CONVERGENCE ANALYSIS OF OPERATOR SPLITTING METHODS FOR THE BURGERS-HUXLEY EQUATION

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ABSTRACT

CONVERGENCE ANALYSIS OF OPERATOR SPLITTING METHODS FOR THE BURGERS-HUXLEY EQUATION

The purpose of this thesis is to investigate the implementation of the two operator splitting methods; Lie-Trotter splitting and Strang splitting method applied to the Burgers-Huxley equation and prove their convergence rates in $H^s(\mathbb{R})$, for $s \geq 1$. The analyses are based on the properties of the Sobolev spaces. The Burgers-Huxley equation is deal with the two parts; linear and non-linear parts. The regularity results are shown by using the same technique in (Holden, Lubich and Risebro, 2013) for both parts. By combining these results with the numerical quadratures and the Peano Kernel theorem error bounds are derived for the first and second order splitting methods. In the computational part, the operator splitting methods are applied to the Burgers-Huxley equation. Finally, the convergence rates for the two splitting methods are checked numerically. These numerical results confirmed the theoretical results.

ÖZET

BURGERS-HUXLEY DENKLEMİ İÇİN OPERATÖR AYIRMA METODLARININ YAKINSAKLIK ANALİZİ

Bu tezin amacı, iki operatör ayırma metodu olan Lie-Trotter ve Strang ayırma metotlarının Burgers-Huxley denklemine uygulanmasını incelemek ve bu methodların yakınsaklık analizlerini, $s \geq 1$ olmak üzere, $H^s(\mathbb{R})$ uzayında kanıtlamaktır. Analizler, Sobolev uzayının özelliklerine dayanmaktadır. Burgers-Huxley denklemi, doğrusal ve doğrusal olmayan olmak üzere iki bölümde ele alınmıştır. Her iki bölüm için de doğruluk sonuçları (Holden, Lubich and Risebro, 2013) da kullanılan tekniğin aynısı kullanılarak gösterilmiştir. Bu sonuçlar, sayısal integrasyon ve Peano Kernel teoremi ile birleştirilerek birinci ve ikinci mertebeden ayırma metotları için hata sınırları elde edilmiştir. Sayısal kısımda, Burgers-Huxley denklemine operatör ayırma metotları uygulanmıştır. Son olarak, bu iki ayırma metodunun yakınsaklık hızları sayısal olarak kontrol edilmiştir. Bu sayısal sonuçlar teorik sonuçlar ile doğrulanmıştır.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

INTRODUCTION

Partial differential equations have great importance in most field of science. In fact, partial differential equations originated from the study of surfaces in geometry and for solving a wide variety of problems in mechanics (Debnath, 2012). Nonlinear partial differential equations (NPDE) are also important in various fields of science and technology. NPDEs has been study of nonlinear wave propagation problems. These problems arise in different areas of applied mathematics, real-world physical systems, including gas dynamics, fluid mechanics, elasticity, relativity, ecology, neurology, thermodynamics and many more are modelled by NPDEs.

Being a NPDE, Burgers-Huxley equation (BHE) has a great importance. It describes the interactions between reaction mechanisms, convection effects and diffusion transports (Satsuma,1987). This equation was firstly introduced by Bateman (Bateman, 1915), then used by Burgers (Burgers, 1939) in a mathematical modelling in turbulance. BHE is a perturbation problem by having a perturbation parameter (say, ϵ).

There are different approaches for solving NPDEs, but there is no general method for finding solutions. Therefore, numerical approximation methods are important in physical problems. BHE has studied by a variety of researchers. The differential transform method is applied to the generalized Burgers-Huxley equation in (Biazar and Mohammadi, 2010). In (Ismail, Raslan and Rabbah, 2004) Adomain decomposition method is applied for approximate solution for the BHE. The iterative differential quadrature method is performed by (Tomosiello, 2010) for BHE. A numerical solution of the generalized BHE is presented in (Javidi, 2006), which is solved by using the spectral collocation method. Also, Javidi (Javidi, 2006) has given a pseudospectral method for generalized BHE. In (Jiwari and Mittal, 2011) they have applied a quasilinearization process which is long and complicated procedure. Then, Zhou and Cheng (Zhou and Cheng, 2011) have applied the operator splitting method to the BHE by solving two nonlinear subproblems.

Operator splitting method is a powerful method for solving complex models. The basic idea of the splitting process is splitting the problem into two subproblems and solving each subproblems iteratively instead of the whole problem. For a detailed explanation and introduction to the operator splitting methods we recommend the study which is worked by (Machlachan, 2002) and master thesis (Yazıcı, 2010).

The idea of the operator splitting, which is Lie-Trotter splitting, dates back to the

1950s. It was in 1957 that this method was first used in the solution of partial differential equations (Bagrinovskii and Gudunov, 1957). The first splitting methods were developed in the 1960s and 1970s and were based on the fundamental results of finite difference methods. The classsical splitting methods are the Lie-Trotter splitting and the Strang splitting method (Dimov, 2001), (Strang, 1968). A renewal of the methods was done in the 1980s while using the methods or complex process underlying partial differential methods in (Crandall, 1980).

The main purpose in this thesis is to apply the Lie-Trotter and Strang splitting method to BHE and prove the convergence rates for these methods in Sobolev spaces. Error estimates for convergence of the Lie-Trotter and Strang splitting methods were studied for the KdV equation in (Holden, Karlsen, Risebro and Tao, 2011). Since the solutions of the KdV equation remain bounded in Sobolev space they compound this with the counter argument guarantees the existence of time step Δt that avoid the solution blowing up. On the other hand, (Holden, Lubich and Risebro, 2013) studied equation with a Burgers type nonlinearity including the KdV equation. They implement an analysis which is based on the statement of the error terms which are accured in the local error as quadrature forms. In (Jahnke and Lubich, 2000) and (Lubich, 2008), similar analyses are studied for linear evolution equations and for nonlinear Schrödinger equations, respectively. We follow the similar approach in (Holden, Karlsen, Risebro and Tao, 2011) to show the convergence rates of the operator splitting methods which are implemented on the BHE in Sobolev spaces.

We focus our attention on the case of linear and nonlinear operators such as,

$$U_t = AU(t) + B(U(t)), \ \ with \ t \in [0,T], \ \ U|_{t=t_0} = U_0 \tag{1.1}$$

We employ Lie-Trotter and Strang splitting methods to the one-dimensional Burgers-Huxley equation,

$$U_t + \alpha U U_x - \epsilon U_{xx} = \beta(1-U)(U-\gamma)U, \tag{1.2}$$

with the initial condition

$$U|_{t=t_0} = U_0 \tag{1.3}$$

where $t > 0$, $\alpha,\beta \geq 0$, $0 < \epsilon \leq 1$ and $0 < \gamma < 1$. When $\alpha = 0$ and $\epsilon = 1$, equation (1.2) reduces to Huxley equation and when $\beta = 0$, reduces to Burgers' equation.

There has been intense research in solving BHE. The recent article (Zhou and Cheng, 2011) focuses on solving BHE by using operator spltting methods. They decompose the equation into two subproblems, i.e. a Burgers equation and a nonlinear ordinary differential equation. In contrast to that approach, in this thesis we break the (1.2) into linear diffusion equation and nonlinear reaction equation. In this latter type of the operator splitting, the simpler equations are solved and then recoupled over the initial conditions in delicate ways to preserve a certain accuracy. We denote by $U(t) = \Omega^t_{A+B}(U_0)$ is the solution at the time t of (3.1) with given initial condition and the approximate split solution is denoted by U_N, at $t = N\Delta t \leq T$, as $\Delta t \to 0$, where the split solutions are denoted as follows,

Lie-Trotter Splitting solution,

$$U_{N+1} = \Psi^{\Delta t}(U_N) = \Omega_A^{\Delta t}\left(\Omega_B^{\Delta t}(U_N)\right), \quad N = 0, 1, 2, ... \tag{1.4}$$

Strang splitting solution,

$$U_{N+1} = \Pi^{\Delta t}(U_N) = \Omega_A^{\Delta t/2}\left(\Omega_B^{\Delta t}\left(\Omega_A^{\Delta t/2}(U_N)\right)\right), \quad N = 0, 1, 2, ... \tag{1.5}$$

In our case we split the equation (1.2) into two subequations,

$$u_t = Au = \epsilon u_{xx}, \tag{1.6}$$

and

$$v_t = B(v) = \beta(1 - v)(v - \gamma)v - \alpha v v_x, \tag{1.7}$$

acting on appropriate Sobolev spaces.

The outline of this thesis can be given as follows:

In Chapter 2, we introduce the operator splitting methods Lie-Trotter and Strang splitting by explaning the basic idea of the splitting and giving their algorithms. Then, we apply the operator splitting methods to the BHE by dividing the equation into linear and nonlinear parts. We start our analysis by giving two hypotheses about the local well-posedness of the solutions to the BHE and boundedness of the solution and the initial condition in Sobolev spaces. Furthermore, we also present and prove the regularity results for both linear and nonlinear parts of the BHE and prove the lemmas which are about the boundedness of the nonlinear part in Sobolev spaces. We also use auxillary lemmas about the Sololev spaces for both Lie-Trotter and Strang splitting methods. The Sobolev spaces and related properties are introduced in Appendix B. Since the proofs are depends on the differential theory in Banach spaces we give the definition of the Fréchet derivative in Appendix C. By using these regularity results, definition of the Fréchet derivative and the quadrature error estimates we get the local error estimates for the two splitting methods. Finally, we add up all the local errors and get the global error estimates for both Lie-Trotter and Strang splitting methods. A brief overview of the concepts of the numerical integration and the Peano Kernel Theorem are given in Appendix A. As a result of these lemmas and properties we prove that Lie-Trotter splitting converges as $O(\Delta t)$, while Strang splitting converges $O((\Delta t)^2)$ in Sobolev spaces.

Chapter 3 deals with the numerical results and simulations for the two operator splitting methods applied to the BHE. We employ various numerical schemes for the space discretization and finally we confirm the convergence results by using the Chebyshev differentiation matrices. These experiments using MATLAB confirm the theoretical results which are shown in Chapter 2.

Finally, in Chapter 4, we summarize the main results in the thesis and given a brief conclusion.

CHAPTER 2

CONVERGENCE ANALYSIS OF THE OPERATOR SPLITTING METHODS

Operator splitting methods are well known in the field of numerical solution of partial differential equations. The technique is generally used in one of the two ways: It is used in methods in which one splits the differential operator such that each split system only involves derivatives along one of the coordinate axes. Alternatively, it is used as a means to split the differential operator into several parts, where each part represents a particular physical phenomenon, such as convection, diffusion, etc. In either case, the corresponding numerical method is defined as a sequence of solves of each of the split problems. This can lead to very efficient methods, since one can treat each part of the original operator independently.

Operator splitting means the spatial differential operator appearing in the equations is split into a sum of different sub-operators having simpler forms, and the corresponding equations can be solved easier. Operator splitting is an attractive technique for solving coupled systems of partial differential equations, since complex equation system maybe split into simpler parts that are easier to solve. Several operator splitting techniques exist, but we will apply a class of methods often referred as fractional step methods.

This work is devoted to analytical prove the converge rates for the two splitting methods; Lie-Trotter and Strang splitting methods using a new framework recently introduced in (Holden, Lubich and Risebro, 2013). In (Holden, Lubich and Risebro, 2013), the correct convergence rate for the Strang splitting in Sobolev spaces is proven, for a large class of partial differential equations. We follow this outline, and in addition we adopt the ideas from the framework to prove the correct convergence rates for these operator splitting methods for the Burgers-Huxley equation.

This chapter is divided as follows: First, we introduce the operator splitting methods for an abstract differential equation, then we apply the Lie-Trotter and Strang splitting methods to the Burgers-Huxley equation. This is put through by some regularity results for the both linear and nonlinear parts of the equation. Then, we find the local error estimates for these splitting methods. Finally, by adding up all the local errors we get the global error.

2.1. Operator Splitting Method

We focus our attention on the following partial differential equation

$$U_t = (A + B)U(t), \tag{2.1}$$

$$U(0) = U_0. \tag{2.2}$$

where $t \in [0, T]$ with $T > 0$. A, B are assumed to be differential operators between some normed spaces, say $\mathbf{X}$ and the initial condition U_0, solution $U(t)$ are also in $\mathbf{X}$. We can write the Taylor series expansion for $U(t)$ as follows,

$$U(t) = U(0) + tU_t(0) + O(t^2). \tag{2.3}$$

By substituting (2.1) into the (2.3) we get

$$U(t) = U(0) + t(A + B)(U_0) + O(t^2). \tag{2.4}$$

The idea of the operator splitting method is dividing the problem into simpler sub-problems and solve them for a small time step Δt. We discretize the time such that $t_N \leq N\Delta t$.

Splitting algorithm is a quite simple procedure. Start with solving the first sub-problem with the operator A by using the original initial condition of the problem, then we solve the second sub-problem with the operator B by using the the first sub-problem's solution as an initial condition. And the procedure is going on this way. We solve the following sub-problems instead of solving the whole problem,

$$u_t = Au(t)\,, \quad t \in [t^N, t^{N+1}] \tag{2.5}$$

$$u(t^N) = u_s^N, \tag{2.6}$$

$$v_t = Bv(t)\,, \quad t \in [t^N, t^{N+1}] \tag{2.7}$$

$$v(t^N) = u(t^{N+1}), \tag{2.8}$$

where split condition is given at $t = 0$ as $u_s^0 = U_0$ in (2.2) and the approximate solution at $t = t^{N+1}$ is $u_s^{N+1} = v(t^{N+1})$, where $t^{N+1} = t^N + \Delta t$, Δt is time step and $N = 0, 1, ...n - 1$ such that

$t^n = T$. Writing out this procedure we get,

$$\begin{aligned} U_{N+1} &= \Omega_A^{\Delta t}(\Omega_B^{\Delta t}(U_N)) \\ &= \Omega_A^{\Delta t} \circ \Omega_B^{\Delta t}(U_N) = [\Omega_A^{\Delta t} \circ \Omega_B^{\Delta t}]^N(U_0). \end{aligned} \tag{2.9}$$

where U_N is the split solution and Ω is the exact solution operator. This is the Lie-Trotter splitting.

Strang splitting algorithm is similar to the Lie-Trotter splitting, but the main difference is we solve the first sub-problem for a half interval with the operator A, then solve for the whole interval with operator B and again solve the half interval with the operator A. The algorithm is given as,

$$u_t = Au(t)\ , \quad t \in [t^N, t^{N+1/2}] \tag{2.10}$$
$$u(t^N) = u_s^N, \tag{2.11}$$
$$v_t = Bv(t)\ , \quad t \in [t^N, t^{N+1}] \tag{2.12}$$
$$v(t^N) = u(t^{N+1/2}), \tag{2.13}$$
$$w_t = Aw(t)\ , \quad t \in [t^N, t^{N+1/2}] \tag{2.14}$$
$$w(t^N) = v(t^{N+1}), \tag{2.15}$$

where $t^{N+1/2} = t^N + 0.5\Delta t$, and the approximate solution at $t = t^{N+1}$ is $u_s^{N+1} = w(t^{N+1})$. Writing out this procedure we get,

$$\begin{aligned} U_{N+1} &= \Omega_A^{\Delta t/2}(\Omega_B^{\Delta t}(\Omega_A^{\Delta t/2}(U_N))) \\ &= \Omega_A^{\Delta t/2} \circ \Omega_B^{\Delta t} \circ \Omega_A^{\Delta t/2}(U_N) = [\Omega_A^{\Delta t/2} \circ \Omega_B^{\Delta t} \circ \Omega_A^{\Delta t/2}]^N(U_0). \end{aligned} \tag{2.16}$$

We need to show both (2.9) and (2.16) converge the exact solution of the given problem when $\Delta t \to 0$. We know Lie-Trotter splitting as first order and Strang splitting as second order splitting methods. The main goal of this chapter is to prove these convergence rates for the Lie-Trotter and Strang splitting methods in Sobolev spaces.

To achieve this goal, we first find the error for one step with splitting methods which is known as the local error, the by summing up all these errors we get the global error. We use the same technique in (Holden, Lubich and Risebro, 2013) to find this estimate. We use the numerical quadratures and the Peano Kernel theorem which are given in the Appendix A and Appendix B for local error estimates in $H^s(\mathbb{R})$, where $H^s(\mathbb{R})$ is the Sobolev space with positive s.

