Derivation of Input/Output Relationships for the Bennett 6R Linkages Based on the Method of Decomposition

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Abstract. The Bennett overconstrained 6R linkages are the double-planar, the double-spherical and the plano-spherical 6R linkages. These mechanisms are obtained by combining simple planar and/or spherical mechanisms and then removing one of the common links. This paper presents the derivation of the input/output relationships for these mechanisms using the decomposition method. This method is based on writing the input/output equations for the two imaginary loops comprising the 6R mechanism and then eliminating the imaginary joint variable. It is found that the resulting input/output equations contain up to 4th power of trigonometric terms, such as $\cos^4\theta$.

Key words: Bennett 6R linkages, passive joints, input/output relationship, method of decomposition

1 Introduction

The first reported overconstrained mechanism is due to Sarrus in 1853 [1]. The Sarrus linkage is a spatial 6R linkage obtained by assembling two planar dyads in perpendicular planes and it can be interpreted as two slider crank mechanisms with a common slider, axis of which is along the intersection of the perpendicular planes. The angle between the intersecting planes is arbitrary and the two triplets of parallel joint axes may be positioned arbitrarily and the linkage will be still mobile. We will call such a generalized Sarrus linkage as the double-planar 6R linkage (Fig. 1a). As far as the authors know, such a form of the general Sarrus linkage as shown in Fig. 1a is presented nowhere else. The relative motion between the links with nonparallel joint axes is linear translation and hence a prismatic joint can be inserted between the two. Such a joint is called a passive joint [2].

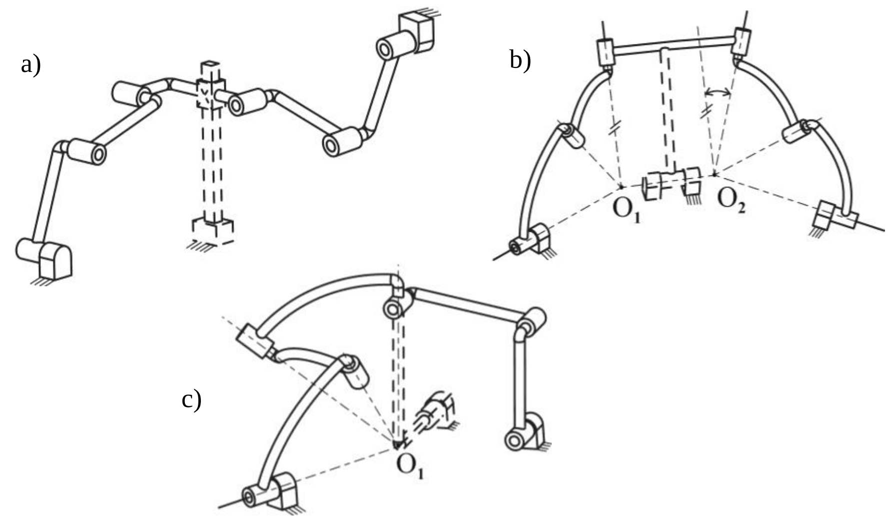

Fig. 1 a) Double-planar (generalized Sarrus) linkage, (b) double-spherical linkage, and (c) plano-spherical linkage together with their passive joints (adapted from [4])

In 1905 Bennett worked on the Sarrus linkage and proposed two new families of overconstrained linkages known as the double-spherical and the plano-spherical 6R linkages [3]. The double-spherical linkage is obtained by merging two spherical four-bar mechanisms with two common links (Fig. 1b) and the plano-spherical linkage is obtained by merging a planar four-bar mechanism and a spherical four-bar mechanism with two common links (Fig. 1c). Both of these linkages have a passive revolute joint. For the double-spherical linkage, the passive joint axis is through the line connecting the sphere centers O_1 and O_2. For the plano-spherical linkage the passive joint axis is normal to the plane and passes through the sphere intersecting the plane. The plano-spherical linkage and the double-planar linkage can be obtained from the double-spherical linkage by sending one or both of the sphere centers to infinity [4]. The Bennett 6R linkages are examples of linkages obtained from intersections of Euclidean subspaces [2].

A special case of the double-spherical 6R linkage is the well known double-Hooke's-joint linkage for which the twist angles are 90°, 90°, 0°, 90°, 90° and arbitrary angle between the fixed joints. Baker [5] derived the input/output (I/O) relationship for the double-spherical 6R linkage starting from the double-Hooke's-joint linkage. In this paper we present an alternative formulation based on the method of decomposition [6].

