Function Synthesis of Bennett 6R Mechanisms Using Chebyshev Approximation

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Abstract

This study focuses on approximate function synthesis of the three types of overconstrained Bennett 6R mechanisms using Chebyshev approximation. The three mechanisms are the double-planar, double-spherical and the plano-spherical 6R linkages. The single-loop 6R mechanisms are dissected into two imaginary loops and function synthesis is performed for both loops. First, the link lengths are employed as construction parameters of the mechanism. Then extra construction parameters for the input or output joint variables are introduced in order to increase the design points and hence enhance the accuracy of approximation. The synthesis formulations are applied computationally as case studies. The case studies illustrate how a designer can compare the three types of Bennett 6R mechanisms for the same function. Also we present a comparison of the spherical four-bar with the double-spherical 6R mechanism and show that the accuracy is improved when the 6R linkage is used.

Keywords: Function synthesis, Bennett 6R mechanisms, method of decomposition, Chebyshev approximation

1. Introduction

When manufactured precisely, overconstrained mechanisms prove themselves to have higher precision compared to simply constrained mechanisms. The motivation of this study is to investigate the approximate synthesis of an overconstrained mechanism which belongs to subspace $\lambda = 5$. A $\lambda = 5$ mechanism means that the position and orientation of all links of the mechanism can be described with at most five independent parameters at any instant during the motion of the mechanism. Two obvious $\lambda = 5$ subspaces are the RRRPP and RRPPP subspaces, for which a translational or rotational degree-of-freedom (dof) is constrained.

The first example of a $\lambda = 5$ mechanism is the Sarrus (1853) linkage. The Sarrus linkage is a spatial 6R linkage (a linkage with six revolute joints) obtained by assembling two planar dyads in perpendicular planes. Later, Bennett (1905) generalized the Sarrus linkage such that the angle between the planes of the dyads is arbitrary (Fig. 1a). For this linkage, no link has a rotational motion about the axis along the intersection of the planes of the dyads, so the linkage belongs to an RRPPP type of $\lambda = 5$ subspace. The generalized Sarrus linkage can be considered as the combination of two planar slider-crank mechanisms in intersecting planes such that the fixed link and the slider link is common to both of the slider cranks. The sliding direction is along the intersection of the planes. In such an assembly, the linkage remains mobile with single dof even if the prismatic joint is removed. Hence a single loop, 6R linkage is obtained. The removed prismatic joint is defined as a passive joint (Selvi, 2012).

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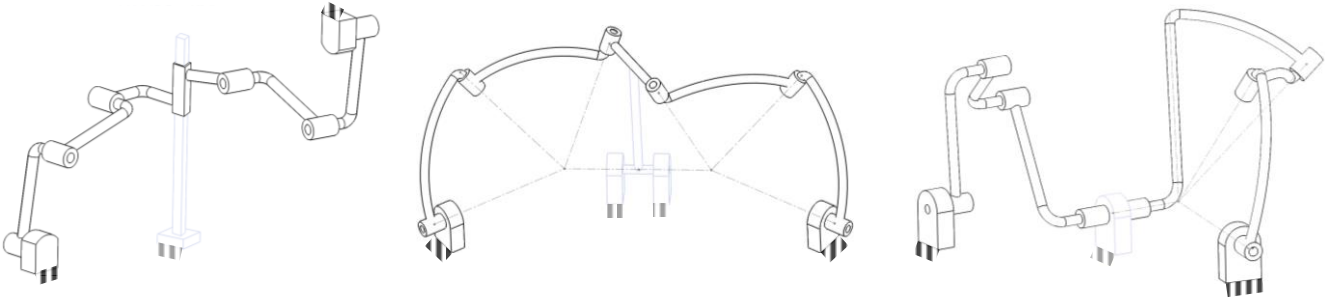E-mail address: gokhankiper@iyte.edu.tr (G. Kiper).

Fig. 1 a) Double-planar 6R (generalized Sarrus) linkage, (b) double-spherical 6R linkage, and (c) plano-spherical 6R linkage with their passive joints

Along with the generalized Sarrus linkage, Bennett (1905) introduced two more $\lambda = 5$ mechanisms. One of these mechanisms is the double-spherical 6R mechanism which is obtained by merging two spherical four-bar linkages with two common links and then removing the joint connecting the common links. The removed revolute joint is the passive joint of this linkage and its axis is along the line connecting the centers of the spherical four-bars (Fig. 1b). It seems not possible to classify this linkage in an RRPPP or an RRRPP type of $\lambda = 5$ subspace. The other mechanism is the plano-spherical 6R mechanism, which is obtained by merging a planar four-bar linkage and a spherical four-bar linkage with two common links. Once again, the revolute joint connecting the common links is removed to obtain the 6R linkage (Fig. 1c). The axis of this passive joint passes through the center of the spherical four-bar and it is in the same direction as all the axes of the planar four-bar. For this linkage, no link has a translational motion along the direction of axes of the planar four-bar, hence the linkage belongs to an RRRPP type of $\lambda = 5$ subspace.

In this study, we investigate function synthesis of the double-planar (generalized Sarrus), the double spherical and the plano-spherical 6R linkages. Since these mechanisms have more construction parameters than simple single-loop mechanisms such as the planar four-bar, spherical four-bar and planar slider-crank mechanisms, we expect to attain more precise function synthesis.

The synthesis of the single-loop planar linkages is vastly studied (Erdman, Sandor, 1997). Synthesis of spherical four-bar mechanism (SFBM) is also widely studied. Hartenberg and Denavit (1964) and Zimmerman (1967) have respectively presented the three- and four-precision-point function generation of the SFBM. Cervantes- Sánchez et al. (2009a) have worked on formulation of the function synthesis problem of the SFBM for three and four precision points and further presented formulations for five and six precision points (Cervantes- Sánchez et al., 2009b). Rao et al. (1973), Farhang et al. (1988, 1999), Alizade et al. (1994, 2005) and Murray and McCarthy (1995) used polynomial approximation method for three, four and five precision points for the function synthesis of the SFBM. Suixian et al. (2009) worked on the optimal selection of precision points. Alizade and Gezgin (2011) have applied interpolation, least squares and Chebyshev approximation methods to solve the six-precision-point function synthesis of the SFBM. Maaroof and Dede (2013, 2014) have worked on the synthesis of the double-spherical 6R linkage using interpolation approximation.

Recently, Alizade et al. (2013) investigated the derivation of the input/output (I/O) relationships of the three types of Bennett 6R linkages using the method of decomposition (Alizade et. al, 1980). In this study, we shall use the I/O equations obtained in that paper.

In Section 2, we outline the Chebyshev approximation method for synthesis of mechanisms. In Section 3, we derive the objective functions for the Bennett mechanisms and formulate equations necessary for the synthesis. In Section 4, we show how to relate the function to be generated to the mechanism joint

parameters. In Section 5, we present some case studies and the discussions section concludes the article.

2. Chebyshev Approximation

In this section we review the polynomial approximation methods first studied by Chebyshev in 1850s (Chebyshev, 1854, 1859). Let f(x) be a continuous function defined on x ∈ [a, b]. A polynomial $P_n(x)$ of degree n is called the best approximation of f(x) if L = max|f(x) − P(x)| is minimum. The alternation theorem of Chebyshev states that for a given function f(x) and order n the best approximation is unique and the maxima L are attained n + 2 times on [a, b] alternately with opposite signs. Although this theorem guaranties the unique existence of the best approximation of a function, neither it, nor its proof leads to a method to find the best approximation. An iterative method is proposed by Remez (1932). The Remez algorithm is as follows (Temes, 1967):

1. Select design points x_i^0 ∈ [a, b], i = 0, .., n + 1 (usually x_0 = a, x_{n+1} = b) and linearly solve for the coefficients of an approximation polynomial $P_n^0(x)$ and L from the n + 2 equations $P_n^0(x_i^0) - f(x_i^0) = (-1)^i L$

2. Find the n + 2 local extrema x_i^1 of $E^0(x) = P_n^0(x_i^0) - f(x_i^0)$ in [a, b].

3. Repeat steps 1 and 2 by replacing x_i^j by x_i^{j+1} until the design points stabilize.

Convergence is not guaranteed in this iterative algorithm. In function synthesis of mechanisms, we derive the input/output (I/O) relationship which is a function of the construction parameters of the mechanism and input and output variables. The I/O equation is written in a polynomial form

$P_n(\mathbf{x}) = \sum_{j=1}^{n} P_j f_j(\mathbf{x})$, called the objective function, where P_j are functions of the construction parameters

and f_j are the functions of the input and output variables represented by **x**. If the number of coefficients P_j (= n) is equal to the number of construction parameters, then given n + 1 many design points $\{\mathbf{x}_i\}_1^{n+1}$,

the coefficients $\{P_j\}_1^n$ and the Chebyshev error L are linearly solved from

$$\sum_{j=1}^{n} P_j f_j(\mathbf{x}_i) - F(\mathbf{x}_i) = (-1)^{i+1} L \text{ for i = 1,...,n + 1} \tag{1}$$

Once $\{P_j\}_1^n$ are solved, the construction parameters are determined from $\{P_j\}_1^n$. If the number of coefficients P_j (= n) is greater than the number of construction parameters, it means that the coefficients P_j are interrelated. For illustration, consider the case where there are n − 1 construction parameters, but P_j = n. Then we choose a coefficient, say, P_n, which is expressible in terms of the others. Let $P_n = \lambda$, $P_j = \ell_j + m_j \lambda$ for j = 1,…,n − 1 and L=ℓ+mλ. Eq. (1) can be reformulated as

$$\sum_{j=1}^{n-1} (\ell_j + m_j \lambda) f_j(\mathbf{x}_i) + \lambda f_n(\mathbf{x}_i) - F(\mathbf{x}_i) = (-1)^{i+1} (\ell + m\lambda) \text{ for i = 1,…,n} \tag{2}$$