2.2. Application to the Burgers-Huxley Equation

We will investigate the Burgers-Huxley equation as follows,

$$U_t + \alpha U U_x - \epsilon U_{xx} = \beta(1 - U)(U - \gamma)U, \tag{2.17}$$

$$U(t_0) = U_0. \tag{2.18}$$

where $x \in \mathbb{R}$, $t \in [0, T]$ for a fixed time $T > 0$, $\alpha, \beta \geq 0$, $0 < \epsilon \leq 1$ and $0 < \gamma < 1$. In this work we will split the complex problem into simpler subequations, each of which solved by an efficient method. With general formulation of the operator splitting method we formulate the problem which we shall delve into. Applying the operator splitting method to (2.17), and splitting it into two subequations gives

$$u_t = A(u) = \epsilon u_{xx}, \tag{2.19}$$

$$v_t = B(v) = \beta(1 - v)(v - \gamma)v - \alpha v v_x. \tag{2.20}$$

We will investigate the Lie-Trotter and Strang splitting numerically for the given Burgers-Huxley equation. In the begining of the analysis, we assume that the solutions to the BHE are locally well-posed and bounded. Thus, the following hypotheses are about the local well-posedness of the solutions to (2.17) and boundedness of the solution and initial condition in Sobolev spaces.

Hypothesis 2.1 *(Nilsen, 2011) For a fixed time T, there exists $M > 0$ such that for all U_0 in $H^k(\mathbb{R})$ with $\|U_0\|_{H_k} \leq M$, there exists a unique strong solution U in $C([0, T], H^k)$ of (2.17). In addition, for the initial data U_0 there exists a constant $K(M, T) < \infty$, such that*

$$\|\tilde{U}(t) - U(t)\|_{H^k} \leq K(M, T)\|\tilde{U}_0 - U_0\|_{H^k}, \tag{2.21}$$

for two arbitrary solutions U and $\tilde{U}$, corresponding to two different initial data $\tilde{U}_0$ and U_0.

Hypothesis 2.2 *The solution $U(t)$ and the initial data U_0 of (3.1) are both in $H^k(\mathbb{R})$, and are bounded as*

$$\|U(t)\|_{H^k} \leq M < \rho \text{ and } \|U_o\|_{H^k} \leq C < \infty, \tag{2.22}$$

for $0 \leq t \leq T$.

We define following set of integers such that,

$$s \geq 1, \quad m = s + 3, \quad n = s + 1 = m - 2. \tag{2.23}$$

We specify for which integers the hypothesis should hold in the lemmas and theorems for the operator splitting methods. (Nilsen, 2011)

2.3. Regularity results for the Burgers-Huxley Equation

In this section, we will present and prove several results to estimate the local error for the operator splitting for the Burgers-Huxley equation. We need to show that there exists a small time step Δt for the solutions $\Omega_A^t(u_0)$ and $\Omega_B^t(v_0)$ in a Sobolev spaces.

2.3.1. Results for the Nonlinear Part

In the previous section we split the Burgers-Huxley equation into linear and nonlinear parts. To prove the convergence of the splitting we need to show that both parts are bounded. We state which properties B must satisfy in the following lemmas.

Lemma 2.1 *For m and n in (2.23) assume the solution $\Omega_B^t(v_0) = v(t)$ of (2.20) with initial data v_0 in $H^m(\mathbb{R})$, satisfies $\|\Omega_B^t(v_0)\|_{H^n} \leq \alpha$ for $0 \leq t \leq \Delta t$. Then $\Omega_B^t(v_0)$ is in $H^m(\mathbb{R})$ and in particular*

$$\|\Omega_B^t(v_0)\|_{H^m} \leq e^{c\alpha_1 t}\|v_0\|_{H^m}, \tag{2.24}$$

where $\alpha_1 = (C + 2C\alpha + C\alpha^2)$, C is a general constant and c is independent of v_0.

Proof We use the definition of the norm $H^m(\mathbb{R})$ in the Appendix C and find that $v(t)$ satisfies the following,

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|\Omega_B^t(v_0)\|_{H^m}^2 \\
= \; & \frac{1}{2}\frac{d}{dt}\|v\|_{H^m}^2 = \frac{1}{2}\frac{d}{dt}\sum_{j=0}^{m}\int_{\mathbb{R}}\partial_x^j w\partial_x^j w_t dx \\
= \; & (v, v_t)_{H^m} = (v, \beta v(1-v)(v-\gamma) - \alpha v v_x)_{H^m} \\
= \; & \beta(1+\gamma)\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{j}{k}\int_{\mathbb{R}}\partial_x^j v\partial_x^k v\partial_x^{j-k} v dx \\
& -\beta\sum_{j=0}^{m}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{j}{k}\binom{k}{l}\int_{\mathbb{R}}\partial_x^j v\partial_x^l w\partial_x^{k-l} v\partial_x^{j-k} v dx \\
& -\beta\gamma\sum_{j=0}^{m}\int_{\mathbb{R}}\partial_x^j v\partial_x^j v dx - \alpha\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{j}{k}\int_{\mathbb{R}}\partial_x^j v\partial_x^{k+1} v\partial_x^{j-k} v dx. \qquad (2.25)
\end{aligned}
$$

We investigate the each parts for different cases.

Case 1: For $j < m$ and $k < j$, we obtain for the first term of (2.25)

$$
\begin{aligned}
\left|\int_{\mathbb{R}}\partial_x^j v\partial_x^k v\partial_x^{j-k} v dx\right| & \leq \int_{\mathbb{R}}|\partial_x^j v\partial_x^k v\partial_x^{j-k} v| dx \\
& \leq \|\partial_x^j v\|_{L^\infty}\|\partial_x^{\max\{k,j-k\}} v\|_{L^2}\|\partial_x^{\min\{k,j-k\}} v\|_{L^2} \\
& \leq C\| v\|_{H^m}\| v\|_{H^m}\| v\|_{H^n} \\
& \leq C\alpha\| v\|_{H^m}^2, \qquad (2.26)
\end{aligned}
$$

where we have used Sobolev inequality and the fact that

$$
\begin{aligned}
& \max\{k, j-k\} \leq j+1 \leq m, \\
& \min\{k, j-k\} \leq \frac{j}{2} \leq \frac{m}{2} = \frac{s-1}{2} + 2 \leq s+1 = m-2 = n,
\end{aligned}
$$

since $m \geq 4$.

For the second term of (2.25),

$$\begin{aligned}\left|\int_{\mathbb{R}} \partial_x^j v \partial_x^l v \partial_x^{k-l} v \partial_x^{j-k} v dx\right| &\leq \int_{\mathbb{R}} |\partial_x^j v \partial_x^l v \partial_x^{k-l} v \partial_x^{j-k} v| dx \\ &\leq ||\partial_x^j v||_{L^\infty} ||\partial_x^{j-k} v||_{L^\infty} \int_{\mathbb{R}} |\partial_x^l v \partial_x^{k-l} v| dx \\ &\leq ||v||_{H^m} ||v||_{H^m} ||\partial_x^l v||_{L^2} ||\partial_x^{k-l} v||_{L^2} \\ &\leq C||v||_{H^m}^2 ||v||_{H^l} ||v||_{H^{k-l}} \\ &\leq C\alpha^2 ||v||_{H^m}^2. \end{aligned} \tag{2.27}$$

If we take $l < k \leq n$ and $k - l < n$.

For the third term of (2.25),

$$\begin{aligned}\left|\int_{\mathbb{R}} \partial_x^j v \partial_x^j v dx\right| &\leq \int_{\mathbb{R}} |\partial_x^j v \partial_x^j v| dx \\ &\leq ||\partial_x^j v||_{L^2} |\partial_x^j v||_{L^2} \\ &\leq C||v||_{H^m}^2. \end{aligned} \tag{2.28}$$

The last term of the (2.25) we have the bound

$$\begin{aligned}\left|\int_{\mathbb{R}} \partial_x^j v \partial_x^{k+1} v \partial_x^{j-k} v dx\right| &\leq ||\partial_x^j v||_{L^\infty} ||v||_{H^m} ||v||_{H^n} \\ &\leq C\alpha ||v||_{H^m}^2, \end{aligned} \tag{2.29}$$

see (Nilsen, 2011).

Case 2: For $j = m$, we obtain for the first term of (2.25)

$$\begin{aligned}\left|\int_{\mathbb{R}} \partial_x^j v \partial_x^k v \partial_x^{j-k} v dx\right| &\leq ||\partial_x^k v||_{L^\infty} ||\partial_x^m v||_{L^2} ||\partial_x^{m-k} v||_{L^2} \\ &\leq C||\partial_x v||_{H^k} ||v||_{H^m} ||v||_{H^{m-k}} \\ &\leq C||v||_{H^{k+1}} ||v||_{H^m}^2. \end{aligned} \tag{2.30}$$

To find a bound we investigate this inequality in two cases; when $k+1 \leq n$ and when $k = n$. For the first case we obtain

$$\left|\int_{\mathbb{R}} \partial_x^j v \partial_x^k v \partial_x^{j-k} v dx\right| \leq C\alpha \|v\|_{H^m}^2. \tag{2.31}$$

For the second case, we get

$$\begin{aligned}\left|\int_{\mathbb{R}} \partial_x^j v \partial_x^k v \partial_x^{j-k} v dx\right| &\leq \|v\|_{H^{n+1}} \|v\|_{H^m} \|v\|_{H^{m-n}} \\ &\leq C\alpha \|v\|_{H^m}^2,\end{aligned} \tag{2.32}$$

here we have used that $n+1 \leq n+2 \leq m$, and $m-n = 2 \leq s+1 = n$.

We are left with 2 cases; $k \leq m$ and $k = m = j$. For the first case we get,

$$\begin{aligned}\left|\int_{\mathbb{R}} \partial_x^j v \partial_x^k v \partial_x^{j-k} v dx\right| &\leq \|\partial_x^m v\|_{L^2} \|\partial_x^k v\|_{L^2} \|\partial_x^{m-k} v\|_{L^\infty} \\ &\leq C\|v\|_{H^m} \|v\|_{H^m} \|v\|_{H^{m-k+1}} \\ &\leq C\alpha \|v\|_{H^m}^2,\end{aligned} \tag{2.33}$$

because $m-k+1 < m-n \leq 2 \leq n$. For the second case, we have

$$\begin{aligned}\left|\int_{\mathbb{R}} \partial_x^m v \partial_x^m v v dx\right| &\leq \|v\|_{L^\infty} \|\partial_x^m v\|_{L^2} \|\partial_x^m v\|_{L^2} \\ &\leq C\|v\|_{H^n} \|v\|_{H^m}^2 \\ &\leq C\alpha \|v\|_{H^m}^2.\end{aligned} \tag{2.34}$$

For the second term of (2.25),

$$\begin{aligned}\left|\int_{\mathbb{R}} \partial_x^m v \partial_x^l v \partial_x^{k-l} v \partial_x^{m-k} v dx\right| &\leq \|\partial_x^l v\|_{L^\infty} \|\partial_x^{k-l} v\|_{L^\infty} \|\partial_x^m v\|_{L^2} \|\partial_x^{m-k} v\|_{L^2} \\ &\leq C\|v\|_{H^{l+1}} \|v\|_{H^{k-l+1}} \|v\|_{H^m} \|v\|_{H^{m-k}}.\end{aligned} \tag{2.35}$$

The above inequality is divided in two cases; when $l+1 \leq n, \quad k-l+1 \leq n$ and $l+1 \leq n, \quad k=n$. For the first case we have

$$\begin{aligned}\left|\int_{\mathbb{R}} \partial_x^m v \partial_x^l v \partial_x^{k-l} v \partial_x^{m-k} v dx\right| &\leq C\|v\|_{H^n}\|v\|_{H^n}\|v\|_{H^m}\|v\|_{H^m} \\ &\leq C\alpha^2\|v\|_{H^m}^2. \end{aligned} \tag{2.36}$$

For the second case we have

$$\begin{aligned}\left|\int_{\mathbb{R}} \partial_x^m v \partial_x^l v \partial_x^{k-l} v \partial_x^{m-k} v dx\right| &\leq C\|v\|_{H^n}\|v\|_{H^m}\|v\|_{H^m}\|v\|_{H^n} \\ &\leq C\alpha^2\|v\|_{H^m}^2. \end{aligned} \tag{2.37}$$

Since, $n-l+1 \leq m$, and $m-n \leq 2 \leq s+1 = n$.

We are left with three cases; $l+1 = k = n, \quad l+1 \leq m$, with $m = n$ and $k = m = j = l$. For the first case, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}} \partial_x^m v \partial_x^l v \partial_x^{k-l} v \partial_x^{m-k} v dx\right| &\leq \|\partial_x^{m-k} v\|_{L^\infty}\|\partial_x^l v\|_{L^\infty}\|\partial_x^m v\|_{L^2}\|\partial_x^{k-l} v\|_{L^2} \\ &\leq C\|v\|_{H^{m-k+1}}\|v\|_{H^{l+1}}\|v\|_{H^m}\|v\|_{H^{k-l}} \\ &\leq C\|v\|_{H^m}\|v\|_{H^n}\|v\|_{H^m}\|v\|_{H^n}. \end{aligned} \tag{2.38}$$

Since, $n-l \leq n, \quad m-k+1 \leq m$. For the second case we get the same result, but now we use that $m-k+1 \leq m-n \leq 2 \leq n$.

For the third case,

$$\begin{aligned}\left|\int_{\mathbb{R}} \partial_x^m v \partial_x^m v v v dx\right| &\leq \int_{\mathbb{R}} |(\partial_x^m v)^2 v^2| dx \leq \|v\|_{L^\infty}^2\|\partial_x^m v\|_{L^2}^2 \\ &\leq C\|v\|_{H^n}^2\|v\|_{H^m}^2 \\ &\leq C\alpha^2\|v\|_{H^m}^2. \end{aligned} \tag{2.39}$$

For the third term of (2.25),

$$\begin{aligned}\left|\int_{\mathbb{R}} \partial_x^m v \partial_x^m v dx\right| &\leq \|\partial_x^m v\|_{L^2}\|\partial_x^m v\|_{L^2} \\ &\leq C\|v\|_{H^m}^2. \end{aligned} \tag{2.40}$$

Finally, the last term of the (2.25) we have the bound

$$\left|\int_{\mathbb{R}} \partial_x^m v \partial_x^{k+1} v \partial_x^{m-k} v dx\right| \quad \leq \quad C\alpha ||v||_{H^m}^2, , \tag{2.41}$$

see (Nilsen, 2011).

All in all we get, by summing up the estimates, the following inequality

$$\frac{d}{dt}||v(t)||_{H^m}^2 = ||v(t)||_{H^m}\frac{d}{dt}||v(t)||_{H^m} \leq c\alpha_1||v(t)||_{H^m}^2 \tag{2.42}$$

which leads to

$$\frac{d}{dt}||v(t)||_{H^m} \leq c\alpha_1||v(t)||_{H^m} \tag{2.43}$$

where $\alpha_1 = (C + 2C\alpha + C\alpha^2)$. This result concludes the proof. (Holden, Lubich and Risebro, 2013) □

Lemma 2.2 *Assume* $||v_0||_{H^k} \leq K$ *for some* $k \geq 1$. *Then there exists* $\bar{t}(K) > 0$ *such that* $||\Omega_B^t(v_0)||_{H^k} \leq 2K$ *for* $0 \leq t \leq \bar{t}(K)$.

Proof By doing the same calculations as in the proof of Lemma (2.1) with k instead of m and using the bound for U_0 in $H^k(\mathbb{R})$, we arrive with the following inequality

$$||v(t)||_{H^k}\frac{d}{dt}||v(t)||_{H^k} \leq c||v(t)||_{H^k}^4, \tag{2.44}$$

which simplifies to

$$\frac{d}{dt}||v(t)||_{H^k} \leq c||v(t)||_{H^k}^3. \tag{2.45}$$

The result follows by comparing with the solution of the differential equation $y' = cy^3$. (Nilsen, 2011). □

Lemma 2.3 *If* $||v_0||_{H^{s+2}} \leq C_0$ *for* $s \geq 1$ *, then there exists* $\bar{t}$ *depending on* C_0*, such that the solution* $v(t)$ *of the (2.20) is* $C^3([0, \bar{t}], H^s)$.

Proof We can define the following equality by using Lemma (2.2) where $t \in [0, \bar{t}]$,

$$\tilde{v}(t) = v_0 + tB(v_0) + \int_0^t (t-s)dB(v(s))[B(v(s))]ds, \tag{2.46}$$

where $dB(.)[.]$ is the Fr*é*chet derivative. If we calculate the second derivative of $\tilde{v}$ we get the following,

$$\begin{aligned}
\tilde{v}_{tt} &= dB(v(s))[B(v(s))] \\
&= -3\beta v^2(-v^3 + (1+\gamma)v^2 - \gamma v) + 3\alpha v^2 v v_x \\
&\quad +2\beta(1+\gamma)v(-v^3 + (1+\gamma)v^2 - \gamma v) - 2\alpha(1+\gamma)v v_x \\
&\quad -\gamma\beta(-v^3 + (1+\gamma)v^2 - \gamma v) + \gamma\alpha v v_x \\
&\quad -\beta v(-3v^2 v_x + 2(1+\gamma)v v_x - \gamma v_x) + \alpha(v v_x^2 + v^2 v_{xx}) \\
&\quad -\beta v_x(-v^3 + (1+\gamma)v^2 - \gamma v) + \alpha v v_x^2.
\end{aligned} \tag{2.47}$$

By differentiation (2.20) with respect to t, we get

$$\begin{aligned}
v_{tt} &= B(v)_t = (-\beta v^3 + \beta(1+\gamma)v^2 - \beta\gamma v - \alpha v v_x)_t \\
&= -3\beta v^2 v_t + 2\beta(1+\gamma)v v_t - \beta\gamma v_t - \alpha v_t v_x - \alpha v v_{xt} \\
&= -3\beta v^2(B(v)) + 2\beta(1+\gamma)v(B(v)) - \beta\gamma(B(v)) - \alpha(B(v))v_x - \alpha v(B(w))_x \\
&= \tilde{v}_{tt},
\end{aligned} \tag{2.48}$$

we see that $\tilde{v}(0) = U_0$ and $\tilde{v}_t(0) = B(U_0) = v_t$. Thus we have shown that $v = \tilde{v}$. The same is also true for $\tilde{v_{ttt}}$. It follows that $\tilde{v}$ is $C^3([0, \bar{t}], H^s)$ (Nilsen, 2011). □

2.3.2. Results for the Linear Part

In this subsection, we need to show that A is continuous and bounded. The critical point for (2.19) in combination with those for (2.20), is that the Sobolev norm do not increase.We state this propoerty in the following lemma (Nilsen, 2011).

