SOLUTION OF DAM BREAK PROBLEM BY USING COMPLEX ANALYSIS

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ABSTRACT

SOLUTION OF DAM BREAK PROBLEM BY USING COMPLEX ANALYSIS

In this thesis, we examine the liquid flow and the free surface shape by using some methods of complex analysis during the early stage of dam breaking. In order to obtain the leading-order uniform solution, we use the asymptotic expansions with respect to a small parameter. The small parameter describe the initial duration of the stage. However, this solution which calls outer solution is not valid in a small vicinity of the intersection point between the vertical free surface and the horizontal rigid bottom.

A local analysis of the outer solution helps to estimate the dimension of this vicinity. In the vicinity of the intersection point, local coordinates are stretched to solve the flow singularity and with the aid of these coordinates the leading order inner solution is obtained which describes the formation of the jet flow along the bottom.

ÖZET

BARAJ YIKIMI PROBLEMİNİN KOMPLEKS ANALİZ KULLANILARAK ÇÖZÜMÜ

Bu çalışmada, baraj yıkılmasının ilk anları esnasında sıvı akışı ve serbest su yüzeyi şeklini bazı kompleks analiz methodları kullanılarak inceliyoruz. Birinci mertebeden çözüm elde etmek için küçük bir parametreye göre asimptotik açılımları kullanıyoruz. Küçük parametre aşamanın ilk anlarını tanımlar. Fakat bu dış çözüm olarak adlandırdığımız çözüm dikey serbest yüzey ve yatay katı zeminin kesişim noktasının küçük bir komşuluğunda geçerli değildir.

Dış çözümün lokal analizi bu komşuluğun boyutunun tahmin edilmesine yardım ediyor. Kesim noktasının komşuluğunda, bölgesel koordinatlar genişletiliyor ve bu koordinarların yardımıyla zemindeki püsküren akışın oluşumunu tanımlayan birinci mertebe iç çözüm elde ediliyor.

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CHAPTER 1

INTRODUCTION

We deal with dam-break problem which is a type of unsteady gravity-driven flow. The problem is started by removing a vertical dam in front of liquid region. At the beginning the liquid is at rest and located in the region $x' > 0, -H < y' < 0$ (Figure 1.1). The upper part of the liquid boundary ,$x' > 0, y' = 0$, is the initial position of the liquid free surface and the vertical wall, $x' = 0, -H < y' < 0$, is a dam. The lower horizontal boundary, $y' = -H, x' > 0$, represents the rigid bottom. The liquid is taken incompressible and inviscid. Initially, pressure distribution in the liquid is hydrostatic , $p'(x', y', 0) = -\rho g y'$, where ρ is the liquid density and g is the gravity acceleration. At the initial time instant, $t' = 0$, the dam is suddenly removed and the gravity-driven flow starts. Primed variables represent dimensional ones.

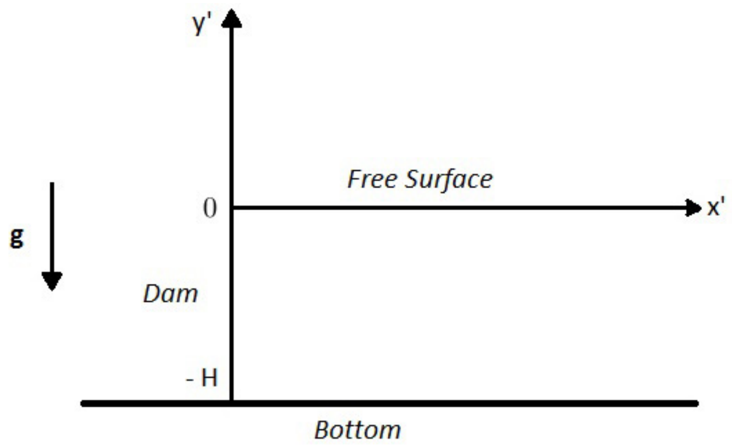


Figure 1.1. Flow region at initial time instant $t' = 0$

During the early stages of the process we will examine the liquid flow and the shape of the free surface which change with time. At the end of the this process, we obtain that flow is potential and two dimensional. We denote the upper part of the free surface as $y' = Y'(x', t'), x' > 0$ and the other one is defined as $x' = X'(y', t'), -H < y' < 0$. Moreover ,the aim of this study is to construct the leading-order uniform solution which is derived by using matched asymptotic expansion.Near the intersection point, solutions should be considered as "outer" solutions and these have to be corrected with "inner" solutions. For the solution which is valid in the whole flow domain, solutions have to be matched which is divided close to the intersection point.

Pohle and Stoker studied dam-breaking problem which leads to gravity-driven flow by using Lagrangian description (FV, 1950) ,(Stoker, 1992). Pohle wrote "Many hydrodynamic problems consider flows in which the region occupied by the fluid is a variable function of time. Euler representation is difficult to apply to such problems. The Lagrangian representation ,however, has the far-reaching advantage that the independent space variables are the initial coordinates of the particles: the region occupied by the fluid is therefore a fixed region." This assertions can be accepted if there are not intersection points between rigid boundaries and free surfaces of the liquid. However, this is not the situation for many important problems such as the water-entry problem, the dam-break problem and problems of floating or free-surface piercing bodies. Near the intersection points, one must determine before which fluid particles originally belonging to the free surface may be found later on the rigid boundary and vice versa. That is to say, the flow region is fixed in the Lagrangian variables but the subdivision of the region boundary into free surface and rigid boundary is unknown and has to be obtained as a part of solution. Because of this, the analysis of dam-break problem in the Lagrangian variables not so useful as it was excepted by Pohle and Stoker.

Pohle use power series to expand the liquid displacement and the hydrodynamic pressure with respect to time t' but only leading-order terms were constructed and analyzed. Pohle wrote that except in the neighborhood of the singular point, where intersection of the free surface and rigid bottom,the calculated profile of the water surface for small times can be assumed to be a reasonable approximation to the physical problem . Therefore, in the Lagrangian variables the small time solution which behaves non-physical shape of water surface is derived near the intersection point. A similar behaviour of the free surface can also be calculated in Eulerian variables. One can get the outer solution in both situation. However, there were no approach to construct the inner solutions of dam break problem either in Lagrangian or Eulerian variables.

For such an inner solution was successfully obtained in a similar problem about a uniformly accelerating wavemaker by King and Needham (King and Needham, 1994). After that Yilmaz and Korobkin solved the dam-break problem in a different way (Korobkin and Yilmaz, 2009). Outer solution is obtained in both leading and second order. Inner solution also calculated by reducing the problem to the well known Cauchy-Poisson problem. In this study, we use some methodology and findings from King and Needham's paper and Yilmaz and Korobkin's paper.

In Chapter 2, some basic concepts in fluid dynamics are given and they may use for the formulations of the problem.

In Chapter 3, we formulate the dam-breaking problem. Boundary and initial conditions are determined then non-linear boundary value problem is constructed. After then, with the appropriate parameters, problem takes the non-dimensional form.

In Chapter 4, we examine the early stage of the problem by using small parameter and get the leading order boundary value problem. Then behaviour of the leading order outer solution is obtained with the singular point near the intersection point with the help of the some methods of complex analysis and decide outer matching conditions.

In Chapter 5, the dimensions of the inner region is indicated and constructed the problem with in inner variables. Then we obtain the leading order inner solution.

CHAPTER 2

FUNDAMENTAL NOTIONS IN FLUID DYNAMICS

2.1. Equations of Motion

2.1.1. Euler's Equation

Our aim is to define the motion of fluid in a region D in two or three dimensional space (Chorin and Marsden, 2000). Let $\mathbf{x}'$ = (x′,y′,z′), in standard Euclidean coordinates in space, is a point in D and consider the particle of fluid moving through $\mathbf{x}'$ at time t' with the velocity $\mathbf{u}'$(x′,y′). Therefore , $\mathbf{u}'$ is a vector field on D for each fixed time. $\mathbf{u}'$ is called "the velocity field of the fluid".

Suppose V is any subregion of D, then the mass of fluid in V at time t' is given by

$$m(V,t') = \int_V \rho(x',t')dV, \tag{2.1}$$

where $\rho(x',t')$ is well-defined mass density of the fluid, dV is the volume element in the plane or in space. In order to apply the standard operations of calculus, we may assume that the functions $\mathbf{u}'$ and ρ are smooth enough and a continuum assumption is that ρ exists.

After that derivation of the equations is based on the following principles :

- Conservation of mass : mass is neither created nor destroyed,
- Balance of momentum (Newton's second law) : the rate of change of momentum of a portion of the fluid equals to the force,
- Conservation of energy : energy is neither created nor destroyed.

1) Conservation of Mass : V is a fixed subregion of D (V does not depend on time), then the rate of change of mass in V is that,

$$\frac{d}{dt'}m(V,t') = \frac{d}{dt'}\int_V \rho(x',t')dV = \int_V \frac{\partial \rho}{\partial t'}(x',t')dV. \tag{2.2}$$

Let S denote the boundary of the region V which is assumed to be smooth and $\mathbf{n}'$ denote the unit normal vector in the outside direction at points of S. $\mathbf{u}'$ = (u′,v′) and ds denote the area element on S. The volume rate across S per unit area is $\mathbf{u}' \cdot \mathbf{n}'$ and the mass of flow rate per unit area is $\rho\ \mathbf{u}' \cdot \mathbf{n}'$. The basic conception of the conservation of mass is that the rate of increase of mass in V equals the rate of mass of flow crossing out of S. "The integral form of the law of conservation of mass" is denoted by

$$\frac{d}{dt'}\int_V \rho dV = -\int_S \rho\mathbf{u}' \cdot \mathbf{n}' dS. \tag{2.3}$$

From the divergence theorem, it is described equivalently,

$$\int_V \left[\frac{d\rho}{dt} + div(\rho\mathbf{u}')\right] dV = 0. \tag{2.4}$$

This is satisfied for all V, then it is equivalent to

$$\frac{d\rho}{dt'} + div(\rho\mathbf{u}') = 0. \tag{2.5}$$

This last equation is the "differential form of the law of conservation of mass", in other words ,"the continuity equation".

2) Balance of Momentum : If V is a region in the fluid at a particular instant time t', the force F_p felt by the fluid inside V as a stress on its boundary is

$$F_p = \text{force on V} = -\int_S p'\mathbf{n}' ds \tag{2.6}$$

where p' is the pressure, and from divergence theorem,

$$F_p = -\int_V \nabla' p' dV. \tag{2.7}$$

In addition, let the given body force per unit mass is denoted by $\mathbf{b}'$(x′,t′), and hence, the total body force F_B is

$$F_B = \int_V \rho b dV. \tag{2.8}$$

As a result of these forces that are acted on any piece of fluid material, Newton's second law, $F = ma$, is written as

$$F = F_p + F_B = -\int_V \nabla' p' dV + \int_V \rho b dV. \tag{2.9}$$

and by rearranging as per unit volume, above expression becomes

$$\rho \frac{D\mathbf{u}'}{Dt} = -\nabla p' + \rho b. \tag{2.10}$$

which is the differential form of the law of "balance of momentum".