The method of decomposition originates from a simple idea: since the above-mentioned mechanisms are obtained as merging two simple loop mechanisms and then removing the passive joint, the original single loop may be decomposed into two imaginary loops. By taking the input and output joints as the fixed joints, the I/O equations for each imaginary loop are obtained. The passive joint is output for the first loop and input for the second loop. Eliminating the passive joint variable from the two I/O equations, the I/O equation for the 6R mechanism is obtained. Also, these linkages prove useful in function synthesis when the method of decomposition is applied. The synthesis methods are left for future studies.

2 The double-planar 6R linkage

Together with the passive prismatic joint, the double-planar 6R linkage may be considered to be composed of a pair of slider-crank mechanisms. The sliding direction in both slider-cranks is common and intuitively it can be verified that the angle between the planes of motion of the two planar mechanisms does not affect the I/O relationship. The planes of motion have to be nonparallel, but, there is no harm in considering the two slider-crank mechanisms in the same plane as long as the prismatic joint is included. Let ϕ be the input ange, θ be the output angle and s be the passive joint variable of the double-planar 6R mechanism shown in Fig. 2.

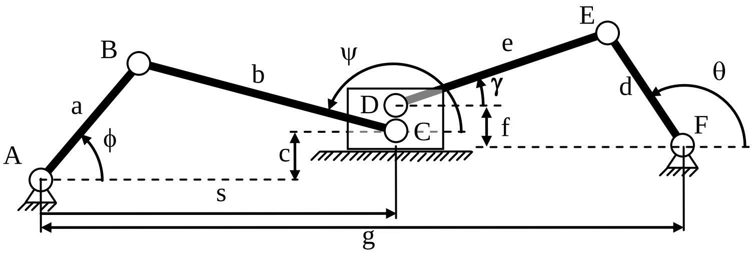

Fig. 2 Double-planar 6R linkage

Loop closure equation for loop ABC:

$$\overrightarrow{AB}=\overrightarrow{AC}+\overrightarrow{CB} \;\Rightarrow\; \begin{array}{c} a\cos\phi=s+b\cos\psi \\ a\sin\phi=c+b\sin\psi \end{array} \tag{1}$$

Eliminating ψ from Eq. (1):

$$b^2=(a\cos\phi-s)^2+(a\sin\phi-c)^2 \Rightarrow s^2=2a\cos\phi s-a^2+b^2-c^2+2ac\sin\phi \tag{2}$$

Loop closure equation for loop DEF:

$$\overrightarrow{DE}=\overrightarrow{DF}+\overrightarrow{FE} \;\Rightarrow\; \begin{array}{c} e\cos\gamma=g-s+d\cos\theta \\ e\sin\gamma=-f+d\sin\theta \end{array} \tag{3}$$

Eliminating γ from Eq. (3):

$$\begin{aligned} e^2 &= (g - s + d\cos\theta)^2 + (-f + d\sin\theta)^2 \\ \Rightarrow s^2 &= 2(g + d\cos\theta)s - d^2 + e^2 - f^2 - g^2 - 2dg\cos\theta + 2df\sin\theta \end{aligned} \tag{4}$$

Equating the right hand sides of Eqs. (2, 4):

$$s = \frac{a^2 - b^2 + c^2 - d^2 + e^2 - f^2 - g^2 - 2ac\sin\phi - 2dg\cos\theta + 2df\sin\theta}{2(a\cos\phi - d\cos\theta - g)} \tag{5}$$

Substituting s from Eq. (5) into Eq. (2):

$$\begin{aligned} &(A - 2ac\sin\phi - 2dg\cos\theta + 2df\sin\theta)^2 + 4(B - 2ac\sin\phi)(a\cos\phi - d\cos\theta - g)^2 \\ &-4a\cos\phi(A - 2ac\sin\phi - 2dg\cos\theta + 2df\sin\theta)(a\cos\phi - d\cos\theta - g) = 0 \end{aligned} \tag{6}$$

where $A = a^2 - b^2 + c^2 - d^2 + e^2 - f^2 - g^2$ and $B = a^2 - b^2 + c^2$. Eq. (6) is the implicit I/O relation of the double-planar 6R linkage, which contains up to 3rd power of trigonometric terms, such as $\cos^3\theta$.

3 The double-spherical 6R linkage

Let ϕ, ψ and θ be the respective input, passive joint and output angle of the double-spherical 6R mechanism shown in Fig. 3. O_1D and O_2E are skew with a twist angle of γ. The radii of the spheres do not affect the I/O relationship, so without loss of generality assume both radii as 1. Also notice that the distance $|O_1O_2|$ has no effect on the I/O relationships.