The multipliers of λ may be collected in Eq. (2):

$$\sum_{j=1}^{n-1} \ell_j f_j(\mathbf{x}_i) - F(\mathbf{x}_i) + \lambda \left[\sum_{j=1}^{n-1} m_j f_j(\mathbf{x}_i) + f_n(\mathbf{x}_i) \right] = (-1)^{i+1} \ell + \lambda (-1)^{i+1} m, \text{ i = 1,…,n} \tag{3}$$

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Equating the coefficients of λ and the remaining parts in Eq. (3) we obtain

$$\sum_{j=1}^{n-1}\ell_j f_j(\mathbf{x_i})+(-1)^i\ell=F(\mathbf{x_i}) \ \ \text{and} \ \ \sum_{j=1}^{n-1}m_j f_j(\mathbf{x_i})+(-1)^i m=-f_n(\mathbf{x_i}), \ i = 1,\dots,n \tag{4}$$

Eq. (4) is a set of 2n linear equations with unknowns $\{\ell_j\}_1^{n-1}$, $\{m_j\}_1^{n-1}$, ℓ and m. After the unknowns are determined, since there is a relationship between $P_n = \lambda$ and some other coefficients, λ is solved from this relationship. $P_j=\ell_j+m_j\lambda$ for j = 1,…,n − 1 and $L=\ell+m\lambda$ are determined. The n − 1 construction parameters are solved from $\{P_j\}_1^{n-1}$.

If there are n − 2 construction parameters for P_j = n, we let $P_{n-1} = \lambda_1$, $P_n = \lambda_2$, $P_j=\ell_j+m_j\lambda_1+n_j\lambda_2$ for j = 1,...,n − 2 and $L=\ell+m\lambda_1+n\lambda_2$. P_{n-1} and P_n should be dependent on other coefficients. Eq. (1) becomes

$$\sum_{j=1}^{n-2}(\ell_j+m_j\lambda_1+n_j\lambda_2)f_j(\mathbf{x_i})+\lambda_1 f_{n-1}(\mathbf{x_i})+\lambda_2 f_n(\mathbf{x_i})-F(\mathbf{x_i})=(-1)^{i+1}(\ell+m\lambda_1+n\lambda_2) \ \text{for i = 1,...,n − 1} \tag{5}$$

Equating the coefficients of λ_1, λ_2 and the remaining parts in Eq. (5) we obtain

$$\sum_{j=1}^{n-2}\ell_j f_j(\mathbf{x_i})+(-1)^i\ell=F(\mathbf{x_i}), \ \sum_{j=1}^{n-2}m_j f_j(\mathbf{x_i})+(-1)^i m=-f_{n-1}(\mathbf{x_i}) \ \text{and} \ \sum_{j=1}^{n-2}n_j f_j(\mathbf{x_i})+(-1)^i n=-f_n(\mathbf{x_i}), \ i = 1,...,n − 1 \tag{6}$$

$\{\ell_j\}_1^{n-2}$, $\{m_j\}_1^{n-2}$, $\{n_j\}_1^{n-2}$, ℓ, m and n are solved linearly from Eq. (6). Then λ_1 and λ_2 are solved from the relationship in between P_{n-1}, P_n and the other coefficients. $P_j=\ell_j+m_j\lambda_1+n_j\lambda_2$ for j = 1,...,n − 2 and $L=\ell+m\lambda_1+n\lambda_2$ are determined. The n − 2 construction parameters are solved from $\{P_j\}_1^{n-1}$.

3. The Objective Functions

The I/O relationships of the Bennett 6R mechanisms are derived in (Alizade et al., 2013). We shall formulate the objective functions based on the I/O relationships.

3.1 The Double-Planar 6R Mechanism with 6 Parameters

Together with the passive prismatic joint, the double-planar 6R linkage may be considered to be composed of a pair of slider-crank mechanisms. Let ϕ be the input angle, θ be the output angle and s be the passive joint variable of the double-planar 6R mechanism shown in Fig. 2. For the time being, assume ϕ_0, s_0 and θ_0 to be zero. The relationship between the input ϕ and the output θ does not change if the mechanism dimensions are scaled, so without loss of generality we may assume the fixed link length g = 1.
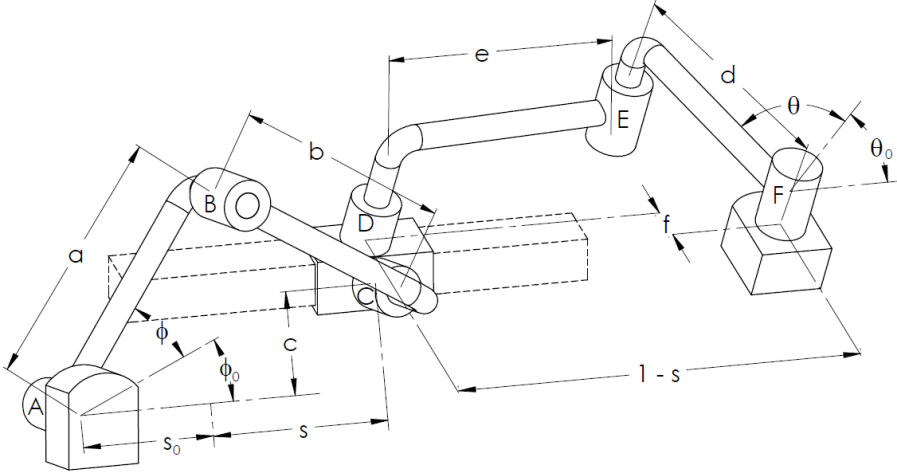
Fig. 2 Kinematic representation of the double-planar 6R linkage

For the double planar 6R mechanism the I/O equation for the imaginary loop ABC is given by (Alizade et al., 2013)

$$b^2 = (a\cos\phi - s)^2 + (a\sin\phi - c)^2 \Rightarrow -a^2 + b^2 - c^2 + 2as\cos\phi + 2ac\sin\phi = s^2 \tag{7}$$

This I/O relationship contains three construction parameters, namely a, b, c. Together with the Chebychev error L, there are four parameters to be determined. Therefore the function synthesis may be performed for four design points. Rewriting Eq. (7) in the polynomial form

$$\delta_i(\mathbf{x_i}) = F(\mathbf{x_i}) - \sum_{j=1}^{3} P_j f_j(\mathbf{x_i}) = (-1)^{i+1} L \text{ for i} = 1,\dots,4 \tag{8}$$

where

$$P_1 = -a^2 + b^2 - c^2, \ f_1(\mathbf{x_i}) = 1, \ P_2 = a, \ f_2(\mathbf{x_i}) = 2s_i\cos\phi_i, \ P_3 = ac, \ f_3(\mathbf{x_i}) = 2\sin\phi_i, \ F(\mathbf{x_i}) = s_i^2 \tag{9}$$

$\mathbf{x_i} = (\phi_i, s_i)$ represents the ith design point and $\delta_i(\mathbf{x_i}) = \pm L$ is the error. The reason for selecting the objective function as $F(\mathbf{x_i}) = s^2$ is that this function solely comprises the output s and the error $\delta_i(\mathbf{x_i})$ better represents the error in the output. Given the design points $\mathbf{x_i} = (\phi_i, s_i)$ for i = 1,…,4, P_1, P_2, P_3 and L are determined via the Remez algorithm as explained in Section 2. Once P_1, P_2, P_3 are determined, the construction parameters a, b and c are found from Eq. (9) as follows:

$$a = P_2, \ c = \frac{P_3}{a}, \ b = \sqrt{P_1 + a^2 + c^2} \tag{10}$$

If any of a or c turns out to be negative, or if b is not real, the limits for ϕ_i and/or s_i shall be altered.