3) Conservation of Energy : As we know from the physics that, kinetic energy is KE= $\frac{1}{2}mV^2$ where m is mass and V is velocity of the particle. For fluid contained in a domain σ, and with having a velocity field $\mathbf{u}'$, kinetic energy takes the form of

$$E_k = \frac{1}{2} \int_\sigma \rho ||\mathbf{u}'||^2 dV \tag{2.11}$$

where $||\mathbf{u}'||$ is length of the $\mathbf{u}'$. To find total energy that represents this fluid in a domain σ, we also have to add potential energy such as intermolecular potentials, and any other remaining kinetic or potential energy contributions such as internal molecular vibrations. We call all of these remaining energies as internal energy E_i, and therefore, we write total energy as

$$E_T = E_k + E_i \tag{2.12}$$

If this fluid does work, or if we pump energy into the fluid, then we will change total energy E_T of the fluid. In order to understand how this total energy changes, we will analyse the rate of change of kinetic energy of fluid which leads to

$$\frac{1}{2}\frac{D}{Dt}||\mathbf{u}'||^2 = \mathbf{u}' \cdot \frac{\partial \mathbf{u}'}{\partial t} + \mathbf{u}' \cdot (\mathbf{u}' \cdot \nabla')\mathbf{u}'. \tag{2.13}$$

2.1.2. Incompressibility

Let S is a fixed closed surface drawn in the fluid and take unit normal $\mathbf{u}'$ in the outward direction (Acheson, 1990). Fluid will be entering and leaving the enclosed region V at some places on S.
If the velocity component along the outward normal is $\mathbf{u}' \cdot \mathbf{n}'$, then the volume of fluid leaving through ∂S in unit time is $\mathbf{u}' \cdot \mathbf{n}'\partial S$, so the net volume rate is

$$\int_S \mathbf{u}' \cdot \mathbf{n}' dS \tag{2.14}$$

and with the help of divergence theorem and by taking into consideration of incompressibility of fluid, conclude that

$$\int_V \nabla' \cdot \mathbf{u}' dV = 0. \tag{2.15}$$

It is true for all regions V, then the "incompressibility condition" is obtained in the following form:

$$\nabla' \cdot \mathbf{u}' = 0 \tag{2.16}$$

in the whole fluid.

2.1.3. Euler's Equations for Incompressible Flows

Euler's equation for an incompressible fluid is defined with the following equations;

$$\frac{D\mathbf{u}'}{Dt} = -\frac{1}{\rho}\nabla p' + \mathbf{g}, \tag{2.17}$$

$$\nabla' \cdot \mathbf{u}' = 0. \tag{2.18}$$

g is the gravitational force which is conservative and it can be defined as the gradient of a potential such that **g** = - $\nabla'\chi$. With the definition of substantial derivative, from (2.17) we have

$$\frac{\partial \mathbf{u}'}{\partial t'} + (\mathbf{u}' \cdot \nabla')\mathbf{u}' = -\nabla'(\frac{p'}{\rho} + \chi), \tag{2.19}$$

where ρ is constant. Moreover, by using the vector identity

$$(\mathbf{u}' \cdot \nabla')\mathbf{u}' = (\nabla' \wedge \mathbf{u}') \wedge \mathbf{u}' + \nabla'(\frac{1}{2}|\mathbf{u}'|^2) \tag{2.20}$$

the momentum equation is obtained in the form

$$\frac{\partial \mathbf{u}'}{\partial t'} + (\nabla' \wedge \mathbf{u}') = -\nabla'(\frac{p'}{\rho} + \frac{1}{2}|\mathbf{u}'|^2 + \chi). \tag{2.21}$$

2.2. Irrotational Flow

2.2.1. Vorticity

The vorticity is represented by ω and defined as

$$\omega = \nabla' \wedge \mathbf{u}'. \tag{2.22}$$

For irrotational flow the vorticity is zero by definition and in two-dimensions, velocity is defined by $\mathbf{u}'$ = $[u'(x', y', t'), v(x', y', t'), 0]$, therefore vorticity $\omega = (0, 0, \omega)$ takes the form

$$\omega = \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'}. \tag{2.23}$$

2.2.2. Potential Flow

Velocity Potential

The velocity potential φ' exists only if the flow is irrotational, i.e, $\nabla' \wedge \mathbf{u}' = 0$.It is defined at any point P by

$$\varphi' = \int_O^P \mathbf{u}' \cdot d\mathbf{x}' \tag{2.24}$$

where O is some arbitrary fixed point. In a simply connected fluid region φ' is independent of the path between O and P, thus a single-valued function of position. Partial differentiation of the equation (2.24) gives

$$\mathbf{u}' = \nabla'\varphi' \tag{2.25}$$

and the vector identity $\nabla' \wedge \nabla'\varphi' = 0$ confirms that this flow is irrotational, as desired. Such an inviscid, irrotational flow is called a "potential flow".

2.2.3. Bernoulli's Equation for Unsteady Irrotational Flow

Since the flow is irrotational, the velocity can be defined by $\mathbf{u}' = \nabla'\varphi'$. By keeping in view Euler's equation (2.21) and condition of irrotational flow, we have

$$\frac{d}{dt'}\nabla'\varphi' = -\nabla'(\frac{p'}{\rho} + \frac{1}{2}|\mathbf{u}'|^2 + \chi). \tag{2.26}$$

Integration of this equation gives,

$$\frac{\partial\varphi'}{dt'} + \frac{p'}{\rho} + \frac{1}{2}|\mathbf{u}'|^2 + \chi = G(t) \tag{2.27}$$

where G(t) is an arbitrary function of time alone. Equation (2.27) is called "Bernoulli's equation for unsteady irrotational flow".

CHAPTER 3

DAM BREAK PROBLEM

3.1. Definition of the Non-Linear Problem

3.1.1. Equation of Motion

To formulate the dam break problem (Korobkin and Yilmaz, 2009), first we use the incompressibility condition in two dimensions. Taking into consideration $u' = \nabla'\varphi'$ and incompressibility condition, we get

$$\nabla'^2\varphi' = 0 \qquad \text{in } \sigma'(t'). \tag{3.1}$$

where $\sigma'(t')$ is a region between $x' > 0$ and $-H < y' < 0$ (Figure 1.1). So, in the region, the velocity potential satisfies Laplace equation. From Bernoulli's equation for unsteady irrotational flow,(2.27),(G(t) = 0) we obtain pressure distribution in the region as,

$$-p' = \rho\varphi'_{t'} + \frac{\rho}{2}|\nabla'\varphi|^2 + \rho g y' \qquad \text{in } \sigma'(t'). \tag{3.2}$$

3.1.2. Boundary Conditions

We have dynamic and kinematic boundary conditions at the free surfaces, the kinematic condition suggest that the fluid particles on the free surface must remain on the surface, hence if we define surface functions $F(x', y', t') = y' - Y'(x', t')$ and $G(x', y', t') = x' - X'(y', t')$ then we may conclude that $F(x', y', t')$ and $G(x', y', t')$ remain constant (zero) for any particular fluid particle on the free surface. Therefore, it is expressed as $\frac{DF}{Dt'} = 0$ on $y' = Y'(x', t')$ and $\frac{DG}{Dt'} = 0$ on $x' = X'(y', t')$, i.e.

$$\frac{\partial F}{\partial t'} + (u'.\nabla')F = 0, \qquad \text{on } y' = Y'(x', t')$$

$$\varphi'_{y'} = \frac{\partial Y'}{\partial t'} + \varphi'_{x'} \frac{\partial Y'}{\partial x'} \qquad \text{on } \ y' = Y'(x', t') \tag{3.3}$$

and

$$\frac{\partial G}{\partial t'} + (u'.\nabla')G = 0, \qquad \text{on } \ x' = X'(y', t')$$

$$\varphi'_{x'} = \frac{\partial X'}{\partial t'} + \varphi'_{y'} \frac{\partial X'}{\partial y'} \qquad \text{on } \ x' = X'(y', t'). \tag{3.4}$$

For the dynamic condition, the pressure on the free surfaces is atmospheric ,

$$p'(x', y', t') = 0 \qquad \text{on } \ y' = Y(x', t') \ \ , \ \ x' = X'(y', t'). \tag{3.5}$$

Moreover, there is a slip boundary condition on $y' = -H$ which means

$$\varphi'_{y'}(x', -H, t') = 0. \tag{3.6}$$

3.1.3. Initial Conditions

Before the dam breaks, the liquid is at rest so the free surfaces which are horizontal and vertical and the velocity potentials are zero and also pressure distribution is hydrostatic at the initial time $(t' = 0)$. Thus, we can define the initial conditions as follows,

$$\varphi(x', y', 0) = 0 \qquad , \qquad X'(y', t') = Y'(x', t') = 0 \qquad , \qquad p'(x', y', 0) = -\rho g y' \tag{3.7}$$

Finally, we construct the mathematical modelling of the non-linear problem in two dimensions as

$$\nabla'^2 \varphi' = 0 \qquad (\text{in } \ \sigma'(t'))$$

$$-p' = \rho \varphi'_{t'} + \frac{\rho}{2} |\nabla' \varphi|^2 + \rho g y' \qquad (\text{in } \ \sigma'(t'))$$

$$\varphi'_{y'} = Y'_{t'} + \varphi'_{x'} Y'_{x'} \qquad (\text{on } \ y' = Y'(x', t'))$$

$$\varphi'_{x'} = X'_{t'} + \varphi'_{y'} X'_{y'} \qquad (\text{on } \ x' = X'(y', t')) \tag{3.8}$$

$$p'(x', y', t') = 0 \qquad (\text{on } \ x' = X'(y', t') \ \text{ and } \ y' = Y'(x', t'))$$

$$\varphi'_{y'}(x', -H, t') = 0$$

$$\varphi'(x', y', 0) = X'(y', 0) = Y'(x', 0) = 0$$

$$\varphi' \to 0 \qquad \text{and} \qquad p' \to -\rho g y' \qquad \text{as} \qquad x' \to \infty.$$

3.2. Equations in Non-dimensional Form

For simplicity, we use non-dimensional variables in asymptotic analysis. So, in order to formulate non-dimensional dam-break problem, following transformations are used;

$$\left.\begin{array}{rcr} & x' = Hx \quad , & y' = Hy \\ X' = SX \quad , & Y' = SY & , p' = \rho g H p \\ & \varphi' = gHT\varphi \quad , & t' = Tt. \end{array}\right\} \tag{3.9}$$

Substituting these transformations into Laplace equation we obtain,

$$\frac{\partial \varphi'}{\partial x'} = \frac{\partial (gHT\varphi)}{\partial (xH)} \qquad , \qquad \frac{\partial \varphi'}{\partial y'} = \frac{\partial (gHT\varphi)}{\partial (yH)}$$

$$\nabla^2 \varphi = 0, \qquad (\text{in } \ \sigma(t)). \tag{3.10}$$

Pressure distribution in the region;

$$\left(\frac{gHT}{T}\right)\frac{\partial\varphi}{\partial t}+\frac{1}{2}\left[\left(\frac{gHT}{T}\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{gHT}{T}\frac{\partial\varphi}{\partial y}\right)^2\right]+gHy.=-\rho gHp$$

Dividing by gH and choosing $\delta=\frac{gT^2}{H}$, we get,

$$-p=\varphi_t+\frac{\delta}{2}|\nabla\varphi|^2+y,\qquad (\text{in }\ \sigma(t)). \tag{3.11}$$

For the dynamic boundary condition;

$$p=0,\qquad (\text{on }\ y=\delta Y(x,t)\ \ ,\ \ x=\delta X(y,t)) \tag{3.12}$$

Kinematic boundary condition on the horizontal free surface is

$$\left(\frac{gHT}{H}\right)\frac{\partial\varphi}{\partial y}=\left(\frac{S}{H}\right)\frac{\partial Y}{\partial x}\left(\frac{gHT}{H}\right)\frac{\partial\varphi}{\partial x}+\left(\frac{S}{T}\right)\frac{\partial Y}{\partial t},$$