Lemma 2.4 *(Nilsen, 2011) Let P be a linear polynomial of degree $l \geq 2$ with constant coefficients, which satisfies*

$$ReP(i\xi) \leq 0, \ \textit{for all}, \ \xi \in \mathbb{R}. \tag{2.49}$$

In addition, let m be a integer such that $m \geq l$, and assume u_0 is in $H^{m+l}(\mathbb{R})$ and the solution $e^t(u_0) = u(t)$ of $u_t = P(\partial_x)u, \ , u|_{t=0} = u_0$ is in $H^m(\mathbb{R})$ and satisfies

$$\int_{\mathbb{R}} (\partial_x^{j+l/2} v)^2 < \infty,$$

for all $j \leq m$ and l even. Then $e^t(u_0)$ has a non-increasing norm in $H^m(\mathbb{R})$, in particular

$$||e^{Pt}(u_0)||_{H^m} \leq ||u_0||_{H^{m+1}}.$$

2.4. Lie-Trotter Splitting

In the previous subsections, we have exhibited results about the linear and nonlinear parts of the BHE. This section is devoted to show the global error of the Lie-Trotter splitting. We first estimate the local error of the method then achieve the global error bound.

2.4.1. Local Error in H^s space

Lemma 2.5 *Let the Hypothesis 2.2 holds for $k = s + 2$ where $s \geq 1$ for the solution of the equation (2.17). The local error of the Lie-Trotter splitting is bounded in the Sobolev norm as follows,*

$$||\Omega_A^{\Delta t}(\Omega_B^{\Delta t}(U_0)) - \Omega_{A+B}^{\Delta t}(U_0)||_{H^s} \leq C\Delta t^2, \tag{2.50}$$

where U_0 is in $H^{s+2}(\mathbb{R})$ and C only depends on $||U_0||_{H^{s+2}}$.

Proof In the following proof, we follow similar way to (Holden, Lubich and Risebro, 2013). Burgers-Huxley equation is in the form

$$U_t = AU + B(U), \tag{2.51}$$

where $AU = (\partial_x^2)U$ and $B(U) = -\beta U^3 + \beta(1+\gamma)U^2 - \beta\gamma U - \alpha U U_x$. The exact solution is $U(t) = \Omega^t_{A+B}(U_0)$, it can be written as follows

$$U(t) = \Omega^t_A U_0 + \int_0^t \Omega_A^{(t-s)}(B(U(s)))ds. \tag{2.52}$$

This is similar to formula $\varphi(t) - \varphi(0) = \int_0^t \dot{\varphi}(s)ds$ when $\varphi(s) = \Omega_A^{(t-s)}(U(s))$.

$$\varphi(t) = U(t), \;\; \varphi(0) = \Omega^t_A U_0, \tag{2.53}$$

$$\varphi'(s) = -A\Omega_A^{(\Delta t-s)}U(s) + \Omega_A^{(\Delta t-s)} \underbrace{U'(s)}_{AU+B(U)} . \tag{2.54}$$

By using the the following formula with $\varphi(\rho) = \Omega_A^{(s-\rho)}(U(\rho))$

$$B(\varphi(s)) - B(\varphi(0)) = \int_0^s dB(\varphi(\rho))[\dot{\varphi}(\rho)]d\rho, \tag{2.55}$$

we get

$$B(U(s)) = B(\Omega^s_A U_0) + \int_0^s dB(\Omega_A^{(s-\rho)}U(\rho))[\Omega_A^{(s-\rho)}B(U(\rho))]d\rho. \tag{2.56}$$

After inserting Equation (2.56) into Equation (2.52) for $t = \Delta t$, we get

$$U(\Delta t) = \Omega^{\Delta t}_A U_0 + \int_0^{\Delta t} \Omega_A^{(\Delta t-s)} B(\Omega^s_A U_0)ds + E_1, \tag{2.57}$$

where

$$E_1 = \int_0^{\Delta t} \int_0^s \Omega_A^{(\Delta t - s)} dB(\Omega_A^{(s-\rho)} U(\rho))[\Omega_A^{(s-\rho)} B(U(\rho))] d\rho ds. \tag{2.58}$$

The Lie-Trotter splitting solution for $[0, \Delta t]$ interval can be written as

$$U_1 = \Omega_A^{\Delta t}(\Omega_B^{\Delta t}(U_0)), \tag{2.59}$$

We use the first-order Taylor expansion with integral remainder term in H^s

$$\Omega_B^{\Delta t}(u) = u + \Delta t B(u) + \Delta t^2 \int_0^1 (1-\theta) dB(\Omega_B^{\Delta t\theta}(u))[B(\Omega_B^{\Delta t\theta}(u))] d\theta. \tag{2.60}$$

By inserting the expansion into (2.59), for $u = U_0$

$$u_1 = \Omega_A^{\Delta t} U_0 + \Delta t \Omega_A^{\Delta t}(B(U_0)) + E_2, \tag{2.61}$$

with

$$E_2 = (\Delta t)^2 \int_0^1 (1-\theta) \Omega_A^{\Delta t} dB(\Omega_B^{\Delta t\theta}(U_0))[B(\Omega_B^{\Delta t\theta}(U_0))] d\theta. \tag{2.62}$$

Thus, the error becomes

$$U_1 - U(\Delta t) = \Delta t \Omega_A^{\Delta t}(B(U_0)) - \int_0^{\Delta t} \Omega_A^{(\Delta t - s)}(B(\Omega_A^s(U_0))) ds + (E_2 - E_1), \tag{2.63}$$

by defining

$$h(s) = \Omega_A^{(\Delta t - s)}(B(\Omega_A^s(U_0))), \tag{2.64}$$

we can rewrite equation (2.63) by using the Peano Kernel for rectangle rule as follows

$$U_1 - U(\Delta t) = \int_0^{\Delta t} K_R(t) h'(t) dt + (E_2 - E_1). \tag{2.65}$$

By using the substitution $\theta = t/\Delta t$, the integral is transformed to

$$\int_0^{\Delta t} K_R(t) h'(t) dt = (\Delta t)^2 \int_0^1 (\theta - 1) h'(\theta \Delta t) d\theta = (\Delta t)^2 \int_0^1 K_R(\theta) h'(\theta \Delta t) d\theta. \tag{2.66}$$

Then, applying the H^s norm and using the triangle inequality,

$$\begin{aligned} \|U_1 - U(\Delta t)\|_{H^s} &\leq (\Delta t)^2 \int_0^1 \|K_R(\theta) h'(\theta \Delta t)\|_{H^s} d\theta + \|(E_2 - E_1)\|_{H^s} \\ &\leq (\Delta t)^2 \int_0^1 \|K_R(\theta) h'(\theta \Delta t)\|_{H^s} d\theta + \|E_2\|_{H^s} + \|E_1\|_{H^s}, \end{aligned} \tag{2.67}$$

where K_R is bounded kernel. Here $h'(s) = -\Omega_A^{(\Delta t - s)}[A, B](\Omega_A^s(U_0))$ with double Lie commutator

$$[A, B] = dA(v)[B(v)] - dB(v)[A(v)]. \tag{2.68}$$

Lemma (2.4) gives that $\Omega_A^t(U_0)$ do not increase the Sobolev norm, and therefore it is sufficient to consider the commutator for a general vector v. Using (2.19) and (2.20), we write

$$\begin{aligned} [A, B](u) &= -6uu_x^2 - 3u^2 u_{xx} + 2(1+\gamma)u_x^2 + 2(1+\gamma)uu_{xx} - \gamma u_{xx} - 2u_x u_{xx} - u_x u_{xx} - uu_{xxx} \\ &\quad -(-3v^2 u_{xx} + 2(1+\gamma)uu_{xx} - \gamma u_{xx} - uu_{xxx} - u_{xx}u_x). \end{aligned} \tag{2.69}$$

Hence we get,

$$\begin{aligned} \|h'(s)\|_{H^s} &= \| -6uu_x^2 + 2(1+\gamma)u_x^2 - 2u_x u_{xx}\|_{H^s} \\ &\leq 6\|u\|_{H^s}\|\partial_x u\|_{H^s}^2 + 2(1+\gamma)\|\partial_x u\|_{H^s}^2 + 2\|\partial_x u\|_{H^s}^2 \|\partial_x^2 u\|_{H^s}^2 \\ &\leq 6\|u\|_{H^s}\|u\|_{H^{s+1}}^2 + (2+2\gamma)\|u\|_{H^{s+1}}^2 + 2\|u\|_{H^{s+1}}\|u\|_{H^{s+2}} \\ &\leq 6\|u\|_{H^{s+2}}^3 + (4+2\gamma)\|u\|_{H^{s+2}}^2 \leq C\|u\|_{H^{s+2}}^3. \end{aligned} \tag{2.70}$$

If we combine the Lemma(2.4) with the fact that $u = \Omega_A^s(U_0)$, we get the following inequality

$$\|h'(s)\|_{H^s} \leq C\|\Omega_A^s(U_0)\|_{H^{s+2}}^3 \leq C\|U_0\|_{H^{s+2}}^3. \tag{2.71}$$

The integral in (2.67) is bounded as

$$(\Delta t)^2 \int_0^1 \|h'(\theta\Delta t)\|_{H^s} d\theta \leq C\|U_0\|_{H^{s+2}}^3 (\Delta t)^2. \tag{2.72}$$

Next, we will find the error bound for E_1 in (2.58),

$$\begin{aligned}
\|E_1\|_{H^s} &\leq \int_0^{\Delta t}\int_0^s \|\Omega_A^{(\Delta t-s)}(dB(\Omega_A^{(s-\rho)}(U(\rho)))[\Omega_A^{(s-\rho)}(B(U(\rho)))]\|_{H^s} d\rho ds \\
&\leq \int_0^{\Delta t}\int_0^s \|dB(\Omega_A^{(s-\rho)})(U(\rho))[\Omega_A^{(s-\rho)}(B(U(\rho)))]\|_{H^s} d\rho ds \\
&\leq \int_0^{\Delta t}\int_0^s \|-3(\Omega_A^{(s-\rho)}(U(\rho)))^2(\Omega_A^{(s-\rho)}(B(U(\rho))))\|_{H^s} d\rho ds \\
&+ 2(1+\gamma)\int_0^{\Delta t}\int_0^s \|(\Omega_A^{(s-\rho)}(U(\rho)))(\Omega_A^{(s-\rho)}(B(U(\rho))))\|_{H^s} d\rho ds \\
&+ \gamma\int_0^{\Delta t}\int_0^s \|(\Omega_A^{(s-\rho)}(B(U(\rho))))\|_{H^s} d\rho ds \\
&+ \int_0^{\Delta t}\int_0^s \|(\Omega_A^{(s-\rho)}(U(\rho))\Omega_A^{(s-\rho)}B(U(\rho)))_x\|_{H^s} d\rho ds.
\end{aligned} \tag{2.73}$$

We can rewrite the above inequality for simplicity,

$$\|E_1\|_{H^s} \leq I_1 + I_2 + I_3 + I_4. \tag{2.74}$$

We obtain the following bounds by using the Banach algebra property of $H^s(\mathbb{R})$ and non-increasing of the solution of (2.19),

$$\begin{aligned}
I_1 &\leq \int_0^{\Delta t}\int_0^s \|U(\rho)\|_{H^s}^2\|B(U(\rho))\|_{H^s}d\rho ds \\
&\leq \int_0^{\Delta t}\int_0^s \|U(\rho)\|_{H^s}^2\left(\|U(\rho)\|_{H^s}^3 + (1+\gamma)\|U(\rho)\|_{H^s}^2 + \gamma\|u(\rho)\|_{H^s} + \|U(\rho)\|_{H^s}\|U(\rho)_x\|_{H^s}\right) \\
&\leq \int_0^{\Delta t}\int_0^s \left(\|U(\rho)\|_{H^s}^5 + (1+\gamma)\|U(\rho)\|_{H^s}^5 + \gamma\|U(\rho)\|_{H^s}^3 + \|U(\rho)\|_{H^s}^3\|U(\rho)\|_{H^{s+1}}\right)d\rho ds \\
&\leq C\int_0^{\Delta t}\int_0^s R^5 d\rho ds = CR^5\int_0^{\Delta t} sds = CR^5(\Delta t)^2, \qquad (2.75)
\end{aligned}$$

we get the following estimate for the second integral

$$\begin{aligned}
I_2 &\leq \int_0^{\Delta t}\int_0^s \|U(\rho)\|_{H^s}\|B(U(\rho))\|_{H^s}d\rho ds \\
&\leq C\int_0^{\Delta t}\int_0^s \|U(\rho)\|_{H^s}\left(\|U(\rho)\|_{H^s}^3 + (1+\gamma)\|U(\rho)\|_{H^s}^2 + \gamma\|u(\rho)\|_{H^s} + \|U(\rho)\|_{H^s}\|U(\rho)_x\|_{H^s}\right) \\
&\leq CR^4(\Delta t)^2, \qquad (2.76)
\end{aligned}$$

we can write the following bound for the third integral

$$\begin{aligned}
I_3 &\leq \int_0^{\Delta t}\int_0^s \|B(U(\rho))\|_{H^s}d\rho ds \\
&\leq CR^3(\Delta t)^2, \qquad (2.77)
\end{aligned}$$

for the last integral, we can write the bound as, (see (Nilsen, 2011)).

$$I_4 \leq CR^3(\Delta t)^2. \qquad (2.78)$$

Finally, we get

$$\|E_1\|_{H^s} \leq C(R^5 + R^4 + 2R^3)(\Delta t)^2 \leq M(\Delta t)^2. \qquad (2.79)$$

The final term is estimated similarly as the second term.

$$\begin{aligned}
\|E_2\|_{H_s} &\leq (\Delta t)^2 \int_0^1 \|(1-\theta)\Omega_A^{\Delta t}(dB(\Omega_B^{\theta\Delta t}(u)))[B(\Omega_B^{\theta\Delta t}(u))]\|_{H_s} d\theta \qquad (2.80)\\
&\leq (\Delta t)^2 \int_0^1 \|dB(\Omega_B^{\theta\Delta t}(U_0))[B(\Omega_B^{\theta\Delta t}(U_0))]\|_{H_s} d\theta\\
&\leq (\Delta t)^2 \int_0^1 \|3(\Omega_B^{\theta\Delta t}(U_0))^2(B(\Omega_B^{\theta\Delta t}(U_0)))\|_{H_s} d\theta\\
&\quad +2(1+\gamma)(\Delta t)^2 \int_0^1 \|(\Omega_B^{\theta\Delta t}(U_0))(B(\Omega_B^{\theta\Delta t}(U_0)))\|_{H_s} d\theta\\
&\quad +(\Delta t)^2 \int_0^1 \|B(\Omega_B^{\theta\Delta t}(U_0))\|_{H_s} d\theta\\
&\quad +(\Delta t)^2 \int_0^1 \|(\Omega_B^{\theta\Delta t}(U_0))(B(\Omega_B^{\theta\Delta t}(U_0)))\|_{H_s} d\theta. \qquad (2.81)
\end{aligned}$$

By doing the similar approach for E_1 we find following bound for E_2. The only difference is the use of the regularity result for the nonlinear part which is given in Lemma (2.2). For a sufficiently small Δt, Lemma (2.2) ensures that $\|(\Omega_B^{\theta\Delta t}(U_0))\|_{H^{s+1}} \leq \|(\Omega_B^{\theta\Delta t}(U_0))\|_{H^{s+2}} \leq R$.Thus, the bound for E_2 is given as follows,

$$\|E_2\|_{H_s} \leq C(\Delta t)^2(M_1 + M_2 + M_3), \qquad (2.82)$$

where $M_1 = (R^5 + R^4 + 2R^3)$, $M_2 = (R^4 + 2R^3 + R^2)$ and $M_3 = (R^3 + 2R^2 + R)$.

