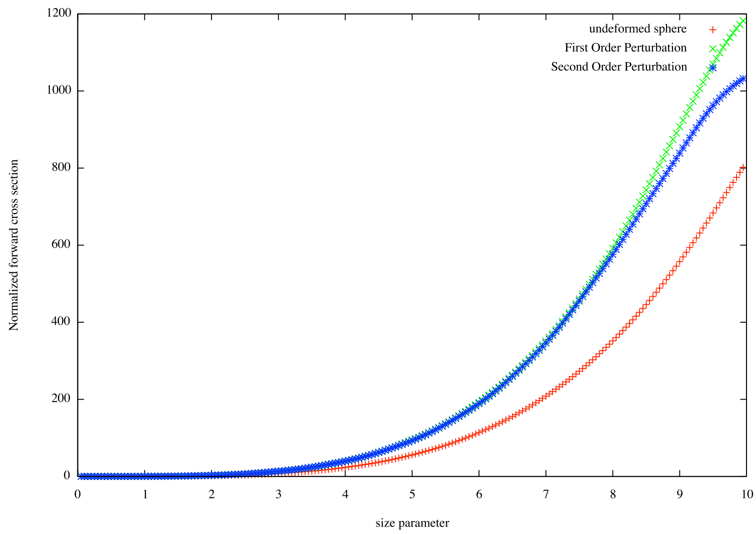

Figure 4.6. Forward scattering.

5) When $m \neq 0$, spherical harmonic functions $Y_n^m(\theta,\varphi)$ are complex quantities, in order to obtain real deformation functions we take real part of spherical harmonics. To do this we use complex conjugation represented by *. The choice of the deformation function,

$$f(\theta,\varphi) = \frac{1}{2}\Big(Y_2^1(\theta,\varphi) + (Y_2^1(\theta,\varphi))^*\Big) \tag{4.4}$$

enables us to obtain a generic object from a sphere given with Figure 4.7.

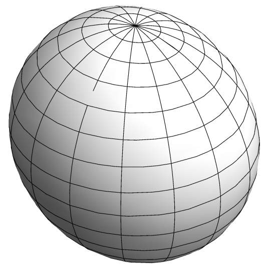

Figure 4.7. Deformation with function $\mathrm{Re}(Y_2^1(\theta,\varphi))$

Choosing deformation parameter $\beta = 0.17$ we obtain Figure 4.8 for scattering from perfect electric conductor.

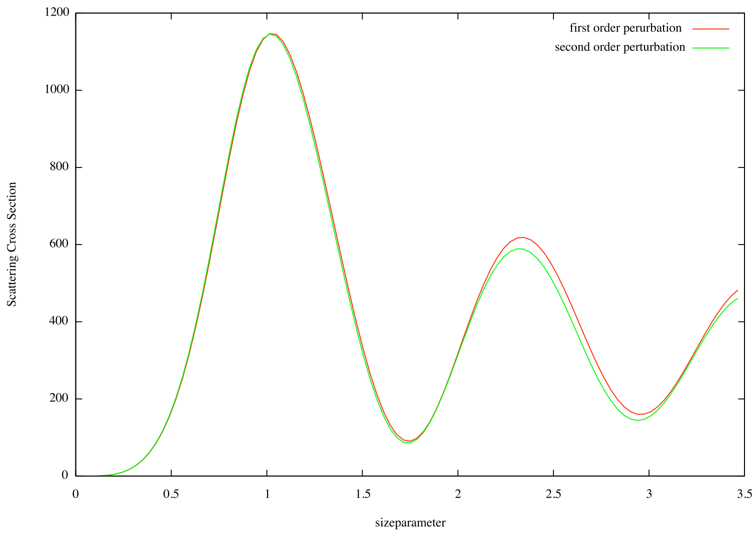

Figure 4.8. Deformation parameter $\beta = 0.17$

6) In order to analyse effect of azimuthal angle dependence, as we did in item 5) we will obtain deformation functions by the following spherical harmonics; $Y_3^0(\theta,\varphi)$, $Y_3^1(\theta,\varphi)$, $Y_3^2(\theta,\varphi)$, $Y_3^3(\theta,\varphi)$. The medium parameters are the ones used in item 2) with deformation parameter $\beta = 0.0929705$.

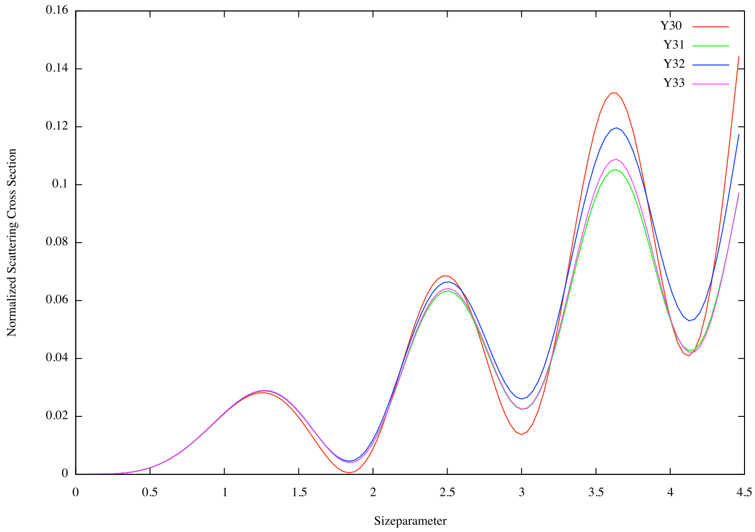

Figure 4.9. Effect of azimuthal dependence

The graph, for example, labelled with $Y32$ is obtained by the second order correction belongs to the deformation function;

$$f(\theta, \varphi) = \frac{1}{2}\Big(Y_3^2(\theta, \varphi) + (Y_3^2(\theta, \varphi))^*\Big). \tag{4.5}$$

We have to notice that, each deformation function leads to a new obstacle. In order to compare the effect of scatterers, we normalize with the scatterer's surface area after evaluating back scattering cross section. Since the scattering cross section is obtained for the second order corrections, on evaluation of surface areas second order corrections are included.

CHAPTER 5

CONCLUSION

In the present work following items have been considered;

1) Solution of Maxwell equation subject to the deformed conducting spherical body is obtained up to the second order in perturbation parameter.

2) Scattering of Electromagnetic plane waves from deformed conducting bodies is discussed and scattered field components are given up to the second order corrections.

3) Scattering of Electromagnetic plane waves from deformed dielectric obstacles is discussed and scattered and transmitted field components are given up to the second order in perturbation parameter.

4) All the angular integrals encountered in item 1) and 2) are solved analytically. These integrals are evaluated numerically in other methods.

5) Analytic solutions are obtained and Fortran code is written for most general deformation function $f = f(\theta, \varphi)$ which enables us to consider not only rotationally symmetric bodies but more realistic ones.

   As shown in Chapter 4, the second order corrections enable us more detailed analysis and more close results to the exact values in the region of large size parameter. For the obstacles much bigger than the incident wave length second order corrections give more accurate results. Besides of the contribution on second order correction, our results are easily applicable to the arbitrarily shaped objects. To apply our results for a new geometry it is enough to write medium parameters and find spherical harmonic expansion coefficient of new geometry. Thus our results are powerful on analysing the effect of the shape of surface. In other methods scattering problem is solved for a specific geometry, for example in T matrix method to see the effect of the shape of new scatterer one must solve the problem from the beginning for the new scatterer (Mischenko, 2004), (Mischenko, 2009). To compare the effect of surface roughness for the following particles

$$\tilde{R}(\theta) = R_0(1 + \beta \cos(n\theta)) \tag{5.1}$$

one has to perform all analytic and numeric calculations from the beginning for each n (Mugnai, 1985), (Rother, 2006). But in perturbation method one can handle all cases in a single code. We believe that because of this property perturbation method will find large application in scalar and Electromagnetic Casimir effect to analyse the relation, between geometry of the structure and sign of the energy.

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APPENDIX A

TRIPLE INTEGRALS OF ASSOCIATED LEGENDRE FUNCTIONS

In this Appendix, we will evaluate the following triple associated Legendre integrals encountered during the calculations;

$$Int_0(n, j, l, m, s, m') = \int_0^\pi P_n^m(\cos\theta) P_j^s(\cos\theta) P_l^{m'}(\cos\theta)\sin\theta d\theta \tag{A.1}$$

$$Int_1(n, j, l, m, s, m') = \int_0^\pi \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\frac{\partial P_j^s(\cos\theta)}{\partial\theta} P_l^{m'}(\cos\theta)\sin\theta d\theta \tag{A.2}$$

$$Int_2(n, j, l, m, s, m') = \int_0^\pi \frac{P_n^m(\cos\theta)}{\sin\theta}\frac{\partial P_j^s(\cos\theta)}{\partial\theta} P_l^{m'}(\cos\theta)\sin\theta d\theta \tag{A.3}$$

$$Int_3(n, j, l, m, s, m') = \int_0^\pi \frac{P_n^m(\cos\theta)}{\sin\theta}\frac{P_j^s(\cos\theta)}{\sin\theta} P_l^{m'}(\cos\theta)\sin\theta d\theta \tag{A.4}$$

$$Int_4(n, j, l, m, s, m') = \int_0^\pi \frac{\partial P_n^m(\cos\theta)}{\partial\theta} P_j^s(\cos\theta) P_l^{m'}(\cos\theta)\sin^2\theta d\theta \tag{A.5}$$

$$Int_5(n, j, l, m, s, m') = \int_0^\pi \frac{\partial P_n^m(\cos\theta)}{\partial\theta} P_j^s(\cos\theta) P_l^{m'}(\cos\theta)\sin\theta d\theta \tag{A.6}$$

$$Int_6(n, j, l, m, s, m') = \int_0^\pi \frac{P_n^m(\cos\theta)}{\sin\theta} P_j^s(\cos\theta) P_l^{m'}(\cos\theta)\sin\theta d\theta \tag{A.7}$$

Here the integers n, j, l, m, s, m' can take any value as far as Associated Legendre Functions allow them. The integral (A.1) is well studied integral and the result of this integral for positive integers n, j, l, m, s, m' is reported in (Dong, 2002). On attempt to solve other integrals the main aim is to reduce them to the integral in A.1, During the calculations the integers could take negative values, to make the lower indices n, j, l positive, we use following function

$$c_2(n) = \begin{cases} |n| - 1 & : n < 0 \\ n & : n \geq 0. \end{cases} \tag{A.8}$$

This functions originate from the fact that Associated Legendre equation is invariant under the transformation $l \to -l - 1$. With the help of $c_2(n)$, we have a new integral with positive lower indices

$$Int_0(n,j,l,m,s,m') = Int_{01}(c_2(n),c_2(j),c_2(l),m,s,m') \tag{A.9}$$

To make upper indices m, s, m' positive, we use the relation $P_n^m(x) = c_1(n,m)P_n^{|m|}(x)$ where

$$c_1(n,m) = \begin{cases} (-1)^m \frac{(n-m)!}{(n+m)!} & : m < 0 \\ 1 & : m \geq 0. \end{cases} \tag{A.10}$$

Together with the function $c_1(n,m)$, we have new integral whose all indices are positive.

$$Int_{01}(n,j,l,m,s,m') = c_1(n,m)c_1(j,s)c_1(l,m')Int_{02}(n,j,l,|m|,|s|,|m'|)$$

Since the all indices are positive, we can use the results of (Dong, 2002) to determine the integral

$$\begin{aligned} Int_{02}(n,j,l,m,s,m') &= \int_0^{\pi} P_n^m(\cos\theta)P_j^s(\cos\theta)P_l^{m'}(\cos\theta)\sin\theta d\theta \\ &= \sqrt{\frac{(n+m)!(j+s)!(l+m')!}{(n-m)!(j-s)!(l-m')!}}\left(\sum_{\rho_1=\rho_{min}^1}^{\rho_{max}^1} C_{m\,s\,M_2}^{n\,j\,\rho_1} C_{0\,0\,0}^{n\,j\,\rho_1}\right) \\ &\times \left(\sum_{\rho_2=\rho_{min}^2}^{\rho_{max}^2} \sqrt{\frac{(\rho_2-M_3)!}{(\rho_2+M_3)!}} C_{M_2 m' M_3}^{\rho_1\,l\,\rho_2} C_{0\,0\,0}^{\rho_1\,l\,\rho_2}\right) \\ &\times \left(\frac{[(-1)^{M_3}+(-1)^{\rho_2}]M_3 2^{M_3-2}\Gamma(\frac{\rho_2}{2})\Gamma(\frac{\rho_2+M_3+1}{2})}{(\frac{\rho_2-M_3}{2})!\Gamma(\frac{\rho_2+3}{2})}\right) \\ &\times [1-mod(2,\rho_1+n+j)]\,[1-mod(2,\rho_2+\rho_1+l)]\,, \end{aligned} \tag{A.11}$$

where

$$M_2 = m+s, \quad M_3 = m+s+m', \quad \rho_{min}^1 = \max(M_2,|n-j|)$$
$$\rho_{max}^1 = n+j, \quad \rho_{max}^2 = \rho_1 + l, \quad \rho_{min}^2 = \max(M_3,|\rho_1 - l|).$$