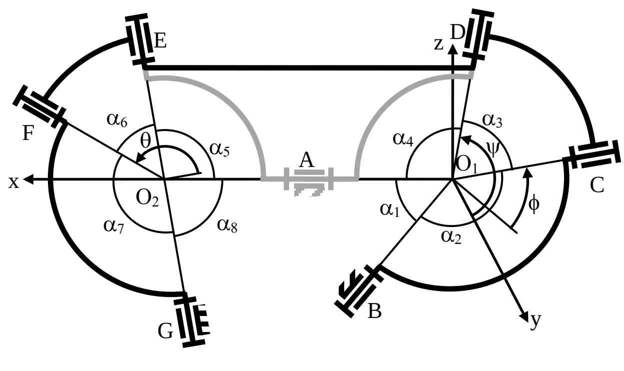

Fig. 3 Double-spherical 6R linkage

Coordinates of B, C, D can be found as follows:

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = Z[\alpha_1] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ s_1 \\ 0 \end{bmatrix} \quad , \quad \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = X[\psi] Z[\alpha_4] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_4 \\ s_4 c\psi \\ s_4 s\psi \end{bmatrix} \quad ,$$
$$\begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} = Z[\alpha_1] X[\phi] Z[\alpha_2] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 c_2 - s_1 s_2 c\phi \\ s_1 c_2 + c_1 s_2 c\phi \\ s_2 s\phi \end{bmatrix} \tag{7}$$

where X[.] and Z[.] represent rotation matrices about x and z axes, respectively and $c_i = \cos\alpha_i$, $s_i = \sin\alpha_i$ for i = 1, 2, 3, 4 and other c and s are short for cosine and sine, respectively. The angle between O_1C and O_1D is α_3, so

$$\overrightarrow{O_1C} \cdot \overrightarrow{O_1D} = c_3 \Rightarrow c_1 c_2 c_4 - c_3 - s_1 s_2 c_4 c\phi + s_1 c_2 s_4 c\psi + c_1 s_2 s_4 c\phi c\psi + s_2 s_4 s\phi s\psi = 0 \tag{8}$$

Eq. (8) gives the I/O equation for the first loop. For the second loop the coordinates of E, F and G with respect to O_2 can be calculated as follows:

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = X[\psi+\gamma] Z[\pi-\alpha_5] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_5 \\ s_5 c\gamma c\psi - s_5 s\gamma s\psi \\ s_5 c\gamma s\psi + s_5 s\gamma c\psi \end{bmatrix} \quad ,$$
$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = Z[\pi-\alpha_8] X[\theta] Z[\alpha_7] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_7 c_8 - s_7 s_8 c\theta \\ c_7 s_8 - s_7 c_8 c\theta \\ s_7 s\theta \end{bmatrix} \quad , \quad \begin{bmatrix} G_x \\ G_y \\ G_z \end{bmatrix} = Z[\pi-\alpha_8] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_8 \\ s_8 \\ 0 \end{bmatrix} \tag{9}$$

The angle between O_2E and O_2F is α_6, so

$$\begin{aligned} \overrightarrow{O_2E} \cdot \overrightarrow{O_2F} = c_6 \Rightarrow\; & c_5 c_7 c_8 - c_6 + c_5 s_7 s_8 c\theta + s_5 c_7 s_8 c\gamma c\psi - s_5 c_7 s_8 s\gamma s\psi \\ & - s_5 s_7 c_8 c\gamma c\theta c\psi + s_5 s_7 c_8 s\gamma c\theta s\psi + s_5 s_7 c\gamma s\theta s\psi + s_5 s_7 s\gamma s\theta c\psi = 0 \end{aligned} \tag{10}$$

ψ is to be eliminated from Eq. (8) and Eq. (10). Notice that Eq. (8) and Eq. (10) are linear in terms of $c\psi$ and $s\psi$. Writing these equations in matrix form:

$$\begin{bmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{bmatrix} \begin{bmatrix} c\psi \\ s\psi \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \tag{11}$$

where the coefficients are functions of link parameters α_i, γ, input angle ϕ and output angle θ only: $P_1 = s_1 c_2 + c_1 s_2 c\phi$, $Q_1 = s_2 s\phi$, $R_1 = (-c_1 c_2 c_4 + c_3 + s_1 s_2 c_4 c\phi)/s_4$,

$P_2 = c_7 s_8 c\gamma - s_7 c_8 c\gamma c\theta + s_7 s\gamma s\theta$, $Q_1 = -c_7 s_8 s\gamma + s_7 c_8 s\gamma c\theta + s_7 c\gamma s\theta$ and $R_2 = (-c_5 c_7 c_8 + c_6 - c_5 s_7 s_8 c\theta)/s_5$. The linear set Eq. (11) can be solved using Cramer's rule and put into $s^2\psi + c^2\psi = 1$ to get

$$c\psi = \frac{\begin{vmatrix} R_1 & Q_1 \\ R_2 & Q_2 \end{vmatrix}}{\begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix}},\ s\psi = \frac{\begin{vmatrix} P_1 & R_1 \\ P_2 & R_2 \end{vmatrix}}{\begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix}} \Rightarrow (R_1Q_2 - R_2Q_1)^2 + (P_1R_2 - P_2R_1)^2 = (P_1Q_2 - P_2Q_1)^2 \quad (12)$$

Eq. (12) is the implicit I/O relation of the double-spherical 6R linkage and it contains up to 4^{th} power of trigonometric terms, such as $\cos^4\theta$.