Dividing the above equation by gT ;

$$\varphi_y=\delta Y_x\varphi_x+Y_t\qquad (\text{on }\ y=\delta Y(x,t,\delta),x>0) \tag{3.13}$$

and similarly for the vertical free surface, we get

$$\varphi_x=\delta\varphi_y X_y+X_t\qquad (\text{on }\ x=\delta X(y,t,\delta),-1<y<\delta Y(0,t,\delta)). \tag{3.14}$$

Slip boundary condition takes the form ,

$$\varphi_y=0\qquad (\text{on }\ y=-1). \tag{3.15}$$

Initial conditions at $t = 0$ are

$$\varphi(x, y, 0, \delta) = 0 \qquad , \qquad Y(x, 0, \delta) = 0 \qquad , \qquad X(y, 0, \delta) = 0. \tag{3.16}$$

Finally, the radiation condition is

$$\varphi \to 0 \qquad (x \to \infty). \tag{3.17}$$

CHAPTER 4

LEADING ORDER VELOCITY POTENTIAL

4.1. Behaviour For The Small Parameter ($\delta \to 0$)

Solution to the boundary value problem (3.10)-(3.17) is sought in the following form, as $\delta \to 0$;

$$\begin{aligned} \varphi(x,y,t,\delta) &= \varphi_0(x,y,t) + \delta\varphi_1(x,y,t) + O(\delta^2), \\ X(y,t,\delta) &= X_0(y,t) + \delta X_1(y,t) + O(\delta^2), \\ Y(x,t,\delta) &= Y_0(x,t) + \delta Y_1(x,t) + O(\delta^2), \\ p(x,y,t,\delta) &= p_0(x,y,t) + \delta p_1(x,y,t) + O(\delta^2). \end{aligned} \tag{4.1}$$

We substitute these asymptotic expansions in non-dimensional boundary value problem (3.10)-(3.17) and let $\delta \to 0$. From Laplace equation we have

$$\Delta(\varphi_0(x,y,t) + \delta\varphi_1(x,y,t) + ...) = 0$$

which gives,

$$\nabla^2\varphi_0(x,y,t) = 0 \qquad \text{(order 1)}, \qquad (\text{in } \ \sigma(t)), \tag{4.2}$$

$$\nabla^2\varphi_1(x,y,t) = 0 \qquad (\text{order } \delta), \qquad (\text{in } \ \sigma(t)). \tag{4.3}$$

From the dynamic boundary condition on the free surfaces, $p = 0$,

$$0 = (\varphi_0 + \delta\varphi_1 + ...)_t + \frac{\delta}{2}[(\varphi_0 + \delta\varphi_1 + ...)_x^2 + (\varphi_0 + \delta\varphi_1 + ...)_y^2] + y$$

we have

$$\varphi_{0,t} + y = 0 \qquad \text{(of order 1)} \tag{4.4}$$

which gives on the horizontal free surface ,

$$\varphi_0 = 0 \qquad (\text{on } \ y = 0, x > 0)$$

and on vertical free surface,

$$\varphi_0 = -yt \qquad (\text{on } \ -1 < y < 0, x = 0).$$

The second order dynamic condition on the free surfaces is

$$\varphi_{1,t} + |\nabla\varphi_0|^2 = 0 \qquad (\text{order } \delta). \tag{4.5}$$

From the kinematic boundary condition on the horizontal free surface, we have

$$(\varphi_0 + \delta\varphi_1 + ...)_y = (Y_0 + \delta Y_1 + ...)_t + \delta[(\varphi_0 + \delta\varphi_1 + ...)_y][(Y_0 + \delta Y_1 + ...)_x],$$

which gives,

$$\varphi_{0,y} = Y_{0,t} \qquad \text{(order 1)}, \qquad (\text{on } \ x > 0, y = 0,) \tag{4.6}$$

$$\varphi_{1,y} = Y_{1,t} + \varphi_{0,x} Y_{0,x} \qquad (\text{order } \delta), \qquad (\text{on } \ x > 0, y = 0). \tag{4.7}$$

and on the vertical free surface, we have

$$(\varphi_0 + \delta\varphi_1 + ...)_x = (X_0 + \delta X_1 + ...)_t + \delta[(\varphi_0 + \delta\varphi_1 + ...)_x][(X_0 + \delta X_1 + ...)_y],$$

which gives,

$$\varphi_{0,x} = X_{0,t} \qquad \text{(order 1)}, \qquad (\text{on } x = 0, -1 < y < 0), \tag{4.8}$$

$$\varphi_{1,y} = X_{1,t} + \varphi_{0,y} X_{0,y} \qquad \text{(order } \delta), \qquad (\text{on } x = 0, -1 < y < 0). \tag{4.9}$$

From slip boundary condition on y = - 1, $\varphi_y = 0$, we have

$$\varphi_{0,y} = 0 \qquad \text{(order 1)}, \qquad (\text{on } y = -1), \tag{4.10}$$

$$\varphi_{1,y} = 0 \qquad \text{(order } \delta), \qquad (\text{on } y = -1). \tag{4.11}$$

Initial conditions at $t = 0$ are

$$\begin{aligned} \varphi(x, y, 0) &= 0 \\ Y(x, 0) &= 0 \\ X(y, 0) &= 0, \end{aligned} \tag{4.12}$$

and we have,

$$\left.\begin{aligned} \varphi_0(x, y, 0) = 0 \\ Y_0(x, 0) = 0 \\ X_0(y, 0) = 0 \end{aligned}\right\} \qquad \text{(order 1)}, \qquad (\text{on } \sigma(t)), \tag{4.13}$$

$$\left.\begin{aligned} \varphi_1(x, y, 0) = 0 \\ Y_1(x, 0) = 0 \\ X_1(y, 0) = 0 \end{aligned}\right\} \qquad \text{(order } \delta), \qquad (\text{on } \sigma(t)). \tag{4.14}$$

Then, we obtain the following leading order boundary value problem,

$$\nabla^2 \varphi_0 = 0 \qquad \text{(in region)},$$

$$\varphi_0 = 0 \qquad , \qquad \varphi_{0,y} = Y_{0,t} \qquad \text{(on } y = 0, x > 0\text{)},$$

$$\varphi_0 = -yt \qquad , \qquad \varphi_{0,x} = X_{0,t} \qquad \text{(on } x = 0, -1 < y < 0\text{)}, \tag{4.15}$$

$$\varphi_{0,y} = 0 \qquad \text{(on } y = -1, x > 0\text{)},$$

$$\varphi_0 \to 0 \qquad \text{as} \qquad x \to \infty.$$

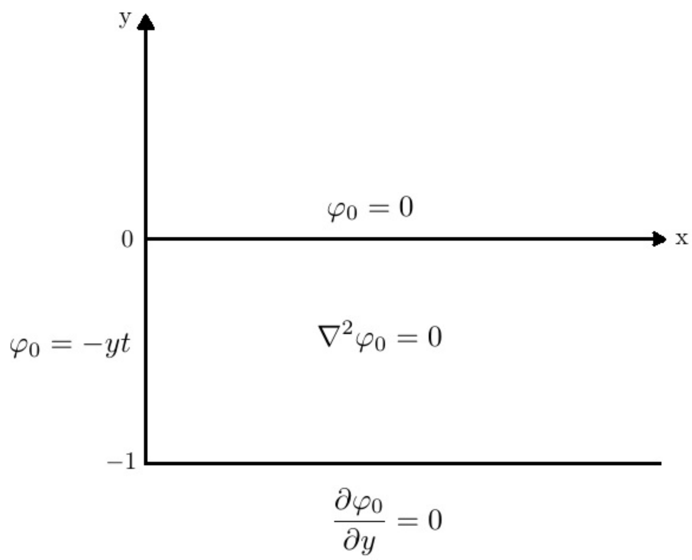


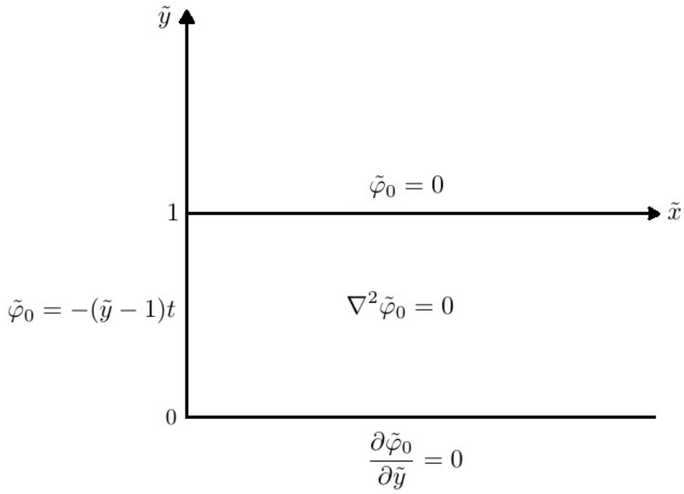
Figure 4.1. Region of the problem for the leading order velocity potential

4.1.1. Mapping Of The Region Onto Half Plane

We prefer to solve the problem on the upper half plane instead of the region $x > 0, -1 < y < 0$. First we use the translation

$$x = \tilde{x} \qquad , \qquad y = \tilde{y} - 1 \tag{4.16}$$

and we shift the region (Figure 4.1). Then by using theorems on transformation of harmonic functions and transformation of boundary conditions (See Appendix A),we rewrite the leading order boundary value problem in the new coordinate system ,



Since the complex potential w_0 which is defined as $\tilde{w_0} = \tilde{\varphi_0} + i\tilde{\psi_0}$ is analytic in the region, from Cauchy-Riemann equations we have,

$$\tilde{\varphi_0}_{,\tilde{x}} = \tilde{\psi_0}_{,\tilde{y}} \qquad , \qquad \tilde{\varphi_0}_{,\tilde{y}} = -\tilde{\psi_0}_{,\tilde{x}} \,. \tag{4.17}$$

We extend the region to $-1 < \tilde{y} < 0, \tilde{x} > 0$ (Figure 4.2) by using the reflection principle (or symmetry principle) (see Appendix B), then we map the region onto upper half plane easily.