With this final result, we have evaluated integral A.1 for arbitrary integers. We will use functions $c_1(n,m)$ and $c_2(n)$ systematically for other integrals. In addition to this, on evaluation of other integrals, we need following recurrence relations for the Associated Legendre functions (Arfken, 2015).

$$\frac{d}{dx}P_n^m(x) = \frac{-nx}{1-x^2}P_n^m(x) + \frac{n+m}{1-x^2}P_{n-1}^m(x), \qquad x = \cos\theta \tag{A.12}$$

$$xP_n^m(x) = \frac{(n-m+1)}{2n+1}P_{n+1}^m(x) + \frac{n+m}{2n+1}P_{n-1}^m(x) \tag{A.13}$$

$$\frac{P_n^m(x)}{\sqrt{1-x^2}} = (2n-1)P_{n-1}^{m-1}(x) + \frac{P_{n-2}^m(x)}{\sqrt{1-x^2}} \tag{A.14}$$

The last identity is a recurrence relation and contains finite number of terms because lower indices are getting smaller and smaller and after enough iteration it will be less than upper indices which results in vanishing of Legendre polynomials. Here, we write this term in a more compact way;

$$\frac{P_n^m(x)}{\sqrt{1-x^2}} = \sum_{\tau=0}^{\lfloor\frac{n-m}{2}\rfloor}(2(n-2\tau)-1)P_{n-1-2\tau}^{m-1}(x) \tag{A.15}$$

where the $\lfloor x \rfloor$ denotes the geratest integer less or equal to x. For the sake of brevity, we will use following abbreviations

$$\Sigma(n,m,\tau)h(n,m,\tau) = \sum_{\tau=0}^{\lfloor\frac{n-m}{2}\rfloor}(2(n-2\tau)-1)h(n,m,\tau) \tag{A.16}$$

with the help of above abbreviations, we found that ;

$$
\begin{aligned}
Int_1(n,j,l,m,s,m') &= Int_{11}(c_2(n),c_2(j),c_2(l),m,s,m') \qquad \text{(A.17)}\\
Int_{11}(n,j,l,m,s,m') &= c_1(n,m)c_1(j,s)c_1(l,m')Int_{12}(n,j,l,|m|,|s|,|m'|)\\
Int_{12}(n,j,l,m,s,s') &= \frac{1}{4}\Big[(n-m+1)(n+m)(j-s+1)(j+s)\\
&\times Int_0(n,j,l,m-1,s-1,m')\\
&-(n-m+1)(n+m)Int_0(n,j,l,m-1,s+1,m')\\
&-(j-s+1)(j+s)Int_0(n,j,l,m+1,s-1,m')\\
&+Int_0(n,j,l,m+1,s+1,m')\Big]
\end{aligned}
$$

In the same manner, we find that

$$
\begin{aligned}
Int_2(n,j,l,m,s,m') &= Int_{21}(c_2(n),c_2(j),c_2(l),m,s,m') \qquad \text{(A.18)}\\
Int_{21}(n,j,l,m,s,m') &= c_1(n,m)c_1(j,s)c_1(l,m')Int_{22}(n,j,l,|m|,|s|,|m'|)\\
Int_{22}(n,j,l,m,s,m') &= \Big[-\Sigma(n,m,\tau_1)\Sigma(j+1,s,\tau_2)a_1(j,s)\\
&\times Int_0(n-1-2\tau_1,j-2\tau_2,l,m-1,s-1,m')\\
&-\Sigma(n,m,\tau_3)\Sigma(j-1,s,\tau_4)a_2(j,s)\\
&\times Int_0(n-1-2\tau_3,j-2-2\kappa_4,l,m-1,s-1,m')\Big],
\end{aligned}
$$

where

$$
a_1(n,m) = \frac{-n(n-m+1)}{2n+1}, \quad a_2(n,m) = \frac{(n+m)(n+1)}{2n+1}. \qquad \text{(A.19)}
$$

$$
\begin{aligned}
Int_3(n,j,l,m,s,m') &= Int_{31}(c_2(n),c_2(j),c_2(l),m,s,m') \qquad \text{(A.20)}\\
Int_{31}(n,j,l,m,s,m') &= c_1(n,m)c_1(j,s)c_1(l,m')Int_{32}(n,j,l,|m|,|s|,|m'|)\\
Int_{32}(n,j,l,m,s,m') &= \Big[\Sigma(n,m,\rho_1)\Sigma(j,s,\rho_2)\\
&\times \; Int_0(n-1-2\rho_1,j-1-2\rho_2,l,m-1,s-1,m')\Big].
\end{aligned}
$$

$$
\begin{aligned}
Int_4(n,j,l,m,s,m') &= Int_{41}(c_2(n),c_2(j),c_2(l),m,s,m') \\
Int_{41}(n,j,l,m,s,m') &= c_1(n,m)c_1(j,s)c_1(l,m')Int_{42}(n,j,l,|m|,|s|,|m'|) \\
Int_{42}(n,j,l,m,s,m') &= -a_1(n,m)\Sigma(j,s,\rho_1)Int_0(n+1,j-1-2\rho_1,l,m,s-1,m') \\
&\quad -a_2(n,m)\Sigma(j,s,\rho_1)Int_0(n-1,j-1-2\rho_1,l,m,s-1,m')
\end{aligned} \tag{A.21}
$$

$$
\begin{aligned}
Int_5(n,j,l,m,s,m') &= Int_{51}(c_2(n),c_2(j),c_2(l),m,s,m') \\
Int_{51}(n,j,l,m,s,m') &= c_1(n,m)c_1(j,s)c_1(l,m')Int_{52}(n,j,l,|m|,|s|,|m'|) \\
Int_{52}(n,j,l,m,s,m') &= \frac{1}{2}(n-m+1)(n+m)Int_0(n,j,l,m-1,s,m') \\
&\quad -\frac{1}{2}Int_0(n,j,l,m+1,s,m')
\end{aligned} \tag{A.22}
$$

$$
\begin{aligned}
Int_6(n,j,l,m,s,m') &= Int_{61}(c_2(n),c_2(j),c_2(l),m,s,m') \\
Int_{61}(n,j,l,m,s,m') &= c_1(n,m)c_1(j,s)c_1(l,m')Int_{62}(n,j,l,|m|,|s|,|m'|) \\
Int_{62}(n,j,l,m,s,m') &= \Sigma(n,m,\rho)Int_0(n-2\rho-1,j,l,m-1,s,m')
\end{aligned} \tag{A.23}
$$

APPENDIX B

EXPLICIT EXPRESSION FOR SUMMATION TERMS IN CONDUCTING SCATTERING

In this Appendix, we give explicit expression for the terms encountered on the scattering electromagnetic plane waves from conducting deformed body.

B.1. Terms Belongs to the First Order Corrections

$$
\begin{aligned}
T_1(n,m,j,s) &= \Bigg\{ -\frac{ib^0_{nm}}{\omega\mu\epsilon} f^s_j \left(r\frac{\partial}{\partial r}[(\frac{\partial^2}{\partial r^2}+k^2)(krz_n(kr))] \right)_{r=R} \\
&\times\delta_{msm'} I(n,j,l,m,s,m') + \frac{im}{\mu} a^0_{nm} kz_n(kR) f^s_j \delta_{msm'} Int_2(n,j,l,m,s,m') \\
&+\frac{ib^0_{nm}}{\omega\mu\epsilon} f^s_j \left[\frac{1}{r}\frac{\partial}{\partial r}(\ kr\, z_n(kr)) \right]_{r=R} \delta_{msm'} Int_1(n,j,l,m,s,m') \\
&\frac{-isa^0_{nm} f^s_j}{\mu}[kz_n(kR)]\delta_{msm'} Int_2(j,n,l,s,m,m') \\
&-\frac{ismf^s_j b^0_{nm}}{\omega\mu\epsilon}[\frac{1}{r}\frac{\partial}{\partial r}(\ kr\, z_n(kr))]_{r=R}\delta_{msm'} Int_3(n,j,l,m,s,m') \Bigg\}_{r=R}.
\end{aligned}
$$

$$
\begin{aligned}
T_2(n,j,s) &= \Bigg\{ \frac{kE_0}{2w\mu} f^s_j i^n (2n+1) r\frac{r}{dr}(\frac{j_n(kr)}{kr}) Int_0(n,l,j,1,s,m')[2\pi\delta_{1+s,m'} - 2\pi\delta_{-1+s,-m'}] \\
&+\frac{kE_0}{2iw\mu} i^n (2n+1) j_n(kr) f^s_j \times [2\pi\delta_{1+s,m'} - 2\pi\delta_{-1+s,-m'}] \\
&\times[a_3(n,0) int5(j,n+1,l,s,0,m') + a_4(n,0) int5(j,n-1,l,s,0,m')] \\
&+\frac{iskE_0}{2w\mu} i^n (2n+1) j_n(kr) f^s_j \\
&\times int6(n,j,l,0,s,m') \times [2\pi\delta_{1+s,m'} + 2\pi\delta_{-1+s,-m'}] \Bigg\}_{r=R},
\end{aligned}
$$

where $\delta_{msm'} = 2\pi\delta_{m+s,m'}$, $a_3(n,m) = \frac{n-m+1}{2n+1}$ and $a_4(n,m) = \frac{n+m}{2n+1}$.

$$\begin{aligned}
T_3(n,m,j,s) &= \frac{ma^0_{nm}f^s_j}{\omega\mu\epsilon}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(\,kr\,z_n(kr))\right)\right]_{r=R}\delta_{msm'}Int_0(n,j,l,m,s,m') \\
&\quad -\frac{b^0_{nm}f^s_j}{\epsilon}\left[r\frac{\partial}{\partial r}(\,k\,z_n(kr))\right]_{r=R}\delta_{msm'}Int_4(n,j,l,m,s,m') \\
&\quad +\frac{a^0_{nm}sf^s_j}{\omega\mu\epsilon}\left[(\frac{\partial^2}{\partial r^2}+k^2)(kr\,z_n(kr))\right]_{r=R}\delta_{msm'}Int_0(n,j,l,m,s,m'),
\end{aligned}$$

$$\begin{aligned}
T_4(n,j,s) &= -\frac{f^s_j kE_0 i^n}{2\omega\mu}r\frac{d}{dr}j_n(kr)\times[2\pi\delta_{1+s,m'}+2\pi\delta_{-1+s,m'}] \\
&\quad \times[Int_0(j,n+1,l,s,0,m')-Int_0(j,n-1,l,s,0,m')] \\
&\quad -\frac{sf^s_jE_0i^n(2n+1)}{2}\frac{j_n(kr)}{kr}Int_0(n,j,l,1,s,m')[2\pi\delta_{1+s,m'}+2\pi\delta_{-1+s,m'}].
\end{aligned}$$

B.2. Terms Belongs to the Second Order Corrections

$$T_5(n,m,j,s) = \frac{-1}{2}\left[r^2\frac{\partial^2}{\partial r^2}(krz_n(kr))\right]_{r=R}\frac{ib^0_{nm}}{\omega\mu\epsilon}k^s_j\delta_{msm'}Int_0(n,j,l,s,m,m')$$

$$T_6(n,m,j,s) = \frac{-ib^1_{nm}}{\omega\mu\epsilon}f^s_j\left[r\frac{\partial}{\partial r}(krz_n(kr))\right]_{r=R}\delta_{msm'}Int_0(n,j,l,s,m,m')$$

$$T_7(n,m,j,s) = \frac{ima^0_{nm}}{2\mu}k^s_j\left[r\frac{\partial}{\partial r}(kz_n(kr))\right]_{r=R}\delta_{msm'}Int_2(n,j,l,s,m,m')$$

$$T_8(n,m,j,s) = \frac{ib^0_{nm}}{2\omega\mu\epsilon}k^s_j\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(krz_n(kr))\right]_{r=R}\delta_{msm'}Int_1(n,j,l,m,s,m')$$

$$T_9(n,m,j,s) = \frac{ima^1_{nm}}{\mu}f^s_j\,[kz_n(kr)]_{r=R}\,\delta_{msm'}Int_2(n,j,l,m,s,m')$$

$$T_{10}(n,m,j,s) = \frac{ib^1_{nm}}{\omega\mu\epsilon}f^s_j\left[\frac{1}{r}\frac{\partial}{\partial r}(krz_n(kr))\right]_{r=R}\delta_{msm'}Int_1(n,j,l,m,s,m')$$

$$T_{11}(n,m,j,s) = \frac{-ima^0_{nm}}{2\mu}k^s_j\,[kz_n(kr)]_{r=R}\,\delta_{msm'}Int_2(n,j,l,,m,s,m')$$

$$T_{12}(n,m,j,s) = -\frac{ib^0_{nm}}{2\omega\mu\epsilon}k^s_j\left[\frac{1}{r}\frac{\partial}{\partial r}(krz_n(kr))\right]_{r=R}\delta_{msm'}Int_1(n,j,l,m,s,m')$$