For the imaginary loop DEF (Alizade et al., 2013)

$$e^2 = (1 - s + d\cos\theta)^2 + (-f + d\sin\theta)^2 \Rightarrow d^2 - e^2 + f^2 + (1-s)^2 + 2d(1-s)\cos\theta - 2df\sin\theta = 0 \tag{11}$$

Eq. (11), once again, contains three construction parameters: d, e, f. Note that Eq. (7) and Eq. (11) have similar form. Again, there are four parameters to be determined and hence four design points are required. The polynomial form in Eq. (8) can be used, but this time

$$P_1 = \frac{d^2 - e^2 + f^2}{2df}, \ f_1(\mathbf{x_i}) = 1, \ P_2 = \frac{1}{f}, \ f_2(\mathbf{x_i}) = (1 - s_i)\cos\theta_i, \ P_3 = \frac{1}{2df}, \ f_3(\mathbf{x_i}) = (1 - s_i)^2, \ F(\mathbf{x_i}) = \sin\theta_i \ ; \ \mathbf{x_i} = (s_i, \theta_i) \ (12)$$

Given the design points $\mathbf{x_i} = (s_i, \theta_i)$ for i = 1,...,4, P_1, P_2, P_3 and L are solved via Remez algorithm. The construction parameters d, e and f are determined from Eq. (12) as follows:

$$f = \frac{1}{P_2}, \ d = \frac{1}{2P_3 f}, \ e = \sqrt{d^2 + f^2 - 2P_1 df} \tag{13}$$

3.2 The Double-Planar 6R Mechanism with 7, 8 and 9 Parameters

When the input and output angles are not measured from the same reference (x-axis in Fig. 2), we can define extra construction parameters as the location of the reference geometry with respect to the fixed frame. In this sense the input angle ϕ may be measured from an initial angle ϕ_0. In this case, Eq. (7) can be modified as

$$-a^2 + b^2 - c^2 + 2as\cos\phi_0\cos\phi - 2as\sin\phi_0\sin\phi + 2ac\cos\phi_0\sin\phi + 2ac\sin\phi_0\cos\phi = s^2 \tag{14}$$

Rewriting Eq. (14) in polynomial form

$$\delta_i(\mathbf{x_i}) = F(\mathbf{x_i}) - \sum_{j=1}^{5} P_j f_j(\mathbf{x_i}) = (-1)^{i+1} L \ \text{ for i = 1,...,5} \tag{15}$$

where

$$P_1 = -a^2 + b^2 - c^2, \ f_1(\mathbf{x_i}) = 1, \ P_2 = a\cos\phi_0, \ f_2(\mathbf{x_i}) = 2s_i\cos\phi, \ P_3 = a\sin\phi_0, \ f_3(\mathbf{x_i}) = -2s_i\sin\phi_i, \ P_4 = ac\cos\phi_0,$$
$$f_4(\mathbf{x_i}) = 2\sin\phi_i, \ P_5 = ac\sin\phi_0, \ f_5(\mathbf{x_i}) = 2\cos\phi_i, \ F(\mathbf{x_i}) = s_i^2 \ \text{ and } \mathbf{x_i} = (\phi_i, s_i) \tag{16}$$

P_2, P_3, P_4 and P_5 are dependent such that

$$P_3 P_4 = P_2 P_5 \tag{17}$$

Let $P_5 = \lambda, P_j = \ell_j \lambda + m_j$ for j = 1,...,4 and $L = \ell\lambda + m$. After finding ℓ_j and m_j as explained in Section 2, λ is determined as follows:

$$P_3 P_4 - P_2 P_5 = (\ell_3\lambda + m_3)(\ell_4\lambda + m_4) - \lambda(\ell_2\lambda + m_2) = (\ell_3\ell_4 - \ell_2)\lambda^2 + (\ell_3 m_4 + m_3\ell_4 - m_2)\lambda + m_3 m_4 = 0$$
$$\Rightarrow \lambda = \frac{-(\ell_3 m_4 + m_3\ell_4 - m_2) \mp \sqrt{(\ell_3 m_4 + m_3\ell_4 - m_2)^2 - 4m_3 m_4(\ell_3\ell_4 - \ell_2)}}{2(\ell_3\ell_4 - \ell_2)} \tag{18}$$

Note that λ has two solutions. The designer can choose the solution which yields the less error in the end. After determining λ, $P_j = \ell_j\lambda + m_j$ for j = 1,...,4 and $L = \ell\lambda + m$ are determined. Using Eq. (16) the construction parameters a, b, c and ϕ_0 in terms of P_j are found as follows:

$$\phi_0 = \text{atan2}(P_2, P_3), \ a = \sqrt{P_2^2 + P_3^2}, \ c = \frac{\sqrt{P_4^2 + P_5^2}}{a}, \ b = \sqrt{P_1 + a^2 + c^2} \tag{19}$$

Note that all the construction parameters are determined uniquely and the solution is guaranteed provided that $P_1 + a^2 + c^2 \geq 0$.

Similarly, one can introduce s_0 into Eq. (7):

$$b^2 = (a\cos\phi - s - s_0)^2 + (a\sin\phi - c)^2 \Rightarrow -a^2 + b^2 - c^2 - s_0^2 + 2ac\sin\phi + 2as\cos\phi - 2s_0 s + 2as_0\cos\phi = s^2 \quad (20)$$

Eq. (20) can be written in the polynomial form given in Eq. (15) where

$$P_1 = -a^2 + b^2 - c^2 - s_0^2, \ f_1(\mathbf{x_i}) = 1, \ P_2 = ac, \ f_2(\mathbf{x_i}) = 2\sin\phi_i, \ P_3 = a, \ f_3(\mathbf{x_i}) = 2s_i\cos\phi_i, \ P_4 = s_0, \ f_4(\mathbf{x_i}) = -2s_i,$$
$$P_5 = as_0, \ f_5(\mathbf{x_i}) = 2\cos\phi_i, \ F(\mathbf{x_i}) = s_i^2 \quad (21)$$

Note that

$$P_5 = P_3 P_4 \quad (22)$$

Once again, let $P_5 = \lambda$, $P_j = \ell_j\lambda + m_j$ for j = 1,...,4 and $L = \ell\lambda + m$. After finding ℓ_j and m_j as explained in Section 2, λ is determined as follows:

$$P_3 P_4 - P_5 = (\ell_3\lambda + m_3)(\ell_4\lambda + m_4) - \lambda = \ell_3\ell_4\lambda^2 + (\ell_3 m_4 + m_3\ell_4 - 1)\lambda + m_3 m_4 = 0$$
$$\Rightarrow \lambda = \frac{-(\ell_3 m_4 + m_3\ell_4 - 1) \mp \sqrt{(\ell_3 m_4 + m_3\ell_4 - 1)^2 - 4\ell_3\ell_4 m_3 m_4}}{2\ell_3\ell_4} \quad (23)$$

λ has two solutions. Having found λ, $P_j = \ell_j\lambda + m_j$ for j = 1,...,4 and $L = \ell\lambda + m$. The construction parameters are solved uniquely from Eq. (21) as:

$$a = P_3, \ s_0 = P_4, \ c = P_2/a, \ b = \sqrt{P_1 + a^2 + c^2 + s_0^2} \quad (24)$$

The solution is feasible if P_2, P_3, $P_4 \geq 0$ and $P_1 + a^2 + c^2 + s_0^2 \geq 0$.

Similar to introducing ϕ_0, we can measure the output θ from a different reference with an initial angle of θ_0 as shown in Fig. 2. Then Eq. (11) yields

$$d^2 - e^2 + f^2 + (1-s)^2 + 2d(1-s)\cos\theta_0\cos\theta - 2d(1-s)\sin\theta_0\sin\theta - 2df\sin\theta_0\cos\theta - 2df\cos\theta_0\sin\theta = 0 \quad (25)$$

Eq. (25) can be written in the polynomial form given in Eq. (15) where

$$P_1 = \frac{d^2 - e^2 + f^2}{2df\cos\theta_0}, \ f_1(\mathbf{x_i}) = 1, \ P_2 = \frac{1}{2df\cos\theta_0}, \ f_2(\mathbf{x_i}) = (1-s_i)^2, \ P_3 = \frac{1}{f}, \ f_3(\mathbf{x_i}) = (1-s_i)\cos\theta_i,$$
$$P_4 = \tan\theta_0, \ f_4(\mathbf{x_i}) = -\cos\theta_i, \ P_5 = \frac{\tan\theta_0}{f}, \ f_5(\mathbf{x_i}) = -(1-s_i)\sin\theta_i, \ F(\mathbf{x_i}) = \sin\theta_i \quad (26)$$

Again $P_5 = P_3 P_4$ and $P_5 = \lambda$ can be solved using Eq. (23). $P_j = \ell_j\lambda + m_j$ for j = 1,...,4 and $L = \ell\lambda + m$ are determined and the construction parameters are solved from Eq. (26) as:

$$\theta_0 = \tan^{-1} P_4, \ f = \frac{1}{P_3}, \ d = \frac{1}{2P_2 f\cos\theta_0}, \ e = \sqrt{d^2 + f^2 - 2df\cos\theta_0 P_1} \quad (27)$$

Also it is possible to consider inclusion of s_0 to loop DEF. However, inclusion of s_0 can be done to either of the loops, not both. The next step is to include of both ϕ_0 and s_0 to loop ABC. In that case, Eq. (14) can be modified as

$$-a^2+b^2-c^2-s_0{}^2-2s_0s+2as\cos\phi_0\cos\phi-2as\sin\phi_0\sin\phi$$
$$+2a(s_0\cos\phi_0+c\sin\phi_0)\cos\phi-2a(s_0\sin\phi_0-c\cos\phi_0)\sin\phi=s^2 \tag{28}$$

Eq. (28) can be written in polynomial form as

$$\delta_i(\mathbf{x_i})=F(\mathbf{x_i})-\sum_{j=1}^{6}P_jf_j(\mathbf{x_i})=(-1)^{i+1}L \text{ for i = 1,...,6} \tag{29}$$

where

$$P_1=-a^2+b^2-c^2-s_0{}^2, \ f_1(\mathbf{x_i})=1, \ P_2=s_0, \ f_2(\mathbf{x_i})=-2s_i, \ P_3=a\cos\phi_0, \ f_3(\mathbf{x_i})=2s_i\cos\phi_i,$$
$$P_4=a\sin\phi_0, \ f_4(\mathbf{x_i})=-2s_i\sin\phi_i, \ P_5=a(s_0\cos\phi_0+c\sin\phi_0), \ f_5(\mathbf{x_i})=2\cos\phi_i, \tag{30}$$
$$P_6=a(s_0\sin\phi_0-c\cos\phi_0), \ f_6(\mathbf{x_i})=-2\sin\phi_i, \ F(\mathbf{x_i})=s_i{}^2$$

There are 5 construction parameters, but 6 polynomial coefficients. The dependency in between P_j is obtained by eliminating c from $P_5 = P_2P_3 + cP_4$ and $P_6 = P_2P_4 - cP_3$:

$$P_3P_5+P_4P_6=P_2\left(P_3{}^2+P_4{}^2\right) \tag{31}$$

Let $P_2 = \lambda$ (any of P_3, ..., P_6 may also be selected as λ), $P_j=\ell_j\lambda+m_j$ for j = 1, 3, .., 6 and $L=\ell\lambda+m$ $L=\ell\lambda+m$. ℓ_j and m_j can be solved linearly as explained in Section 2. Then, Eq. (31) is a cubic equation in terms of λ and the three solutions for λ can be found analytically. It is guaranteed that at least one of the solutions is real. In case of three real solutions, the designer can pick the solution which yields lesser error.