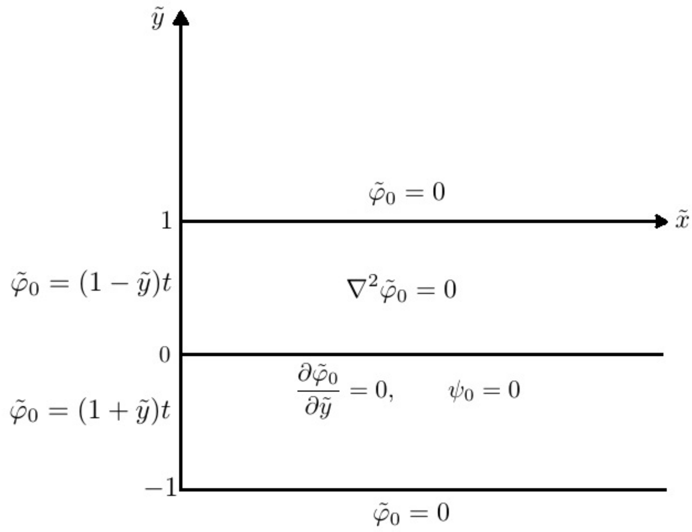


Figure 4.2. Application of the reflection principle on the region

By using transformation $\tilde{\zeta} = i\sinh(\frac{\pi\tilde{z}}{2})$ where $\tilde{\zeta} = \tilde{u} + i\tilde{v}$ and $\tilde{z} = \tilde{x} + i\tilde{y}$,

$$\begin{aligned}\tilde{u} &= -\sin(\frac{\pi\tilde{y}}{2})\cosh(\frac{\pi\tilde{x}}{2}) \\ \tilde{v} &= \sinh(\frac{\pi\tilde{x}}{2})\cos(\frac{\pi\tilde{y}}{2})\end{aligned} \tag{4.18}$$

we carry the region onto the upper half plane (Figure 4.3). By using theorems on transformation of harmonic functions and boundary conditions (See Appendix A), we get

$$\nabla^2\Phi_0 = 0 \qquad \text{(in the region)}, \tag{4.19}$$

$$\Phi_0 = 0 \qquad (\text{on } \tilde{u} > 1 \ , \tilde{u} < -1, \tilde{v} = 0) \tag{4.20}$$

and on the vertical boundary, $-1 < \tilde{y} < 1, \tilde{x} = 0$, velocity potential is transformed as

$$\Phi_0 = (1 - \frac{2}{\pi}|\arcsin(\tilde{u})|)t \qquad (\text{on } -1 < \tilde{u} < 1, \tilde{v} = 0). \tag{4.21}$$

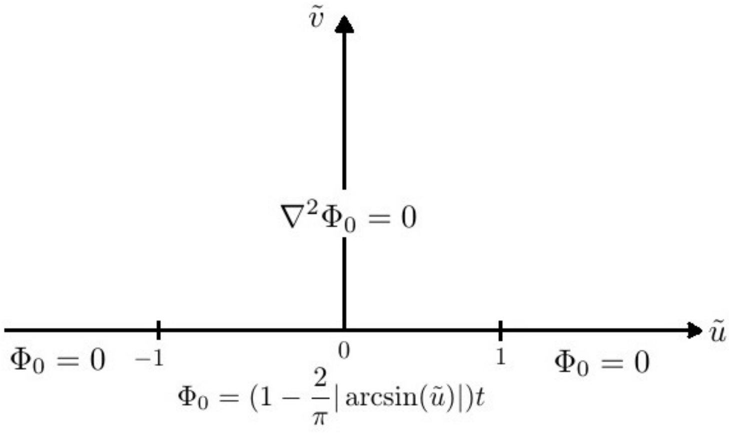


Figure 4.3. Flow region on the upper half plane

4.1.2. The Leading Order Solution and Singularity

To obtain the velocities, we take derivative of velocity potential with respect to $\tilde{u}$ on $\tilde{v} = 0$ and we find horizontal velocity component $\Phi_{0,\tilde{u}}$ (Figure 4.4).

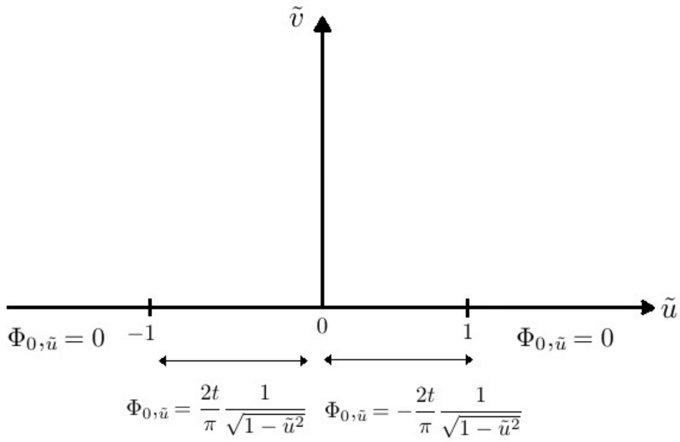


Figure 4.4. Derivative of the velocity potential on the real axis

For vertical velocity component, $\Phi_{0,\tilde{v}}$, we use Hilbert inversion formula (See Appendix C) then we have,

$$\Phi_{0,\tilde{v}} = \frac{2t}{\pi^2\sqrt{1-\tilde{u}^2}} \ln\left|\frac{1-\sqrt{1-\tilde{u}^2}}{1+\sqrt{1-\tilde{u}^2}}\right| \quad , \quad -1 < \tilde{u} < 1, \tilde{v} = 0 \tag{4.22}$$

$$\Phi_{0,\tilde{v}} = \frac{4t}{\pi^2\sqrt{\tilde{u}^2-1}} \arctan\left(\frac{1}{\sqrt{\tilde{u}^2-1}}\right) \quad , \quad \tilde{u} < -1, \;\; \tilde{u} > 1, \tilde{v} = 0. \tag{4.23}$$

At the point $\tilde{\zeta} = 0$, by using Poisson's integral formula (See Appendix D),(Figure 4.3)

$$\Phi_0(\tilde{u}, \tilde{v}) = \frac{\tilde{v}t}{\pi} \int_{-1}^{1} \frac{1 - \frac{2}{\pi}|\arcsin\xi|}{(\tilde{u}-\xi)^2 + \tilde{v}^2} d\xi.$$

We take derivative of Φ_0 with respect to $\tilde{v}$ for vertical velocity component at point $(0, 0)$, then we have

$$\Phi_{0,\tilde{v}} = \frac{t}{\pi} \int_{-1}^{1} \frac{1 - \frac{2}{\pi}|\arcsin\xi|}{(\tilde{u}-\xi)^2 + \tilde{v}^2} d\xi + \frac{\tilde{v}t}{\pi} \int_{-1}^{1} \frac{-2\tilde{v}[1 - \frac{2}{\pi}|\arcsin\xi|]}{[(\tilde{u}-\xi)^2 + \tilde{v}^2]^2} d\xi,$$

$$\Phi_{0,\tilde{v}}(0, 0) = \frac{t}{\pi} \int_{-1}^{1} \frac{1 - \frac{2}{\pi}|\arcsin\xi|}{\xi^2} d\xi \tag{4.24}$$

By taking this integral we find,

$$\Phi_{0,\tilde{v}}(0, 0) = -\frac{4t}{\pi^2} \left[1 + \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|\right] \Bigg|_0^1 = \infty. \tag{4.25}$$

Therefore, there is a logarithmic singularity at point $(0, 0)$.
We go back to the original region and write velocity components. We know that

$$\Phi_0(\tilde{u}, \tilde{v}) = \tilde{\varphi}_0(\tilde{x}(\tilde{u}, \tilde{v}), \tilde{y}(\tilde{u}, \tilde{v}))$$

thus we write velocities at $\tilde{x} = 0, 0 < \tilde{y} < 1$ corresponding to the line segment $-1 < \tilde{u} < 1, \tilde{v} = 0$

$$\tilde{\varphi}_{0,\tilde{x}} = \Phi_{0,\tilde{u}}\, \tilde{u}_{\tilde{x}} + \Phi_{0,\tilde{v}}\, \tilde{v}_{\tilde{x}} \tag{4.26}$$

from (4.17) we have,

$$\tilde{u} = -\sin(\frac{\pi\tilde{y}}{2}) \quad , \quad \tilde{v} = 0 \quad , \quad \tilde{u}_{\tilde{x}} = 0 \quad , \quad \tilde{v}_{\tilde{x}} = \frac{\pi}{2}\cos(\frac{\pi\tilde{y}}{2})$$

then if we substitute these equations with (4.18) into (4.22) we obtain

$$\tilde{\varphi}_{0,\tilde{x}} = \frac{2t}{\pi^2\sqrt{1-\sin^2(\frac{\pi\tilde{y}}{2})}} \ln\left|\frac{1-\sqrt{1-\sin^2(\frac{\pi\tilde{y}}{2})}}{1+\sqrt{1-\sin^2(\frac{\pi\tilde{y}}{2})}}\right|\left[\frac{\pi}{2}\cos(\frac{\pi\tilde{y}}{2})\right],$$

$$\tilde{\varphi}_{0,\tilde{x}} = \frac{t}{\pi\cos(\frac{\pi\tilde{y}}{2})} \ln\left|\frac{1-\cos(\frac{\pi\tilde{y}}{2})}{1+\cos(\frac{\pi\tilde{y}}{2})}\right|\left[\cos(\frac{\pi\tilde{y}}{2})\right].$$

By using the half angle formula of cosine, $\cos(\frac{\pi\tilde{y}}{2}) = \cos^2(\frac{\pi\tilde{y}}{4}) - \sin(\frac{\pi\tilde{y}}{4})$,

$$\tilde{\varphi}_{0,\tilde{x}} = \frac{t}{\pi} \ln\left|\frac{1-\cos^2(\frac{\pi\tilde{y}}{4})+\sin(\frac{\pi\tilde{y}}{4})}{1+\cos^2(\frac{\pi\tilde{y}}{4})-\sin(\frac{\pi\tilde{y}}{4})}\right|,$$

$$\tilde{\varphi}_{0,\tilde{x}} = \frac{2t}{\pi}\ln[\tan(\frac{\pi\tilde{y}}{4})].$$

From transformation $\tilde{x} = x$, $\tilde{y} = y + 1$, horizontal velocity component is obtained in the form,

$$\tilde{\varphi}_{0,x}\,(0, y, t) = \frac{2t}{\pi}\ln[\tan\left(\frac{\pi(y+1)}{4}\right)]. \tag{4.27}$$

At $\tilde{u} > 1$, $\tilde{u} < -1$ corresponding to the line segment $\tilde{y} = 1$, $\tilde{x} > 0$

$$\tilde{\varphi}_{0,\tilde{y}} = \Phi_{0,\tilde{u}}\, \tilde{u}_{\tilde{y}} + \Phi_{0,\tilde{v}}\, \tilde{v}_{\tilde{y}} \tag{4.28}$$

similarly to the above derivation we have

$$\tilde{u} = -\cosh(\frac{\pi\tilde{x}}{2}) \qquad , \qquad \tilde{v} = 0 \qquad , \qquad \tilde{u}_{\tilde{y}} = 0 \qquad , \qquad \tilde{v}_{\tilde{y}} = -\frac{\pi}{2}\sinh(\frac{\pi\tilde{x}}{2}).$$

By substituting these equations and (4.18) into (4.23) we obtain,

$$\tilde{\varphi}_{0,\tilde{y}} = \frac{4t}{\pi^2\sqrt{\cosh^2(\frac{\pi\tilde{x}}{2})-1}} \arctan\left(\frac{1}{\sqrt{\cosh^2(\frac{\pi\tilde{x}}{2})-1}}\right)\Big[-\frac{\pi}{2}\sinh(\frac{\pi\tilde{x}}{2})\Big],$$

$$\tilde{\varphi}_{0,\tilde{y}} = -\frac{2t}{\pi}\arctan\left(\frac{1}{\sinh(\frac{\pi\tilde{x}}{2})}\right).$$

From the property of the arctangent function , $\arctan(x) = 2\arctan(\frac{x}{1+\sqrt{1+x^2}})$ and by using $x = \frac{1}{\sinh(\frac{\pi\tilde{x}}{2})}$ we obtain

$$\tilde{\varphi}_{0,\tilde{y}} = -\frac{4t}{\pi}\arctan\left(\frac{1}{\sinh(\frac{\pi\tilde{x}}{2})+\cosh\frac{\pi\tilde{x}}{2})}\right).$$

After applying the transformation $\tilde{x} = x$, $\tilde{y} = y+1$, we find vertical velocity component at $y = 0$,

$$\varphi_{0,y}(x,0,t) = -\frac{4t}{\pi}\arctan\big(\exp(-\frac{\pi x}{2})\big). \tag{4.29}$$

Correspondingly, from (4.29) and (4.27) , we obtain free surface equations in the form,

$$Y_0(x,t) = -\frac{2t^2}{\pi}\arctan\big(\exp(-\frac{\pi x}{2})\big) \tag{4.30}$$

and

$$X_0(y,t) = \frac{t^2}{\pi}\ln[\tan(\frac{\pi(1+y)}{4})]. \tag{4.31}$$

We see that X_0 has a logarithmic singularity at the intersection point , as $y \to -1$.