$$T_{13}(n,m,j,s) = \frac{-isa^0_{nm}}{2\mu}k^s_j\left[r\frac{\partial}{\partial r}(kz_n(kr))\right]_{r=R}\delta_{msm'}Int_2(j,n,l,s,m,m')$$

$$T_{14}(n,m,j,s) = -\frac{ismb^0_{nm}}{2\omega\mu\epsilon}k^s_j\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(krz_n(kr))\right]_{r=R}\delta_{msm'}Int_3(n,j,l,m,s,m')$$

$$T_{15}(n,m,j,s) = \frac{-isa^1_{nm}}{\mu}f^s_j\,[kz_n(kr)]_{r=R}\,\delta_{msm'}Int_2(j,n,l,s,m,m')$$

$$T_{16}(n,m,j,s) = -\frac{ismb^1_{nm}}{\omega\mu\epsilon} f^s_j \left[\frac{1}{r}\frac{\partial}{\partial r}(krz_n(kr))\right]_{r=R} \delta_{msm'} Int_3(n,j,l,m,s,m')$$

$$T_{17}(n,m,j,s) = \frac{isa^0_{nm}}{2\mu} k^s_j \left[kz_n(kr)\right]_{r=R} \delta_{msm'} Int_2(j,n,l,s,m,m')$$

$$T_{18}(n,m,j,s) = \frac{ismb^0_{nm}}{2\omega\mu\epsilon} k^s_j \left[\frac{\partial}{r\partial r}(krz_n(kr))\right]_{r=R} \delta_{msm'} Int_3(n,j,l,m,s,m')$$

$$T_{19}(n,j,s) = \frac{k^s_j k E_0 i^n (2n+1)}{4w\mu} \left[r^2 \frac{d^2}{dr^2}(\frac{j_n(kr)}{kr})\right]_{r=R} Int_0(n,j,l,1,s,m')(S\delta)$$

$$T_{20}(n,j,s) = -\frac{ik^s_j k E_0 i^n (2n+1)}{4w\mu} \left[r\frac{d}{dr} j_n(kr)\right]_{r=R} (S\delta)$$

$$\times\Big[a_3(n,0) int_5(j,n+1,l,s,0,m') + a_4(n,0) int_5(j,n-1,l,s,0,m')\Big]$$

$$T_{21}(n,j,s) = -\frac{ik^s_j k E_0 i^n (2n+1)}{4w\mu} j_n(kR)(S\delta)$$

$$\times\Big[a_3(n,0) int_5(j,n+1,l,s,0,m') + a_4(n,0) int_5(j,n-1,l,s,0,m')\Big]$$

$$T_{22}(n,j,s) = -\frac{isk^s_j k E_0 i^n (2n+1)}{4w\mu} j_n(kR) Int_0(n,j,l,0,s,m')(C\delta)$$

$$T_{23}(n,m,j,s) = \frac{-a^0_{nm} s k^s_j}{2\omega\mu\epsilon}\left[(\frac{\partial^2}{\partial r^2} + k^2)(krz_n(kr))\right]_{r=R} \delta_{msm'} I(n,j,l,m,s,m')$$

$$T_{24}(n,m,j,s) = \frac{a^0_{nm} s k^s_j}{2\omega\mu\epsilon}\left[r\frac{\partial}{\partial r}(\frac{\partial^2}{\partial r^2} + k^2)(krz_n(kr))\right]_{r=R} \delta_{msm'} I(n,j,l,m,s,m')$$

$$T_{25}(n,m,j,s) = \frac{a^1_{nm} s f^s_j}{\omega\mu\epsilon}\left[(\frac{\partial^2}{\partial r^2} + k^2)(krz_n(kr))\right]_{r=R} \delta_{msm'} I(n,j,l,m,s,m')$$

$$T_{26}(n,m,j,s) = \frac{ma^0_{nm} k^s_j}{2\omega\mu\epsilon}\left[r^2\frac{\partial^2}{\partial r^2}\frac{1}{r}\frac{\partial}{\partial r}(krz_n(kr))\right]_{r=R} \delta_{msm'} I(n,j,l,m,s,m')$$

$$T_{27}(n,m,j,s) = -\frac{b^0_{nm} k^s_j}{2\epsilon}\left[r^2\frac{\partial^2}{\partial r^2}(kz_n(kr))\right]_{r=R} \delta_{msm'} Int_4(n,j,l,m,s,m')$$

$$T_{28}(n,m,j,s) = \frac{ma^1_{nm} f^s_j}{\omega\mu\epsilon}\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(krz_n(kr))\right]_{r=R} \delta_{msm'} I(n,j,l,m,s,m')$$

$$T_{29}(n,m,j,s) = -\frac{b^1_{nm} f^s_j}{\epsilon}\left[r\frac{\partial}{\partial r}(kz_n(kr))\right]_{r=R} \delta_{msm'} Int_4(n,j,l,m,s,m')$$

$$T_{30}(n,m,j,s) = \frac{-i^{(n+1)}(2n+1)k^s_j E_0}{4}\left[\frac{j_n(kr)}{kr}\right]_{r=R} int4(j,n,l,s,1,m')(C\delta)$$

$$T_{31}(n,m,j,s) = \frac{i^{(n+1)}(2n+1)k^s_j E_0}{4}\left[r\frac{d}{dr}(\frac{j_n(kr)}{kr})\right]_{r=R} int4(j,n,l,s,1,m')(C\delta)$$

$$
\begin{aligned}
T_{32}(n,m,j,s) = \frac{-1}{4}k_j^s E_0 i^n (2n+1) j_n(kR)(C\delta) \\
\times\Big\{a_3(n,0)\frac{1}{2l+1}int_0(n+1,j,l+1,0,s,m'+1) \\
-a_3(n,0)\frac{1}{2l+1}int_0(n+1,j,l-1,0,s,m'+1) \\
+a_4(n,0)\frac{1}{2l+1}int_0(n-1,j,l+1,0,s,m'+1) \\
-a_4(n,0)\frac{1}{2l+1}int_0(n-1,j,l-1,0,s,m'+1)\Big\}
\end{aligned}
$$

where

$$
\begin{aligned}
(C\delta) &= (2\pi\delta_{1+s,m'} + 2\pi\delta_{-1+s,m'}) \\
(S\delta) &= (2\pi\delta_{1+s,m'} - 2\pi\delta_{-1+s,m'})
\end{aligned}
$$

APPENDIX C

EXPLICIT EXPRESSION FOR SUMMATION TERMS IN DIELECTRIC SCATTERING

In this appendix, we give explicit expressions of v_1-v_8. The following abbreviations have been used through this chapter;

$$\begin{aligned}
(C\delta) &= (2\pi\delta_{1+s,m'} + 2\pi\delta_{-1+s,m'}) \\
(S\delta) &= (2\pi\delta_{1+s,m'} - 2\pi\delta_{-1+s,m'}) \\
a_n^{inc} &= -i\frac{2n+1}{n(n+1)}
\end{aligned}$$

$$v_1 = \sum_{k=1}^{5}\sum_{n,j,s} T_{1,1,k}(n,j,s) + \sum_{k=1}^{11}\sum_{n,m,j,s} T_{1,2,k}(n,m,j,s)$$

where

$$\begin{aligned}
T_{1,1,1}(n,j,s) &= \frac{-1}{2}m' s f_j^s a_n^{inc} E_0 \left[\frac{j_n(k_1 r)}{k_1 r}\right]_{r=R} Int_3(n,j,l,1,s,m')(C\delta) \\
T_{1,1,2}(n,j,s) &= \frac{-1}{2}im' f_j^s E_0 a_n^{inc} \left[r\frac{\partial j_n(k_1 r)}{\partial r}\right]_{r=R} Int_6(n,j,l,0,s,m')(S\delta) \\
T_{1,1,3}(n,j,s) &= \frac{-i}{2}a_n^{inc} E_0 f_j^s \left[r\frac{\partial j_n(k_1 r)}{\partial r}\right]_{r=R} (a_3(n,0)Int_5(l,j,n+1,m',s,0))(C\delta) \\
T_{1,1,4}(n,j,s) &= \frac{-i}{2}a_n^{inc} E_0 f_j^s \left[r\frac{\partial j_n(k_1 r)}{\partial r}\right]_{r=R} (a_4(n,0)Int_5(l,j,n-1,m',s,0))(C\delta) \\
T_{1,1,5}(n,j,s) &= \frac{-1}{2}a_n^{inc} E_0 f_j^s \left[\frac{j_n(k_1 r)}{k_1 r}\right]_{r=R} Int_1(l,j,n,m',s,1)(C\delta) \\
T_{1,2,1}(n,m,j,s) &= \frac{m' s\, a_{nm}^{0s} 2\pi\delta_{m+s,m'} f_j^s}{\omega\epsilon_1\mu_1} Int_3(n,j,l,m,s,m') \left[(\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z_n^s(k_1 r))\right]_{r=R} \\
T_{1,2,2}(n,m,j,s) &= -\frac{m' s\, a_{nm}^{0t} 2\pi\delta_{m+s,m'} f_j^s}{\omega\epsilon_2\mu_2} Int_3(n,j,l,m,s,m') \left[(\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z_n^t(k_2 r))\right]_{r=R} \\
T_{1,2,3}(n,m,j,s) &= \frac{mm' a_{nm}^{0s} 2\pi\delta_{m+s,m'} f_j^s}{\omega\epsilon_1\mu_1} Int_3(n,j,l,m,s,m') \left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_1 r z_n^s(k_1 r))\right]_{r=R}
\end{aligned}$$

$$T_{1,2,4}(n,m,j,s) = -\frac{m' 2\pi\delta_{m+s,m'} b^{0s}_{nm} f^s_j}{\epsilon_1} Int_2(l,n,j,m',m,s)\left[r\frac{\partial}{\partial r}(k_1 z^s_n(k_1 r))\right]_{r=R}$$

$$T_{1,2,5}(n,m,j,s) = -\frac{mm' a^{0t}_{nm} 2\pi\delta_{m+s,m'} f^s_j}{\omega\epsilon_2\mu_2} Int_3(n,j,l,m,s,m')\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_2 r z^t_n(k_2 r))\right]_{r=R}$$

$$T_{1,2,6}(n,m,j,s) = \frac{m' 2\pi\delta_{m+s,m'} b^{0t}_{nm} f^s_j}{\epsilon_2} Int_2(l,n,j,m',m,s)\left[r\frac{\partial}{\partial r}(k_2 z^t_n(k_2 r))\right]_{r=R}$$

$$T_{1,2,6}(n,m,j,s) = \frac{a^{0s}_{nm} 2\pi\delta_{m+s,m'} f^s_j}{\omega\epsilon_1\mu_1} Int_1(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_1 r z^s_n(k_1 r))\right]_{r=R}$$

$$T_{1,2,7}(n,m,j,s) = -\frac{m 2\pi\delta_{m+s,m'} b^{0s}_{nm} f^s_j}{\epsilon_1} Int_2(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}(k_1 z^s_n(k_1 r))\right]_{r=R}$$

$$T_{1,2,8}(n,m,j,s) = -\frac{a^{0t}_{nm} 2\pi\delta_{m+s,m'} f^s_j}{\omega\epsilon_2\mu_2} Int_1(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_2 r z^t_n(k_2 r))\right]_{r=R}$$

$$T_{1,2,9}(n,m,j,s) = \frac{m 2\pi\delta_{m+s,m'} b^{0t}_{nm} f^s_j}{\epsilon_2} Int_2(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}(k_2 z^t_n(k_2 r))\right]_{r=R}$$

$$T_{1,2,10}(n,m,j,s) = \frac{a^{0s}_{nm} 2\pi\delta_{m+s,m'} f^s_j}{\omega\epsilon_1\mu_1} Int_1(l,j,n,m',s,m)\left[(\frac{\partial^2}{\partial r^2}+k_1^2)(k_1 r\, z^s_n(k_1 r))\right]_{r=R}$$

$$T_{1,2,11}(n,m,j,s) = -\frac{a^{0t}_{nm} 2\pi\delta_{m+s,m'} f^s_j}{\omega\epsilon_2\mu_2} Int_1(l,j,n,m',s,m)\left[(\frac{\partial^2}{\partial r^2}+k_2^2)(k_2 r\, z^t_n(k_2 r))\right]_{r=R}$$

$$v_2 = \sum_{k=1}^{5}\sum_{n,j,s} T_{2,1,k}(n,j,s) + \sum_{k=1}^{12}\sum_{n,m,j,s} T_{2,2,k}(n,m,j,s)$$

$$T_{2,1,1}(n,j,s) = \frac{is\, a^{inc}_n k_1 E_0 f^s_j}{2\omega\mu_1} Int_2(n,l,j,1,m',s)\frac{j_n(k_1 r)}{k_1 r}(S\delta)$$

$$T_{2,1,2}(n,j,s) = -\frac{a^{inc}_n k_1 E_0 f^s_j}{2\omega\mu_1} Int_5(l,j,n,m',s,0)\left[r\frac{\partial j_n(k_1 r)}{\partial r}\right]_{r=R}(C\delta)$$