As an alternative, it is possible to introduce s_0 and θ_0 simultaneously to loop DEF. This case is similar to loop ABC and again it is necessary to solve a cubic equation.

3.3 The Double-Spherical 6R Mechanism with 8 Parameters

A double-spherical 6R mechanism is shown in Fig. 3. The imaginary spherical 4-bar loops ABCD and AEFG share the common passive joint axis A, which is along the line connecting the spherical loop centers O_1 and O_2. The x-axis of the fixed coordinate system is chosen along O_1O_2 and the y axis is selected such that the O_1B revolute joint axis remains on the xy-plane. ϕ, ψ and θ are the respective input, passive joint and output angle. O_1D and O_2E axes are in general skew with a twist angle of γ. The radii of the spheres do not affect the I/O relationship, so without loss of generality, assume both radii as 1. Also, notice that the distance $|O_1O_2|$ has no effect on the I/O relationship. For the time being assume $\phi_0 = \gamma = \theta_0 = 0$.
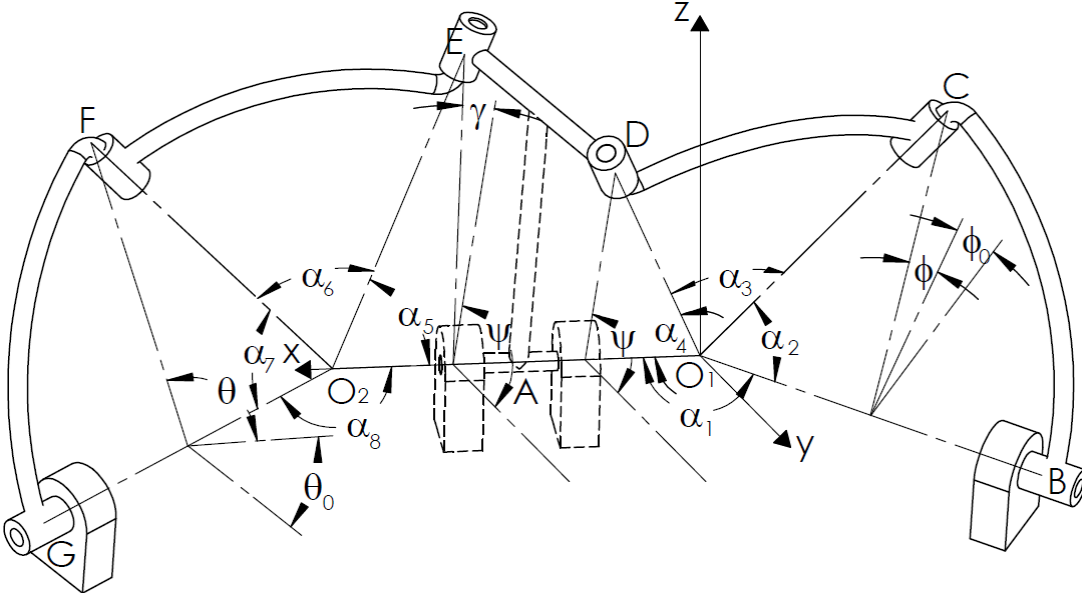
Fig. 3 Kinematic representation of the double-spherical 6R linkage

For the imaginary loop ABCD the I/O equation reads (Alizade et al., 2013)

$$\cos\alpha_1\cos\alpha_2\cos\alpha_4-\cos\alpha_3-\sin\alpha_1\sin\alpha_2\cos\alpha_4\cos\phi+\sin\alpha_1\cos\alpha_2\sin\alpha_4\cos\psi$$
$$+\cos\alpha_1\sin\alpha_2\sin\alpha_4\cos\phi\cos\psi+\sin\alpha_2\sin\alpha_4\sin\phi\sin\psi=0 \tag{32}$$

Eq. (32) contains four construction parameters: α_1, α_2, α_3, α_4. Together with the Chebychev error L, there are five parameters to be determined. The function synthesis may be performed for five design points. Writing Eq. (32) in polynomial form:

$$\delta_i(\mathbf{x_i})=F(\mathbf{x_i})-\sum_{j=1}^{4}P_jf_j(\mathbf{x_i})=(-1)^{i+1}L \ \text{ for } \ i=1,...,5 \tag{33}$$

where

$$P_1=\frac{\cos\alpha_3-\cos\alpha_1\cos\alpha_2\cos\alpha_4}{\sin\alpha_1\cos\alpha_2\sin\alpha_4}, \ f_1(\mathbf{x_i})=1, \ P_2=\frac{\tan\alpha_2}{\tan\alpha_4}, \ f_2(\mathbf{x_i})=\cos\phi_i, \ P_3=\frac{\tan\alpha_2}{\tan\alpha_1},$$

$$f_3(\mathbf{x_i})=-\cos\phi_i\cos\psi_i, \ P_4=\frac{\tan\alpha_2}{\sin\alpha_1}, \ f_4(\mathbf{x_i})=-\sin\phi_i\sin\psi_i, \ F(\mathbf{x_i})=\cos\psi_i \tag{34}$$

$\mathbf{x_i}=(\phi_i,\psi_i)$ represents the ith design point and $\delta_i(\mathbf{x_i})$. P_j are solved linearly from Eq. (33) using the Remez algorithm and then the construction parameters are solved from Eq. (34) as follows:

$$\alpha_1=\cos^{-1}\frac{P_3}{P_4}, \ \alpha_2=\tan^{-1}(P_3\tan\alpha_1), \ \alpha_4=\tan^{-1}\frac{\tan\alpha_2}{P_2},$$

$$\alpha_3=\cos^{-1}(\cos\alpha_1\cos\alpha_2\cos\alpha_4+P_1\sin\alpha_1\cos\alpha_2\sin\alpha_4) \tag{35}$$

Notice that the solution for the construction parameters is not unique. For example there are two alternative solutions for α_1: $\cos^{-1}(P_3/P_4)$ and $-\cos^{-1}(P_3/P_4)$. In Eq. (35) three are 16 possible solutions. The solution/solutions which yield the desired function generation should be determined by either constructing a virtual/actual model of the mechanism or checking the input/output values.

For loop AEFG the I/O equation for $\gamma = 0$ is given by (Alizade et al., 2013)

$$\cos\alpha_5\cos\alpha_7\cos\alpha_8 - \cos\alpha_6 + \sin\alpha_5\cos\alpha_7\sin\alpha_8\cos\psi - \sin\alpha_5\sin\alpha_7\cos\alpha_8\cos\theta\cos\psi$$
$$+\sin\alpha_5\sin\alpha_7\sin\theta\sin\psi + \cos\alpha_5\sin\alpha_7\sin\alpha_8\cos\theta = 0 \tag{36}$$

Eq. (36) can be written in polynomial form given in Eq. (33) where

$$P_1 = \frac{\cos\alpha_6 - \cos\alpha_5\cos\alpha_7\cos\alpha_8}{\cos\alpha_5\sin\alpha_7\sin\alpha_8},\ f_1(\mathbf{x_i})=1,\ P_2=-\frac{\tan\alpha_5}{\tan\alpha_7},\ f_2(\mathbf{x_i})=\cos\psi_i,\ P_3=\frac{\tan\alpha_5}{\tan\alpha_8},\ f_3(\mathbf{x_i})=\cos\theta_i\cos\psi_i,$$

$$P_4 = -\frac{\tan\alpha_5}{\sin\alpha_8},\ f_4(\mathbf{x_i})=\sin\theta_i\sin\psi_i,\ F(\mathbf{x_i})=\cos\theta_i\ \text{ and } \mathbf{x_i}=(\psi_i,\theta_i) \tag{37}$$

For given 5 design points, $\mathbf{x_i}=(\phi_i,\psi_i)$, P_j are solved linearly from Eq. (33) using the Remez algorithm and then, the construction parameters are solved from Eq. (37) as follows:

$$\alpha_8=\cos^{-1}\left(-\frac{P_3}{P_4}\right),\ \alpha_5=\tan^{-1}(P_3\tan\alpha_8),\ \alpha_7=\tan^{-1}-\frac{\tan\alpha_5}{P_2},$$

$$\alpha_6=\cos^{-1}(\cos\alpha_5\cos\alpha_7\cos\alpha_8+P_1\cos\alpha_5\sin\alpha_7\sin\alpha_8) \tag{38}$$