4.1.2.1. Outer Matching Conditions

Because of the singularity at the intersection point, the formal asymptotic expansions (4.1) are not valid so they should be considered as outer expansions to this problem. To determine the behaviour in the neighborhood of the point $(0,-1)$ we need an inner region which is defined by stretching coordinates $x = a\xi$ and $y = a\eta - 1$, a is small positive constant $(a << 1)$ and $\xi = O(1)$,$\eta = O(1)$ and to constitute the form of the inner expansion we need the local behaviour of the velocity potential in the new coordinate axis as $(\xi^2 + \eta^2) \to 0$.

4.1.3. Behaviour Of The Leading Order Solution Near The Singular Point

In the new coordinates, we have horizontal velocity component in the form,

$$\varphi_{0,x}(0,\eta,t) = \frac{2t}{\pi} \ln\left[\tan\frac{\pi}{4}a\eta\right].$$

Since we investigate the local behaviour of velocities, we expand $\tan(\frac{\pi}{4}a\eta)$ into Taylor series as $a \to 0$,

$$\varphi_{0,x}(0,\eta,t) = \frac{2t}{\pi} \ln[\frac{\pi}{4}a\eta + O(a^2)]$$

$$\varphi_{0,x}(0,\eta,t) = \frac{2t}{\pi} \ln[\frac{\pi}{4}a\eta] + O(a^2). \qquad (4.32)$$

We know that complex potential is defined as $w_0(z,t) = \varphi_0(x,y,t) + i\psi_0(x,y,t)$ and consider the leading order complex velocity of the flow,

$$\frac{dw_0}{dz} = \frac{\partial\varphi_0}{\partial x} + i\frac{\partial\psi_0}{\partial x}.$$

With the help of the analicity of w_0, Cauchy-Riemann equations are satisfied,

$$\varphi_{0,x} = \psi_{0,y} \qquad , \qquad \varphi_{0,y} = -\psi_{0,x}\,.$$

Then, we can write

$$\frac{dw_0}{dz} = \frac{\partial \varphi_0}{\partial x} - i\frac{\partial \varphi_0}{\partial y}. \tag{4.33}$$

At $x = 0$, $-1 < y < 0$ which coincides with $\xi = 0$, $\eta > 0$, we have

$$\varphi_{0,y} = -t \qquad \text{and} \qquad \varphi_{0,x} = \frac{2t}{\pi}\ln[\frac{\pi}{4}a\eta] + O(a^2). \tag{4.34}$$

Substituting these equations into (4.33) we get

$$w_{0,z} = \frac{2t}{\pi}\ln(\frac{\pi}{4}a\eta) + it + O(a^2) \tag{4.35}$$

$$w_{0,z} = \frac{2t}{\pi}\left[\ln(\frac{\pi}{4}a\eta) + i\frac{\pi}{2}\right] + O(a^2)$$

By choosing the branch of logarithmic function as log1 = 0 ,we write log i instead of $i\frac{\pi}{2}$,

$$w_{0,z} = \frac{2t}{\pi}\left[\ln(\frac{\pi}{4}ia\eta)\right] + O(a^2)$$

For $\zeta = \xi + i\eta$ and on $\xi = 0$, $\eta > 0$, we obtain complex velocity in the form,

$$w_{0,\zeta} = \frac{2at}{\pi}\left[\ln(\frac{\pi}{4}a\zeta)\right] + O(a^3). \tag{4.36}$$

The boundary conditions in (4.15) give $w_0(-i,t) = t$ at the intersection point and integration of (4.36) give ,as $a \to 0$,

$$w_0(z,t) = t + \frac{2t}{\pi}\zeta a\log a + \frac{2t}{\pi}a[\log\zeta + \log\frac{\pi}{4} - 1]\zeta + O(a^3). \tag{4.37}$$

The asymptotic behaviour of the leading-order velocity potential close to the intersection point is obtained by the real part of the complex potential,thus we get

$$\varphi_0(z,t) = t + \frac{2t}{\pi}\xi a \log a + \frac{2t}{\pi} a[\log \rho + \log \frac{\pi}{4} - 1]\xi - \frac{2t}{\pi} a\eta\theta + O(a^3). \tag{4.38}$$

where $\xi = \rho\cos\theta$, $\eta = \rho\sin\theta$. Moreover, on the vertical free surface, behaviour of the leading-order surface shape is found from the condition $\varphi_{0,x} = X_{0,t}$, as $a \to 0$,

$$X_0(\eta,t) = \frac{t^2}{\pi}\log\left[\frac{\pi a\eta}{4}\right] + O(a^2), \qquad \text{(on } x = 0, -1 < y < 0). \tag{4.39}$$

CHAPTER 5

THE INNER VARIABLES

5.1. Inner Region Problem

For an inner solution of this problem when $\xi = O(1)$, $\eta = O(1)$ as $a \to 0$, we must check the order of retained terms and neglected terms in boundary conditions which we investigated in chapter 4, as $\rho = (\xi^2 + \eta^2)^{\frac{1}{2}} \to 0$. In order to be consistent, their orders must be matched. Therefore,in dynamic condition, the neglected term $\frac{\delta}{2}|\nabla\varphi|^2$ = O($\delta log^2 a$) is equal to order of retained term φ_t = O($a \log a$) if and only if $\delta = -\frac{a}{\log a}$. With these predictions, the relation between the free surface expressions in the inner and outer regions is found in the following form,

$$\xi = -\frac{1}{\log a}X. \tag{5.1}$$

5.1.1. Equations of the Inner Region

The inner velocity potential $\varphi_i(\xi, \eta, t, \delta)$ and the free surface shape which are based on the asymptotic formulas are sought as,

$$\varphi_i(\xi, \eta, t, \delta) = t + a \log a \varphi_{i0} + \varphi_{i1} + O(\frac{a}{\log a}) \tag{5.2}$$

$$\xi = -\frac{t^2}{\pi} - \frac{1}{\log a}H(\eta, t) + O(\frac{1}{\log^2 a}) \qquad \text{and so} \qquad X = \frac{t^2}{\pi}\log a + H(\eta, t) + O(\frac{1}{\log a}) \tag{5.3}$$

where X(y,t,δ) describes the shape of the free surface. These inner region equations are substituted in (3.10)-(3.12),(3.14),(3.15) then new conditions are found as follows:
Laplace equation is satisfied by both φ_{i0} and φ_{i1}.

$$\nabla^2\varphi_{i0} = 0 \qquad \text{and} \qquad \nabla^2\varphi_{i1} = 0 \qquad \text{(in the region)} \tag{5.4}$$

At the bottom, $\eta = 0$, $\varphi_{i},_{\eta} = 0$ thus,

$$\varphi_{i0},_{\eta} = 0 \qquad \text{and} \qquad \varphi_{i1},_{\eta} = 0. \tag{5.5}$$

Kinematic boundary condition is

$$\varphi_{i},_{\xi} = \xi_{\eta}\varphi_{i},_{\eta} - a\log a\xi_{t} \qquad \text{at } \xi = -\frac{t^2}{\pi} - \frac{1}{\log a}H(\eta,t) \tag{5.6}$$

$$\begin{aligned}(a\log a\varphi_{i0},_{\xi} + a\varphi_{i1},_{\xi} + O(\frac{a}{\log a}) = [-\frac{t^2}{\pi} - \frac{1}{\log a}H(\eta,t) + O(\frac{1}{\log^2 a})]_{\eta}[t \\ + a\log a\varphi_{i0} + \varphi_{i1} + O(\frac{a}{\log a})]_{\eta} - a\log a[-\frac{t^2}{\pi} - \frac{1}{\log a}H(\eta,t) + O(\frac{1}{\log^2 a})]_{t}\end{aligned} \tag{5.7}$$

$$a\log a\varphi_{i0},_{\xi} + \varphi_{i1},_{\xi} = -aH_{\eta}\varphi_{i0},_{\eta} - \frac{a}{\log a}H_{\eta}\varphi_{i1},_{\eta} + a\log a\frac{2t}{\pi} + aH_{t}. \tag{5.8}$$

When Taylor expansion is applied around $\xi = -\frac{t^2}{\pi}$, one gets,

$$\varphi_{i0},_{\xi} = \frac{2t}{\pi} \qquad (\text{order } a\log a), \qquad (\text{at } \xi = -\frac{t^2}{\pi}) \tag{5.9}$$

$$-\varphi_{0},_{\xi\xi} H + \varphi_{i1},_{\xi} = \varphi_{i0},_{\eta} H_{\eta} + H_{t} \qquad (\text{order a}), \qquad (\text{at } \xi = -\frac{t^2}{\pi}) \tag{5.10}$$

Dynamic boundary condition is

$$\varphi_{i},_{t} - \frac{a}{2\log a}|\nabla\varphi_{i}|^2 + a\eta - 1 = 0 \qquad \text{at } \xi = -\frac{t^2}{\pi} - \frac{1}{\log a}H(\eta,t) \tag{5.11}$$

$$\begin{aligned}(t + a\log a\varphi_{i0} + a\varphi_{i1} + ...)_{t} - \frac{a}{2\log a}\left[\frac{1}{a^2}\left(t + a\log a\varphi_{i0} + a\varphi_{i1} + ...\right)_{\xi}^{2} \right. \\ \left. + \frac{1}{a^2}\left(t + a\log a\varphi_{i0} + a\varphi_{i1} + ...\right)_{\eta}^{2}\right] + a\eta - 1 = 0\end{aligned} \tag{5.12}$$

From Taylor expansion around $\xi = -\frac{t^2}{\pi}$ we get,

$$\varphi_{i0,t} - \frac{1}{2}|\nabla\varphi_{i0}|^2 = 0 \qquad \text{(order } a\log a\text{)}, \qquad (\text{at } \xi = -\frac{t^2}{\pi}) \tag{5.13}$$

$$\varphi_{i0,t\xi} H + \varphi_{i1,t} - \frac{2t}{\pi}\varphi_{i1,\xi} + \eta = 0 \qquad \text{(order a)}, \qquad (\text{at } \xi = -\frac{t^2}{\pi}). \tag{5.14}$$

Equations of order $a\log a$ of (5.6) and (5.10) leads to,

$$\varphi_{i0} = \frac{2t}{\pi}\xi + \frac{4t^3}{3\pi^2} \tag{5.15}$$

at $\xi = -\frac{t^2}{\pi}$. For the function φ_{i1} of order a, on the free surfaces, kinematic boundary condition gives

$$\varphi_{i1,\xi} = H_t, \qquad (\text{at } \xi = -\frac{t^2}{\pi}), \tag{5.16}$$

$$\varphi_{i1,t} - \frac{2}{\pi}H - \frac{2t}{\pi}\varphi_{i1,\xi} + \eta = 0, \qquad (\text{at } \xi = -\frac{t^2}{\pi}). \tag{5.17}$$