Hence, by combining the estimates, we obtain the following bound for the local error,

$$\|U_1 - U(\Delta t)\|_{H^s} \leq c(\Delta t)^2, \qquad (2.83)$$

where c depends only on the initial condition and Δt is sufficiently small. □

2.4.2. Global Error in H^s space

Theorem 2.1 *Suppose that the exact solution $U(\cdot, t)$ of Equation (2.17) is in H^{s+2} for $0 \leq t \leq T$. Then Lie-Trotter splitting solution U_N has first order global error for $\Delta t < \bar{\Delta}t$ where $\bar{\Delta}t > 0$*

and $t_N = N\Delta t \leq T$,

$$\|U_N - U(\cdot, t_N)\|_{H^s} \leq G\Delta t, \tag{2.84}$$

where G only depends on $\|U_0\|_{H^{s+2}}$ *and* T.

Proof By using the local error estimate in (2.5), we determine the global error in $H^s(\mathbb{R})$ and prove the first order convergence for the Lie-Trotter splitting. We need to show that split solution is bounded at each step in $H^{s+2}(\mathbb{R})$. To achieve this result we have to use the regularity results for both linear and nonlinear parts of the Burgers-Huxley equation.

For the linear part, we know that A do not increase the Sobolev norm, Lemma (2.4). To show that the Lie-Trotter splitting has a first order convergence, we need to prove the boundedness of the split solution U_N at each time step. For a better understanding of the proof we use an induction argument. We begin with assuming that Hypothesis (2.1) and Hypothesis (2.2) holds for $k = s$. We use the same notation as in (Holden, Lubich and Risebro, 2013); we take

$$U_N^k = \Omega_{A+B}^{(N-k)\Delta t}(U_k) = \Omega^{(N-k)\Delta t}(U_k), \tag{2.85}$$

as the exact solution to (2.17) and we assume that

$$\|U_k\|_{H^s} \leq M, \tag{2.86}$$

$$\|U_k\|_{H^{s+2}} \leq C_1, \tag{2.87}$$

$$\|U_k - U(t_k)\|_{H^s} \leq \zeta\Delta t, \tag{2.88}$$

is true for $k \leq N-1$. We need to show that the above inequalities are true for $k = N$ where C_1 is a constant from Lemma (2.1) and $\zeta = K(M,T)c_s(C_1)$ where $K(M,T)$ is given in (2.21) and $c_l(C_1)$ is a constant from Lemma (2.5). Using the telescope sum and the triangle inequality, we write the error as follows,

$$\|U_N - U(t_N)\|_{H^s} = \left\| \sum_{k=0}^{N-1} U_N^{k+1} - U_N^k \right\|_{H^s} \leq \sum_{k=0}^{N-1} \|U_N^{k+1} - U_N^k\|_{H^s}, \tag{2.89}$$

by using the notation we get,

$$\|U_N - U(\cdot, t_N)\|_{H^s} \leq \sum_{k=0}^{N-1} \|\Omega^{(N-k-1)\Delta t}(\Psi^{\Delta t}(U_k) - (\Omega^{\Delta t}(U_k))\|_{H^s}, \tag{2.90}$$

For $k \leq N-2$ we get by using the Hypothesis (2.2),

$$\|\Psi^{\Delta t}(U_k)\|_{H^s} = \|U_{k+1}\|_{H^s} \leq M \tag{2.91}$$

and the exact solution,

$$\|\Omega^{\Delta t}(U_k)\|_{H^s} \leq \|\Omega^{\Delta t}(U_k) - \Omega^{\Delta t}(U(t_k))\|_{H^s} + \|\Omega^{\Delta t}(U(t_k))\|_{H^s}, \tag{2.92}$$

by using (2.21) we get,

$$\begin{aligned} \|\Omega^{\Delta t}(U_k)\|_{H^s} &\leq K(M,T)\|U_k - U(t_k)\|_{H^s} + \|U(t_{k+1})\|_{H^s} \leq K(M,T)\zeta\Delta t + \rho, \quad (2.93) \\ &\leq M \quad (2.94) \end{aligned}$$

Hence using the Hypothesis (2.1) and the results in (2.154) and (2.156) for $k \leq N-1$ we obtain,

$$\|\Omega^{(N-k-1)\Delta t}(\Psi^{\Delta t}(U_k) - (\Omega^{\Delta t}(U_k))\|_{H^s} \leq K(M,T)c_l(C_1)(\Delta t)^3. \tag{2.95}$$

By using $N\Delta t \leq T$ and adding up all term we get,

$$\|U_N - U(t_N)\|_{H^s} \leq NK(M,T)c_l(C_1)(\Delta t)^3 \leq \zeta(\Delta t)^3. \tag{2.96}$$

We also need to prove the boundedness U_N. If we choose $\zeta\Delta t \leq M-\rho$ and use the Hypothesis (2.2),

$$\|U_N\|_{H^s} = \|U_N - U(t_N)\|_{H^s} + \|U(t_N)\|_{H^s} \leq M - \rho + \rho \leq M. \tag{2.97}$$

To show that U_N is bounded in $H^{s+2}(\mathbb{R})$,we write

$$||U_N||_{H^{s+2}} = ||\Omega_A^{\Delta t} \circ \Omega_B^{\Delta t}(U_{N-1})||_{H^{s+2}} \leq ||\Omega_B^{\Delta t}(U_{N-1})||_{H^{s+2}}, \tag{2.98}$$

where we have used the boundedness of the linear solution and Lemma (2.2) such that $||\Omega_B^{\Delta t}(U_{N-1})||_{H^{s+2}} \leq 2M$ as long as $||U_{N-1}||_{H^{s+2}}$ is bounded. Thus using the Lemma (2.1) we get the following result,

$$||U_N||_{H^{s+2}} \leq e^{2\alpha_1 M\Delta t}||U_{N-1}||_{H^{s+2}} \leq C. \tag{2.99}$$

If we combine all results, we achieve the main goal as follows,

$$\begin{aligned} ||U_N - U(\cdot, t_N)||_{H^s} &\leq \sum_{k=0}^{N-1} ||\Omega^{(N-k-1)\Delta t}(\Psi^{\Delta t}(U(t_k)) - (\Omega^{\Delta t}(U(t_k)))||_{H^s} \\ &\leq \sum_{k=0}^{N-1} K(R,T)||\Psi^{\Delta t}(U(t_k)) - \Omega^{\Delta t}(U(t_k))||_{H^s} \\ &\leq NK(R,T)c_1(C_0)(\Delta t)^2 \\ &\leq TK(R,T)c_1(C_0)(\Delta t) \\ &\leq G(\Delta t), \end{aligned} \tag{2.100}$$

this completes the proof. □

2.5. Strang Splitting

We use the same technique as in the Lie-Trotter splitting method to show that the Strang splitting method has a second order convergence. The main distinction from the Lie-Trotter is we use second order midpoint rule and higher order series expansion. First, we show the local error estimates then by using these results prove the global error estimate for Strang splitting method.

2.5.1. Local error in H^s space

Lemma 2.6 *Let $s \geq 1$ be an integer and hypothesis 2.2 holds for $k = m$ for the solution $U(t) = \Omega^{\Delta t}_{A+B}(U_0)$ of (2.17). If the initial data U_0 is in $H^m(\mathbb{R})$, then the local error of the Strang splitting is bounded in $H^s(\mathbb{R})$ by*

$$\|\Omega_A^{\Delta t/2}(\Omega_B^{\Delta t}(\Omega_A^{\Delta t/2}(U_0))) - \Omega^{\Delta t}_{A+B}(U_0)\|_{H^s} \leq c_s(\Delta t)^3, \tag{2.101}$$

where c_s only depends on $\|U_0\|_{H^m}$.

Proof We start with

$$B(\varphi(s)) - B(\varphi(0)) = \int_0^s dB(\varphi(\rho))[\dot{\varphi}(\rho)]d\rho, \tag{2.102}$$

where $\varphi(\rho) = \Omega_A^{(s-\rho)}(U(\rho))$.

Hence we get,

$$B(u(s)) - B(\Omega_A^s(U_0)) = \int_0^s dB(\Omega_A^{(s-\rho)}U(\rho))[\Omega_A^{(s-\rho)}(B(U(\rho)))]d\rho. \tag{2.103}$$

From the variation of constants formula we have the exact solution of (2.17) such that,

$$\Omega^t_{A+B}(U_0) = \Omega^t_A(U_0) + \int_0^t \Omega_A^{(t-s)}(B(U(s)))ds. \tag{2.104}$$

To find the exact solution after one step, we insert (2.103) into (2.104) and get the following,

$$\begin{aligned} U(\Delta t) &= \Omega_A^{\Delta t}(U_0) + \int_0^{\Delta t} \Omega_A^{(\Delta t-s)}(B(\Omega_A^s(U(s))))ds \\ &\quad + \int_0^{\Delta t}\int_0^s \Omega_A^{(\Delta t-s)}(H(U(\rho)))d\rho ds, \end{aligned} \tag{2.105}$$

where $H(U(\rho))$ is defined for a general vector v as

$$H(v) = dB(\Omega_A^{(s-\rho)}(v))[\Omega_A^{(s-\rho)}(B(v))]. \tag{2.106}$$

Using (2.103), we get

$$H(U(\rho))) = H(\Omega_A^\rho(U_0)) + \int_0^\rho dH(\Omega_A^{(\rho-\tau)}(U(\tau)))[\Omega_A^{(\rho-\tau)}(B(U(\tau)))]d\tau. \tag{2.107}$$

where

$$\begin{aligned} dH(v)[w] &= d^2B(\Omega_A^{(s-\rho)}(v))[\Omega_A^{(s-\rho)}(w), \Omega_A^{(s-\rho)}(B(v))] \\ &\quad + dB(\Omega_A^{(s-\rho)}(v))[\Omega_A^{(s-\rho)}(dB(v)[w])]. \end{aligned} \tag{2.108}$$

Substituting the integral formula for H into the (2.105) we get,

$$U(\Delta t) = \Omega_A^{\Delta t}(U_0) + \int_0^{\Delta t} \Omega_A^{(\Delta t - s)}(B(\Omega_A^s(U(s))))ds + S1, \tag{2.109}$$

where

$$\begin{aligned} S1 &= \int_0^{\Delta t}\int_0^s \Omega_A^{(\Delta t-s)}\left(dB(\Omega_A^s(U_0))[\Omega_A^{(s-\rho)}(B(\Omega_A^s(U_0)))]\right)d\rho ds \\ &\quad + \int_0^{\Delta t}\int_0^s\int_0^\rho dH(\Omega_A^{(\rho-\tau)}(U(\tau)))[\Omega_A^{(\rho-\tau)}(B(U(\tau)))]d\tau d\rho ds. \end{aligned} \tag{2.110}$$

One step with Strang splitting is

$$U_1 = \Omega_A^{\Delta t/2}(\Omega_B^{\Delta t}(\Omega_A^{\Delta t/2}(U_0))). \tag{2.111}$$

We can write the second order Taylor expansion for the $\Omega_B^{\Delta t}$ as follows,

$$\begin{aligned} \Omega_B^{\Delta t}(v) &= v + \Delta t B(v) + \frac{1}{2}\Delta t^2 dB(v)[B(v)] \\ &\quad + (\Delta t)^3 \int_0^1 \frac{1}{2}(1-\theta)^2 \left(d^2B(\Omega_B^{\theta\Delta t}(v))[B(\Omega_B^{\theta\Delta t}(v)), B(\Omega_B^{\theta\Delta t}(v))]\right. \\ &\quad \left. + dB(\Omega_B^{\theta\Delta t}(v))\left[dB(\Omega_B^{\theta\Delta t}(v))[B(\Omega_B^{\theta\Delta t}(v))]\right]\right) d\theta. \end{aligned} \tag{2.112}$$

After simplification the notation we rewrite the integrand as

$$\begin{aligned}\Omega_B^{\Delta t}(v) &= v + \Delta t B(v) + \frac{1}{2}\Delta t^2 dB(v)[B(v)] \\ &\quad + (\Delta t)^3 \int_0^1 \frac{1}{2}(1-\theta)^2 \left(d^2B(B,B) + dBdBB)(\Omega_B^{\theta\Delta t}(v)\right) d\theta. \end{aligned} \quad (2.113)$$

Inserting this series expansion into (2.111), we obtain

$$\begin{aligned} U_1 &= \Omega^{\Delta t A}(U_0) + \Delta t \Omega_A^{\Delta t/2}(B(\Omega_A^{\Delta t/2}(U_0))) \\ &\quad + \frac{1}{2}\Delta t^2 \Omega_A^{\Delta t/2}(dB(\Omega_A^{\Delta t/2}(U_0))[B(\Omega_A^{\Delta t/2}(U_0))]) + S2, \end{aligned} \quad (2.114)$$

where

$$S2 = (\Delta t)^3 \int_0^1 \frac{1}{2}(1-\theta)^2 \Omega_A^{\Delta t/2}(d^2B(B,B) + dBdBB)(\Omega_B^{\theta\Delta t}(\Omega_A^{\Delta t/2}(U_0)))d\theta. \quad (2.115)$$

The local error after one step is,

$$\begin{aligned} U_1 - U(\Delta t) &= \Delta t \Omega_A^{\Delta t/2}\left(B(\Omega_A^{\Delta t/2}(U_0))\right) - \int_0^{\Delta t} \Omega_A^{(\Delta t - s)}\left(B(\Omega_A^s(U(s)))\right) ds \\ &+ \frac{1}{2}(\Delta t)^2 \Omega_A^{\Delta t/2}\left(dB(\Omega_A^{\Delta t/2}(U_0))[B(\Omega_A^{\Delta t/2}(U_0))]\right) + (S2 - S1). \end{aligned} \quad (2.116)$$

We can rewrite the above expression in a simpler form by defining,

$$h(s,\rho) = \Omega_A^{(\Delta t - s)}\left(dB(\Omega_A^s(U_0))[\Omega_A^{(s-\rho)}(B(\Omega_A^s(U_0)))]\right), \quad (2.117)$$

$$f(s) = \Omega_A^{(\Delta t - s)}\left(B(\Omega_A^s(U_0))\right). \quad (2.118)$$

Hence we can write the local error in a simplified form,

$$\begin{aligned} U_1 - U(\Delta t) &= \Delta t f(\Delta t/2) - \int_0^{\Delta t} f(s) ds \\ &\quad + \frac{1}{2}(\Delta t)^2 h(\Delta t/2, \Delta t/2) - \int_0^{\Delta t}\int_0^s h(s,\rho) d\rho ds \\ &\quad + S3 - S4, \end{aligned} \quad (2.119)$$

where

$$S3 = (\Delta t)^3 \int_0^1 \frac{1}{2}(1-\theta)^2 \Omega_A^{\Delta t/2}(d^2B(B,B) + dBdBB)(\Omega_B^{\theta\Delta t}(\Omega_A^{\Delta t/2}(U_0)))d\theta, \tag{2.120}$$

and

$$S4 = \int_0^{\Delta t}\int_0^{s}\int_0^{\rho} dH(\Omega_A^{(\rho-\tau)}(U(\tau)))[\Omega_A^{(\rho-\tau)}(B(U(\tau)))]d\tau d\rho ds. \tag{2.121}$$

The difference of the first two term is the error of the midpoint rule and the second line of (2.119) is the error of the two dimensional quadrature rule. We can rewrite the equation (2.119) by using these error rules and triangle rule

$$\begin{aligned} \|U_1 - U(\Delta t)\|_{H^s} &\le \int_0^{\Delta t} \|k(t)f''(t)\|_{H^s} ds \\ &+ \left\| \frac{1}{2}(\Delta t)^2 h(\Delta t/2, \Delta t/2) - \int_0^{\Delta t}\int_0^{s} h(s,\rho)d\rho ds \right\|_{H^s} \\ &+ \|S3\|_{H^s} + \|S4\|_{H^s}, \end{aligned} \tag{2.122}$$

where $k(t)$ is bounded kernel and $f''(t)$ is the Fr*é*chet derivative given as

$$\begin{aligned} f''(s) &= \Omega_A^{(\Delta t-s)}\Big((dA(v))^2[B(v)] - dA(v)[dB(v)[A(v)]] - d^2A(v)[B(v),A(v)] \\ &- dA(v)[dB(v)[A(v)]] + d^2B(v)[A(v)]^2 + dB(v)[dA(v)[A(v)]]\Big), \end{aligned} \tag{2.123}$$

for $v = \Omega_A^s(U_0)$. We know that $\Omega_A^{(\Delta t-s)}$ is bounded in Sobolev norm. We find the Fr*é*chet derivatives of the given operators A and B in (2.19) and (2.20),

$$\begin{aligned} &dA(v)[h] = A(h), \\ &d^2A(v)[h,k] = 0, \\ &dB(v)[h] = -3\beta v^2 h + 2\beta(1+\gamma)vh - \beta\gamma h - \alpha(vh)_x, \\ &d^2B(v)[h,k] = -6\beta vkh + 2\beta(1+\gamma)kh - \alpha(kh)_x. \end{aligned} \tag{2.124}$$