3.4 The Double-Spherical 6R Mechanism with 9, 10 and 11 Parameters

When extra construction parameter ϕ_0 is added into Eq. (32):

$$\cos\alpha_1\cos\alpha_2\cos\alpha_4 - \cos\alpha_3 - \sin\alpha_1\sin\alpha_2\cos\alpha_4\cos\phi_0\cos\phi + \sin\alpha_1\sin\alpha_2\cos\alpha_4\sin\phi_0\sin\phi$$
$$+\sin\alpha_1\cos\alpha_2\sin\alpha_4\cos\psi + \cos\alpha_1\sin\alpha_2\sin\alpha_4\cos\phi_0\cos\phi\cos\psi \tag{39}$$
$$-\cos\alpha_1\sin\alpha_2\sin\alpha_4\sin\phi_0\sin\phi\cos\psi + \sin\alpha_2\sin\alpha_4\sin\phi_0\cos\phi\sin\psi + \sin\alpha_2\sin\alpha_4\cos\phi_0\sin\phi\sin\psi = 0$$

Eq. (39) can be written in polynomial form as

$$\delta_i(\mathbf{x_i})=F(\mathbf{x_i})-\sum_{j=1}^{7}P_jf_j(\mathbf{x_i})=(-1)^{i+1}L\ \text{ for } i=1,...,6 \tag{40}$$

where

$$P_1=\frac{\cos\alpha_1\cos\alpha_2\cos\alpha_4-\cos\alpha_3}{\cos\alpha_1\sin\alpha_2\sin\alpha_4\sin\phi_0},\ f_1(\mathbf{x_i})=1,\ P_2=\frac{\tan\alpha_1}{\tan\alpha_4},\ f_2(\mathbf{x_i})=\sin\phi_i,\ P_3=\frac{\tan\alpha_1}{\tan\alpha_2\sin\phi_0},$$

$$f_3(\mathbf{x_i})=\cos\psi_i,\ P_4=\frac{1}{\tan\phi_0},\ f_4(\mathbf{x_i})=\cos\phi_i\cos\psi_i,\ P_5=\frac{1}{\cos\alpha_1},\ f_5(\mathbf{x_i})=\cos\phi_i\sin\psi_i, \tag{41}$$

$$P_6=-\frac{\tan\alpha_1}{\tan\alpha_4\tan\phi_0},\ f_6(\mathbf{x_i})=\cos\phi_i,\ P_7=\frac{1}{\cos\alpha_1\tan\phi_0},\ f_7(\mathbf{x_i})=\sin\phi_i\sin\psi_i,\ F(\mathbf{x_i})=\sin\phi_i\cos\psi_i$$

There are five construction parameters, i.e. α_1, α_2, α_3, α_4 and ϕ_0, however there are seven P_j. Indeed, two of the P_j can be represented in terms of other as

$$P_6 = -P_2P_4 \text{ and } P_7 = P_4P_5 \tag{42}$$

Let $P_6=\lambda_1$, $P_7=\lambda_2$, $P_j=\ell_j+m_j\lambda_1+n_j\lambda_2$ for $j=1,...,5$ and $L=\ell+m\lambda_1+n\lambda_2$. ℓ_j, m_j and n_j are found as

explained in Section 2. Writing Eq. (42) in terms of λ_1 and λ_2:

$$P_2P_4+P_6=m_2m_4\lambda_1^2+n_2n_4\lambda_2^2+(m_2n_4+n_2m_4)\lambda_1\lambda_2+(\ell_2m_4+m_2\ell_4+1)\lambda_1+(\ell_2n_4+n_2\ell_4)\lambda_2+\ell_2\ell_4=0$$
$$P_4P_5-P_7=m_4m_5\lambda_1^2+n_4n_5\lambda_2^2+(m_4n_5+n_4m_5)\lambda_1\lambda_2+(\ell_4m_5+m_4\ell_5-1)\lambda_1+(\ell_4n_5+n_4\ell_5)\lambda_2+\ell_4\ell_5=0$$

(43)

As shown in (Alizade, Kilit, 2005), when λ_2 is eliminated from Eq. (43) to obtain a third order equation in terms of λ_1, which can be solved analytically. At this point, it should be emphasized that the P_j in Eq. (41) are selected as such on purpose so that Eq. (42) is obtained and hence, there are three solutions. For other cases, the fourth order term which appears after elimination in Eq. (43), does not vanish, thus analytical solution is not possible and it is possible that there are no real solutions.

After determining λ_1 and λ_2, $P_j=\ell_j+m_j\lambda_1+n_j\lambda_2$ for j = 1,...,5 and $L=\ell+m\lambda_1+n\lambda_2$ are determined. From Eq. (41) the construction parameters α_1, α_2, α_3, α_4 and ϕ_0 are found as follows:

$$\phi_0=\tan^{-1}\frac{1}{P_4},\ \alpha_1=\cos^{-1}\frac{1}{P_5},\ \alpha_2=\tan^{-1}\frac{\tan\alpha_1}{P_3\sin\phi_0},\ \alpha_4=\tan^{-1}\frac{\tan\alpha_1}{P_2},$$
$$\alpha_3=\cos^{-1}(\cos\alpha_1\cos\alpha_2\cos\alpha_4-P_1\cos\alpha_1\sin\alpha_2\sin\alpha_4\sin\phi_0)$$

(44)

When the twist angle γ in between O_1D and O_2E in Fig. 3 is taken into account Eq. (32) becomes

$$\cos\alpha_1\cos\alpha_2\cos\alpha_4-\cos\alpha_3-\sin\alpha_1\sin\alpha_2\cos\alpha_4\cos\phi+\sin\alpha_1\cos\alpha_2\sin\alpha_4\cos\gamma\cos\psi$$
$$-\sin\alpha_1\cos\alpha_2\sin\alpha_4\sin\gamma\sin\psi+\cos\alpha_1\sin\alpha_2\sin\alpha_4\cos\gamma\cos\phi\cos\psi$$
$$-\cos\alpha_1\sin\alpha_2\sin\alpha_4\sin\gamma\cos\phi\sin\psi+\sin\alpha_2\sin\alpha_4\sin\gamma\sin\phi\cos\psi+\sin\alpha_2\sin\alpha_4\cos\gamma\sin\phi\sin\psi=0$$

(45)

Eq. (45) can be written in polynomial form in Eq. (40) where

$$P_1=\frac{\cos\alpha_1\cos\alpha_2\cos\alpha_4-\cos\alpha_3}{\cos\alpha_1\sin\alpha_2\sin\alpha_4\sin\gamma},\ f_1(\mathbf{x_i})=1,\ P_2=\frac{\tan\alpha_1}{\tan\alpha_4\sin\gamma},\ f_2(\mathbf{x_i})=-\cos\phi_i,\ P_3=\frac{\tan\alpha_1}{\tan\alpha_2},\ f_3(\mathbf{x_i})=-\sin\psi_i,$$

$$P_4=\frac{1}{\tan\gamma},\ f_4(\mathbf{x_i})=\cos\phi_i\cos\psi_i,\ P_5=\frac{1}{\cos\alpha_1},\ f_5(\mathbf{x_i})=\sin\phi_i\cos\psi_i,\ P_6=-\frac{\tan\alpha_1}{\tan\alpha_2\tan\gamma},\ f_6(\mathbf{x_i})=-\cos\psi_i,$$

(46)

$$P_7=\frac{1}{\cos\alpha_1\tan\gamma},\ f_7(\mathbf{x_i})=\sin\phi_i\sin\psi_i,\ F(\mathbf{x_i})=\cos\phi_i\sin\psi_i$$

P_6 and P_7 depend on other P_j as

$$P_6=-P_3P_4 \text{ and } P_7=P_4P_5$$

(47)

which has the same form as in Eq. (42) and yield there solutions similar to the previous case. The construction parameters are obtained from Eq. (46) as follows:

$$\gamma=\tan^{-1}\frac{1}{P_4},\ \alpha_1=\cos^{-1}\frac{1}{P_5},\ \alpha_2=\tan^{-1}\frac{\tan\alpha_1}{P_3},\ \alpha_4=\tan^{-1}\frac{\tan\alpha_1}{P_2\sin\gamma},$$
$$\alpha_3=\cos^{-1}(\cos\alpha_1\cos\alpha_2\cos\alpha_4-P_1\cos\alpha_1\sin\alpha_2\sin\alpha_4\sin\gamma)$$

(48)

The twist angle γ can also be included in Eq. (36) to replace ψ by $\gamma + \psi$:

$$\cos\alpha_5\cos\alpha_7\cos\alpha_8-\cos\alpha_6+\cos\alpha_5\sin\alpha_7\sin\alpha_8\cos\theta+\sin\alpha_5\cos\alpha_7\sin\alpha_8\cos\gamma\cos\psi$$
$$-\sin\alpha_5\cos\alpha_7\sin\alpha_8\sin\gamma\sin\psi-\sin\alpha_5\sin\alpha_7\cos\alpha_8\cos\gamma\cos\theta\cos\psi$$
$$+\sin\alpha_5\sin\alpha_7\cos\alpha_8\sin\gamma\cos\theta\sin\psi+\sin\alpha_5\sin\alpha_7\cos\gamma\sin\theta\sin\psi+\sin\alpha_5\sin\alpha_7\sin\gamma\sin\theta\cos\psi=0$$

(49)