At $\eta = 0$ slip boundary condition becomes

$$\varphi_{i1,\eta} = 0. \tag{5.18}$$

The boundary value problem for φ_{i1} is depicted in Figure 5.1.
Furthermore, the functions $\varphi_{i1}(\xi,\eta,t)$ and $H(\eta,t)$ must be matched with the outer solution. At the leading order, matching conditions are, as $\rho \to \infty$,

$$\varphi_{i1}(\xi,\eta,t) = \varphi_{i1}^{\infty}(\xi,\eta,t) + o(\rho) \tag{5.19}$$

$$H(\eta,t) = \frac{t^2}{\pi}\log(\frac{\pi}{4}\eta) + o(1). \tag{5.20}$$

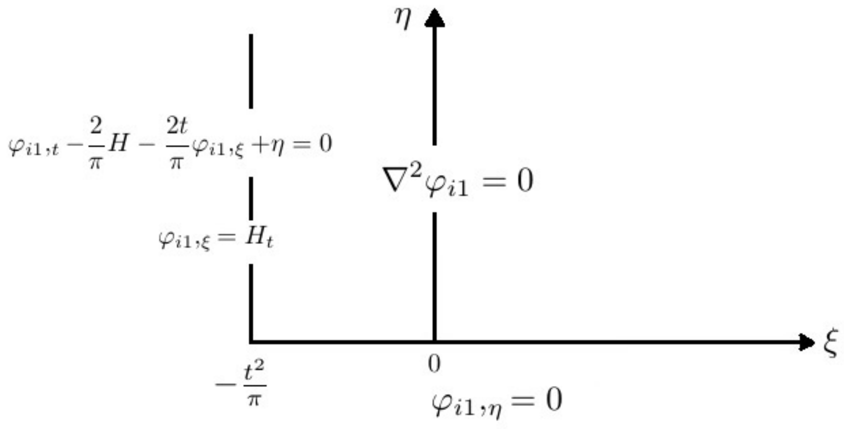


Figure 5.1. Second order inner velocity potential

The function $\varphi_{i1}^{\infty}(\xi, \eta, t)$ is given by (4.38)

$$\varphi_{i1}^{\infty}(\xi, \eta, t) = \frac{2t}{\pi}\left[\xi[\log\rho + \log(\frac{\pi}{4}) - 1] - \eta\theta\right] \tag{5.21}$$

and it is harmonic.

5.1.2. Solution of the Second Order Inner Region Problem

We carry the region to the first quarter plane by using the translation (King and Needham, 1994)

$$\xi = x - \frac{t^2}{\pi} \qquad , \qquad \eta = y. \tag{5.22}$$

Then, the boundary value problem takes the form in new variables with the new unknown functions $\varphi(x, y, t) = \varphi_{i1}(x - \frac{t^2}{\pi}, y, t)$ and $H(\eta, t) = \tilde{H}(y, t)$;

$$\nabla^2\varphi = 0 \qquad (\text{in } \ x > 0, y > 0)$$

$$\varphi_y = 0 \qquad (\text{on } \ y = 0)$$

$$\varphi_x = \tilde{H}_t \qquad (\text{on } \ x = 0) \tag{5.23}$$

$$\varphi_t - \frac{2}{\pi}\tilde{H} + y = 0 \qquad (\text{on } \ x = 0)$$

$$\varphi(0, y, 0) = 0 \qquad \text{and} \qquad \frac{\partial \varphi}{\partial t}(0, y, 0) = -y$$

Matching conditions follow from (5.20) and (5.21)

$$\varphi(x, y, t) \sim \frac{2t}{\pi}\left[x\left[\log[x^2 + y^2]^{\frac{1}{2}} + \log\frac{\pi}{4} - 1\right] - y \arctan\left(\frac{y}{x}\right)\right] + o((x^2 + y^2)^{\frac{1}{2}}) \tag{5.24}$$

$$\tilde{H}(y, t) \sim \frac{t^2}{\pi}\log(\frac{\pi y}{4}) + o(1) \tag{5.25}$$

as $(x^2 + y^2)^{\frac{1}{2}} \to \infty$. In polar coordinates , $x = r\cos\bar{\theta}$ and $y = r\sin\bar{\theta}$,

$$\varphi(r, \bar{\theta}, t) \sim \frac{2t}{\pi}\left[r\cos\bar{\theta}\left[\log r + \log\frac{\pi}{4} - 1\right] - \bar{\theta} r \sin\bar{\theta}\right] + o(r) \qquad , \qquad r \to \infty \tag{5.26}$$

$$\tilde{H}(r, \bar{\theta}, t) \sim \frac{t^2}{\pi}\log(\frac{\pi r}{4}) + o(1) \qquad , \qquad r \to \infty \tag{5.27}$$

To solve this linear problem , we may use integral transform but we see that φ is unbounded as $r \to \infty$ in these matching conditions. Thus, some operations are applied and we obtain

$$\left.\begin{aligned} \varphi = &\ \ \bar{\varphi} + \frac{2t}{\pi}\left[r\cos\bar{\theta}\left[\log r + \log\frac{\pi}{4} - 1\right] - \bar{\theta} r \sin\bar{\theta}\right] \\ \tilde{H} = &\ \ \bar{H} + \frac{t^2}{\pi}\log(\frac{\pi r}{4}) \end{aligned}\right\} \tag{5.28}$$

Therefore, the boundary value problem in the new dependent variables $\bar{\varphi}$ and $\bar{H}$ become,

$$\nabla^2 \bar{\varphi} = 0 \qquad (\text{in } \ 0 < \bar{\theta} < \frac{\pi}{2},\ r > 0)$$

$$\bar{\varphi}_{\bar{\theta}} = 0 \qquad \text{on} \qquad \bar{\theta} = 0$$

$$\frac{1}{r}\bar{\varphi}_{\bar{\theta}} = -\bar{H}_t \qquad \text{on} \qquad \bar{\theta} = \frac{\pi}{2} \tag{5.29}$$

$$\bar{\varphi}_t - \frac{2}{\pi}\bar{H} - \frac{2t^2}{\pi^2}\log\frac{\pi r}{4} = 0 \qquad \text{on} \qquad \bar{\theta} = \frac{\pi}{2}.$$

Initial conditions are

$$\bar{\varphi}(r, \frac{\pi}{2}, 0) = 0 \qquad \text{and} \qquad \bar{\varphi}_t(r, \frac{\pi}{2}, 0) = 0.$$

Functions, $\bar{H}$ and $\bar{\varphi}$, have the properties $\bar{H} = o(1)$, $\bar{\varphi} = o(r)$ as $r \to \infty$. However, $\bar{\varphi} = o(r)$ is not sufficient to apply an integral transform so we use a coordinate expansion by choosing $\bar{\varphi} = A(t)\log r + B(t) + C(t)\frac{\sin(\bar{\theta})}{r} + D(t)\frac{\cos(\bar{\theta})}{r} + O(\frac{1}{r^2})$, $r >> 1$ and substituting these expansion into the boundary value problem, we obtain as $r \to \infty$,

$$\bar{\varphi} = \frac{2t^3}{3\pi^2}\log r + \frac{2t^3}{3\pi^2}\log\frac{\pi}{4} + O(\frac{1}{r}) \qquad \text{and} \qquad \bar{H} = O(\frac{1}{r^2}).$$

We define the potential again because we have to get free surface and potential vanishing as $r \to \infty$

$$\bar{\varphi} = \varphi^\star + \frac{t^3}{3\pi^2}\log(1 + r^2) + \frac{2t^3}{3\pi^2}\log\frac{\pi}{4} \qquad , \qquad \bar{H} = H^\star. \tag{5.30}$$

We choose $\log(1 + r^2)$ instead of $\log r$ to avoid a singularity for $r \to 0$. With $\varphi^\star = O(\frac{1}{r})$ as $r \to \infty$, we get the following boundary value problem :

$$\nabla^2\varphi^\star \;\; = -\tfrac{t^3}{3\pi^2}\nabla^2\log(1 + r^2) \tag{5.31}$$

$$\varphi^\star_{\bar{\theta}} \;\; = 0 \qquad (\text{on} \qquad \bar{\theta} = 0) \tag{5.32}$$

$$\frac{1}{r}\varphi^\star_{\bar{\theta}} \;\; = -H^\star_t \qquad (\text{on} \qquad \bar{\theta} = \tfrac{\pi}{2}) \tag{5.33}$$

$$\varphi^\star_t \;\; = \tfrac{2}{\pi}H^\star + \tfrac{2t^2}{\pi^2}\log r - \tfrac{t^2}{\pi^2}\log(1 + r^2) \qquad (\text{on} \qquad \bar{\theta} = \tfrac{\pi}{2}) \tag{5.34}$$

$$\varphi^\star(r,\frac{\pi}{2},0)=0 \qquad , \qquad \varphi^\star_t(r,\frac{\pi}{2},0)=0. \tag{5.35}$$

With the help of the conditions, as $r\to 0$

$$\varphi^\star = O(1) \qquad \text{and} \qquad H^\star = O(\log r)$$

and as $r\to\infty$

$$\varphi^\star = O(\frac{1}{r}) \qquad \text{and} \qquad H^\star = O(\frac{1}{r^2})$$

we can use the "Mellin transform" (See Appendix E) and find a solution for the problem. Hence, the Mellin transform of the unknown functions is that

$$\mathcal{M}[\varphi^\star(r,\bar\theta,t)] = \hat\varphi(p,\bar\theta,t) = \int_0^\infty r^{p-1}\varphi^\star(r,\bar\theta,t)dr, \tag{5.36}$$

$$\mathcal{M}[H^\star(r,t)] = \hat H(p,t) = \int_0^\infty r^{p-1}H^\star(r,t)dr. \tag{5.37}$$

From the behaviour of the $\varphi^\star$ and $H^\star$ at 0 and ∞, we determine fundamental strip of the functions where they exist and analytic. Then, for velocity potential fundamental strip is the interval $0 < Re(p) < 1$. For the free surface, it is defined as $0 < Re(p) < 2$ in complex p-plane.