We get the following,

$$
\begin{aligned}
f''(s) \;=\;& -\partial_x^4 \beta v^3 + \beta(1+\gamma)\partial_x^4 v^2 + \partial_x^4(-3\beta v^2 + 2\beta\gamma(1+\gamma) - 4v + 2 - \gamma) \\
& +6\partial_x^2(v^2\partial_x^2 v) - 4(1+\gamma)\partial_x^2(vv_x) + \alpha\partial_x^4(vv_x) \\
& -2\alpha\partial_x^2((v\partial_x^2(v))_x) + ((\partial_x^2(v))^2)_x + (v\partial_x^4(v))_x. \qquad (2.125)
\end{aligned}
$$

Writing out the argument using the Leibniz' rule gives

$$
\begin{aligned}
\|f''(s)\|_{H^s} \;\leq\;& \beta\sum_{k=1}^{4}\binom{4}{k}\left\|\partial_x^k v^2 \partial_x^{4-k} v\right\|_{H^s} + (1+\gamma)\beta\sum_{k=1}^{4}\binom{4}{k}\left\|\partial_x^k v \partial_x^{4-k} v\right\|_{H^s} \\
& +6\beta\sum_{k=1}^{2}\binom{2}{k}\left\|\partial_x^k v^2 \partial_x^{2-k} v\right\|_{H^s} + 4(1+\gamma)\sum_{k=1}^{2}\binom{2}{k}\left\|\partial_x^k v \partial_x^{2-k} v\right\|_{H^s} \\
& +\sum_{k=1}^{2}\binom{4}{k}\left\|\partial_x^{4-k} v \partial_x^{k+1} v\right\|_{H^s} + 2\sum_{k=0}^{1}\binom{3}{k}\left\|\partial_x^{3-k} v \partial_x^{2+k} v\right\|_{H^s} \\
& +(4+\gamma)\|\partial_x^4 v\|_{H^s} + \|2\partial_x^2 v \partial_x^3 v\|_{H^s}. \qquad (2.126)
\end{aligned}
$$

Using the Banach algebra property in Appendix(B) and $v = \Omega_A^s(U_0)$ we get,

$$
\|f''(s)\|_{H^s} \;\leq\; C\|\Omega_A^s U_0\|_{H^m}^3 \leq C\|U_0\|_{H^m}^3. \qquad (2.127)
$$

The bound for the first term of (2.122) is obtained as,

$$
\begin{aligned}
\int_0^{\Delta t} \|k(t)f''(t)\|_{H^s} ds \;\leq\;& (\Delta t)^3 \int_0^1 \|k(\theta)f''(\theta\Delta t)\|_{H^s} d\theta \\
\leq\;& (\Delta t)^3 \int_0^1 \|f''(\theta\Delta t)\|_{H^s} d\theta \leq C\|U_0\|_{H^m}^3 (\Delta t)^3. \qquad (2.128)
\end{aligned}
$$

For the second line of (2.122) we use the two dimensional quadrature formula and get the following result,

$$
\left\|\frac{1}{2}\Delta t^2 h(\Delta t/2, \Delta t/2) - \int_0^{\Delta t}\int_0^s h(s,\rho) d\rho ds\right\|_{H^s} \leq C\Delta t^3 (max\|\frac{\partial h}{\partial s}\|_{H^s} + max\|\frac{\partial h}{\partial \rho}\|_{H^s}), \quad (2.129)
$$

We need to find the bound for the partial derivatives of h. Let's define the following equalities,

$$
\begin{aligned}
v(s) &= \Omega_A^s(U_0), \\
w(s,\rho) &= \Omega_A^{(s-\rho)}(B(v(\rho))).
\end{aligned}
\tag{2.130}
$$

Now we can write,

$$
h(s,\rho) = \Omega_A^{(\Delta t - s)}(dB(v(s))[w(s,\rho)]). \tag{2.131}
$$

Start with the first derivative,

$$
\begin{aligned}
\left\|\frac{\partial h}{\partial s}\right\|_{H^s} &= \|\Omega_A^{(\Delta t-s)}\left(-A(dB(v)[w]) + d^2B(v)[A(v), w] + dB(v)[A(w)]\right)\|_{H^s} \\
&\leq \| - A(dB(v)[w]) + d^2B(v)[A(v), w] + dB(v)[A(w)]\|_{H^s} \\
&\leq \| - A\beta(-3v^2w) - 6\beta vA(v)w - 3\beta v^2A(w)\|_{H^s} \\
&\quad + \| - A(2\beta(1+\gamma)vw) + 2\beta(1+\gamma)A(v)w + 2\beta(1+\gamma)vA(w)\|_{H^s} \\
&\quad + \|A(vw)_x - (A(v)w)_x - (vA(w))_x\|_{H^s}
\end{aligned}
\tag{2.132}
$$

By using the Fr*é*chet derivative and Leibniz' rule we get the following result,

$$
\begin{aligned}
\left\|\frac{\partial h}{\partial s}\right\|_{H^s} &\leq C_1\|u_0\|^3_{H^{s+3}} + C_2\|U_0\|^4_{H^{s+3}} \\
&\leq C\|U_0\|^4_{H^m}.
\end{aligned}
\tag{2.133}
$$

Now for the other derivative we use the similar approach to previous one and get the result,

$$
\begin{aligned}
\left\|\frac{\partial h}{\partial \rho}\right\| &= \|\Omega_A^{(\Delta t-s)}\left(dB(v)[\Omega_A^{(s-\rho)}(-A(B(v)) + dB(v)[A(v)])]\right)\|_{H^s} \\
&\leq C\|U_0\|^5_{H^m}.
\end{aligned}
\tag{2.134}
$$

For the third term in (2.122) we use the triangle inequality and definition of the second order Fr$\acute{e}$chet derivative of the operator B which is defined in (2.20),

$$
\begin{aligned}
\|S3\|_{H^s} &\leq \|(\Delta t)^3 \int_0^1 \Omega_A^{\Delta t/2}(d^2B(B,B) + dBdBB)(\Omega_B^{\theta\Delta t}(\Omega_A^{\Delta t/2}(U_0)))\|_{H^s} d\theta \\
&\leq (\Delta t)^3 \int_0^1 \left\| d^2B(B,B)(w) + dBdBB(w) \right\|_{H^s} \\
&\leq (\Delta t)^3 \left(\|d^2B(B,B)(w)\|_{H^s} + \|dBdBB(w)\|_{H^s} \right), \qquad (2.135)
\end{aligned}
$$

where w is redefined as follows,

$$w = \Omega_B^{\theta\Delta t}(\Omega_A^{\Delta t/2}(U_0)). \qquad (2.136)$$

The bound for the first term is obtained easily,

$$\|d^2B(B,B)(w)\|_{H^s} \leq C\|U_0\|_{H^m}^7, \qquad (2.137)$$

and the second term can be bounded as

$$\|dBdBB(w)\|_{H^s} \leq C\|U_0\|_{H^m}^8. \qquad (2.138)$$

Hence by using these two result, we get

$$\|S3\|_{H^s} \leq C(\Delta t)^3\|U_0\|_{H^p}^8 \leq C(\Delta t)^3. \qquad (2.139)$$

For the last term in (2.122) we get,

$$\|S4\|_{H^s} \leq \int_0^{\Delta t}\int_0^s\int_0^\rho \|dH(v)[w]\|_{H^s} d\tau d\rho ds, \qquad (2.140)$$

where H is defined in (2.106) and we redefine v and w as follows,

$$v = \Omega_A^{(\rho-\tau)}(U(\tau)) \quad \text{and} \quad w = \Omega_A^{(\rho-\tau)}(B(U(\tau))). \qquad (2.141)$$

We need to find a bound for the integrand,

$$\begin{aligned} \|dH(v)[w]\|_{H^s} &\leq \|d^2B(\Omega_A^{(s-\rho)}(v))[\Omega_A^{(s-\rho)}(w), \Omega_A^{(s-\rho)}(B(v))]\|_{H^s} \\ &+ \|dB(\Omega_A^{(s-\rho)}(v))[\Omega_A^{(s-\rho)}(dB(v)[w])]\|_{H^s}. \end{aligned} \tag{2.142}$$

For the first term, we find a bound by using the same technique as the previous one and get the following estimate,

$$\begin{aligned} &\|d^2B(\Omega_A^{(s-\rho)}(v))[\Omega_A^{(s-\rho)}(w), \Omega_A^{(s-\rho)}(B(v))]\|_{H^s} \\ \leq\ &\| - 6\Omega_A^{(s-\rho)}(v)(\Omega_A^{(s-\rho)}(w)\Omega_A^{(s-\rho)}(B(v)))\|_{H^s} \\ +\ &2(1+\gamma)\|\Omega_A^{(s-\rho)}(w)\Omega_A^{(s-\rho)}(B(v))\|_{H^s} + \|(\Omega_A^{(s-\rho)}(w)\Omega_A^{(s-\rho)}(B(v)))_x\|_{H^s}. \end{aligned} \tag{2.143}$$

We find the following by using the Lemma (2.1),

$$\begin{aligned} &\|d^2B(\Omega_A^{(s-\rho)}(v))[\Omega_A^{(s-\rho)}(w), \Omega_A^{(s-\rho)}(B(v))]\|_{H^s} \\ \leq\ &C(\|U_0\|_{H^m}^3 + \|U_0\|_{H^m}^4) \end{aligned} \tag{2.144}$$

and the second term is bounded by,

$$\|dB(\Omega_A^{(s-\rho)}(v))[\Omega_A^{(s-\rho)}(dB(v)[w])]\|_{H^s} \leq C\|U_0\|_{H^m}^5 \tag{2.145}$$

Using (2.144) and (2.145) we obtain the following result,

$$\|S4\|_{H^s} \leq \int_0^{\Delta t}\int_0^s\int_0^\rho C(\|U_0\|_{H^m}^3 + \|U_0\|_{H^m}^4 + \|U_0\|_{H^m}^5)d\tau d\rho ds \leq C(\Delta t)^3. \tag{2.146}$$

Finally, we get the local error in 2.122 and this completes the proof. □

2.5.2. Global error in H^s space

Theorem 2.2 *Assume there exists a solution of (2.17). If the Hypothesis (2.1) holds for* $k = s+1$ *and Hypothesis (2.2) holds for* $k = s+3$ *then there exists* $\bar{\Delta}t > 0$ *such that for all* $\Delta t \leq \bar{\Delta}t$.

$$\|U_N - U(\cdot, t_N)\|_{H^s} \leq C_s(\Delta t)^2, \tag{2.147}$$

where U_N *is Strang splitting solution and* Δt *and* C_s *depends on* $\|U_0\|_{H^{s+3}}$ *and* T *where* $T \geq n\Delta t$.

Proof To prove the global error in $H^s(\mathbb{R})$ we use the local error estimate and results for linear and nonlinear parts of the BHE. The proof relies on the induction argument.

We start with assuming that Hypothesis (2.1) and Hypothesis (2.2) holds for $k = s$. We use the same notation as in ((Holden, Lubich and Risebro, 2013)), we take

$$U_N^k = \Omega_{A+B}^{(N-k)\Delta t}(U_k) = \Omega^{(N-k)\Delta t}(U_k) \tag{2.148}$$

as the exact solution to (3.1) and we assume that

$$\|U_k\|_{H^s} \leq M, \tag{2.149}$$

$$\|U_k\|_{H^{s+3}} \leq C_1, \tag{2.150}$$

$$\|U_k - u(t_k)\|_{H^s} \leq \zeta\Delta t, \tag{2.151}$$

holds for all $k \leq N-1$. We need to show that the above inequalities are true for $k = N$ where C_1 is a constant from Lemma (2.1) and $\zeta = K(M,T)c_s(C_1)$ where $K(M,T)$ is given in (2.21) and $c_s(C_1)$ is a constant from Lemma (2.5). Using the telescope sum and the triangle inequality, we write the error as follows,

$$\|U_N - U(t_N)\|_{H^s} = \|\sum_{k=0}^{N-1} u_N^{k+1} - u_N^k\|_{H^s} \leq \sum_{k=0}^{N-1} \|u_N^{k+1} - u_N^k\|_{H^s}, \tag{2.152}$$

by using the notation we get,

$$\|U_N - U(\cdot, t_N)\|_{H^s} \leq \sum_{k=0}^{N-1} \|\Omega^{(N-k-1)\Delta t}(\Pi^{\Delta t}(U_k) - (\Omega^{\Delta t}(U_k))\|_{H^s}. \tag{2.153}$$

For $k \leq N-2$ we get by using the Hypothesis (2.2),

$$\|\Pi^{\Delta t}(U_k)\|_{H^s} = \|U_{k+1}\|_{H^s} \leq M, \tag{2.154}$$

and the exact solution,

$$\|\Omega^{\Delta t}(U_k)\|_{H^s} \leq \|\Omega^{\Delta t}(U_k) - \Omega^{\Delta t}(U(t_k))\|_{H^s} + \|\Omega^{\Delta t}(U(t_k))\|_{H^s}, \tag{2.155}$$

by using (2.21) we get,

$$\begin{aligned} \|\Omega^{\Delta t}(U_k)\|_{H^s} &\leq K(M,T)\|U_k - U(t_k)\|_{H^s} + \|U(t_{k+1})\|_{H^s} \leq K(M,T)k\Delta t + \rho, && (2.156) \\ &\leq M. && (2.157) \end{aligned}$$

Hence using the Hypothesis (2.1) and the results in (2.154) and (2.156) for $k \leq N-1$ we obtain,

$$\|\Omega^{(N-k-1)\Delta t}(\Pi^{\Delta t}(U_k) - (\Omega^{\Delta t}(U_k))\|_{H^s} \leq K(M,T)c_s(C_1)(\Delta t)^3. \tag{2.158}$$

Summing up all term and using $N\Delta t = T$,

$$\|U_N - U(t_N)\|_{H^s} \leq NK(M,T)c_s(C_1)(\Delta t)^3 \leq k(\Delta t)^3. \tag{2.159}$$

We also need to prove the boundedness U_N. If we choose $k\Delta t \leq M - \rho$ and use the Hypothesis (2.2),

$$\|U_N\|_{H^s} = \|U_N - U(t_N)\|_{H^s} + \|U(t_N)\|_{H^s} \leq M - \rho + \rho \leq M. \tag{2.160}$$

To show that U_N is bounded in $H^{s+3}(\mathbb{R})$,we write

$$\|U_N\|_{H^{s+3}} = \|\Omega_A^{\Delta t/2} \circ \Omega_B^{\Delta t} \circ \Omega_A^{\Delta t/2}(U_{N-1})\|_{H^{s+3}} \leq \|\Omega_B^{\Delta t}(U_{N-1})\|_{H^{s+3}}, \tag{2.161}$$

where we have used the boundedness of the linear solution and Lemma (2.2) such that $\|\Omega_B^{\Delta t}(U_{N-1})\|_{H^{s+3}} \leq 2M$ as long as $\|U_{N-1}\|_{H^{s+3}}$ is bounded. Thus using the Lemma (2.1) we get the following result,

$$\|U_N\|_{H^{s+3}} \leq e^{2\alpha_1 M \Delta t}\|U_{N-1}\|_{H^{s+3}} \leq C. \tag{2.162}$$

Hence we get the following result,

$$\begin{aligned}
\|U_N - U(\cdot, t_N)\|_{H^s} &\leq \sum_{k=0}^{N-1} \|\Omega^{(n-k-1)\Delta t}(\Pi^{\Delta t}(U(t_k)) - (\Omega^{\Delta t}(U(t_k)))\|_{H^s} \\
&\leq \sum_{k=0}^{N-1} K(R,T)\|\Pi^{\Delta t}(U(t_k)) - \Omega^{\Delta t}(U(t_k))\| \\
&\leq NK(R,T)c_1(C_0)(\Delta t)^3 \\
&\leq TK(R,T)c_1(C_0)(\Delta t)^2 \\
&\leq C(\Delta t)^2, \qquad (2.163)
\end{aligned}$$

this completes the proof. □

CHAPTER 3

NUMERICAL RESULTS

In this chapter, we present numerical experiments for the Lie-Trotter and Strang splitting methods applied to the BHE. In the previous chapter, we have proved the theoretical results for these operator splitting methods, we need to show that we have the correct numerical convergence rates for Δt.

In the numerical examination, we also consider the CPU runtimes and accuracy of the errors. First, we need to introduce the methods which we use for the time and space discretizations. Then, we will give the results for the BHE with given initial and boundary conditions for the different diffusion constants.