Writing Eq. (49) in polynomial form Eq. (40), the P_j are selected as

$$P_1 = \frac{\cos\alpha_5 \cos\alpha_7 \cos\alpha_8 - \cos\alpha_6}{\sin\alpha_5 \sin\alpha_7 \cos\alpha_8 \cos\gamma},\ f_1(\mathbf{x_i}) = 1,\ P_2 = \frac{\tan\alpha_8}{\tan\alpha_7},\ f_2(\mathbf{x_i}) = \cos\psi_i,\ P_3 = \tan\gamma,\ f_3(\mathbf{x_i}) = \cos\theta_i \sin\psi_i,$$

$$P_4 = \frac{\tan\alpha_8}{\tan\alpha_5 \cos\gamma},\ f_4(\mathbf{x_i}) = \cos\theta_i,\ P_5 = \frac{1}{\cos\alpha_8},\ f_5(\mathbf{x_i}) = \sin\theta_i \sin\psi_i,\ P_6 = -\frac{\tan\alpha_8 \tan\gamma}{\tan\alpha_7},\ f_6(\mathbf{x_i}) = \sin\psi_i, \quad (50)$$

$$P_7 = \frac{\tan\gamma}{\cos\alpha_8},\ f_7(\mathbf{x_i}) = \sin\theta_i \cos\psi_i,\ F(\mathbf{x_i}) = \cos\theta_i \cos\psi_i$$

P_6 and P_7 depend on other P_j as

$$P_6 = -P_2 P_3 \text{ and } P_7 = P_3 P_5 \quad (51)$$

which has the same form as in Eq. (42) and yield three solutions. The construction parameters are obtained from Eq. (50) as follows:

$$\gamma = \tan^{-1} P_3,\ \alpha_8 = \cos^{-1}\frac{1}{P_5},\ \alpha_7 = \tan^{-1}\left(-\frac{\tan\alpha_8 \tan\gamma}{P_6}\right),\ \alpha_5 = \tan^{-1}\frac{\tan\alpha_8}{P_4 \cos\gamma},$$

$$\alpha_6 = \cos^{-1}(\cos\alpha_5 \cos\alpha_7 \cos\alpha_8 - P_1 \sin\alpha_5 \sin\alpha_7 \cos\alpha_8 \cos\gamma) \quad (52)$$

Next, we can introduce the extra parameter θ_0 in order to replace θ by $\theta_0 + \theta$ in Eq. (36):

$$\cos\alpha_5 \cos\alpha_7 \cos\alpha_8 - \cos\alpha_6 + \sin\alpha_5 \cos\alpha_7 \sin\alpha_8 \cos\psi - \sin\alpha_5 \sin\alpha_7 \cos\alpha_8 \cos\theta_0 \cos\theta \cos\psi$$

$$+\sin\alpha_5 \sin\alpha_7 \cos\alpha_8 \sin\theta_0 \sin\theta \cos\psi + \sin\alpha_5 \sin\alpha_7 \cos\theta_0 \sin\theta \sin\psi + \sin\alpha_5 \sin\alpha_7 \sin\theta_0 \cos\theta \sin\psi \quad (53)$$

$$+\cos\alpha_5 \sin\alpha_7 \sin\alpha_8 \cos\theta_0 \cos\theta - \cos\alpha_5 \sin\alpha_7 \sin\alpha_8 \sin\theta_0 \sin\theta = 0$$

Writing Eq. (53) in polynomial form Eq. (40), the P_j are selected as

$$P_1 = \frac{\cos\alpha_5 \cos\alpha_7 \cos\alpha_8 - \cos\alpha_6}{\sin\alpha_5 \sin\alpha_7 \cos\alpha_8 \cos\theta_0},\ f_1(\mathbf{x_i}) = 1,\ P_2 = \frac{\tan\alpha_8}{\tan\alpha_7 \cos\theta_0},\ f_2(\mathbf{x_i}) = \cos\psi_i,\ P_3 = \tan\theta_0,$$

$$f_3(\mathbf{x_i}) = \cos\psi_i \sin\theta_i,\ P_4 = \frac{1}{\cos\alpha_8},\ f_4(\mathbf{x_i}) = \sin\psi_i \sin\theta_i,\ P_5 = \frac{\tan\alpha_8}{\tan\alpha_5},\ f_5(\mathbf{x_i}) = \cos\theta_i,\ P_6 = -\frac{\tan\alpha_8 \tan\theta_0}{\tan\alpha_5}, \quad (54)$$

$$f_6(\mathbf{x_i}) = \sin\theta_i,\ P_7 = \frac{\tan\theta_0}{\cos\alpha_8},\ f_7(\mathbf{x_i}) = \sin\psi_i \cos\theta_i,\ F(\mathbf{x_i}) = \cos\psi_i \cos\theta_i$$

P_6 and P_7 depend on other P_j as

$$P_6 = -P_3 P_5 \text{ and } P_7 = P_3 P_4 \quad (55)$$

which has the same form as in Eq. (42) and yield three solutions. The construction parameters are obtained from Eq. (54) as follows:

$$\theta_0 = \tan^{-1} P_3,\ \alpha_8 = \cos^{-1}\frac{1}{P_4},\ \alpha_7 = \tan^{-1}\frac{\tan\alpha_8}{P_2 \cos\theta_0},\ \alpha_5 = \tan^{-1}\frac{\tan\alpha_8}{P_5},$$

$$\alpha_6 = \cos^{-1}(\cos\alpha_5 \cos\alpha_7 \cos\alpha_8 - P_1 \sin\alpha_5 \sin\alpha_7 \cos\alpha_8 \cos\theta_0) \quad (56)$$

It is also possible to include both of ϕ_0 and γ in loop ABCD and both of γ and θ_0 in loop AEFG. This case is already studied in (Alizade, Gezgin, 2011), where it is necessary to define six nonlinear parameters λ_k for $k = 1,...,6$. The solution for λ_k necessitates numerical solution.

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3.5 Plano-Spherical 6R Mechanism with 7 Parameters

A plano-spherical 6R linkage is depicted in Fig. 4, where ϕ is the input and θ is the output, or vice versa. The link length definitions for the spherical and planar parts are the same as for the corresponding imaginary loops of the double-planar and double-spherical linkages. In general, there is a twist angle γ in between O_1D' and AE directions. The x-axis direction of the fixed coordinate system is selected to be parallel to the joint axis directions of the planar loop. The y-axis is selected such that the xy-plane includes the O_1B axis of the spherical loop and the z-axis is along the common normal of O_1B and G rotation axes. For the time being assume $\phi_0 = \gamma = \theta_0 = 0$.
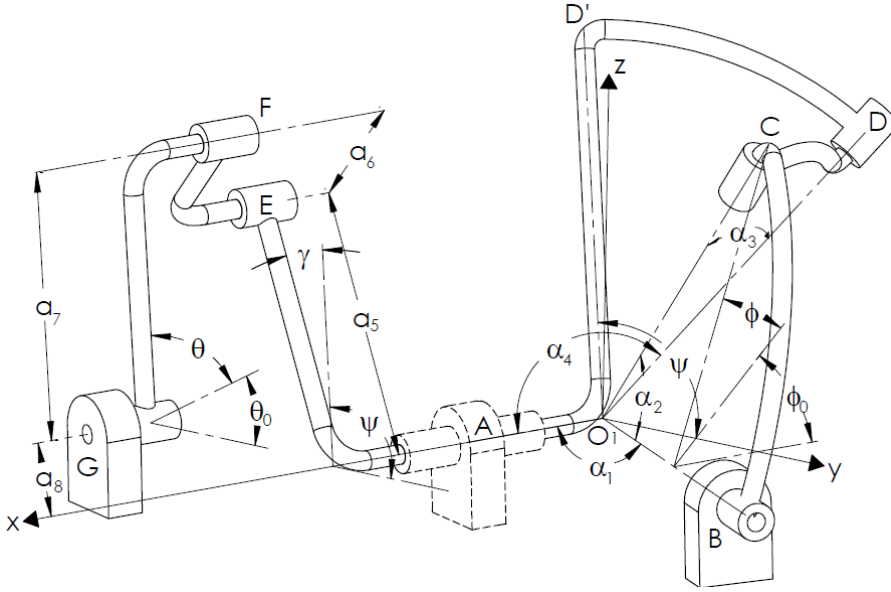

Fig. 4 Kinematic representation of the plano-spherical 6R linkage

The I/O relationship for the spherical loop is the same as for the first loop of double-spherical 6R linkage and is given by Eq. (32). All links of the planar loop move parallel to the yz-plane, so the x coordinates are irrelevant when analyzing this loop. Scaling all the link lengths of the planar four bar mechanism has no effect on the function synthesis task, so without loss of generality we will assume $a_5 = 1$. For $\gamma = 0$, the I/O equation for the planar loop can be derived as (Alizade et al., 2013):

$$1 - a_6{}^2 + a_7{}^2 + a_8{}^2 - 2a_8 \sin\psi - 2a_7 \cos(\psi - \theta) + 2a_7 a_8 \sin\theta = 0 \tag{57}$$

This I/O relationship contains three construction parameters: a_6, a_7, a_8. Together with the Chebychev error L, there are four parameters to be determined. Therefore, the function synthesis may be performed for four design points. Eq. (57) can be written in the polynomial form of Eq. (8), where

$$P_1 = \frac{1 - a_6{}^2 + a_7{}^2 + a_8{}^2}{2a_7 a_8}, \ f_1(\mathbf{x_i}) = 1, \ P_2 = \frac{1}{a_7}, \ f_2(\mathbf{x_i}) = -\sin\psi_i, \ P_3 = \frac{1}{a_8}, \ f_3(\mathbf{x_i}) = -\cos(\psi_i - \theta_i), \ F(\mathbf{x_i}) = -\sin\theta_i \tag{58}$$