Mellin transform is applied for the first equation , $\nabla^2\varphi^\star = -\frac{t^3}{3\pi^2}\nabla^2\log(1+r^2)$, thus an ordinary differential equation is obtained as follows (Gradshteyn and Ryzhik, 2000),

$$\left[\frac{\partial^2}{\partial\bar\theta^2}+(p-2)^2\right]\hat\varphi(p-2,\bar\theta,t) = \frac{t^3}{3\pi}(p-2)\frac{1}{\sin(\frac{\pi p}{2})}$$

or equivalently it can be written as $p\to p+2$,

$$\left[\frac{\partial^2}{\partial\bar\theta^2}+p^2\right]\hat\varphi(p,\bar\theta,t) = -\frac{t^3}{3\pi}p\frac{1}{\sin(\frac{\pi p}{2})}, \qquad (\text{in } 0<\bar\theta<\frac{\pi}{2}). \tag{5.38}$$

This equation is solved by "variation of parameters" and general solution is found in the form:

$$\hat{\varphi}(p,\bar{\theta},t) = c_1(p,t)\cos(p\bar{\theta}) + c_2(p,t)\sin(p\bar{\theta}) - \frac{t^3}{3\pi p\sin(\frac{\pi p}{2})} \tag{5.39}$$

where $c_1(p,t)$ and $c_2(p,t)$ are to be determined from the boundary conditions.
Similarly, Mellin transform of the boundary conditions ;

$$\hat{\varphi}_{\bar{\theta}}(p,0,t) = 0 \tag{5.40}$$

$$\hat{\varphi}_{\bar{\theta}}(p-1,\frac{\pi}{2},t) = -\hat{H}_t(p,t) \tag{5.41}$$

$$\hat{\varphi}_t(p,\frac{\pi}{2},t) = \frac{2}{\pi}\hat{H}(p,t) - \frac{t^2}{\pi p\sin(\frac{\pi p}{2})}. \tag{5.42}$$

Initial conditions are

$$\hat{\varphi}(p,\frac{\pi}{2},0) = 0 \qquad , \qquad \hat{\varphi}_t(p,\frac{\pi}{2},0) = 0 \tag{5.43}$$

From (5.40), on $\bar{\theta} = 0$ we find $c_2(p,t) = 0$ and hence

$$\hat{\varphi}(p,\bar{\theta},t) = c_1(p,t)\cos(p\bar{\theta}) - \frac{t^3}{3\pi p\sin(\frac{\pi p}{2})}. \tag{5.44}$$

If we combine kinematic and dynamic boundary condition, we get on $\bar{\theta} = \frac{\pi}{2}$

$$\hat{\varphi}_{tt}(p) + \frac{2}{\pi}\hat{\varphi}_{\bar{\theta}}(p-1) + \frac{2t}{\pi p\sin(\frac{\pi p}{2})} = 0. \tag{5.45}$$

Finally, by substituting (5.44) into this condition ,one gets

$$c_{1,tt}(p,t) + \frac{2}{\pi}(p-1)c_1(p-1,t) = 0 \tag{5.46}$$

with conditions

$$c_1(p,0) = 0 \qquad \text{and} \qquad c_{1,t}(p,0) = 0. \tag{5.47}$$

In order to solve this difference-differential equation, Laplace transform is used :

$$\mathcal{L}[c_1(p,t)] = C_1(p,\omega) = \int_0^\infty e^{-\omega t} c_1(p,t)dt$$

For the equation (5.46) if Laplace transform is applied, we get the following difference equation,

$$\omega^2 C_1(p,\omega) + \frac{2}{\pi}(p-1)C_1(p-1,\omega) = 0 \tag{5.48}$$

which can be written equivalently as $p \to p+1$,

$$\omega^2 C_1(p+1,\omega) + \frac{2}{\pi}(p)C_1(p,\omega) = 0 \tag{5.49}$$

The solution of this difference equation is obtained as (See)

$$C_1(p,\omega) = a(p)(-1)^p \left[\frac{2}{\pi\omega^2}\right]^p \Gamma(p) \tag{5.50}$$

where $a(p)$ is a solution of $\frac{a(p)}{a(p-1)} = 1$.
Then, by using inverse Laplace transform

$$c_1(p,t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\omega t} C_1(p,\omega)d\omega$$

we find the function $c_1(p,t)$ in the form by choosing $0 < Re(p) < \frac{1}{2}$,

$$c_1(p,t) = a(p)(-1)^p \sin(2\pi p)\left[\frac{2^p}{\pi^{p+1}}\right]\Gamma(1-2p)\Gamma(p)t^{2p-1} \tag{5.51}$$

CHAPTER 6

CONCLUSION

In this thesis, we studied two dimensional dam-break problem. We considered asymptotic solution and solution was obtained in the leading order. We followed the same methodology (Korobkin and Yilmaz, 2009) for outer solution and (King and Needham, 1994) for inner solution.

Firstly, the leading order dam-break problem is constructed and to solve the dam-break problem, some methods from complex analysis are used. A singularity is found at the intersection point and it leads to an inner solution. Near the intersection point, the behaviour of the outer solution is analysed as to constitute an inner region and an inner solution.

After that, we investigate the dimensions of the inner region appropriately. The leading order inner velocity potential is obtained by using the corresponding asymptotic matching conditions of the outer solution. To solve the second order inner region problem, Mellin integral transform and Laplace transform methods are used. However, after taking the inverse Laplace transform, solution that we have found does not satisfy the initial conditions. Therefore, we are not able to continue to find an exact solution for the inner region problem. Consequently, we only solve the dam-break problem properly for the outer region near the intersection point.

REFERENCES

Acheson, D. J. (1990). *Elementary fluid dynamics*. Clarendon press Oxford.

Ahlfors, L. V. (1979). *Complex analysis: an introduction to the theory of analytic functions of one complex variable* (3 ed.). McGraw-Hil Inc.

Brown, J. W. and R. V. Churchill (2009). *Complex variables and applications* (8 ed.). McGraw-Hill Higher Education.

Chorin, A. J. and J. E. Marsden (2000). *A mathematical introduction to fluid mechanics*.

Debnath, L. and D. Bhatta (2010). *Integral transforms and their applications*. CRC press.

FV, P. (1950). The lagrangian equations of hydrodynamics : solutions which are analytic functions of time. phd dissertation.

Gakhov, F. D. (1990). *Boundary value problems*. Courier Dover Publications.

Gradshteyn, I. S. and I. M. Ryzhik (2000). *Table of integrals, series, and products*. Academic Press.

Kelley, W. G. and A. C. Peterson (2001). *Difference equations: an introduction with applications*. Academic press.

King, A. and D. Needham (1994). The initial development of a jet caused by fluid, body and free-surface interaction. part 1. a uniformly accelerating plate. *Journal of Fluid Mechanics 268*, 89–101.

Korobkin, A. and O. Yilmaz (2009). The initial stage of dam-break flow. *Journal of Engineering Mathematics 63*(2-4), 293–308.

Stoker, J. (1992). *Water Waves: The Mathematical Theory with Applications*, Volume 4. John Wiley & Sons.

APPENDIX A

TRANSFORMATIONS OF HARMONIC FUNCTIONS AND BOUNDARY CONDITIONS

Theorem 1 *Suppose that a domain D_z in the z-plane is mapped onto a domain D_w in the w-plane with the help of an analytic function $w = f(z) = u(x, y) + iv(x, y)$. On D_w, if h(u,v) is a harmonic function , then (Brown and Churchill, 2009)*

$$H(x, y) = h[u(x, y), v(x, y)] \tag{A.1}$$

is harmonic in D_w

Proof

By using chain rule we can check whether H(x,y) is harmonic or not.From (A.1)

$$H_x = h_u u_x + h_v v_x$$

$$H_{xx} = h_{uu}u_x^2 + h_{uv}v_x u_x + h_u u_{xx} + h_{vu}u_x v_x + h_{vv}v_x^2 + h_v v_{xx} \tag{A.2}$$

$$H_{yy} = h_{uu}u_y^2 + h_{uv}v_y u_y + h_u u_{yy} + h_{vu}u_y v_y + h_{vv}v_y^2 + h_v v_{yy} \tag{A.3}$$

Taking into consideration f is analytic, Cauchy-Riemann equations $u_x = v_y, u_y = -v_x$ hold and functions u,v satisfies Laplace equations so we get the equation

$$H_{xx} + H_{yy} = [h_{uu} + h_{vv}]|f'(z)|^2 \tag{A.4}$$

Then, we conclude that H(x,y) is harmonic in D_z when h(u,v) is harmonic in D_w.

Theorem 2 *A smooth arc C is transformed onto arc γ by a transformation $w = f(z) = u(x, y) + iv(x, y)$ which is conformal. If a function $h(u, v)$ satisfies one of the following* *conditions (Brown and Churchill, 2009),*

$$h = h_0 \qquad or \qquad \frac{dh}{dn} = 0 \tag{A.5}$$

along γ,where h_0 is real constant, $\frac{dh}{dn}$ derivatives normal to γ, then $H(x, y) = h[u(x, y), v(x, y)]$ satisfies the corresponding condition

$$H = h_0 \qquad or \qquad \frac{dH}{dN} = 0 \tag{A.6}$$

along C, where $\frac{dH}{dN}$ denotes derivatives normal to C.

Proof

Firstly, since $H(x, y) = h[u(x, y), v(x, y)]$,the value of H at any point (x,y) on C equals the value of h at (u,v), image of (x,y), under transformation $f(z)$. The image point (u,v) lies on γ and $h = h_0$ along that curve, thus $H = h_0$ along C.
Assume that $\frac{dh}{dn} = 0$ on γ and we know directional derivative is described by

$$\frac{dh}{dn} = (gradh) \cdot \mathbf{n} \tag{A.7}$$

where $gradh$ denotes the gradient of h at point (u,v) on γ and **n** is a unit normal vector to γ at again (u,v). We see from these informations $gradh$ is orthogonal to **n** at (u,v).Therefore, $gradh$ is tangent to γ. At the same time , since gradient are orthogonal to the level curves, γ is orthogonal to a level curve $h(u, v) = c$ passing through (u,v). In addition, we can write $H(x, y) = c$ under transformation $f(z)$. Since C is transformed to γ and it is orthogonal to the level curve $h(u, v) = c$, from the conformality of transformation, C is orthogonal to the level curve $H(x, y) = c$ at (x,y) corresponding to (u,v). It follows that $gradH$ is tangent to C at (x,y) and if **N** is a unit normal vector to C at point (x,y), then $gradH$ is orthogonal to **N**.

$$(gradH) \cdot \mathbf{N} = \frac{dH}{dN} = 0 \tag{A.8}$$

While we are doing these calculations ,we assume that $gradh \neq 0$, and $gradh$, $gradH$ alwas exists. The level curve $H(x, y) = c$ is smooth when $gradh \neq 0$ at (u,v).

APPENDIX B

REFLECTION PRINCIPLE

Suppose $f(z) = u(x, y) + iv(x, y)$ is analytic function in a region D which is symmetric with respect to a real axis. D^+ is the part of D in the upper half plane and σ is the part of real axis. Then (Ahlfors, 1979)

$$\overline{f(\bar{z})} = f(z) \tag{B.1}$$

for each point z in the domain if and only if $f(x)$ is real for each point x on σ.

APPENDIX C

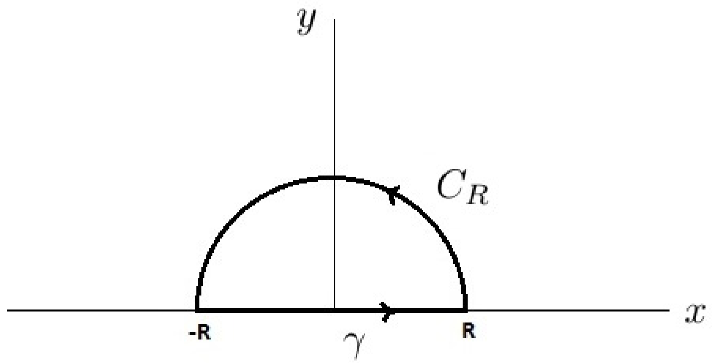
HILBERT INVERSION FORMULA

Suppose that $f(z) = u(x,y) + iv(x,y)$ is an analytic function int he upper half plane. If $u(x)$ and $v(x)$ denote the limiting values of its real and imaginary parts on the real axis, then Hilbert inversion formulae are defined as (Gakhov, 1990)

$$v(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(\xi)}{\xi - x}d\xi + v_{\infty} \tag{C.1}$$

$$u(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{v(\xi)}{\xi - x}d\xi + u_{\infty}, \tag{C.2}$$

where u_{∞} and v_{∞} are the values of the real and imaginary parts at infinity.
Proof Let f be an analytic function of z throughout the upper half plane(Im(z)$\geq$ 0).For a fixed point z above the real axis, C_R denote the upper half of a positively oriented circle with radius R and oriented at the origin, where $R > |z|$.