$$T_{2,1,3}(n,j,s) = -\frac{m' a^{inc}_n k_1 E_0 f^s_j}{2\omega\mu_1}\left[a_3(n,0) Int_6(n+1,j,l,0,s,m')\right]\left[r\frac{\partial j_n(k_1 r)}{\partial r}\right]_{r=R}(S\delta)$$

$$T_{2,1,4}(n,j,s) = -\frac{m' a^{inc}_n k_1 E_0 f^s_j}{2\omega\mu_1}\left[a_4(n,0) Int_6(n-1,j,l,0,s,m')\right]\left[r\frac{\partial j_n(k_1 r)}{\partial r}\right]_{r=R}(S\delta)$$

$$T_{2,1,5}(n,j,s) = \frac{im' a^{inc}_n k_1 E_0 f^s_j}{2\omega\mu_1} Int_2(n,j,l,1,s,m')\left[\frac{j_n(k_1 r)}{k_1 r}\right]_{r=R}(S\delta)$$

$$T_{2,2,1}(n,m,j,s = \frac{s 2\pi\delta_{m+s,m'} b^{0s}_{nm} f^s_j}{\omega\epsilon_1\mu_1} Int_2(n,l,j,m,m',s)\left[(\frac{\partial^2}{\partial r^2}+k_1^2)(k_1 r\, z^s_n(k_1 r))\right]_{r=R}$$

$$
\begin{aligned}
T_{2,2,2}(n,m,j,s &= -\frac{s2\pi\delta_{m+s,m'}b^{0t}_{nm}f^s_j}{\omega\epsilon_2\mu_2}Int_2(n,l,j,m,m',s)\left[(\frac{\partial^2}{\partial r^2}+k_2^2)(k_2 r\, z^t_n(k_2 r))\right]_{r=R}\\
T_{2,2,3}(n,m,j,s &= \frac{a^{0s}_{nm}2\pi\delta_{m+s,m'}f^s_j}{\mu_1}Int_1(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}(k_1 z^s_n(k_1 r))\right]_{r=R}\\
T_{2,2,4}(n,m,j,s &= \frac{m2\pi\delta_{m+s,m'}f^s_j b^{0s}_{nm}}{\omega\epsilon_1\mu_1}Int_2(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_1 r z^s_n(k_1 r))\right]_{r=R}\\
T_{2,2,5}(n,m,j,s &= -\frac{a^{0t}_{nm}2\pi\delta_{m+s,m'}f^s_j}{\mu_2}Int_1(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}(k_2 z^t_n(k_2 r))\right]_{r=R}\\
T_{2,2,6}(n,m,j,s &= -\frac{m2\pi\delta_{m+s,m'}b^{0t}_{nm}f^s_j}{\omega\epsilon_2\mu_2}Int_2(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_2 r z^t_n(k_2 r))\right]_{r=R}\\
T_{2,2,7}(n,m,j,s &= \frac{mm'a^{0s}_{nm}2\pi\delta_{m+s,m'}f^s_j}{\mu_1}Int_3(n,j,l,m,s,m')\left[r\frac{\partial}{\partial r}(k_1 z^s_n(k_1 r))\right]_{r=R}\\
T_{2,2,8}(n,m,j,s &= \frac{m'2\pi\delta_{m+s,m'}b^{0s}_{nm}f^s_j}{\omega\epsilon_1\mu_1}Int_2(l,n,j,m',m,s)\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_1 r z^s_n(k_1 r))\right]_{r=R}\\
T_{2,2,9}(n,m,j,s &= -\frac{mm'a^{0t}_{nm}2\pi\delta_{m+s,m'}f^s_j}{\mu_2}Int_3(n,j,l,m,s,m')\left[r\frac{\partial}{\partial r}(k_2 z^t_n(k_2 r))\right]_{r=R}\\
T_{2,2,10}(n,m,j,s &= -\frac{m'2\pi\delta_{m+s,m'}b^{0t}_{nm}f^s_j}{\omega\epsilon_2\mu_2}Int_2(l,n,j,m',m,s)\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_2 r z^t_n(k_2 r))\right]_{r=R}\\
T_{2,2,11}(n,m,j,s &= \frac{m'2\pi\delta_{m+s,m'}b^{0s}_{nm}f^s_j}{\omega\epsilon_1\mu_1}Int_2[n,j,l,m,s,m']\left[(\frac{\partial^2}{\partial r^2}+k_1^2)(k_1 r\, z^s_n(k_1 r))\right]_{r=R}\\
T_{2,2,12}(n,m,j,s &= -\frac{m'2\pi\delta_{m+s,m'}b^{0t}_{nm}f^s_j}{\omega\epsilon_2\mu_2}Int_2[n,j,l,m,s,m']\left[(\frac{\partial^2}{\partial r^2}+k_2^2)(k_2 r\, z^t_n(k_2 r))\right]_{r=R}
\end{aligned}
$$

$$
v_3 = \sum_{k=1}^{5}\sum_{n,j,s}T_{3,1,k}(n,j,s)+\sum_{k=1}^{12}\sum_{n,m,j,s}T_{3,2,k}(n,m,j,s)
$$

$$
\begin{aligned}
T_{3,1,1}(n,j,s) &= \frac{-1}{2}s\,a^{inc}_n E_0 f^s_j\, Int2(n,l,j,1,m',s)\left[\frac{j_n(k_1 r)}{k_1 r}\right]_{r=R}(C\delta)\\
T_{3,1,2}(n,j,s) &= -\frac{1}{2}ia^{inc}_n E_0 f^s_j Int_5(l,j,n,m',s,0)\left[r\frac{\partial j_n(k_1 r)}{\partial r}\right]_{r=R}(S\delta)\\
T_{3,1,3}(n,j,s) &= -\frac{1}{2}im'a^{inc}_n E_0 f^s_j[a_3(n,0)Int_6(n+1,j,l,0,s,m')]\left[r\frac{\partial j_n(k_1 r)}{\partial r}\right]_{r=R}(C\delta)\\
T_{3,1,4}(n,j,s) &= -\frac{1}{2}im'a^{inc}_n E_0 f^s_j[a_4(n,0)Int_6(n-1,j,l,0,s,m')]\left[r\frac{\partial j_n(k_1 r)}{\partial r}\right]_{r=R}(C\delta)\\
T_{3,1,5}(n,j,s) &= \frac{-1}{2}m'a^{inc}_n Int2(n,j,l,1,s,m')\left[\frac{j_n(k_1 r)}{k_1 r}\right]_{r=R}E_0 f^s_j(C\delta)
\end{aligned}
$$

$$T_{3,2,1}(n,m,j,s) = \frac{sa^{0s}_{nm}2\pi\delta_{m+s,m'}f^s_j}{\omega\epsilon_1\mu_1}Int_2(n,l,j,m,m',s)\left[(\frac{\partial^2}{\partial r^2}+k_1^2)(k_1 r\, z^s_n(k_1 r))\right]_{r=R}$$

$$T_{3,2,2}(n,m,j,s) = -\frac{sa^{0t}_{nm}2\pi\delta_{m+s,m'}f^s_j}{\omega\epsilon_2\mu_2}Int_2(n,l,j,m,m',s)\left[(\frac{\partial^2}{\partial r^2}+k_2^2)(k_2 r\, z^t_n(k_2 r))\right]_{r=R}$$

$$T_{3,2,3}(n,m,j,s) = \frac{ma^{0s}_{nm}2\pi\delta_{m+s,m'}f^s_j}{\omega\epsilon_1\mu_1}Int_2(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_1 r z^s_n(k_1 r))\right]_{r=R}$$

$$T_{3,2,4}(n,m,j,s) = -\frac{2\pi\delta_{m+s,m'}b^{0s}_{nm}f^s_j}{\epsilon_1}Int_1(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}(k_1 z^s_n(k_1 r))\right]_{r=R}$$

$$T_{3,2,5}(n,m,j,s) = -\frac{ma^{0t}_{nm}2\pi\delta_{m+s,m'}f^s_j}{\omega\epsilon_2\mu_2}Int_2(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_2 r z^t_n(k_2 r))\right]_{r=R}$$

$$T_{3,2,6}(n,m,j,s) = \frac{2\pi\delta_{m+s,m'}b^{0t}_{nm}f^s_j}{\epsilon_2}Int_1(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}(k_2 z^t_n(k_2 r))\right]_{r=R}$$

$$T_{3,2,7}(n,m,j,s) = \frac{m' a^{0s}_{nm}2\pi\delta_{m+s,m'}f^s_j}{\omega\epsilon_1\mu_1}Int_2(l,n,j,m',m,s)\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_1 r z^s_n(k_1 r))\right]_{r=R}$$

$$T_{3,2,8}(n,m,j,s) = -\frac{mm'2\pi\delta_{m+s,m'}b^{0s}_{nm}f^s_j}{\epsilon_1}Int_3(n,j,l,m,s,m')\left[r\frac{\partial}{\partial r}(k_1 z^s_n(k_1 r))\right]_{r=R}$$

$$T_{3,2,9}(n,m,j,s) = -\frac{m' a^{0t}_{nm}2\pi\delta_{m+s,m'}f^s_j}{\omega\epsilon_2\mu_2}Int_2(l,n,j,m',m,s)\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_2 r z^t_n(k_2 r))\right]_{r=R}$$

$$T_{3,2,10}(n,m,j,s) = \frac{mm'2\pi\delta_{m+s,m'}b^{0t}_{nm}f^s_j}{\epsilon_2}Int_3(n,j,l,m,s,m')\left[r\frac{\partial}{\partial r}(k_2 z^t_n(k_2 r))\right]_{r=R}$$

$$T_{3,2,11}(n,m,j,s) = \frac{m' a^{0s}_{nm}2\pi\delta_{m+s,m'}f^s_j}{\omega\epsilon_1\mu_1}Int_2[n,j,l,m,s,m']\left[(\frac{\partial^2}{\partial r^2}+k_1^2)(k_1 r\, z^s_n(k_1 r))\right]_{r=R}$$

$$T_{3,2,12}(n,m,j,s) = -\frac{m' a^{0t}_{nm}2\pi\delta_{m+s,m'}f^s_j}{\omega\epsilon_2\mu_2}Int_2[n,j,l,m,s,m']\left[(\frac{\partial^2}{\partial r^2}+k_2^2)(k_2 r\, z^t_n(k_2 r))\right]_{r=R}$$

$$v_4 = \sum_{k=1}^{5}\sum_{n,j,s}T_{4,1,k}(n,j,s)+\sum_{k=1}^{12}\sum_{n,m,j,s}T_{4,2,k}(n,m,j,s)$$

$$T_{4,1,1}(n,j,s) = -\frac{im' sa^{inc}_n k_1 E_0 f^s_j}{2\omega\mu_1}Int_3(n,j,l,1,s,m')\left[\frac{j_n(k_1 r)}{k_1 r}\right]_{r=R}(S\delta)$$

$$T_{4,1,2}(n,j,s) = \frac{m' a^{inc}_n k_1 E_0 f^s_j}{2\omega\mu_1}Int_6(n,j,l,0,s,m')\left[r\frac{\partial j_n(k_1 r)}{\partial r}\right]_{r=R}(C\delta)$$

$$T_{4,1,3}(n,j,s) = \frac{a^{inc}_n k_1 E_0 f^s_j}{2\omega\mu_1}[a_3(n,0)Int_5(l,j,n+1,m',s,0)]\left[r\frac{\partial j_n(k_1 r)}{\partial r}\right]_{r=R}(S\delta)$$

$$T_{4,1,4}(n,j,s) = \frac{a^{inc}_n k_1 E_0 f^s_j}{2\omega\mu_1}[a_4(n,0)Int_5(l,j,n-1,m',s,0)]\left[r\frac{\partial j_n(k_1 r)}{\partial r}\right]_{r=R}(S\delta)$$

$$T_{4,1,5}(n,j,s) = -\frac{ia_n^{inc}k_1E_0f_j^s}{2\omega\mu_1}Int_1(l,j,n,m',s,1)\left[\frac{j_n(k_1r)}{k_1r}\right]_{r=R}(S\delta) \quad (C.1)$$

$$T_{4,2,1}(n,m,j,s) = -\frac{m's2\pi\delta_{m+s,m'}b_{nm}^{0s}f_j^s}{\omega\epsilon_1\mu_1}Int_3(n,j,l,m,s,m')\left[(\frac{\partial^2}{\partial r^2}+k_1^2)(k_1r\,z_n^s(k_1r))\right]_{r=R}$$

$$T_{4,2,2}(n,m,j,s) = \frac{m's2\pi\delta_{m+s,m'}b_{nm}^{0t}f_j^s}{\omega\epsilon_2\mu_2}Int_3(n,j,l,m,s,m')\left[(\frac{\partial^2}{\partial r^2}+k_2^2)(k_2r\,z_n^t(k_2r))\right]_{r=R}$$

$$T_{4,2,3}(n,m,j,s) = -\frac{m'a_{nm}^{0s}2\pi\delta_{m+s,m'}f_j^s}{\mu_1}Int_2(l,n,j,m',m,s)\left[r\frac{\partial}{\partial r}(k_1z_n^s(k_1r))\right]_{r=R}$$