P_j are solved linearly from Eq. (8) using the Remez algorithm and then the construction parameters are solved from Eq. (58) as follows:

$$a_7 = \frac{1}{P_2}, \ a_8 = \frac{1}{P_3}, \ a_6 = \sqrt{1 + a_7^2 + a_8^2 - 2P_1 a_7 a_8} \tag{59}$$

3.6 Plano-Spherical 6R Mechanism with 8, 9 and 10 Parameters

When we include the twist angle γ to the planar loop Eq. (57) becomes

$$1 - a_6^2 + a_7^2 + a_8^2 - 2a_8 \cos\gamma \sin\psi - 2a_8 \sin\gamma \cos\psi - 2a_7 \cos\gamma \cos(\psi - \theta)$$
$$+ 2a_7 \sin\gamma \sin(\psi - \theta) + 2a_7 a_8 \sin\theta = 0 \tag{60}$$

Eq. (60) can be written in polynomial form in Eq. (15) where

$$P_1 = \frac{1 - a_6^2 + a_7^2 + a_8^2}{2a_7 a_8}, \ f_1(\mathbf{x_i}) = 1, \ P_2 = \frac{\cos\gamma}{a_7}, \ f_2(\mathbf{x_i}) = -\sin\psi_i, \ P_3 = \frac{\sin\gamma}{a_7}, \ f_3(\mathbf{x_i}) = -\cos\psi_i, \ P_4 = \frac{\cos\gamma}{a_8},$$

$$f_4(\mathbf{x_i}) = -\cos(\psi_i - \theta_i), \ P_5 = \frac{\sin\gamma}{a_8}, \ f_5(\mathbf{x_i}) = \sin(\psi_i - \theta_i), \ F(\mathbf{x_i}) = -\sin\theta_i \tag{61}$$

The dependency in between P_j is the same as Eq. (17). We assign $P_5 = \lambda$, $P_j = \ell_j \lambda + m_j$ for j = 1,...,4 and $L = \ell\lambda + m$. After finding ℓ_j and m_j as explained in Section 2, λ is determined as in Eq. (18). The construction parameters are determined as

$$\gamma = \mathrm{atan2}(P_2, P_3), \ a_7 = \frac{\cos\gamma}{P_2}, \ a_8 = \frac{\cos\gamma}{P_4}, \ a_6 = \sqrt{1 + a_7^2 + a_8^2 - P_1 2a_7 a_8} \tag{62}$$

When we use $\theta_0 + \theta$ instead of θ in Eq. (57), we obtain

$$1 - a_6^2 + a_7^2 + a_8^2 - 2a_8 \sin\psi - 2a_7 \cos\theta_0 \cos(\psi - \theta) - 2a_7 \sin\theta_0 \sin(\psi - \theta)$$
$$+ 2a_7 a_8 \sin\theta_0 \cos\theta + 2a_7 a_8 \cos\theta_0 \sin\theta = 0 \tag{63}$$

Eq. (60) can be written in polynomial form in Eq. (15) where

$$P_1 = \frac{1 - a_6^2 + a_7^2 + a_8^2}{2a_7 a_8 \cos\theta_0}, \ f_1(\mathbf{x_i}) = 1, \ P_2 = \frac{1}{a_7 \cos\theta_0}, \ f_2(\mathbf{x_i}) = -\sin\psi_i, \ P_3 = \tan\theta_0, \ f_3(\mathbf{x_i}) = \cos\theta_i, \ P_4 = \frac{1}{a_8},$$

$$f_4(\mathbf{x_i}) = -\cos(\psi_i - \theta_i), \ P_5 = \frac{\tan\theta_0}{a_8}, \ f_5(\mathbf{x_i}) = -\sin(\psi_i - \theta_i), \ F(\mathbf{x_i}) = -\sin\theta_i \tag{64}$$

The dependency in between P_j is the same as Eq. (22). We assign $P_5 = \lambda$, $P_j = \ell_j \lambda + m_j$ for j = 1,...,4 and $L = \ell\lambda + m$. After finding ℓ_j and m_j as explained in Section 2, λ is determined as in Eq. (23). The construction parameters are determined as

$$\theta_0 = \begin{cases} \tan^{-1}P_3 & \text{if } P_2 > 0 \\ \tan^{-1}P_3 + \pi & \text{if } P_2 < 0 \end{cases}, \ a_8 = \frac{1}{P_4}, \ a_7 = \frac{1}{P_2 \cos\theta_0}, \ a_6 = \sqrt{1 + a_7^2 + a_8^2 - 2P_1 a_7 a_8 \cos\theta_0} \tag{65}$$

Finally we can add both of γ and θ_0 to Eq. (57) to obtain

$$1-a_6{}^2+a_7{}^2+a_8{}^2-2a_8\cos\gamma\sin\psi-2a_8\sin\gamma\cos\psi-2a_7\cos(\gamma-\theta_0)\cos(\psi-\theta)$$
$$+2a_7\sin(\gamma-\theta_0)\sin(\psi-\theta)+2a_7a_8\sin\theta_0\cos\theta+2a_7a_8\cos\theta_0\sin\theta=0 \tag{66}$$

Eq. (66) can be written in polynomial form in Eq. (29) where

$$P_1=\frac{1-a_6{}^2+a_7{}^2+a_8{}^2}{2a_7a_8\cos\theta_0},\ f_1(\mathbf{x_i})=1,\ P_2=\frac{\cos\gamma}{a_7\cos\theta_0},\ f_2(\mathbf{x_i})=-\sin\psi_i,\ P_3=\tan\theta_0,\ f_3(\mathbf{x_i})=\cos\theta_i,\ P_4=\frac{\sin\gamma}{a_7\cos\theta_0},$$

$$f_4(\mathbf{x_i})=-\cos\psi_i,\ P_5=\frac{\cos(\gamma-\theta_0)}{a_8\cos\theta_0},\ f_5(\mathbf{x_i})=-\cos(\psi_i-\theta_i),\ P_6=\frac{\sin(\gamma-\theta_0)}{a_8\cos\theta_0},\ f_6(\mathbf{x_i})=\sin(\psi_i-\theta_i),\ F(\mathbf{x_i})=-\sin\theta_i \tag{67}$$

There are five construction parameters, whereas there are six P_j. The relationship between P_j can be found to be

$$P_3\left(P_2P_5+P_4P_6\right)=P_4P_5-P_2P_6 \tag{68}$$

Let $P_6=\lambda$, $P_j=\ell_j\lambda+m_j$ for j = 1,...,5 and $L=\ell\lambda+m$. ℓ_j and m_j can be solved linearly as explained in Section 2. Eq. (68) is a cubic equation in terms of λ and can be solved analytically. The construction parameters are solved from Eq. (67) as

$$\theta_0=\tan^{-1}P_3,\ \gamma=\mathrm{atan2}(P_2\cos\theta_0,P_4\cos\theta_0),\ a_7=\frac{\cos\gamma}{P_2\cos\theta_0},\ a_8=\frac{\cos(\gamma-\theta_0)}{P_5\cos\theta_0},\ a_6=\sqrt{1+a_7{}^2+a_8{}^2-2P_1a_7a_8\cos\theta_0}\ (69)$$

4. The Function Synthesis Problem

Let the function to be generated be z = f(x) for $x_{min}\le x\le x_{max}$ and $z_{min}\le z\le z_{max}$. The independent variable x should be related to the mechanism input ϕ and the dependent variable z should be related to the mechanism output θ. However, since we are making use of the method of decomposition, we have an intermediate joint variable ψ for the double-spherical and plano-spherical mechanisms and s for the double-planar mechanism. We shall introduce y = g(x) for $y_{min}\le y\le y_{max}$ such that z = h(y) and hence f(x) = h(g(x)). Depending on the application some or all of z = f(x), x_{min}, x_{max}, z_{min} and z_{max} may be demanded by the specific task. However, the designer can freely select y = g(x), y_{min} and y_{max}. The design may be enhanced via different selection of the function g and also the boundaries of y.

We shall linearly relate x to ϕ, y to ψ (or s) and z to θ as

$$\frac{x-x_{min}}{x_{max}-x_{min}}=\frac{\phi-\phi_{min}}{\phi_{max}-\phi_{min}}\ ,\ \frac{y-y_{min}}{y_{max}-y_{min}}=\frac{\psi-\psi_{min}}{\psi_{max}-\psi_{min}}\ ,\ \frac{z-z_{min}}{z_{max}-z_{min}}=\frac{\theta-\theta_{min}}{\theta_{max}-\theta_{min}} \tag{70}$$

Then desired ψ and θ values for given input ϕ are found as follows:

$$\phi=\frac{\phi_{max}-\phi_{min}}{x_{max}-x_{min}}(x-x_{min})+\phi_{min}\ ,\ \psi=\frac{\psi_{max}-\psi_{min}}{y_{max}-y_{min}}(g(x)-y_{min})+\psi_{min}\ ,\ \theta=\frac{\theta_{max}-\theta_{min}}{z_{max}-z_{min}}(f(x)-z_{min})+\theta_{min} \tag{71}$$

and conversely

$$x=\frac{x_{max}-x_{min}}{\phi_{max}-\phi_{min}}(\phi-\phi_{min})+x_{min}\ ,\ y=\frac{y_{max}-y_{min}}{\psi_{max}-\psi_{min}}(\psi-\psi_{min})+y_{min}\ ,\ z=\frac{z_{max}-z_{min}}{\theta_{max}-\theta_{min}}(\theta-\theta_{min})+z_{min} \tag{72}$$

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We make use of Eq. (71) when determining the design points $\{\phi_i\}_1^n$, $\{\psi_i\}_1^n$ and $\{\theta_i\}_1^n$ from $\{x_i\}_1^n$, $\{y_i\}_1^n = \{g(x_i)\}_1^n$ and $\{z_i\}_1^n = \{f(x_i)\}_1^n$. Selection of $\{x_i\}_1^n$ may be done with equal spacing, Chebyshev spacing, or any other type of spacing depending of the application. Our experience is that the Remez algorithm converges faster (in 3 or 4 iterations) when the initial spacing is selected as the Chebyshev spacing.