3.1. Numerical Results for the Burgers-Huxley Equation

We consider the Burgers-Huxley equation in the form,

$$U_t + \alpha U U_x - \epsilon U_{xx} = \beta(1-U)(U-\gamma)U, \tag{3.1}$$

for $\alpha = \beta = 1$, $\gamma = 0.5$, with initial and boundary conditions as follows (Jiwari and Mittal, 2011),

$$\begin{aligned} U(x,0) &= sin(\pi x), \quad 0 \le x \le 1 \\ U(0,t) &= U(1,t) = 0, \quad 0 \le t \le T. \end{aligned} \tag{3.2}$$

When we apply the operator splitting on Burgers-Huxley equation, we obtain the two subequations as follows,

$$u_t = A(u) = \epsilon u_{xx}, \tag{3.3}$$

$$v_t = B(v) = \beta(1-v)(v-\gamma)v - \alpha v v_x, \tag{3.4}$$

which are solved subequently for small time steps Δt. We will use the Chebyshev Differentiation Matrices for the first and the second derivative of u in (3.3) and (3.4). To find these matrices, we will give the following theorem,

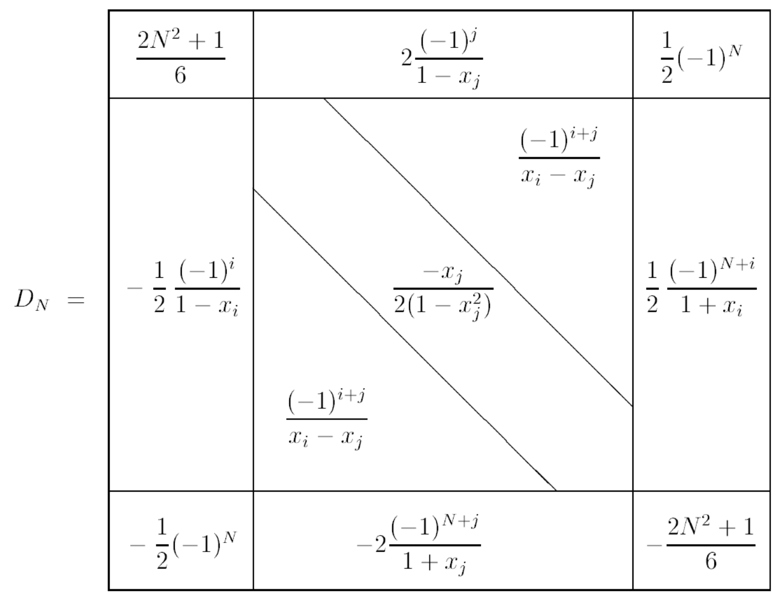
Theorem 3.1 *(Trefethen) For each $N \geq 1$, let the rows and columns of the $(N+1) \times (N+1)$ Chebyshev spectral differentiation matrix D_N be indexed from 0 to N. The entries of this matrix are*

$$
\begin{aligned}
(D_N)_{00} &= \frac{2N^2+1}{6}, \quad (D_N)_{NN} = -\frac{2N^2+1}{6}\,, \\
(D_N)_{jj} &= \frac{-x_j}{2(1-x_j)^2}, \quad j = 1, \ldots, N-1\,, \\
(D_N)_{ij} &= \frac{c_i}{c_j}\frac{(-1)^{i+j}}{(x_i - x_j)}, \quad i \neq j, \quad i, j = 1, \ldots, N-1\,,
\end{aligned}
$$

where

$$
c_i = \begin{cases} 2 & \textit{i=0 or N}, \\ 1 & \textit{otherwise}. \end{cases}
$$

A picture makes the pattern clearer



The jth column of D_N contains the derivative of the degree N polynomial interpolant $p_j(x)$ to the delta function supported at x_j, sampled at the grid points x_i.

For the time integration of the nonlinear part, we consider the semi-implicit RK method

$$w_t = B(w) = \beta(1-w)(w-\gamma)w - \alpha w w_x. \tag{3.5}$$

The semi-implicit Runge-Kutta method is given as follows,

$$\begin{aligned} w_i^{n+1} &= w_i^n + b_1 k_1 + b_2 k_2, \\ k_1 &= \Delta t(1 - a_1 F(w_i)\Delta t)^{-1} B(w_i), \\ k_2 &= \Delta t(1 - a_2 F(w_i + ak_1)\Delta t)^{-1} B(w_i + ak_1), \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} F(w) &= B(w)_w = -3\beta w^2 + 2\beta(1+\gamma)w - \gamma, \\ b_1 &= -0.41315432, \quad b_2 = 1 - b_1, \\ a_1 &= 1 + \frac{\sqrt{6}}{6}, \quad a_2 = 1 - \frac{\sqrt{6}}{6}, \\ a &= \frac{-6 - \sqrt{6} + \sqrt{58 + 20\sqrt{6}}}{6 + 2\sqrt{6}}. \end{aligned} \tag{3.7}$$

Therefore we obtain for the first subequation,

$$\frac{du}{dt} = Au, \tag{3.8}$$

where A is the Chebyshev differentiation matrix for u_{xx}. For the second part (nonlinear part), we apply the semi-implicit RK scheme, which is well-known for the numerical stability and less computational cost.

3.1.1. Lie-Trotter Splitting Solutions

Since there is no exact solution to (3.1), we compare the results to the higher order exponential method to prove convergence of the Lie-Trotter splitting and show the correct convergence rates. The time step length $\Delta t = 0.001$ is used for the numerical experiment. The Figure 3.1 and Figure 3.2 show the layer behaviour of the problem at different values of time t and ϵ.

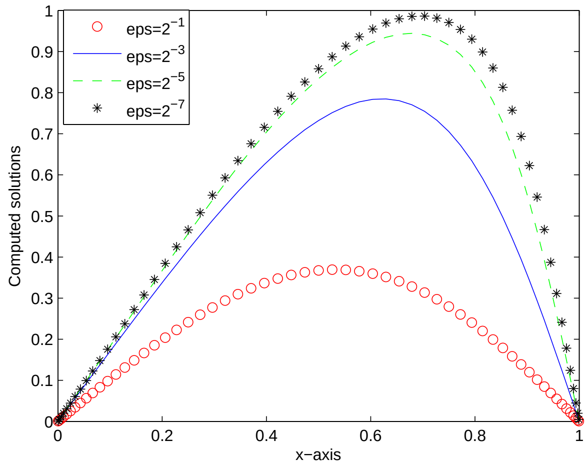


Figure 3.1. Lie-Trotter splitting solutions of BHE for different values of ϵ at T= 0.2.

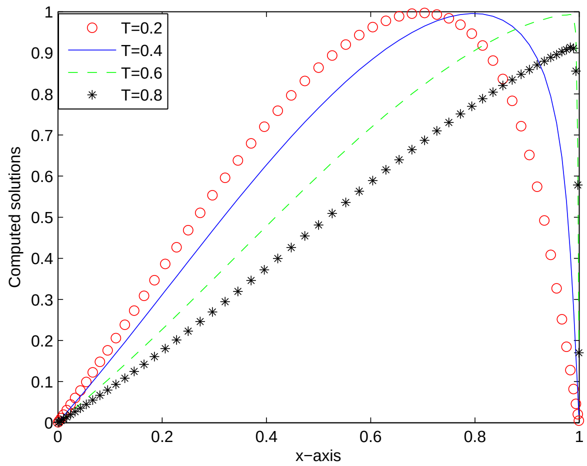


Figure 3.2. Lie-Trotter splitting solutions of BHE for different values of time at $\epsilon = 2^{-9}$.

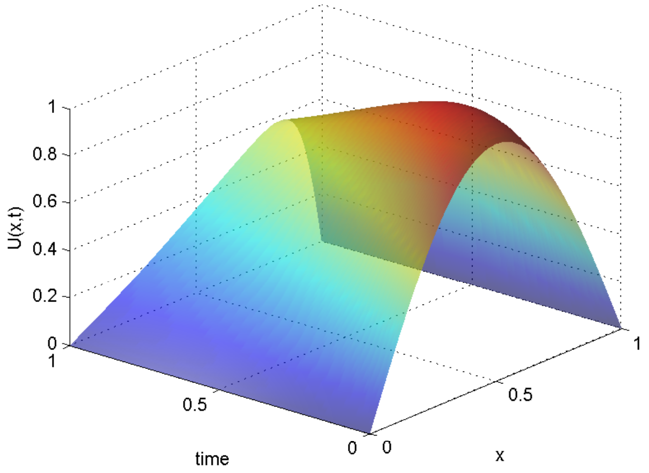


Figure 3.3. Lie-Trotter splitting solutions of BHE for $\Delta t = 0.001$ and $\epsilon = 2^{-5}$.

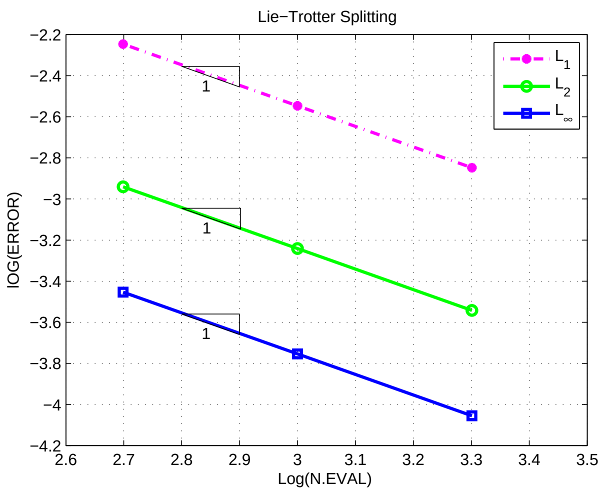


Figure 3.4. Order of L_1, L_2 and L_∞ errors.

The errors are given in Table 3.1 and in the Figure 3.3 Lie-Trotter solution is given for $\Delta t = 0.001$. Finally in Figure 3.4, we give the expected orders. We observe that Lie-Trotter splitting obtain numerical convergence results which is correct with the theoretical results. We also check the running times for Lie-Trotter splitting and nonsplit solution in Table 3.2. We observe that, Lie-Trotter splitting results in faster CPU runtimes.

	$\epsilon = 2^{-3}$			$\epsilon = 2^{-7}$		
T	$\Delta t = 0.001$	$\Delta t = 0.002$	$Order$	$\Delta t = 0.001$	$\Delta t = 0.002$	$Order$
0.2	$8.0990e-04$	0.0016	0.9823	0.0016	0.0031	0.9542
0.4	$8.2993e-04$	0.0017	1.0345	0.0109	0.0212	0.9597
0.6	$5.3939e-04$	0.0011	1.0281	0.0137	0.0263	0.9409
0.8	$3.1277e-04$	$6.2554e-04$	1	0.0096	0.0186	0.9542
1	$1.7619e-04$	$3.5230e-04$	0.9997	0.0065	0.0126	0.9549

Table 3.1. Estimated errors and convergence rates for $\epsilon = 2^{-3}$ and $\epsilon = 2^{-7}$.

$time\ step$	L_1	L_2	L_∞	SR	NR
0.02	0.0566	0.0113	0.0035	0.5858	2.0430
0.01	0.0284	0.0057	0.0018	0.7763	4.0540
0.002	0.0057	0.0011	$3.5230e-04$	1.1576	5.3499
0.001	0.0028	$5.7345e-04$	$1.7619e-04$	2.1738	15.5342
0.0005	0.0014	$2.8577e-04$	$8.8101e-05$	4.0955	16.1621

Table 3.2. Estimated errors and convergence rates for $\epsilon = 2^{-3}$ at fixed time T. (SR=Splitting Runtime, NR=Nonsplitting Runtime)

3.1.2. Strang Splitting Solutions

The numerical convergence rates for Δt are found similar as for the Lie-Trotter splitting method. Since there is no exact solution we use a reference solution and prove the second order convergence rates by comparing the split solution to the solution which is found by using the semi-implicit RK method.

The numerical results are presented in Figures 3.5, Figure 3.6 and Figure 3.7 for different values of ϵ and T. Figure 3.8 shows the computed solution in $x - t$ plane. Expected orders are shown in Figure 3.9.

The convergence of the Strang splitting solution is given by Table 3.3. It shows that if the grid point increase, we get the stable solution. Since there is no exact solution to BHE for given initial and boundary condition, we compare the numerical results with the reference solutions which are obtained by using the Differential Quadrature method (Jiwari and Mittal, 2011). These solutions for given time T and space x are given in Table 3.4

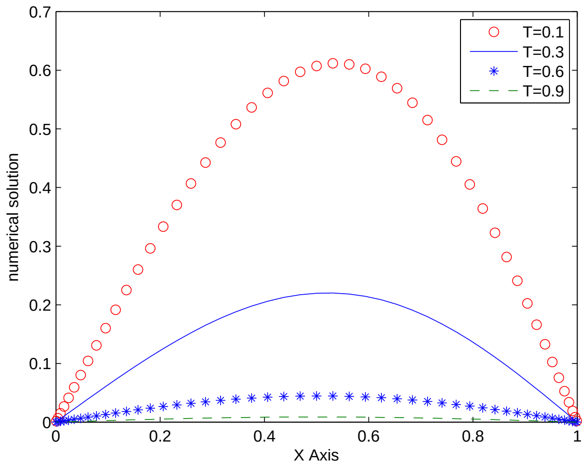


Figure 3.5. Strang splitting solutions of BHE for different values of time at $\epsilon = 2^{-1}$.

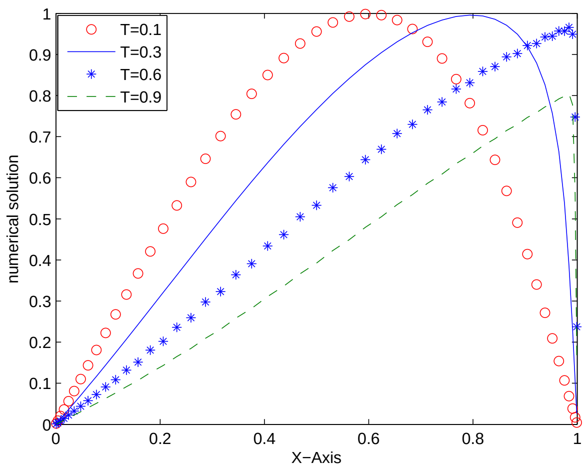


Figure 3.6. Strang splitting solutions of BHE for different values of time at $\epsilon = 2^{-9}$.

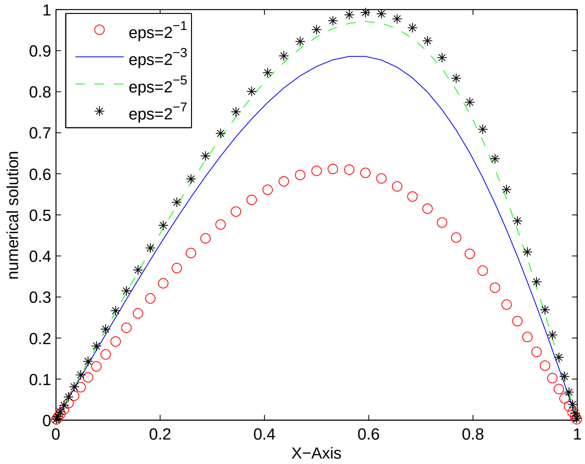


Figure 3.7. Strang splitting solutions of BHE for different values of ϵ at time T=0.1.

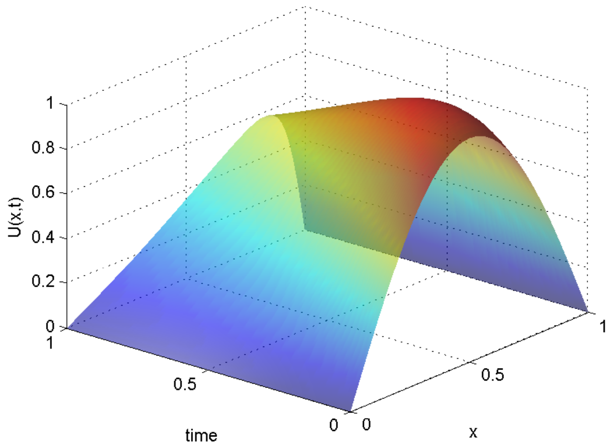


Figure 3.8. Strang splitting solutions of BHE for $\Delta t = 0.001$ and $\epsilon = 2^{-5}$.

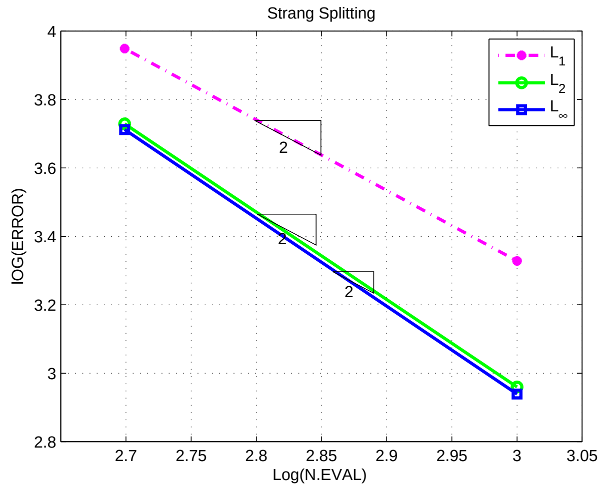


Figure 3.9. Order of L_1, L_2 and L_∞ errors.