4 The plano-spherical 6R linkage

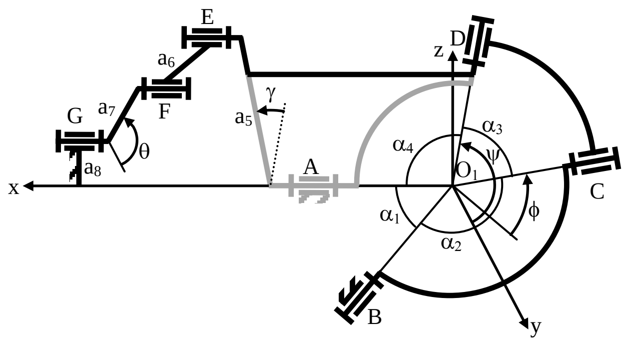

Fig. 4 Plano-spherical 6R linkage

Fig. 4 illustrates a plano-spherical 6R linkage. ϕ is the input and ψ is the output, or vice versa. The I/O relationship for the spherical loop is the same as for the double-spherical 6R linkage and is given by Eq. (8). The planar loop moves parallel to the yz-plane, so the x coordinates are irrelevant. The y, z coordinates for the joints E and F of the planar four-bar loop are found as

$$\begin{bmatrix} E_y \\ E_z \end{bmatrix} = X[\psi+\gamma]\begin{bmatrix} a_5 \\ 0 \end{bmatrix} = \begin{bmatrix} a_5\, c(\psi+\gamma) \\ a_5\, s(\psi+\gamma) \end{bmatrix} \quad , \quad \begin{bmatrix} F_y \\ F_z \end{bmatrix} = \begin{bmatrix} 0 \\ a_8 \end{bmatrix} + X[\theta]\begin{bmatrix} a_7 \\ 0 \end{bmatrix} = \begin{bmatrix} a_7\, c\theta \\ a_8 + a_7\, s\theta \end{bmatrix} \qquad (13)$$

The distance between E and F is a_6, so

$$\begin{aligned}&(E_y-F_y)^2+(E_z-F_z)^2=\left[a_5c(\psi+\gamma)-a_7c\theta\right]^2+\left[a_5s(\psi+\gamma)-a_8-a_7s\theta\right]^2=a_6^2\\&\Rightarrow a_5^2-a_6^2+a_7^2+a_8^2-2a_7a_8s\theta-2a_5a_7c(\psi+\gamma-\theta)-2a_5a_8s(\psi+\gamma)=0\qquad(14)\\&\Rightarrow 2a_5\left[a_7c(\theta-\gamma)+a_8s\gamma\right]c\psi+2a_5\left[a_7s(\theta-\gamma)+a_8c\gamma\right]s\psi=a_5^2-a_6^2+a_7^2+a_8^2-2a_7a_8s\theta\end{aligned}$$

Eqs. (8) and (14) constitute a linear set of equations in terms of $c\psi$ and $s\psi$ as in Eq. (11), but this time $P_2=2a_5\left[a_7c(\theta-\gamma)+a_8s\gamma\right]$, $Q_2=2a_5\left[a_7s(\theta-\gamma)+a_8c\gamma\right]$, and $R_2=a_5^2-a_6^2+a_7^2+a_8^2-2a_7a_8s\theta$. $c\psi$ and $s\psi$ are linearly solved and ψ is eliminated as in Section 3 to obtain the I/O relation of the plano-spherical 6R linkage and once again it contains up to 4th power of trigonometric terms in terms of θ and ψ, such as $\cos^4\theta$.

5 Conclusions and Discussions

The Bennett 6R mechanisms are special overconstrained 6R mechanisms in that they can be dissected into planar slider crank or planar four bar or spherical four bar loops once the imaginary joint is inserted in between the loops. This allows us to formulate the I/O equations for the loops separately and then eliminate the passive joint variable to obtain the I/O relation for the 6R mechanism without conducting spatial kinematics calculations. The method of decomposition is applied to Bennett 6R mechanisms for the first time. This method is not only useful in analysis, but also it has numerous advantages in synthesis. Synthesis methods based on the formulations in this paper will be the scope of further studies.

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