$$T_{4,2,4}(n,m,j,s) = -\frac{mm'2\pi\delta_{m+s,m'}b_{nm}^{0s}f_j^s}{\omega\epsilon_1\mu_1}Int_3(n,j,l,m,s,m')\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_1rz_n^s(k_1r))\right]_{r=R}$$

$$T_{4,2,5}(n,m,j,s) = \frac{m'a_{nm}^{0t}2\pi\delta_{m+s,m'}f_j^s}{\mu_2}Int_2(l,n,j,m',m,s)\left[r\frac{\partial}{\partial r}(k_2z_n^t(k_2r))\right]_{r=R}$$

$$T_{4,2,6}(n,m,j,s) = \frac{mm'2\pi\delta_{m+s,m'}b_{nm}^{0t}f_j^s}{\omega\epsilon_2\mu_2}Int_3(n,j,l,m,s,m')\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_2rz_n^t(k_2r))\right]_{r=R}$$

$$T_{4,2,7}(n,m,j,s) = -\frac{ma_{nm}^{0s}2\pi\delta_{m+s,m'}f_j^s}{\mu_1}Int_2(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}(k_1z_n^s(k_1r))\right]_{r=R}$$

$$T_{4,2,8}(n,m,j,s) = -\frac{2\pi\delta_{m+s,m'}b_{nm}^{0s}f_j^s}{\omega\epsilon_1\mu_1}Int_1(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_1rz_n^s(k_1r))\right]_{r=R}$$

$$T_{4,2,9}(n,m,j,s) = \frac{ma_{nm}^{0t}2\pi\delta_{m+s,m'}f_j^s}{\mu_2}Int_2(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}(k_2z_n^t(k_2r))\right]_{r=R}$$

$$T_{4,2,10}(n,m,j,s) = \frac{2\pi\delta_{m+s,m'}b_{nm}^{0t}f_j^s}{\omega\epsilon_2\mu_2}Int_1(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_2rz_n^t(k_2r))\right]_{r=R}$$

$$T_{4,2,11}(n,m,j,s) = -\frac{2\pi\delta_{m+s,m'}b_{nm}^{0s}f_j^s}{\omega\epsilon_1\mu_1}Int_1(l,j,n,m',s,m)\left[(\frac{\partial^2}{\partial r^2}+k_1^2)(k_1r\,z_n^s(k_1r))\right]_{r=R}$$

$$T_{4,2,12}(n,m,j,s) = \frac{2\pi\delta_{m+s,m'}b_{nm}^{0t}f_j^s}{\omega\epsilon_2\mu_2}Int_1(l,j,n,m',s,m)\left[(\frac{\partial^2}{\partial r^2}+k_2^2)(k_2r\,z_n^t(k_2r))\right]_{r=R}$$

$$v_5 = \sum_{k=1}^{7}\sum_{n,j,s}T_{5,1,k}(n,j,s)+\sum_{k=1}^{27}\sum_{n,m,j,s}T_{5,2,k}(n,m,j,s)$$

$$T_{5,1,1}(n,j,s) = \frac{1}{4}m'sa_n^{inc}E_0k_j^sInt_3(n,j,l,1,s,m')\left[\frac{j_n(k_1r)}{k_1r}\right]_{r=R}(C\delta)$$

$$T_{5,1,2}(n,j,s) = \frac{-1}{4}m'sa_n^{inc}E_0k_j^sInt_3(n,j,l,1,s,m')\left[r\frac{\partial}{\partial r}(\frac{j_n(k_1r)}{k_1r})\right]_{r=R}(C\delta)$$

$$T_{5,1,3}(n,j,s) = -\frac{1}{4}im'a_n^{inc}E_0k_j^sInt_6(n,j,l,0,s,m')\left[r^2\frac{\partial^2j_n(k_1r)}{\partial r^2}\right]_{r=R}(S\delta)$$

$$T_{5,1,4}(n,j,s) = \frac{1}{4}a_n^{inc}E_0k_j^s Int_1(l,j,n,m',s,1)\left[\frac{j_n(k_1r)}{k_1r}\right]_{r=R}(C\delta)$$

$$T_{5,1,5}(n,j,s) = \frac{-1}{4}a_n^{inc}E_0k_j^s Int_1(l,j,n,m',s,1)\left[r\frac{\partial}{\partial r}(\frac{j_n(k_1r)}{k_1r})\right]_{r=R}(C\delta)$$

$$T_{5,1,6}(n,j,s) = -\frac{1}{4}ia_n^{inc}E_0k_j^s[a_3(n,0)Int_5(l,j,n+1,m',s,0)]\left[r^2\frac{\partial^2 j_n(k_1r)}{\partial r^2}\right]_{r=R}(C\delta)$$

$$T_{5,1,7}(n,j,s) = -\frac{1}{4}ia_n^{inc}E_0k_j^s[a_4(n,0)Int_5(l,j,n-1,m',s,0)]\left[r^2\frac{\partial^2 j_n(k_1r)}{\partial r^2}\right]_{r=R}(C\delta)\quad\text{(C.2)}$$

$$T_{5,2,1}(n,m,j,s) = -\frac{m' s a_{nm}^{0s}2\pi\delta_{m+s,m'}k_j^s}{2\omega\epsilon_1\mu_1}Int_3(n,j,l,m,s,m')\left[(\frac{\partial^2}{\partial r^2}+k_1^2)(k_1r\,z_n^s(k_1r))\right]_{r=R}$$

$$T_{5,2,2}(n,m,j,s) = \frac{m' s a_{nm}^{0t}2\pi\delta_{m+s,m'}k_j^s}{2\omega\epsilon_2\mu_2}Int_3(n,j,l,m,s,m')\left[(\frac{\partial^2}{\partial r^2}+k_2^2)(k_2r\,z_n^t(k_2r))\right]_{r=R}$$

$$T_{5,2,3}(n,m,j,s) = \frac{m' s a_{nm}^{0s}2\pi\delta_{m+s,m'}k_j^s}{2\omega\epsilon_1\mu_1}Int_3(n,j,l,m,s,m')\left[r\frac{\partial}{\partial r}(\frac{\partial^2}{\partial r^2}+k_1^2)(k_1r\,z_n^s(k_1r))\right]_{r=R}$$

$$T_{5,2,4}(n,m,j,s) = -\frac{m' s a_{nm}^{0t}2\pi\delta_{m+s,m'}k_j^s}{2\omega\epsilon_2\mu_2}Int_3(n,j,l,m,s,m')\left[r\frac{\partial}{\partial r}(\frac{\partial^2}{\partial r^2}+k_2^2)(k_2r\,z_n^t(k_2r))\right]_{r=R}$$

$$T_{5,2,5}(n,m,j,s) = \frac{m' s a_{nm}^{1s}2\pi\delta_{m+s,m'}f_j^s}{\omega\epsilon_1\mu_1}Int_3(n,j,l,m,s,m')\left[(\frac{\partial^2}{\partial r^2}+k_1^2)(k_1r\,z_n^s(k_1r))\right]_{r=R}$$

$$T_{5,2,6}(n,m,j,s) = -\frac{m' s a_{nm}^{1t}2\pi\delta_{m+s,m'}f_j^s}{\omega\epsilon_2\mu_2}Int_3(n,j,l,m,s,m')\left[(\frac{\partial^2}{\partial r^2}+k_2^2)(k_2r\,z_n^t(k_2r))\right]_{r=R}$$

$$T_{5,2,7}(n,m,j,s) = \frac{mm' a_{nm}^{0s}2\pi\delta_{m+s,m'}k_j^s}{2\omega\epsilon_1\mu_1}Int_3(n,j,l,m,s,m')\left[r^2\frac{\partial^2}{\partial r^2}\frac{1}{r}\frac{\partial}{\partial r}(k_1rz_n^s(k_1r))\right]_{r=R}$$

$$T_{5,2,8}(n,m,j,s) = -\frac{m'2\pi\delta_{m+s,m'}b_{nm}^{0s}k_j^s}{2\epsilon_1}Int_2(l,n,j,m',m,s)\left[r^2\frac{\partial^2}{\partial r^2}(k_1z_n^s(k_1r))\right]_{r=R}$$

$$T_{5,2,9}(n,m,j,s) = -\frac{mm' a_{nm}^{0t}2\pi\delta_{m+s,m'}k_j^s}{2\omega\epsilon_2\mu_2}Int_3(n,j,l,m,s,m')\left[r^2\frac{\partial^2}{\partial r^2}\frac{1}{r}\frac{\partial}{\partial r}(k_2rz_n^t(k_2r))\right]_{r=R}$$

$$T_{5,2,10}(n,m,j,s) = \frac{m'2\pi\delta_{m+s,m'}b_{nm}^{0t}k_j^s}{2\epsilon_2}Int_2(l,n,j,m',m,s)\left[r^2\frac{\partial^2}{\partial r^2}(k_2z_n^t(k_2r))\right]_{r=R}$$

$$T_{5,2,11}(n,m,j,s) = \frac{mm' a_{nm}^{1s}2\pi\delta_{m+s,m'}f_j^s}{\omega\epsilon_1\mu_1}Int_3(n,j,l,m,s,m')\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_1rz_n^s(k_1r))\right]_{r=R}$$

$$T_{5,2,12}(n,m,j,s) = -\frac{m'2\pi\delta_{m+s,m'}b_{nm}^{1s}f_j^s}{\epsilon_1}Int_2(l,n,j,m',m,s)\left[r\frac{\partial}{\partial r}(k_1z_n^s(k_1r))\right]_{r=R}$$

$$T_{5,2,13}(n,m,j,s) = -\frac{mm' a_{nm}^{1t}2\pi\delta_{m+s,m'}f_j^s}{\omega\epsilon_2\mu_2}Int_3(n,j,l,m,s,m')\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_2rz_n^t(k_2r))\right]_{r=R}$$

$$T_{5,2,14}(n,m,j,s) = \frac{m'2\pi\delta_{m+s,m'}b_{nm}^{1t}f_j^s}{\epsilon_2}Int_2(l,n,j,m',m,s)\left[r\frac{\partial}{\partial r}(k_2z_n^t(k_2r))\right]_{r=R}$$

$$T_{5,2,15}(n,m,j,s) = -\frac{a_{nm}^{0s}2\pi\delta_{m+s,m'}k_j^s}{2\omega\epsilon_1\mu_1}Int_1(l,j,n,m',s,m)\left[(\frac{\partial^2}{\partial r^2}+k_1^2)(k_1r\,z_n^s(k_1r))\right]_{r=R}$$

$$T_{5,2,16}(n,m,j,s) = \frac{a_{nm}^{0t}2\pi\delta_{m+s,m'}k_j^s}{2\omega\epsilon_2\mu_2}Int_1(l,j,n,m',s,m)\left[(\frac{\partial^2}{\partial r^2}+k_2^2)(k_2r\,z_n^t(k_2r))\right]_{r=R}$$

$$T_{5,2,17}(n,m,j,s) = \frac{a^{0s}_{nm}2\pi\delta_{m+s,m'}k^s_j}{2\omega\epsilon_1\mu_1}Int_1(l,j,n,m',s,m)\left[r\frac{\partial}{\partial r}(\frac{\partial^2}{\partial r^2}+k_1^2)(k_1 r\, z^s_n(k_1 r))\right]_{r=R}$$

$$T_{5,2,18}(n,m,j,s) = -\frac{a^{0t}_{nm}2\pi\delta_{m+s,m'}k^s_j}{2\omega\epsilon_2\mu_2}Int_1(l,j,n,m',s,m)\left[r\frac{\partial}{\partial r}(\frac{\partial^2}{\partial r^2}+k_2^2)(k_2 r\, z^t_n(k_2 r))\right]_{r=R}$$

$$T_{5,2,18}(n,m,j,s) = \frac{a^{1s}_{nm}2\pi\delta_{m+s,m'}f^s_j}{\omega\epsilon_1\mu_1}Int_1(l,j,n,m',s,m)\left[(\frac{\partial^2}{\partial r^2}+k_1^2)(k_1 r\, z^s_n(k_1 r))\right]_{r=R}$$

$$T_{5,2,19}(n,m,j,s) = -\frac{a^{1t}_{nm}2\pi\delta_{m+s,m'}f^s_j}{\omega\epsilon_2\mu_2}Int_1(l,j,n,m',s,m)\left[(\frac{\partial^2}{\partial r^2}+k_2^2)(k_2 r\, z^t_n(k_2 r))\right]_{r=R}$$

$$T_{5,2,20}(n,m,j,s) = \frac{a^{0s}_{nm}2\pi\delta_{m+s,m'}k^s_j}{2\omega\epsilon_1\mu_1}Int_1(n,l,j,m,m',s)\left[r^2\frac{\partial^2}{\partial r^2}\frac{1}{r}\frac{\partial}{\partial r}(k_1 r z^s_n(k_1 r))\right]_{r=R}$$