T	ϵ	$N=8$	$N=10$	$N=20$	$N=50$
0.2	2^{-2}	0.593005	0.593038	0.593037	0.593037
	2^{-16}	0.870798	0.865167	0.865763	0.865763
0.4	2^{-2}	0.353251	0.353265	0.353265	0.353265
	2^{-16}	0.502698	0.502690	0.502690	0.502690
0.6	2^{-2}	0.206562	0.206568	0.206568	0.206568
	2^{-16}	0.023272	0.023861	0.023860	0.023860
0.8	2^{-2}	0.118291	0.118294	0.118294	0.118294
	2^{-16}	0.030732	0.030741	0.030741	0.030741
1.0	2^{-2}	0.066753	0.066752	0.066752	0.066752
	2^{-16}	0.029940	0.029945	0.029945	0.029945

Table 3.3. Convergence of Strang splitting for BHE at different values of ϵ and time.

We solve the Burgers-Huxley equation in (3.1) by without splitting using the semi-implicit Runge Kutta method and compare the Strang splitting solution with these solutions and get the following results.

The errors are given in Table 3.5 and in the Figure 3.9 we see the expected orders for the Strang splitting method. In the Table 3.6, numerical convergence rates are given for $\Delta t = 0.001$ and $\Delta t = 0.0005$.

T	x	ϵ	Strang splitting solution	Reference solution
0.1	0.25	2^{-3}	0.52586	0.52588
		2^{-7}	0.56849	0.56854
0.9	0.25	2^{-3}	0.15065	0.15062
		2^{-7}	0.18190	0.18172

Table 3.4. Comparison between the split and the reference solution for $N = 21$ and $\Delta t = 0.001$.

Time Step	L_1	L_2	L_∞
10^{-3}	0.0489	0.0103	0.0038
$10^{-3}/2$	0.0238	0.0050	0.0018
$10^{-3}/4$	0.0118	0.0025	$9.0087e-04$
$10^{-3}/8$	0.0059	0.0012	$4.4465e-04$
$10^{-3}/10$	0.0047	$9.495e-04$	$3.5469e-04$

Table 3.5. Estmated errors for $\epsilon = 2^{-9}$ at a fixed time T .

	$\epsilon = 2^{-9}$		
T	$\Delta t = 0.001$	$\Delta t = 0.0005$	*Order*
0.2	0.087082	0.022781	1.9345
0.4	0.075154	0.020701	1.8601
0.6	0.064051	0.018097	1.8234
0.8	0.56221	0.015092	1.8976
1	0.048867	0.010773	2.1818

Table 3.6. Estimated errors and convergence rates for $\epsilon = 2^{-9}$.

CHAPTER 4

CONCLUSION

In this thesis, we investigated the convergence of the operator splitting methods, namely Lie-Trotter and Strang splitting method in Sobolev spaces for BHE. The analyses depend on the differential theory of operators in Banach spaces. Numerical quadratures are used for the error terms. We adopted the same idea in (Holden, Lubich and Risebro, 2013) for the proof of the local and global errors of the splitting methods. We proved first and second order convergence of the Lie-Trotter and Strang splitting methods in $H^s(\mathbb{R})$.

In the numerical experimentation, since there is no exact solution of the BHE for given initial and boundary conditions, we compare the numerical results with the reference solution. We divide the problem into linear and nonlinear parts and solve each subproblems connected the via-initial conditions. We implement different schemes for the subproblems from the splitting process, and test them to find the best combination for the operator splitting. We observe that Chebyshev grid points produced accurate solutions. Comparing the technique in (Jiwari and Mittal, 2011), we solve the BHE in a simpler way by using the operator splitting methods. We presented the errors of the splitting process measured by L_1, L_2 and L_∞ norms and investigateed the numerical convergence rates for Δt. Finally, numerical results show that, expected order of the accuracy for Lie-Trotter and Strang splitting methods are confirmed.

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APPENDIX A

NUMERICAL NTEGRATION

Numerical integration computes the approximations to the definite integrals by using numerical techniques. Consider the definite integral,

$$I(f) \equiv \int_a^b f(x)dx. \tag{A.1}$$

If the function $f(x)$ is continuous on the closed interval $[a,b]$, so that $I(f)$ exists.

In this appendix, we introduce the one dimensional quadrature formulas and give the Peano Kernel theorem. We will use this theorem to show the error terms in a compact form.

A.1. One Dimensional Quadratures

Quadrature formula is a method for approximate of the definite integrals. We consider a quadrature formula as follows (Valeov, 2010),

$$Q_n(f) = \sum_{k=1}^{n} \alpha_i f(x_i), \tag{A.2}$$

where x_i are called nodes of the quadrature formula Q_n and α_i are called coefficients of the quadrature formula for $k = 1, ...n$. If $f \in \mathbb{C}^{n+1}([a,b])$ and Q_n be a quadrature formula of degree n then the error functional is given as,

$$E_n(f) = Q_n(f) - \int_a^b f(x)dx, \tag{A.3}$$

There are different forms for (A.3) exist, but we deal with the Peano Kernel form, which is defined as follows.

A.1.1. The Peano Kernel Theorem

We begin with a verification of the expansion of $f(x)$ as a Taylor polynomial and error term expressed as an integral. Suppose that f^{n+1} exists on $[a, b]$, (Phillips, 2003) then

$$f(x) = f(a) + f'(a)(x-a) + ... + \frac{f^{(n)}(a)(x-a)}{n!} + R_n(f), \tag{A.4}$$

for $a \leq x \leq b$, where

$$R_n(f) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt. \tag{A.5}$$

By using integration by parts in (A.5) we get,

$$R_n(f) = -\frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n-1}(f). \tag{A.6}$$

A second application of this recurrence relation yields,

$$R_n(f) = -\frac{f^{(n)}(a)}{n!}(x-a)^n - \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + R_{n-2}(f). \tag{A.7}$$

Applying the same recurrence relation n times and noting that,

$$R_0(f) = \int_a^b f'(t)dt = f(x) - f(a). \tag{A.8}$$

We can write if $f^{(n+1)}$ is continuous,

$$R_n(f) = \frac{f^{(n+1)}(\xi_x)}{n!}\int_a^x (x-t)^n dt = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-a)^{n+1}. \tag{A.9}$$

where $a < \xi_x < x$.

Definition A.1 *For any fixed real number x and any nonnegative integer n, we write $(x-t)_+^n$ to be denote the function of $+$ for $-\infty < t < \infty$ as follows,*

$$(x-t)_+^n = \begin{cases} (x-t)^n & , \quad -\infty < t \le x, \\ 0 & , \quad t > x. \end{cases}$$

This is called a truncated power function.

With the definition of the truncated power function, the expansion of f as a Taylor polynomial plus remainder can be written as (A.4) where,

$$R_n(f) = \frac{1}{n!}\int_a^b f^{(n+1)}(t)(x-t)_+^n dt. \tag{A.10}$$

Definition A.2 *Let $f \in \mathbb{C}^{n+1}([a,b])$, Q_n be a quadrature formula of degree $n \ge 0$, let the error function E_n, be defined in (A.3). Then the function $K_n(t) = E_n(x-t)_+^n$ is called the Peano kernel of the quadrature formula Q_n of degree n.*

Theorem A.1 *Let $f \in \mathbb{C}^{n+1}([a,b])$ where $n \ge 0$ and Q_n be a quadrature formula. Then the error functional E_n can be represented as*

$$E_n(f) = \frac{1}{n!}\int_a^b K_n(t) f^{(n+1)}(t) dt, \tag{A.11}$$

where $K_n(t)$ is the peano kernel of the quadrature Q_n of degree n (Valeov, 2010).

A.1.2. Peano Kernel for The Rectangle Rule

Suppose that we want to approximate the integral as follows,

$$\int_{t_0}^{\Delta t} f(x)dx \approx f(t_0)\Delta t, \tag{A.12}$$

which is known as one dimensional rectangle rule where $t_0 = 0$. The error is given as,

$$E(f) = \Delta t f(t_0) - \int_{t_0}^{\Delta t} f(x)dx, \tag{A.13}$$

we get the Peano kernel for the rectangle rule as follows,

$$K(t) = E((x-t)_+^0) = \Delta t(0-t)_+^0 - \int_{t_0}^{\Delta t} (x-t)_+^0 dx = t - \Delta t. \tag{A.14}$$

Hence the error can be written as,

$$E(f) = \int_{t_0}^{\Delta t} K(t)f'(t)dt. \tag{A.15}$$

A.1.3. Peano Kernel for The Midpoint Rule

The midpoint rule is given as,

$$\int_{t_0}^{\Delta t} f(x)dx \approx f\left(\frac{\Delta t}{2}\right)\Delta t. \tag{A.16}$$

We obtain the Peano kernels as follows,

$$K_1(t) = E((x-t)_+^0) \quad = \quad \Delta t\left(\frac{\Delta t}{2} - t\right)_+^0 - \int_{t_0}^{\Delta t} (x-t)_+^0 dx = t, \tag{A.17}$$

$$K_2(t) = E((x-t)_+^1) \quad = \quad \Delta t\left(\frac{\Delta t}{2} - t\right)_+ - \int_{t_0}^{\Delta t} (x-t)_+ dx \tag{A.18}$$

$$= \quad \Delta t\left(\frac{\Delta t}{2} - t\right) - \frac{(\Delta t - t)^2}{2}. \tag{A.19}$$

A.2. Two Dimensional Quadratures

We will derive a two dimensional midpoint rule for the double integral (Nilsen, 2011)

$$\int_0^h \int_0^x f(x, y) dy dx. \tag{A.20}$$

The Taylor series expansion for $F(x)$ in $C^{n+1}([0, 1])$ is given as

$$F(1) = F(0) + F'(0) + \frac{F''(0)}{2!} + ... + \frac{F^{(n)}(0)}{n!} + \frac{F^{(n+1)}(\xi)}{(n + 1)!}, \tag{A.21}$$

for $\xi \in [0, 1]$. Define the parametrization of F as

$$F(t) = f(a + th, b + tk), \tag{A.22}$$

for some $f(x, y)$ and t in $[0, 1]$. We assume that $f(x, y)$ has continuous partial derivatives up to order $n + 1$ at all points in an open set containing the line segment joining the points (a, b) and $(a + h, b + k)$ in its domain. We only derive the formula for $n = 1$ (see (Adams, 2003), for n). The derivatives of $F(t)$ is given as

$$\begin{aligned} F'(t) &= h f_x(x + th, y + tk) + k f_y(x + th, y + tk), \\ F''(t) &= h^2 f_{xx}(x + th, y + tk) + 2hk f_{xy}(x + th, y + tk) + k^2 f_{yy}(x + th, y + tk). \end{aligned} \tag{A.23}$$

Thus by using (A.21) and (A.22) we obtain

$$\begin{aligned} F(1) = \quad & f(a + h, b + k) = f(a, b) + h f_x(a + h, b + k) + k f_y(a + h, b + k) \\ & + \frac{1}{2}\left(h^2 f_{xx}(a + \xi h, b + \xi k) + 2hk f_{xy}(a + \xi h, b + \xi k)\right) \\ & + k^2 f_{yy}(a + \xi h, b + \xi k) + ... \end{aligned} \tag{A.24}$$

Letting $h = x - a$ and $k = y - b$, we obtain the second order Taylor formula for $f(x, y)$,

$$\begin{aligned} f(x,y) = \quad & f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) \\ & + \frac{1}{2}\left(h^2 f_{xx}(a + \xi(x-a), b + \xi(y-b))\right) \\ & + 2hk f_{xy}(a + \xi(x-a), b + \xi(y-b)) \\ & + k^2(h^2 f_{yy}(a + \xi(x-a), b + \xi(y-b)) \end{aligned} \tag{A.25}$$

If we return to the double integral, we obtain using (A.25)

$$\int_0^h \int_0^x f(x,y)dydx. = \int_0^h \int_0^x f(a.b)dydx + R(f), \tag{A.26}$$

where

$$\begin{aligned} R(f) = \quad & \int_0^h \int_0^x \left((x-a)f_x(a,b) + (y-b)f_y(a,b)\right) dydx \\ & + \int_0^h \int_0^x \frac{1}{2}\left(h^2 f_{xx}(a + \xi(x-a), b + \xi(y-b))\right) \\ & + 2hk f_{xy}(a + \xi(x-a), b + \xi(y-b)) \\ & + k^2(h^2 f_{yy}(a + \xi(x-a), b + \xi(y-b))dydx. \end{aligned} \tag{A.27}$$

The integral on the right-hand-side in (A.26) is just the area of integration domain times the function itself. Thus, an approximation for the double integral is given as

$$\int_0^h \int_0^x f(x,y)dydx = \frac{h}{2} f(a,b) + R(f), \tag{A.28}$$

and the error is given as

$$E(f) = \int_0^h \int_0^x f(x,y)dydx - \frac{h^2}{2} f(a,b) = R(f), \tag{A.29}$$

which by (A.27) is bounded as

$$\begin{aligned}|E(f)| \leq \quad & \max_T |f_x| \left| \int_0^h \int_0^x (x-a)dydx \right| + \max_T |f_y| \left| \int_0^h \int_0^x (y-b)dydx \right| \\ & + \frac{h^2}{2} \max_T |f_{xx}| \int_0^h \int_0^x (x-a)^2 dydx \\ & + 2hk \max_T |f_{xy}| \int_0^h \int_0^x (x-a)(y-b)dydx \\ & + \frac{h^2}{2} \max_T |f_{yy}| \int_0^h \int_0^x (y-b)^2 dydx \end{aligned} \tag{A.30}$$

where $T = (x, y) : 0 \leq y \leq x \leq h$. By evaluating all of the above integrals we get the following

$$\begin{aligned}|E(f)| \leq \quad & \max_T \left| \frac{\partial f}{\partial x} \right| \left| \frac{1}{3}h^3 - \frac{a}{2}h^2 \right| + \max_T \left| \frac{\partial f}{\partial y} \right| \left| \frac{1}{6}h^3 - \frac{2}{2}h^2 \right| \\ & + \frac{h^2}{2} \max_T \left| \frac{\partial^2 f}{\partial x^2} \right| \left| \frac{1}{4}h^4 - \frac{2a}{3}h^3 + \frac{a^2}{2}h^2 \right| \\ & + hk \max_T \left| \frac{\partial^2 f}{\partial x \partial y} \right| \left| \frac{1}{4}h^4 - \frac{2b}{3}h^3 - \frac{a}{3}h^3 + abh^2 \right| \\ & + \frac{k^2}{2} \max_T \left| \frac{\partial^2 f}{\partial y^2} \right| \left| \frac{1}{12}h^4 - \frac{b}{3}h^3 + \frac{b^2}{2}h^2 \right|, \end{aligned} \tag{A.31}$$

which is the error bound for the two dimensional midpoint rule in (A.28).

APPENDIX B

SOBOLEV SPACES

The Sobolev spaces are important in analysis and the theory of the partial differential equations. The aim of this appendix is to give a brief overview on basic results of the theory of the Sobolev spaces with the imbedding lemmas and give the definition of weak derivative.

B.1. Sobolev Spaces

Let us denote the linear space of $p-th$ order integrable functions on a bounded domain Ω in $\mathbb{R}^n$ as $L^p(\Omega)$, and $L^\infty(\Omega)$ as the linear space of essentially bounded functions which are Banach space with respect to norms (Hoppe, 2010),

$$||u||_p := \left(\int_\Omega |u(x)|^p dx\right)^{1/p}, \tag{B.1}$$

$$||u||_\infty := esssup_{x\in\Omega}|u(x)|. \tag{B.2}$$

For $p = 2$, the space $L^2(\Omega)$ is a Hilbert space with respect to the inner product,

$$(u, v) := \int_\Omega uvdx. \tag{B.3}$$

Since the Sobolev spaces are based on the weak derivative we need to give the definition of the weak derivative.

Definition B.1 *(Hoppe, 2010) Let $u \in L^1(\Omega)$ and $\alpha \in \mathbb{N}_0^d$. The function u is said to have a* weak derivative *$D_w^\alpha u$, if there exists a function $v \in L^1(\Omega)$ such that,*

$$\int_\Omega uD^\alpha\theta dx = (-1)^{|\alpha|}\int_\Omega v\theta dx, \tag{B.4}$$

where $\theta \in C_0^\infty(\Omega)$. We then set $D_w^\alpha u := v$.