Then, from Cauchy integral formula which we know from complex analysis,

$$f(z) = \frac{1}{2\pi i}\int_{C_R\cup\gamma}\frac{f(\tau)}{\tau - z}d\tau = \frac{1}{2\pi i}\int_{C_R}\frac{f(\tau)}{\tau - z}d\tau + \frac{1}{2\pi i}\int_{-R}^{R}\frac{f(\xi)}{\xi - z}d\xi \tag{C.3}$$

$\tau = \xi + i\psi$ is a point on the upper part of the semi-circle. Then, we can write for the upper half plane, as R tends to ∞

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(\xi)}{\xi - z}d\xi + \frac{1}{2\pi i}\lim_{R\to\infty}\int_{C_R}\frac{f(\tau)}{\tau - z}d\tau \tag{C.4}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(\xi)}{\xi - z}d\xi + \frac{1}{2\pi i}f(\infty)\lim_{R\to\infty}\int_{C_R}\frac{1}{\tau - z}d\tau$$

To take the integral over the line C_R, we use polar coordinates $\tau = R\exp(i\theta)$ and obtain

$$f(z) = \frac{1}{2}f(\infty) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(\xi)}{\xi - z}d\xi \tag{C.5}$$

For the integral

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(\xi)}{\xi - z}d\xi,$$

if we use Sokhotski formula,as $z \to x$;

$$f^{+}(x) = \frac{1}{2}f(\infty) + \frac{1}{2}f(x) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(\xi)}{\xi - x}d\xi \tag{C.6}$$

$$u(x) + iv(x) = u(\infty) + iv(\infty) - \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{u(\xi) + iv(\xi)}{\xi - x}d\xi$$

the limiting values are derived as the following form :

$$\left.\begin{aligned} u(x) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{v(\xi)}{\xi - x}d\xi + u_{\infty} \\ v(x) &= -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(\xi)}{\xi - x}d\xi + v_{\infty} \end{aligned}\right\} \tag{C.7}$$

APPENDIX D

POISSON INTEGRAL FORMULA FOR UPPER HALF PLANE

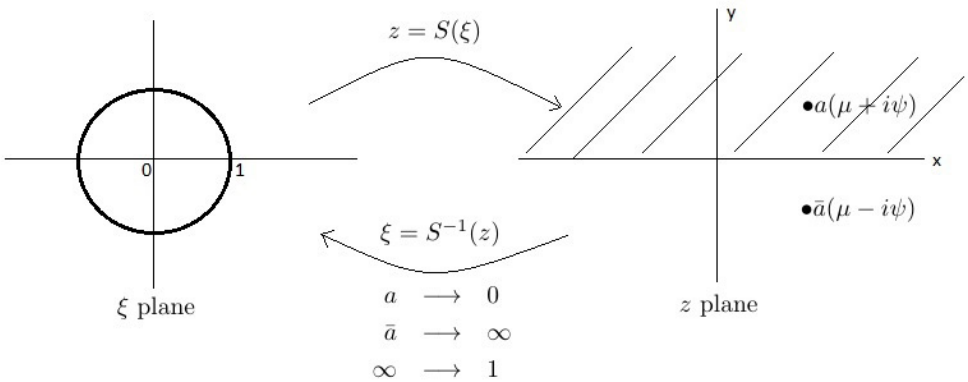
Suppose that real valued function $\phi(\xi)$ is piecewise continuous and bounded for all real ξ. Then the function (Ahlfors, 1979)

$$\Phi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2+y^2}\phi(\xi)d\xi$$

is harmonic in the upper half plane with boundary values $\phi(\xi)$ at points of continuity.

Proof

$\Phi(z)$ is harmonic in the upper half plane. The linear transformation



$$z = S(\xi) = \frac{\bar{a}\xi - a}{\xi - 1} \tag{D.1}$$

maps $|\xi| \leq 1$ onto the upper half plane where $a = \mu + i\psi$ is any point in the upper half plane and $\bar{a} = \mu - i\psi$ is symmetric point of a with respect to real axis. The function $\Phi(S(\xi))$ is harmonic in $|\xi| \leq 1$ and from the arithmetic mean of harmonic functions, we have

$$\Phi(a) = \frac{1}{2\pi}\int_{|\xi|=1} \Phi(S(\xi))darg\xi \tag{D.2}$$

From

$$\xi = \frac{z-a}{z-\bar{a}} \tag{D.3}$$

we can compute

$$darg\xi = -i\frac{d\xi}{\xi} = -i\left[\frac{z-\bar{a}}{z-a}\right]\left(\frac{a-\bar{a}}{(z-\bar{a})^2}\right)dz \tag{D.4}$$

$$= \frac{2\psi}{(z-a)(z-\bar{a})}dz \tag{D.5}$$

and substituting this equation into (D.2), we get

$$\Phi(a) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\psi}{(x-\eta)^2+\psi^2}\phi(x)dx. \tag{D.6}$$

We prove this formula for arbitrary point a in the upper plane so if we write for all point in upper half plane, general case, we obtain the Poisson integral formula for the upper half plane in the following form ;

$$\Phi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-\xi)^2+y^2}\phi(\xi)d\xi. \tag{D.7}$$

APPENDIX E

MELLIN TRANSFORM

E.1. Derivation of Mellin Transform

A natural way of defining this transformation can be provided by using the complex Fourier transform (Debnath and Bhatta, 2010). Let $f : \mathbb{R} \to \mathbb{C}$ be an integrable function and its Fourier transform :

$$\mathcal{F}[f(x)] = F(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixu} f(x) dx \tag{E.1}$$

If we change the variables, $r = e^x$ and $iu = \alpha - p$ where α is constant and p is complex, then we obtain

$$F(ip - i\alpha) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} r^{p-\alpha-1} f(\log r) dr \tag{E.2}$$

By choosing $\frac{1}{\sqrt{2\pi}} r^{-\alpha} f(\log r) \equiv \tilde{f}(r)$ and $F(ip - i\alpha) \equiv \tilde{F}(p)$, we can define the Mellin transform of f(p) as

$$\mathcal{M}[\tilde{f}(r)] = \tilde{F}(p) = \int_{0}^{\infty} r^{p-1} \tilde{f}(r) dr. \tag{E.3}$$

In a similar way, we can introduce inverse Mellin transform by using inverse Fourier transform,

$$\mathcal{F}^{-1}[F(u)] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixu} F(u) du \tag{E.4}$$

With the same variables, $r = e^x$ and $iu = \alpha - p$, we get

$$f(\log r) = \frac{1}{\sqrt{2\pi}} \int_{c-i\infty}^{c+i\infty} r^{\alpha-p} F(ip - i\alpha) dp \tag{E.5}$$

thus, we find inverse Mellin transform in the following form,

$$\mathcal{M}^{-1}[\tilde{F}(p)] = \tilde{f}(r) = \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} r^{-p}\tilde{F}(p)dp. \tag{E.6}$$

E.2. Application of the Mellin Transform for the Equation : $\nabla^2\varphi^\star = -\frac{t^3}{3\pi^2}\nabla^2\log(1+r^2)$

For the left hand side, Laplace equation in polar form is defined by

$$\nabla^2\varphi^\star(r,\bar{\theta},t) = \frac{\partial^2\varphi^\star}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi^\star}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi^\star}{\partial\bar{\theta}^2} \tag{E.7}$$

and apply the Mellin transform, we get

$$\mathcal{M}^{-1}[\nabla^2\varphi^\star] = \underbrace{\int_0^\infty r^{p-1}\frac{\partial^2\varphi^\star}{\partial r^2}dr}_{I_1} + \underbrace{\int_0^\infty r^{p-2}\frac{\partial\varphi^\star}{\partial r}dr}_{I_2} + \underbrace{\int_0^\infty r^{p-3}\frac{\partial^2\varphi^\star}{\partial\bar{\theta}^2}dr}_{I_3}. \tag{E.8}$$

From the method of integration by part;

$$\begin{aligned} I_1 &= \int_0^\infty r^{p-1}\frac{\partial^2\varphi^\star}{\partial r^2}dr \\ &= \lim_{x\to\infty}\left(r^{p-1}\varphi_r^\star\right)\Big|_0^x - (p-1)\int_0^\infty \varphi_r^\star r^{p-2}dr \\ &= (p-1)(p-2)\int_0^\infty \varphi^\star r^{p-3}dr \\ &= (p-1)(p-2)\hat{\varphi}(p-2,\bar{\theta}) \end{aligned} \tag{E.9}$$

Similar way,

$$
\begin{aligned}
I_2 &= \int_0^\infty r^{p-2}\frac{\partial \varphi^\star}{\partial r}dr \\
&= \lim_{x\to\infty}\left(r^{p-2}\varphi_r^\star\right)\Big|_0^x - (p-2)\int_0^\infty \varphi_r^\star r^{p-3}dr \\
&= -(p-2)\hat{\varphi}(p-2,\bar{\theta}) \qquad \text{(E.10)}
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \int_0^\infty r^{p-3}\frac{\partial^2 \varphi^\star}{\partial \bar{\theta}^2}dr \\
&= \frac{\partial^2}{\partial \bar{\theta}^2}\int_0^\infty r^{p-3}\varphi^\star dr \\
&= \frac{\partial^2}{\partial \bar{\theta}^2}\hat{\varphi}(p-2,\bar{\theta}) \qquad \text{(E.11)}
\end{aligned}
$$

For the right hand side of the our main equation, $\frac{-t^3}{3\pi^2}\nabla^2\log(1+r^2) = \frac{-t^3}{3\pi^2}\frac{4}{(1+r^2)^2}$, if we take the transform, we have

$$
\mathcal{M}\left[\frac{-t^3}{3\pi^2}\frac{4}{(1+r^2)^2}\right] = \frac{t^3}{3\pi}(p-2)\frac{1}{\sin(\frac{\pi p}{2})} \qquad \text{(E.12)}
$$

for $0 < Re(p) < 4$.

Finally ,we conclude that the Mellin transform of the equation $\nabla^2\varphi^\star = -\frac{t^3}{3\pi^2}\nabla^2\log(1+r^2)$ find as

$$
\left[\frac{\partial^2}{\partial\theta^2} + (p-2)^2\right]\hat{\varphi}(p-2,\bar{\theta}) = \frac{t^3}{3\pi}\frac{(p-2)}{\sin(\frac{\pi p}{2})}. \qquad \text{(E.13)}
$$

APPENDIX F

DIFFERENCE EQUATIONS

The first order linear difference equation is defined as (Kelley and Peterson, 2001)

$$y(t+1) - p(t)y(t) = r(t) \tag{F.1}$$

where $p(t)$ and $r(t)$ are given functions with $p(t) \neq 0$ for all t.
Solution of the difference equations which are defined as the following type

$$y(t+1) = a\frac{(t-r_1)\dots(t-r_n)}{(t-s_1)\dots(t-s_m)}y(t) \tag{F.2}$$

where $a, r_1, ..., r_n, s_1, ...s_m$ are constants , can be found as follows:

$$y(t) = e^{\sum \log a + \sum[\log(t-r_1)+...+\log(t-r_n)-\log(t-s_1)-...-\log(t-s_m)]}C(t)$$

From the identity $\sum \log(t) = \log\Gamma(t) + C(t)$, we obtain the solution of the equation (F.2),

$$y(t) = C(t)a^t\frac{\Gamma(t-r_1)\dots\Gamma(t-r_n)}{\Gamma(t-s_1)\dots\Gamma(t-s_m)} \tag{F.3}$$

where $\Delta C(t)$ = 0.