$$T_{5,2,21}(n,m,j,s) = -\frac{m2\pi\delta_{m+s,m'}b^{0s}_{nm}k^s_j}{2\epsilon_1}Int_2(n,l,j,m,m',s)\left[r^2\frac{\partial^2}{\partial r^2}(k_1 z^s_n(k_1 r))\right]_{r=R}$$

$$T_{5,2,22}(n,m,j,s) = -\frac{a^{0t}_{nm}2\pi\delta_{m+s,m'}k^s_j}{2\omega\epsilon_2\mu_2}Int_1(n,l,j,m,m',s)\left[r^2\frac{\partial^2}{\partial r^2}\frac{1}{r}\frac{\partial}{\partial r}(k_2 r z^t_n(k_2 r))\right]_{r=R}$$

$$T_{5,2,23}(n,m,j,s) = \frac{m2\pi\delta_{m+s,m'}b^{0t}_{nm}k^s_j}{2\epsilon_2}Int_2(n,l,j,m,m',s)\left[r^2\frac{\partial^2}{\partial r^2}(k_2 z^t_n(k_2 r))\right]_{r=R}$$

$$T_{5,2,24}(n,m,j,s) = \frac{a^{1s}_{nm}2\pi\delta_{m+s,m'}f^s_j}{\omega\epsilon_1\mu_1}Int_1(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_1 r z^s_n(k_1 r))\right]_{r=R}$$

$$T_{5,2,25}(n,m,j,s) = -\frac{m2\pi\delta_{m+s,m'}b^{1s}_{nm}f^s_j}{\epsilon_1}Int_2(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}(k_1 z^s_n(k_1 r))\right]_{r=R}$$

$$T_{5,2,26}(n,m,j,s) = -\frac{a^{1t}_{nm}2\pi\delta_{m+s,m'}f^s_j}{\omega\epsilon_2\mu_2}Int_1(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_2 r z^t_n(k_2 r))\right]_{r=R}$$

$$T_{5,2,27}(n,m,j,s) = \frac{m2\pi\delta_{m+s,m'}b^{1t}_{nm}f^s_j}{\epsilon_2}Int_2(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}(k_2 z^t_n(k_2 r))\right]_{r=R}$$

$$v_6 = \sum_{k=1}^{7}\sum_{n,j,s}T_{6,1,k}(n,j,s) + \sum_{k=1}^{28}\sum_{n,m,j,s}T_{6,2,k}(n,m,j,s)$$

$$T_{6,1,1}(n,j,s) = -\frac{is\, a^{inc}_n k_1 E_0 k^s_j}{4\omega\mu_1}Int_2(n,l,j,1,m',s)\left[\frac{j_n(k_1 r)}{k_1 r}\right]_{r=R}(S\delta)$$

$$T_{6,1,2}(n,j,s) = \frac{is\, a^{inc}_n k_1 E_0 k^s_j}{4\omega\mu_1}Int_2(n,l,j,1,m',s)\left[r\frac{\partial}{\partial r}(\frac{j_n(k_1 r)}{k_1 r})\right]_{r=R}(S\delta)$$

$$T_{6,1,3}(n,j,s) = -\frac{a^{inc}_n k_1 E_0 k^s_j}{4\omega\mu_1}Int_5(l,j,n,m',s,0)\left[r^2\frac{\partial^2 j_n(k_1 r)}{\partial r^2}\right]_{r=R}(C\delta)$$

$$T_{6,1,4}(n,j,s) = -\frac{im' a^{inc}_n k_1 E_0 k^s_j}{4\omega\mu_1}Int_2(n,j,l,1,s,m')\left[\frac{j_n(k_1 r)}{k_1 r}\right]_{r=R}(S\delta)$$

$$T_{6,1,5}(n,j,s) = \frac{im' a_n^{inc} k_1 E_0 k_j^s}{4\omega\mu_1} Int_2(n,j,l,1,s,m') \left[r\frac{\partial}{\partial r}(\frac{j_n(k_1 r)}{k_1 r}) \right]_{r=R} (S\delta)$$

$$T_{6,1,6}(n,j,s) = -\frac{m' a_n^{inc} k_1 E_0 k_j^s}{4\omega\mu_1} \left[r^2 \frac{\partial^2 j_n(k_1 r)}{\partial r^2} \right]_{r=R} [a_1(n,0) Int_6(n+1,j,l,0,s,m')](S\delta)$$

$$T_{6,1,7}(n,j,s) = -\frac{m' a_n^{inc} k_1 E_0 k_j^s}{4\omega\mu_1} \left[r^2 \frac{\partial^2 j_n(k_1 r)}{\partial r^2} \right]_{r=R} [a_2(n,0) Int_6(n-1,j,l,0,s,m')](S\delta)$$

$$T_{6,2,1}(n,m,j,s) = -\frac{s 2\pi \delta_{m+s,m'} b_{nm}^{0s} k_j^s}{2\omega\epsilon_1\mu_1} Int_2(n,l,j,m,m',s) \left[(\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z_n^s(k_1 r)) \right]_{r=R}$$

$$T_{6,2,2}(n,m,j,s) = \frac{s 2\pi \delta_{m+s,m'} b_{nm}^{0t} k_j^s}{2\omega\epsilon_2\mu_2} Int_2(n,l,j,m,m',s) \left[(\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z_n^t(k_2 r)) \right]_{r=R}$$

$$T_{6,2,3}(n,m,j,s) = \frac{s 2\pi \delta_{m+s,m'} b_{nm}^{0s} k_j^s}{2\omega\epsilon_1\mu_1} Int_2(n,l,j,m,m',s) \left[\frac{\partial}{\partial r}(\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z_n^s(k_1 r)) \right]_{r=R}$$

$$T_{6,2,4}(n,m,j,s) = -\frac{s 2\pi \delta_{m+s,m'} b_{nm}^{0t} k_j^s}{2\omega\epsilon_2\mu_2} Int_2(n,l,j,m,m',s) \left[\frac{\partial}{\partial r}(\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z_n^t(k_2 r)) \right]_{r=R}$$

$$T_{6,2,5}(n,m,j,s) = \frac{s 2\pi \delta_{m+s,m'} b_{nm}^{1s} f_j^s}{\omega\epsilon_1\mu_1} Int_2(n,l,j,m,m',s) \left[(\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z_n^s(k_1 r)) \right]_{r=R}$$

$$T_{6,2,6}(n,m,j,s) = -\frac{s 2\pi \delta_{m+s,m'} b_{nm}^{1t} f_j^s}{\omega\epsilon_2\mu_2} Int_2(n,l,j,m,m',s) \left[(\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z_n^t(k_2 r)) \right]_{r=R}$$

$$T_{6,2,7}(n,m,j,s) = \frac{a_{nm}^{0s} 2\pi \delta_{m+s,m'} k_j^s}{2\mu_1} Int_1(n,l,j,m,m',s) \left[r^2 \frac{\partial^2}{\partial r^2}(k_1 z_n^s(k_1 r)) \right]_{r=R}$$

$$T_{6,2,8}(n,m,j,s) = \frac{m 2\pi \delta_{m+s,m'} b_{nm}^{0s} k_j^s}{2\omega\epsilon_1\mu_1} Int_2(n,l,j,m,m',s) \left[r^2 \frac{\partial^2}{\partial r^2} \frac{1}{r} \frac{\partial}{\partial r}(k_1 r z_n^s(k_1 r)) \right]_{r=R}$$

$$T_{6,2,9}(n,m,j,s) = -\frac{a_{nm}^{0t} 2\pi \delta_{m+s,m'} k_j^s}{2\mu_2} Int_1(n,l,j,m,m',s) \left[r^2 \frac{\partial^2}{\partial r^2}(k_2 z_n^t(k_2 r)) \right]_{r=R}$$

$$T_{6,2,10}(n,m,j,s) = -\frac{m 2\pi \delta_{m+s,m'} b_{nm}^{0t} k_j^s}{2\omega\epsilon_2\mu_2} Int_2(n,l,j,m,m',s) \left[r^2 \frac{\partial^2}{\partial r^2} \frac{1}{r} \frac{\partial}{\partial r}(k_2 r z_n^t(k_2 r)) \right]_{r=R}$$

$$T_{6,2,11}(n,m,j,s) = \frac{a_{nm}^{1s} 2\pi \delta_{m+s,m'} f_j^s}{\mu_1} Int_1(n,l,j,m,m',s) \left[r\frac{\partial}{\partial r}(k_1 z_n^s(k_1 r)) \right]_{r=R}$$

$$T_{6,2,12}(n,m,j,s) = \frac{m 2\pi \delta_{m+s,m'} b_{nm}^{1s} f_j^s}{\omega\epsilon_1\mu_1} Int_2(n,l,j,m,m',s) \left[r\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r}(k_1 r z_n^s(k_1 r)) \right]_{r=R}$$

$$T_{6,2,13}(n,m,j,s) = -\frac{a_{nm}^{1t} 2\pi \delta_{m+s,m'} f_j^s}{\mu_2} Int_1(n,l,j,m,m',s) \left[r\frac{\partial}{\partial r}(k_2 z_n^t(k_2 r)) \right]_{r=R}$$

$$T_{6,2,14}(n,m,j,s) = -\frac{m 2\pi \delta_{m+s,m'} b_{nm}^{1t} f_j^s}{\omega\epsilon_2\mu_2} Int_2(n,l,j,m,m',s) \left[r\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r}(k_2 r z_n^t(k_2 r)) \right]_{r=R}$$

$$T_{6,2,15}(n,m,j,s) = -\frac{m' 2\pi \delta_{m+s,m'} b_{nm}^{0s} k_j^s}{2\omega\epsilon_1\mu_1} Int_2[n,j,l,m,s,m'] \left[(\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z_n^s(k_1 r)) \right]_{r=R}$$

$$T_{6,2,16}(n,m,j,s) = \frac{m' 2\pi \delta_{m+s,m'} b_{nm}^{0t} k_j^s}{2\omega\epsilon_2\mu_2} Int_2[n,j,l,m,s,m'] \left[(\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z_n^t(k_2 r)) \right]_{r=R}$$

$$
\begin{aligned}
T_{6,2,17}(n,m,j,s) &= \frac{m' 2\pi\delta_{m+s,m'} b_{nm}^{0s} k_j^s}{2\omega\epsilon_1\mu_1} Int_2[n,j,l,m,s,m'] \left[r\frac{\partial}{\partial r}(\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z_n^s(k_1 r)) \right]_{r=R} \\
T_{6,2,18}(n,m,j,s) &= -\frac{m' 2\pi\delta_{m+s,m'} b_{nm}^{0t} k_j^s}{2\omega\epsilon_2\mu_2} Int_2[n,j,l,m,s,m'] \left[r\frac{\partial}{\partial r}(\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z_n^t(k_2 r)) \right]_{r=R} \\
T_{6,2,19}(n,m,j,s) &= \frac{m' 2\pi\delta_{m+s,m'} b_{nm}^{1s} f_j^s}{\omega\epsilon_1\mu_1} Int_2[n,j,l,m,s,m'] \left[(\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z_n^s(k_1 r)) \right]_{r=R} \\
T_{6,2,20}(n,m,j,s) &= -\frac{m' 2\pi\delta_{m+s,m'} b_{nm}^{1t} f_j^s}{\omega\epsilon_2\mu_2} Int_2[n,j,l,m,s,m'] \left[(\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z_n^t(k_2 r)) \right]_{r=R} \quad \text{(C.3)} \\
T_{6,2,21}(n,m,j,s) &= \frac{m m' a_{nm}^{0s} 2\pi\delta_{m+s,m'} k_j^s}{2\mu_1} Int_3(n,j,l,m,s,m') \left[r^2\frac{\partial^2}{\partial r^2}(k_1 z_n^s(k_1 r)) \right]_{r=R} \\
T_{6,2,22}(n,m,j,s) &= \frac{m' 2\pi\delta_{m+s,m'} b_{nm}^{0s} k_j^s}{2\omega\epsilon_1\mu_1} Int_2(l,n,j,m',m,s) \left[r^2\frac{\partial^2}{\partial r^2}\frac{1}{r}\frac{\partial}{\partial r}(k_1 r z_n^s(k_1 r)) \right]_{r=R} \\
T_{6,2,23}(n,m,j,s) &= -\frac{m m' a_{nm}^{0t} 2\pi\delta_{m+s,m'} k_j^s}{2\mu_2} Int_3(n,j,l,m,s,m') \left[r^2\frac{\partial^2}{\partial r^2}(k_2 z_n^t(k_2 r)) \right]_{r=R} \\
T_{6,2,24}(n,m,j,s) &= -\frac{m' 2\pi\delta_{m+s,m'} b_{nm}^{0t} k_j^s}{2\omega\epsilon_2\mu_2} Int_2(l,n,j,m',m,s) \left[r^2\frac{\partial^2}{\partial r^2}\frac{1}{r}\frac{\partial}{\partial r}(k_2 r z_n^t(k_2 r)) \right]_{r=R} \\
T_{6,2,25}(n,m,j,s) &= \frac{m m' a_{nm}^{1s} 2\pi\delta_{m+s,m'} f_j^s}{\mu_1} Int_3(n,j,l,m,s,m') \left[r\frac{\partial}{\partial r}(k_1 z_n^s(k_1 r)) \right]_{r=R} \\
T_{6,2,26}(n,m,j,s) &= \frac{m' 2\pi\delta_{m+s,m'} b_{nm}^{1s} f_j^s}{\omega\epsilon_1\mu_1} Int_2(l,n,j,m',m,s) \left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_1 r z_n^s(k_1 r)) \right]_{r=R} \\
T_{6,2,27}(n,m,j,s) &= -\frac{m m' a_{nm}^{1t} 2\pi\delta_{m+s,m'} f_j^s}{\mu_2} Int_3(n,j,l,m,s,m') \left[r\frac{\partial}{\partial r}(k_2 z_n^t(k_2 r)) \right]_{r=R} \\
T_{6,2,28}(n,m,j,s) &= -\frac{m' 2\pi\delta_{m+s,m'} b_{nm}^{1t} f_j^s}{\omega\epsilon_2\mu_2} Int_2(l,n,j,m',m,s) \left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_2 r z_n^t(k_2 r)) \right]_{r=R}
\end{aligned}
$$