Definition B.2 *The linear space $W^{m,p}(\Omega)$ where $p \in [1, \infty]$ is given by,*

$$W^{m,p}(\Omega) := u \in L^p(\Omega) | D_w^\alpha u \in L^p, \ |\alpha| \leq m \tag{B.5}$$

is called a Sobolev space. It is a Banach space with the norm,

$$\|u\|_{m,p,\Omega} := \left(\sum_{|\alpha| \leq m} \|D_w^\alpha u\|_{p,\Omega}^p \right)^{1/p}, \quad p \in [1, \infty), \tag{B.6}$$

$$\|u\|_{m,p,\Omega} := \max_{|\alpha| \leq m} \|D_w^\alpha u\|_{\infty,\Omega}. \tag{B.7}$$

We see that $W^{m,2}(\Omega)$ is a Hilbert space with respect to the inner product,

$$(u, v)_{m,2,\Omega} := \sum_{|\alpha| \leq m} \int_\Omega D_w^\alpha u D_w^\alpha v dx. \tag{B.8}$$

These spaces are denoted as $H^m(\Omega) = W^{m,2}$.

B.1.1. Sobolev Spaces of Integer Order

Definition B.3 *We call Sobolev space of order* 1 *on* Ω *the space*

$$H^1(\Omega) = u \in L^2(\Omega), \ \partial_{x_i} u \in L^2(\Omega), \ 1 \leq i \leq d. \tag{B.9}$$

where $\Omega \subset \mathbb{R}^d$.

Definition B.4 *Let $m \in \mathbb{N}$. A function $u \in L^2(\Omega)$ belongs to the Sobolev space of order m, denoted $H^m(\Omega)$, if all the derivatives of u up to the order m, in the distributional sense, belong to $L^2(\Omega)$. By convention, we note that $H^0 = L^2(\Omega)$.*

Theorem B.1 *The spaces $H^m(\Omega)$, $m \geq 0$ with the following inner product are Hilbert spaces:*

$$(u, v)_{H^m} = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u(x) \partial^\alpha \bar{u}(x) dx, \tag{B.10}$$

with the associated norm

$$||u||_{H^m} = \left(\sum_{|\alpha| \le m} ||\partial^\alpha u||^2_{L^2(\Omega)} \right)^{1/2}. \tag{B.11}$$

Definition B.5 *For every* $1 \le p \le \infty$ *and for every* $m \in \mathbb{N},\ m \ge 1$*, the Sobolev spaces as,*

$$W^{m,p} = \{v \in L^p(\Omega),\ \ \partial^\alpha v \in L^p(\Omega),\ \forall \alpha \in \mathbb{N}^d, |\alpha| \le m\}, \tag{B.12}$$

endowed with the norm,

$$||v||_{W^{1,p}} = \left(\sum_{|\alpha| \le m} \int_\Omega |\partial^\alpha v|^p \right)^{1/p}. \tag{B.13}$$

We will consider here the spaces $W^{m,2}(\Omega) = H^m(\Omega)$. We will use the auxilary lemmas in the Sobolev spaces which are about the imbedding results and the Banach algebra property. It is clear from the definition of the Hilbert space, we can say that $H^m(\mathbb{R})$ is imbedded in $H^n(\mathbb{R})$ for $m > n$ in the following way,

$$||v||_{H^n} \le ||v||_{H^m}, \tag{B.14}$$

for $v \in H^n(\mathbb{R})$.

The first lemma shows that $H^m(\mathbb{R})$ is imbedded in $L^\infty(\mathbb{R})$ for $m \ge 1$.

Lemma B.1 *If* $v \in H^m(\mathbb{R})$ *for* $m \ge 1$*, then* $v \in L^\infty(\mathbb{R})$*. Hence we can write the following,*

$$||v||_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2}} ||v||_{H^1} \le C_m ||v||_{H^m}, \tag{B.15}$$

where C_m *depends only on m.*

The proof of this lemma is given in (Nilsen, 2011) in detailed.

Lemma B.2 *The space H^m is a Banach algebra for $m \geq 1$. In particular, if u, v are in H^m then,*

$$||uv||_{H^m} \leq C_m ||u||_{H^m} ||v||_{H^m}. \tag{B.16}$$

We refer (Nilsen, 2011) for the detailed proof.

APPENDIX C

DERIVATIVE IN BANACH SPACES

In this appendix, we give the definition of the Fréchet differential since we will consider the differentiability of the operators in Banach spaces. Then we will explain the rules for this differential.

C.1. The Fréchet Derivative

We introduce the Fréchet differential which is a map between the Banach spaces. Then we derive the elementary rules for differentiation.

Definition C.1 *(Jost, 2010) Let U and V be Banach spaces, $\Omega \subset U$ open, $f : \Omega \to V$ a map and $x_0 \in \Omega$. The map f is said to be differentiable at x_0 if there is a continuous linear map $L := Df(x_0) : U \to V$ such that*

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{||f(x) - f(x_0) - L(x - x_0)||}{||x - x_0||} = 0. \tag{C.1}$$

$Df(x_0)$ is called the derivative of f at x_0. f is said to be Fréchet differentiable in Ω if it is differentiable at every $x_0 \in \Omega$. L is called Fréchet differential of f at point x_0, and is denoted by,

$$L = Df(x_0). \tag{C.2}$$

Lemma C.1 *(Jost, 2010) Let f be a differentiable at x_0. Then $Df(x_0)$ is uniquely determined (Jost, 2010).*

Theorem C.1 *Let U, V, W be Banach spaces, $\Omega \subset U$ open, $x_0 \in \Omega$, $f : \Omega \to V$ differentiable at x_0, and $\Lambda \subset V$ open with $y_0 := f(x_0) \in \Lambda$ and $g : \Lambda \to W$ differentiable at y_0. Then $g \circ f$ is defined in an open neighborhood of x_0 and it is differentiable at x_0 with*

$$D(g \circ f)(x_0) = Dg(y_0) \circ Df(x_0). \tag{C.3}$$

Proof of this theorem is given in detailed in (Jost, 2010).

Definition C.2 *Let U, V be Banach spaces, $\Omega \subset U$ open, $x_0 \in \Omega$, $f : \Omega \to V$ differentiable. If the derivative Df is differentiable at x_0 then f is said to be twice differentiable at x_0 and the derivative of Df in x_0 is denoted by $D^2 f(x_0)$.*

Note that f is a function from Ω into V, hence Df is a map from Ω to $B(U, V)$, which is again a Banach space. Therefore, $D^2 f(x_0) \in B(U, B(U, V))$.

APPENDIX D

MAT-LAB CODES FOR BURGERS-HUXLEY EQUATION

D.1. Codes for Lie-Trotter Splitting

```
    %%%% Burger huxley u0=sin(pi*x) Lie Trotter splitting
tic
clear all
close all
clc
%for k=1:4
alpha=1;
beta=1;
gama=0.5;
%eps=2^((-k-1));
eps=2^((-2*5)+1);
xp=0;X=1;tp=0;T=1;N=50;
%Nt=500;
dt=0.001;
Nt=(T-tp)/dt
dx=1/N;
%dt=(T-tp)/Nt;
w1=1+(6^(1/2)/6);
w2=1-(6^(1/2)/6);
b1=-0.41315432;
b2=1-b1;
a=(-6-sqrt(6)+sqrt(58+20*sqrt(6)))/(6+2*sqrt(6));
g1(N-1,1)=gama;
[D,x]=chebab(N,0,1);
D2=D^2;
D2=D2(2:N,2:N);
D1=D(2:N,2:N);
```

```
x1=x(2:N);
x=x';
t=tp:dt:T;
%u0=(1-cos(x1));
u0=sin(pi*x1);
u_sol(:,1)=u0;
D3=diag(diag(D1));
for i=1:Nt
    u1=expm(eps*D2*dt)*u0;
    k1=dt*(-alpha*u1.*(D1*u1)-beta*(u1.^3)
+(1+gama)*beta*u1.^2-gama*u1)./
(1-w1*(-alpha*D1*u1-D3*u1-3*beta*u1.^2
+2*(1+gama)*beta*u1-g1)*dt);
    k2=dt*(-alpha*(u1+a*k1).*(D1*(u1+a*k1))
-beta*((u1+a*k1).^3)+beta*(1+gama)*(u1+a*k1).^2-gama*(u1+a*k1))./
(1-w2*(-alpha*D1*(u1+a*k1)-3*beta*(u1+a*k1).^2
+2*(1+gama)*beta*(u1+a*k1)-g1)*dt);
    u2=u1+b1*k1+b2*k2;
    u0=u2;
    u_sol(:,i+1)=u0;
end
[D,x]=chebab(N,0,1);
AA=D^2;
A=AA(2:N,2:N);
B=D(2:N,2:N);
x1=x';
f(1:N+1,1)=sin(pi*x1);
u(:,1)=f(2:N,1)';
 for i=1:Nt
     jac=eps*A-diag(u(:,i))*B-B*diag(u(:,i))-3*diag((u(:,i)).^2)
+3*diag(u(:,i))-(0.5)*eye(N-1,N-1);
     [V,D]=eig(dt*jac);
     d=diag(D);
     g(:,i)=eps*A*u(:,i)-u(:,i).*(B*u(:,i)) - (u(:,i)).^3
+(1.5)*(u(:,i)).^2-(0.5)*(u(:,i));
     u(:,i+1)=expm(jac*dt)*u(:,i)+dt*V*diag(phi1(d,dt,1))
*inv(V)*(g(:,i)-jac*u(:,i));
```

```
 end
u1=real(u);
v1(:,1:Nt+1)=0;
v2(:,1:Nt+1)=0;
usol=vertcat(v1,u1,v2);
K=vertcat(v1,u_sol,v2);
litrotter=K;
exp=usol;
er1=max(abs(usol(:,201)-K(:,201)));
er2=max(abs(usol(:,401)-K(:,401)));
er3=max(abs(usol(:,601)-K(:,601)));
er4=max(abs(usol(:,801)-K(:,801)));
er5=max(abs(usol(:,1001)-K(:,1001)))
save('lie.mat','litrotter');
save('exp.mat','exp');
plot(x,K(:,201),'ro')
hold all
plot(x,K(:,401),'b-')
hold all
plot(x,K(:,601),'--')
hold all
plot(x,K(:,801),'k*')
toc
 function [D,x] = chebab(N,a,b)
if N==0, D=0; x=1; return, end
  x = cos(pi*(0:N)/N)';
  c = [2; ones(N-1,1); 2].*(-1).^(0:N)';
  X = repmat(x,1,N+1);
  dX = X-X';
  D  = (c*(1./c)')./(dX+(eye(N+1)));
  D  = D - diag(sum(D'));
  D = D*2/(b-a); x = a+(b-a)/2*(1+x);
 Lie-Trotter Solution with different times
clear all
close all
clc
%for k=1:4
```

```
alpha=1;
beta=1;
gama=0.5;
%eps=2^((-k-1));
eps=2^((-2*5)+1);
xp=0;X=1;tp=0;T=1;N=50;Nt=1000;
dx=1/N;
dt=1/Nt;
w1=1+(6^(1/2)/6);
w2=1-(6^(1/2)/6);
b1=-0.41315432;
b2=1-b1;
a=(-6-sqrt(6)+sqrt(58+20*sqrt(6)))/(6+2*sqrt(6));
g1(N-1,1)=gama;
[D,x]=chebab(N,0,1);
D2=D^2;
D2=D2(2:N,2:N);
D1=D(2:N,2:N);
x1=x(2:N);
x=x';
t=tp:dt:T;
%u0=(1-cos(x1));
u0=sin(pi*x1);
u_sol(:,1)=u0;
D3=diag(diag(D1));
for i=1:Nt
    u1=expm(eps*D2*dt)*u0;
    k1=dt*(-alpha*u1.*(D1*u1)-beta*(u1.^3)
+(1+gama)*beta*u1.^2-gama*u1)./
(1-w1*(-alpha*D1*u1-D3*u1-3*beta*u1.^2
+2*(1+gama)*beta*u1-g1)*dt);
    k2=dt*(-beta*(u1+a*k1).*(D1*(u1+a*k1))
-beta*((u1+a*k1).^3)+beta*(1+gama)*(u1+a*k1).^2
-gama*(u1+a*k1))./(1-w2*(-alpha*D1*(u1+a*k1)
-3*beta*(u1+a*k1).^2+2*(1+gama)*beta*(u1+a*k1)-g1)*dt);
    u2=u1+b1*k1+b2*k2;
    u0=u2;
```

```
    u_sol(:,i+1)=u0;
end
plot(x1,u_sol(:,201),'ro')
hold all
plot(x1,u_sol(:,301),'b-')
hold all
plot(x1,u_sol(:,501),'--')
hold all
plot(x1,u_sol(:,701),'k*')
```

D.2. Codes for Strang Splitting

```
      %%%% Burger huxley u0=sin(pi*x)full_semi_implicit
%tic
clear all
close all
clc
alpha=1;
beta=1;
gama=0.5;
%eps=2^((-k-1));
eps=2^((-2*5)+1);
xp=0;X=1;tp=0;T=1;N=50;
%Nt=500;
dt=0.0001;
Nt=(T-tp)/dt;
dx=1/N;
%dt=(T-tp)/Nt;
w1=1+(6^(1/2)/6);
w2=1-(6^(1/2)/6);
b1=-0.41315432;
b2=1-b1;
a=(-6-sqrt(6)+sqrt(58+20*sqrt(6)))/(6+2*sqrt(6));
g1(N-1,1)=gama;
[D,x]=chebab(N,0,1);
```

```
D2=D^2;
D2=D2(2:N,2:N);
D1=D(2:N,2:N);
x1=x(2:N);
x=x';
t=tp:dt:T;
u0=sin(pi*x1);
u_sol(:,1)=u0;
D3=diag(diag(D1));
g2=eig(D2);
for i=1:Nt
    u1=u0;
    k1=dt*(eps*D2*u1-alpha*u1.*(D1*u1)
-beta*(u1.^3)+(1+gama)*beta*u1.^2-gama*u1)./
(1-w1*(eps*g2-alpha*D1*u1-D3*u1-3*beta*u1.^2
+2*(1+gama)*beta*u1-g1)*dt);
    k2=dt*(eps*D2*(u1+a*k1)-alpha*(u1+a*k1)
.*(D1*(u1+a*k1))-beta*((u1+a*k1).^3)
+beta*(1+gama)*(u1+a*k1).^2-gama*(u1+a*k1))
./(1-w2*(eps*g2-alpha*D1*(u1+a*k1)-3*beta*(u1+a*k1).^2
+2*(1+gama)*beta*(u1+a*k1)-g1)*dt);
    u2=u1+b1*k1+b2*k2;
    u0=u2;
    u_sol(:,i+1)=u0;
end
v1(:,1:Nt+1)=0;
v2(:,1:Nt+1)=0;
K=vertcat(v1,u_sol,v2);
 plot(x,K(:,end),'ro')
   %Strang Splitting Solution
 dtt=dt/2;
 for i=1:Nt
     u1=expm(eps*D2*dtt)*u0;
     k1=dt*(-alpha*u1.*(D1*u1)-beta*(u1.^3)
+(1+gama)*beta*u1.^2-gama*u1)
./(1-w1*(-alpha*D1*u1-3*beta*u1.^2
+2*(1+gama)*beta*u1-g1)*dt);
```

```
   k2=dt*(-beta*(u1+a*k1).*(D1*(u1+a*k1))-
beta*((u1+a*k1).^3)+beta*(1+gama)*(u1+a*k1).^2
-gama*(u1+a*k1))./(1-w2*(-alpha*D1*(u1+a*k1)
-3*beta*(u1+a*k1).^2+2*(1+gama)*beta*(u1+a*k1)-g1)*dt);
     u2=u1+b1*k1+b2*k2;
     u3=expm(eps*D2*dtt)*u2;
     u0=u3;
     u_sol(:,i+1)=u0;
 end
 v1(:,1:Nt+1)=0;
 v2(:,1:Nt+1)=0;
 K=vertcat(v1,u_sol,v2);
 litrotter=K(end,:);
 save('lie.mat','litrotter');
 plot(x,K(:,201),'ro')
 hold all
 plot(x,K(:,401),'b-')
 hold all
 plot(x,K(:,601),'--')
 hold all
 plot(x,K(:,801),'k*')
 function ordertriangle(order, varargin)
     if(nargin==2)
         b_loglog = varargin{1};
     else
         b_loglog = false;
     end
     if(nargin==3)
         color = varargin{2};
     else
         color = 'k';
     end
     [x y] = ginput(2);
     posinit = struct('x', x(1), 'y', y(1));
     width   = x(2)-x(1);

     if(b_loglog)
```

```
        a = y(1)/( x(1)^order);
        posy = a* x(2)^order;
        posxt= sqrt(x(2)*x(1));
        posyt= a* (posxt)^order;

    else
        posy = (posinit.y+width*order);
        posxt = posinit.x+width/2;
        posyt = posinit.y+width/2*order;
    end

    if(order>0)
        text(posxt, posyt, sprintf('%i', order),...
                'VerticalAlignment','bottom',...
                'HorizontalAlignment','right');
    else
        text(posxt, posyt, sprintf('%i', -order),...
            'VerticalAlignment','top',...
            'HorizontalAlignment','right');
    end
    line([posinit.x (posinit.x+width)
   (posinit.x+width)       posinit.x],...
         [posinit.y posy (posinit.y) posinit.y],...
         'Color', color);
end
```

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