$$
\nu_7 = \sum_{k=1}^{7}\sum_{n,j,s} T_{7,1,k}(n,j,s) + \sum_{k=1}^{28}\sum_{n,m,j,s} T_{7,2,k}(n,m,j,s)
$$

$$
\begin{aligned}
T_{7,1,1}(n,j,s) &= \frac{1}{4} s a_n^{inc} E_0 k_j^s Int2(n,l,j,1,m',s) \left[\frac{j_n(k_1 r)}{k_1 r} \right]_{r=R} \quad (C\delta) \\
T_{7,1,2}(n,j,s) &= \frac{-1}{4} s a_n^{inc} E_0 k_j^s Int_2(n,l,j,1,m',s) \left[r\frac{\partial}{\partial r}(\frac{j_n(k_1 r)}{k_1 r}) \right]_{r=R} \quad (C\delta) \\
T_{7,1,3}(n,j,s) &= -\frac{1}{4} i a_n^{inc} E_0 k_j^s Int_5(l,j,n,m',s,0) \left[r^2\frac{\partial^2 j_n(k_1 r)}{\partial r^2} \right]_{r=R} \quad (S\delta) \\
T_{7,1,4}(n,j,s) &= \frac{1}{4} m' a_n^{inc} Int2(n,j,l,1,s,m') E_0 k_j^s \left[\frac{j_n(k_1 r)}{k_1 r} \right]_{r=R} \quad (C\delta)
\end{aligned}
$$

$$T_{7,1,5}(n,j,s) = \frac{-1}{4} m' a_n^{inc} E_0 k_j^s Int_2(n,j,l,1,s,m') \left[r \frac{\partial}{\partial r} (\frac{j_n(k_1 r)}{k_1 r}) \right]_{r=R} (C\delta)$$

$$T_{7,1,6}(n,j,s) = -\frac{1}{4} i m' a_n^{inc} E_0 k_j^s \left[r^2 \frac{\partial^2 j_n(k_1 r)}{\partial r^2} \right]_{r=R} [a_3(n,0) Int_6(n+1,j,l,0,s,m')](C\delta)$$

$$T_{7,1,7}(n,j,s) = -\frac{1}{4} i m' a_n^{inc} E_0 k_j^s \left[r^2 \frac{\partial^2 j_n(k_1 r)}{\partial r^2} \right]_{r=R} [a_4(n,0) Int_6(n-1,j,l,0,s,m')](C\delta)$$

$$T_{7,2,1}(n,m,j,s) = -\frac{s a_{nm}^{0s} 2\pi \delta_{m+s,m'} k_j^s}{2\omega\epsilon_1\mu_1} Int_2(n,l,j,m,m',s) \left[(\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z_n^s(k_1 r)) \right]_{r=R}$$

$$T_{7,2,2}(n,m,j,s) = \frac{s a_{nm}^{0t} 2\pi \delta_{m+s,m'} k_j^s}{2\omega\epsilon_2\mu_2} Int_2(n,l,j,m,m',s) \left[(\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z_n^t(k_2 r)) \right]_{r=R}$$

$$T_{7,2,3}(n,m,j,s) = \frac{s a_{nm}^{0s} 2\pi \delta_{m+s,m'} k_j^s}{2\omega\epsilon_1\mu_1} Int_2(n,l,j,m,m',s) \left[r \frac{\partial}{\partial r} (\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z_n^s(k_1 r)) \right]_{r=R}$$

$$T_{7,2,4}(n,m,j,s) = -\frac{s a_{nm}^{0t} k_j^s 2\pi \delta_{m+s,m'}}{2\omega\epsilon_2\mu_2} Int_2(n,l,j,m,m',s) \left[r \frac{\partial}{\partial r} (\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z_n^t(k_2 r)) \right]_{r=R}$$

$$T_{7,2,5}(n,m,j,s) = \frac{s a_{nm}^{1s} 2\pi \delta_{m+s,m'} f_j^s}{\omega\epsilon_1\mu_1} Int_2(n,l,j,m,m',s) \left[(\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z_n^s(k_1 r)) \right]_{r=R}$$

$$T_{7,2,6}(n,m,j,s) = -\frac{s a_{nm}^{1t} 2\pi \delta_{m+s,m'} f_j^s}{\omega\epsilon_2\mu_2} Int_2(n,l,j,m,m',s) \left[(\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z_n^t(k_2 r)) \right]_{r=R}$$

$$T_{7,2,7}(n,m,j,s) = \frac{m a_{nm}^{0s} 2\pi \delta_{m+s,m'} k_j^s}{2\omega\epsilon_1\mu_1} Int_2(n,l,j,m,m',s) \left[r^2 \frac{\partial^2}{\partial r^2} \frac{1}{r} \frac{\partial}{\partial r} (k_1 r z_n^s(k_1 r)) \right]_{r=R}$$

$$T_{7,2,8}(n,m,j,s) = -\frac{2\pi \delta_{m+s,m'} b_{nm}^{0s} k_j^s}{2\epsilon_1} Int_1(n,l,j,m,m',s) \left[r^2 \frac{\partial^2}{\partial r^2} (k_1 z_n^s(k_1 r)) \right]_{r=R}$$

$$T_{7,2,9}(n,m,j,s) = -\frac{m a_{nm}^{0t} 2\pi \delta_{m+s,m'} k_j^s}{2\omega\epsilon_2\mu_2} Int_2(n,l,j,m,m',s) \left[r^2 \frac{\partial^2}{\partial r^2} \frac{1}{r} \frac{\partial}{\partial r} (k_2 r z_n^t(k_2 r)) \right]_{r=R}$$

$$T_{7,2,10}(n,m,j,s) = \frac{2\pi \delta_{m+s,m'} b_{nm}^{0t} k_j^s}{2\epsilon_2} Int_1(n,l,j,m,m',s) \left[r^2 \frac{\partial^2}{\partial r^2} (k_2 z_n^t(k_2 r)) \right]_{r=R}$$

$$T_{7,2,11}(n,m,j,s) = \frac{m a_{nm}^{1s} 2\pi \delta_{m+s,m'} f_j^s}{\omega\epsilon_1\mu_1} Int_2(n,l,j,m,m',s) \left[r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} (k_1 r z_n^s(k_1 r)) \right]_{r=R}$$

$$T_{7,2,12}(n,m,j,s) = -\frac{2\pi \delta_{m+s,m'} b_{nm}^{1s} f_j^s}{\epsilon_1} Int_1(n,l,j,m,m',s) \left[r \frac{\partial}{\partial r} (k_1 z_n^s(k_1 r)) \right]_{r=R}$$

$$T_{7,2,13}(n,m,j,s) = -\frac{m a_{nm}^{1t} 2\pi \delta_{m+s,m'} f_j^s}{\omega\epsilon_2\mu_2} Int_2(n,l,j,m,m',s) \left[r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} (k_2 r z_n^t(k_2 r)) \right]_{r=R}$$

$$T_{7,2,14}(n,m,j,s) = \frac{2\pi \delta_{m+s,m'} b_{nm}^{1t} f_j^s}{\epsilon_2} Int_1(n,l,j,m,m',s) \left[r \frac{\partial}{\partial r} (k_2 z_n^t(k_2 r)) \right]_{r=R}$$

$$T_{7,2,15}(n,m,j,s) = -\frac{m' a_{nm}^{0s} 2\pi \delta_{m+s,m'} k_j^s}{2\omega\epsilon_1\mu_1} Int_2[n,j,l,m,s,m'] \left[(\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z_n^s(k_1 r)) \right]_{r=R}$$

$$T_{7,2,16}(n,m,j,s) = \frac{m' a^{0t}_{nm} 2\pi\delta_{m+s,m'} k^s_j}{2\omega\epsilon_2\mu_2} Int_2[n,j,l,m,s,m'] \left[(\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z^t_n(k_2 r)) \right]_{r=R}$$

$$T_{7,2,17}(n,m,j,s) = \frac{m' a^{0s}_{nm} 2\pi\delta_{m+s,m'} k^s_j}{2\omega\epsilon_1\mu_1} Int_2[n,j,l,m,s,m'] \left[r\frac{\partial}{\partial r}(\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z^s_n(k_1 r)) \right]_{r=R}$$

$$T_{7,2,18}(n,m,j,s) = -\frac{m' a^{0t}_{nm} 2\pi\delta_{m+s,m'} k^s_j}{2\omega\epsilon_2\mu_2} Int_2[n,j,l,m,s,m'] \left[r\frac{\partial}{\partial r}(\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z^t_n(k_2 r)) \right]_{r=R}$$

$$T_{7,2,19}(n,m,j,s) = \frac{m' a^{1s}_{nm} 2\pi\delta_{m+s,m'} f^s_j}{\omega\epsilon_1\mu_1} Int_2[n,j,l,m,s,m'] \left[(\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z^s_n(k_1 r)) \right]_{r=R}$$

$$T_{7,2,20}(n,m,j,s) = -\frac{m' a^{1t}_{nm} 2\pi\delta_{m+s,m'} f^s_j}{\omega\epsilon_2\mu_2} Int_2[n,j,l,m,s,m'] \left[(\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z^t_n(k_2 r)) \right]_{r=R}$$

$$T_{7,2,21}(n,m,j,s) = \frac{m' a^{0s}_{nm} 2\pi\delta_{m+s,m'} k^s_j}{2\omega\epsilon_1\mu_1} Int_2(l,n,j,m',m,s) \left[r^2\frac{\partial^2}{\partial r^2}\frac{1}{r}\frac{\partial}{\partial r}(k_1 r z^s_n(k_1 r)) \right]_{r=R}$$

$$T_{7,2,22}(n,m,j,s) = -\frac{mm' 2\pi\delta_{m+s,m'} b^{0s}_{nm} k^s_j}{2\epsilon_1} Int_3(n,j,l,m,s,m') \left[r^2\frac{\partial^2}{\partial r^2}(k_1 z^s_n(k_1 r)) \right]_{r=R}$$

$$T_{7,2,23}(n,m,j,s) = -\frac{m' a^{0t}_{nm} 2\pi\delta_{m+s,m'} k^s_j}{2\omega\epsilon_2\mu_2} Int_2(l,n,j,m',m,s) \left[r^2\frac{\partial^2}{\partial r^2}\frac{1}{r}\frac{\partial}{\partial r}(k_2 r z^t_n(k_2 r)) \right]_{r=R}$$

$$T_{7,2,24}(n,m,j,s) = \frac{mm' 2\pi\delta_{m+s,m'} b^{0t}_{nm} k^s_j}{2\epsilon_2} Int_3(n,j,l,m,s,m') \left[r^2\frac{\partial^2}{\partial r^2}(k_2 z^t_n(k_2 r)) \right]_{r=R}$$

$$T_{7,2,25}(n,m,j,s) = \frac{m' a^{1s}_{nm} 2\pi\delta_{m+s,m'} f^s_j}{\omega\epsilon_1\mu_1} Int_2(l,n,j,m',m,s) \left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_1 r z^s_n(k_1 r)) \right]_{r=R}$$

$$T_{7,2,26}(n,m,j,s) = -\frac{mm' 2\pi\delta_{m+s,m'} b^{1s}_{nm} f^s_j}{\epsilon_1} Int_3(n,j,l,m,s,m') \left[r\frac{\partial}{\partial r}(k_1 z^s_n(k_1 r)) \right]_{r=R}$$

$$T_{7,2,27}(n,m,j,s) = -\frac{m' a^{1t}_{nm} 2\pi\delta_{m+s,m'} f^s_j}{\omega\epsilon_2\mu_2} Int_2(l,n,j,m',m,s) \left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_2 r z^t_n(k_2 r)) \right]_{r=R}$$

$$T_{7,2,28}(n,m,j,s) = \frac{mm' 2\pi\delta_{m+s,m'} b^{1t}_{nm} f^s_j}{\epsilon_2} Int_3(n,j,l,m,s,m') \left[r\frac{\partial}{\partial r}(k_2 z^t_n(k_2 r)) \right]_{r=R}$$

$$\nu_8 = \sum_{k=1}^{7}\sum_{n,j,s} T_{8,1,k}(n,j,s) + \sum_{k=1}^{28}\sum_{n,m,j,s} T_{8,2,k}(n,m,j,s) T_{8,2,k}(n,m,j,s))$$

$$T_{8,1,1}(n,j,s) = \frac{im' s\, a^{inc}_n k_1 E_0 k^s_j}{4\omega\mu_1} Int_3(n,j,l,1,s,m') \left[\frac{j_n(k_1 r)}{k_1 r} \right] (S\,\delta)$$

$$T_{8,1,2}(n,j,s) = -\frac{im' s a^{inc}_n k_1 E_0 k^s_j}{4\omega\mu_1} Int_3(n,j,l,1,s,m') \left[r\frac{\partial}{\partial r}(\frac{j_n(k_1 r)}{k_1 r}) \right]_{r=R} (S\,\delta)$$

$$
\begin{aligned}
T_{8,1,3}(n,j,s) &= \frac{m' a_n^{inc} k_1 E_0 k_j^s}{4\omega\mu_1} Int_6(n,j,l,0,s,m') \left[r^2 \frac{\partial^2 j_n(k_1 r)}{\partial r^2} \right]_{r=R} (C\delta) \\
T_{8,1,4}(n,j,s) &= \frac{i a_n^{inc} k_1 E_0 k_j^s}{4\omega\mu_1} Int_1(l,j,n,m',s,1) \left[\frac{j_n(k_1 r)}{k_1 r} \right]_{r=R} (S\delta) \\
T_{8,1,5}(n,j,s) &= -\frac{i a_n^{inc} k_1 E_0 k_j^s}{4\omega\mu_1} Int_1(l,j,n,m',s,1) \left[r \frac{\partial}{\partial r} (\frac{j_n(k_1 r)}{k_1 r}) \right]_{r=R} (S\delta) \\
T_{8,1,6}(n,j,s) &= \frac{a_n^{inc} k_1 E_0 k_j^s}{4\omega\mu_1} \left[r^2 \frac{\partial^2 j_n(k_1 r)}{\partial r^2} \right]_{r=R} [a_3(n,0) Int_5(l,j,n+1,m',s,0)](S\delta) \\
T_{8,1,7}(n,j,s) &= \frac{a_n^{inc} k_1 E_0 k_j^s}{4\omega\mu_1} \left[r^2 \frac{\partial^2 j_n(k_1 r)}{\partial r^2} \right]_{r=R} [a_4(n,0) Int_5(l,j,n-1,m',s,0)](S\delta) \\
T_{8,2,1}(n,m,j,s) &= \frac{m' s 2\pi \delta_{m+s,m'} b_{nm}^{0s} k_j^s}{2\omega\epsilon_1\mu_1} Int_3(n,j,l,m,s,m') \left[(\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z_n^s(k_1 r)) \right]_{r=R} \\
T_{8,2,2}(n,m,j,s) &= -\frac{m' s 2\pi \delta_{m+s,m'} b_{nm}^{0t} k_j^s}{2\omega\epsilon_2\mu_2} Int_3(n,j,l,m,s,m') \left[(\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z_n^t(k_2 r)) \right]_{r=R} \\
T_{8,2,3}(n,m,j,s) &= -\frac{m' s 2\pi \delta_{m+s,m'} b_{nm}^{0s} k_j^s}{2\omega\epsilon_1\mu_1} Int_3(n,j,l,m,s,m') \left[r \frac{\partial}{\partial r} (\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z_n^s(k_1 r)) \right]_{r=R} \\
T_{8,2,4}(n,m,j,s) &= \frac{m' s 2\pi \delta_{m+s,m'} b_{nm}^{0t} k_j^s}{2\omega\epsilon_2\mu_2} Int_3(n,j,l,m,s,m') \left[r \frac{\partial}{\partial r} (\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z_n^t(k_2 r)) \right]_{r=R} \\
T_{8,2,5}(n,m,j,s) &= -\frac{m' s 2\pi \delta_{m+s,m'} b_{nm}^{1s} f_j^s}{\omega\epsilon_1\mu_1} Int_3(n,j,l,m,s,m') \left[(\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z_n^s(k_1 r)) \right]_{r=R} \\
T_{8,2,6}(n,m,j,s) &= \frac{m' s 2\pi \delta_{m+s,m'} b_{nm}^{1t} f_j^s}{\omega\epsilon_2\mu_2} Int_3(n,j,l,m,s,m') \left[(\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z_n^t(k_2 r)) \right]_{r=R} \\
T_{8,2,7}(n,m,j,s) &= -\frac{m' a_{nm}^{0s} 2\pi \delta_{m+s,m'} k_j^s}{2\mu_1} Int_2(l,n,j,m',m,s) \left[r^2 \frac{\partial^2}{\partial r^2} (k_1 z_n^s(k_1 r)) \right]_{r=R} \\
T_{8,2,8}(n,m,j,s) &= -\frac{m m' 2\pi \delta_{m+s,m'} b_{nm}^{0s} k_j^s}{2\omega\epsilon_1\mu_1} Int_3(n,j,l,m,s,m') \left[r^2 \frac{\partial^2}{\partial r^2} \frac{1}{r} \frac{\partial}{\partial r} (k_1 r z_n^s(k_1 r)) \right]_{r=R} \\
T_{8,2,9}(n,m,j,s) &= \frac{m' a_{nm}^{0t} 2\pi \delta_{m+s,m'} k_j^s}{2\mu_2} Int_2(l,n,j,m',m,s) \left[r^2 \frac{\partial^2}{\partial r^2} (k_2 z_n^t(k_2 r)) \right]_{r=R} \\
T_{8,2,10}(n,m,j,s) &= \frac{m m' 2\pi \delta_{m+s,m'} b_{nm}^{0t} k_j^s}{2\omega\epsilon_2\mu_2} Int_3(n,j,l,m,s,m') \left[r^2 \frac{\partial^2}{\partial r^2} \frac{1}{r} \frac{\partial}{\partial r} (k_2 r z_n^t(k_2 r)) \right]_{r=R} \\
T_{8,2,11}(n,m,j,s) &= -\frac{m' a_{nm}^{1s} 2\pi \delta_{m+s,m'} f_j^s}{\mu_1} Int_2(l,n,j,m',m,s) \left[r \frac{\partial}{\partial r} (k_1 z_n^s(k_1 r)) \right]_{r=R} \\
T_{8,2,12}(n,m,j,s) &= -\frac{m m' 2\pi \delta_{m+s,m'} b_{nm}^{1s} f_j^s}{\omega\epsilon_1\mu_1} Int_3(n,j,l,m,s,m') \left[r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} (k_1 r z_n^s(k_1 r)) \right]_{r=R}
\end{aligned}
$$

$$T_{8,2,13}(n,m,j,s) = \frac{m' a^{1t}_{nm} 2\pi\delta_{m+s,m'} f^s_j}{\mu_2} Int_2(l,n,j,m',m,s)\left[r\frac{\partial}{\partial r}(k_2 z^t_n(k_2 r))\right]_{r=R}$$

$$T_{8,2,14}(n,m,j,s) = \frac{mm' 2\pi\delta_{m+s,m'} b^{1t}_{nm} f^s_j}{\omega\epsilon_2\mu_2} Int_3(n,j,l,m,s,m')\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_2 r z^t_n(k_2 r))\right]_{r=R}$$

$$T_{8,2,15}(n,m,j,s) = \frac{2\pi\delta_{m+s,m'} b^{0s}_{nm} k^s_j}{2\omega\epsilon_1\mu_1} Int_1(l,j,n,m',s,m)\left[(\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z^s_n(k_1 r))\right]_{r=R}$$

$$T_{8,2,16}(n,m,j,s) = -\frac{2\pi\delta_{m+s,m'} b^{0t}_{nm} k^s_j}{2\omega\epsilon_2\mu_2} Int_1(l,j,n,m',s,m)\left[(\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z^t_n(k_2 r))\right]_{r=R}$$

$$T_{8,2,17}(n,m,j,s) = -\frac{2\pi\delta_{m+s,m'} b^{0s}_{nm} k^s_j}{2\omega\epsilon_1\mu_1} Int_1(l,j,n,m',s,m)\left[r\frac{\partial}{\partial r}(\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z^s_n(k_1 r))\right]_{r=R}$$

$$T_{8,2,18}(n,m,j,s) = \frac{2\pi\delta_{m+s,m'} b^{0t}_{nm} k^s_j}{2\omega\epsilon_2\mu_2} Int_1(l,j,n,m',s,m)\left[r\frac{\partial}{\partial r}(\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z^t_n(k_2 r))\right]_{r=R}$$

$$T_{8,2,19}(n,m,j,s) = -\frac{2\pi\delta_{m+s,m'} b^{1s}_{nm} f^s_j}{\omega\epsilon_1\mu_1} Int_1(l,j,n,m',s,m)\left[(\frac{\partial^2}{\partial r^2} + k_1^2)(k_1 r\, z^s_n(k_1 r))\right]_{r=R}$$

$$T_{8,2,20}(n,m,j,s) = \frac{2\pi\delta_{m+s,m'} b^{1t}_{nm} f^s_j}{\omega\epsilon_2\mu_2} Int_1(l,j,n,m',s,m)\left[(\frac{\partial^2}{\partial r^2} + k_2^2)(k_2 r\, z^t_n(k_2 r))\right]_{r=R}$$

$$T_{8,2,21}(n,m,j,s) = -\frac{m a^{0s}_{nm} 2\pi\delta_{m+s,m'} k^s_j}{2\mu_1} Int_2(n,l,j,m,m',s)\left[r^2\frac{\partial^2}{\partial r^2}(k_1 z^s_n(k_1 r))\right]_{r=R}$$

$$T_{8,2,22}(n,m,j,s) = -\frac{2\pi\delta_{m+s,m'} b^{0s}_{nm} k^s_j}{2\omega\epsilon_1\mu_1} Int_1(n,l,j,m,m',s)\left[r^2\frac{\partial^2}{\partial r^2}\frac{1}{r}\frac{\partial}{\partial r}(k_1 r z^s_n(k_1 r))\right]_{r=R}$$

$$T_{8,2,23}(n,m,j,s) = \frac{m a^{0t}_{nm} 2\pi\delta_{m+s,m'} k^s_j}{2\mu_2} Int_2(n,l,j,m,m',s)\left[r^2\frac{\partial^2}{\partial r^2}(k_2 z^t_n(k_2 r))\right]_{r=R}$$

$$T_{8,2,24}(n,m,j,s) = \frac{2\pi\delta_{m+s,m'} b^{0t}_{nm} k^s_j}{2\omega\epsilon_2\mu_2} Int_1(n,l,j,m,m',s)\left[r^2\frac{\partial^2}{\partial r^2}\frac{1}{r}\frac{\partial}{\partial r}(k_2 r z^t_n(k_2 r))\right]_{r=R}$$

$$T_{8,2,25}(n,m,j,s) = -\frac{m a^{1s}_{nm} 2\pi\delta_{m+s,m'} f^s_j}{\mu_1} Int_2(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}(k_1 z^s_n(k_1 r))\right]_{r=R}$$

$$T_{8,2,26}(n,m,j,s) = -\frac{2\pi\delta_{m+s,m'} b^{1s}_{nm} f^s_j}{\omega\epsilon_1\mu_1} Int_1(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_1 r z^s_n(k_1 r))\right]_{r=R}$$

$$T_{8,2,27}(n,m,j,s) = \frac{m a^{1t}_{nm} 2\pi\delta_{m+s,m'} f^s_j}{\mu_2} Int_2(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}(k_2 z^t_n(k_2 r))\right]_{r=R}$$

$$T_{8,2,28}(n,m,j,s) = \frac{2\pi\delta_{m+s,m'} b^{1t}_{nm} f^s_j}{\omega\epsilon_2\mu_2} Int_1(n,l,j,m,m',s)\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(k_2 r z^t_n(k_2 r))\right]_{r=R}$$

VITA

Date and Place of Birth: 15.05.1979, Kars-TURKEY

EDUCATION

2008 - 2015 Doctor of Philosophy in Mathematics
Graduate School of Engineering and Sciences, İzmir Institute of Technology,
İzmir -Turkey
Thesis Title: SOLUTION OF MAXWELL EQUATIONS
ON DEFORMED SPHERICAL DOMAINS:
APPLICATIONS TO THE SCATTERING PROBLEMS
Supervisor: Prof. Dr. Oğuz YILMAZ

2003 - 2008 Master of Science in Mathematics
Graduate School of Engineering and Sciences, İzmir Institute of Technology
İzmir -Turkey
Thesis Title: NONLINEAR EULER POISSON DARBOUX EQUATIONS
EXACTLY SOLVABLE IN MULTIDIMENSIONS
Supervisor: Prof. Dr. Oktay PASHAEV

1997 - 2003 Bachelor of Mathematics
Department of Mathematics, Faculty of Science, Marmara University
İstanbul- Turkey.