We make use of Eq. (72) after the synthesis is done and we need to check the error in between the desired z = f(x) and the generated z with the mechanism. At this step, one shall determine the output values of the mechanism loops for several given input values, say 100 values, by solving the I/O relationship.

5. Numerical examples

We have implemented all the formulations in the previous sections in Microsoft Excel® and performed several different function synthesis tasks. Here we present some of the results for illustration.

First, we work on power functions of type $z = x^a$. As an example, consider $z = x^{0.5}$ for $1 \leq x \leq 5$, which is decomposed as $y = x^{0.6}$, $z = y^{5/6}$. For $130° \geq \phi \geq 50°$ input range and $210° \leq \theta \leq 270°$ output range, we performed synthesis for all of the double-planar, double-spherical and plano-spherical 6-R linkages with no extra parameters ϕ_0, θ_0, etc. in order to compare the resultant error. The intermediate joint variable limits are chosen as $0.3 \leq s \leq 0.9$ for the double-planar linkage and $110° \leq \psi \leq 200°$ for the double-spherical and the plano-spherical linkages. The initial design points in between the limits are determined using Chebyshev spacing. The percentage error variation in the z values defined as

$$\% \, \text{Error} = \frac{z_{desired} - z_{mechanism}}{z_{desired}} \times 100 \tag{73}$$

is given in Fig. 5. As seen from Fig. 5, the lowest percentage error values are obtained with the double-spherical linkage. The maximum error magnitude is 0.123% for the double-spherical linkage, 0.291% for the plano-spherical linkage and 1.54% for the double-planar linkage. So, for $z = x^{0.5}$ it seems like the double-spherical linkage provides the best approximation. Of course it is possible to change the intermediate function y = f(x) and the boundaries for the input, output and intermediate joint variables. For different selections, some other linkage may be better than the others. Similarly, the designer may test several different mechanisms for the same function in order to determine the best-suited mechanism for that specific function.
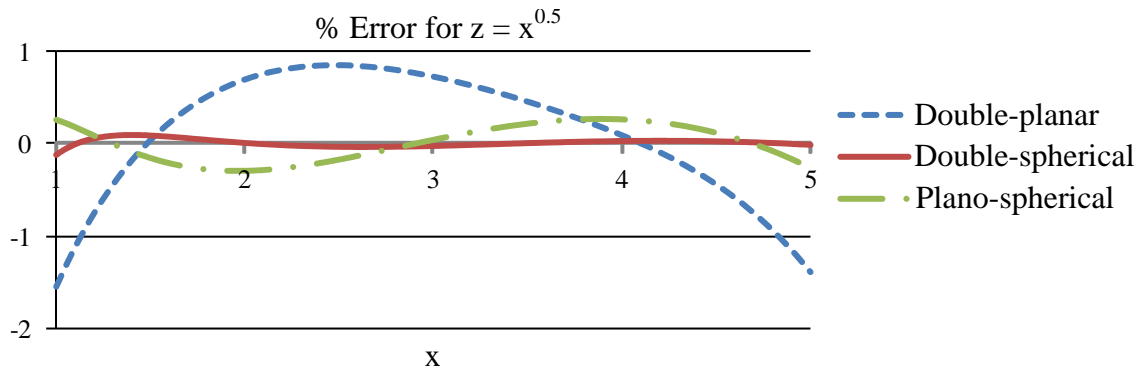

Fig. 5 Percentage error variations for $z = x^{0.5}$

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It is important to note that not every function can be generated with all of the Bennett linkages for arbitrary selections of the joint variable limits. In the above example, we found a proper set of joint variable limits so that the synthesis can be performed for all of the three linkages. If one is to work on just one of the linkages for a specific function, it is possible to decrease the maximum error to the levels of 0.01%. For example, using a double-spherical 6R linkage, for the same function as above, i.e. for $z = x^{0.5}$ for $1 \le x \le 5$, if we select $y(x) = x^{0.8}$, $126° \ge \phi \ge 59°$, $193° \le \theta \le 260°$ and $94° \le \psi \le 199°$, the maximum absolute error is as low as 0.0074%.

The second function we worked on is the $\sin(x)$ function for $45° \le x \le 60°$, which is decomposed as $y = \tan(x/2)$ and $z = 2y/(1+y^2)$. The reason we have such a short range for x is the constraint that we want to compare the three linkages for the same function and same joint variable limits. In this case, we employed the extra parameters ϕ_0 and θ_0 for the input and output angles, respectively. We used the same joint variable limits as above: $130° \ge \phi \ge 50°$, $210° \le \theta \le 270°$, $0.6 \le s \le 0.9$ and $110° \le \psi \le 200°$. In this case, the error values for the three mechanisms are very low and we do not need to present the error variation graph. The maximum error magnitude is $9.3 \times 10^{-4}\%$ for the double-spherical linkage, $9.2 \times 10^{-3}\%$ for the plano-spherical linkage and 0.042% for the double-planar linkage. As expected, as the number of design points increases, the accuracy of approximation increases. The construction parameters designed for the function generation of $z = x^{0.5}$ and $z = \sin(x)$ are listed in Table 1. Note that some link lengths for the plano-spherical linkage turn out to be negative. This does not mean that the construction of the mechanism is not possible. Referring to Fig. 4, a_8 is a directed dimension, so it can be negative, anyway. On the other hand, a_7 being computed negative means that the angle of this link should not be measured from +y direction to GF direction as in Fig. 4, but in the opposite direction. Practically this does not cause any problems. One can simply extend this link and get the measurement from the extension.

Table 1. Designed construction parameters for generation of $z = x^{0.5}$ and $z = \sin(x)$

Linkage	Function	ϕ_0	θ_0	Link lengths	$\lvert\%\,\text{Error}\rvert_{max}$
Double-planar	$z = x^{0.5}$	-	-	a = 0.45044, b = 0.6757, c = 0.65565, d = 0.32562, e = 0.575, f = 0.23706, g = 1	1.54%
Double-spherical	$z = x^{0.5}$	-	-	$\alpha_1 = 158.40°$, $\alpha_2 = 129.13°$, $\alpha_3 = 65.34°$, $\alpha_4 = 94.45°$, $\alpha_5 = 150.67°$, $\alpha_6 = 82.36°$, $\alpha_7 = 93.03°$, $\alpha_8 = 159.25°$	0.123%
Plano-spherical	$z = x^{0.5}$	-	-	$\alpha_1 = 158.40°$, $\alpha_2 = 129.13°$, $\alpha_3 = 65.34°$, $\alpha_4 = 94.45°$, $a_5 = 1$, $a_6 = 1.1770$, $a_7 = -0.5488$, $a_8 = -0.1790$	0.291%
Double-spherical*	$z = x^{0.5}$	-	-	$\alpha_1 = 156.20°$, $\alpha_2 = 274.79°$, $\alpha_3 = 77.03°$, $\alpha_4 = 323.18°$, $\alpha_5 = 351.84°$, $\alpha_6 = 93.15°$, $\alpha_7 = 279.21°$, $\alpha_8 = 172.37°$	0.0074%
Double-planar	$z = \sin(x)$	90.96°	–13.84°	a = 0.3854, b = 1.4708, c = 1.0943, d = 0.0708, e = 4.0310, f = 3.9615, g = 1	0.042%
Double-spherical	$z = \sin(x)$	172.88°	43.25°	$\alpha_1 = 194.54°$, $\alpha_2 = 62.24°$, $\alpha_3 = 126.61°$, $\alpha_4 = 51.52°$, $\alpha_5 = 174.46°$, $\alpha_6 = 171.44°$, $\alpha_7 = 172.04°$, $\alpha_8 = 175.94°$	$9.3 \times 10^{-4}\%$
Plano-spherical	$z = \sin(x)$	172.88°	135.52°	$\alpha_1 = 194.54°$, $\alpha_2 = 62.24°$, $\alpha_3 = 126.61°$, $\alpha_4 = 51.52°$, $a_5 = 1$, $a_6 = 1.3908$, $a_7 = 0.4618$, $a_8 = -0.0899$	$9.2 \times 10^{-3}\%$

*First three rows correspond to synthesis of different linkages for the same range of input/output variables for comparison reasons, whereas row 4 corresponds to synthesis of a double-spherical linkage for further minimized error by changing the input/output ranges.

As a further case study, we worked on a function used in Alizade and Kilit (2005) for comparison. In the paper, the authors generate $z = x^{0.6}$ for $1 \le x \le 5$ by means of a spherical four-bar mechanism with

five construction parameters: the four link lengths α_1, α_2, α_3, α_4 and the extra parameter θ_0 for the output angle. They used interpolation approximation with equally spaced five precision points and the limits are selected as $8° \leq \phi \leq 80°$, $5° \leq \theta \leq 160°$ for the input and the output joint variables, respectively. The variation in the percentage error is depicted in Fig. 6. In this case, the maximum absolute percentage error is 2.229%.
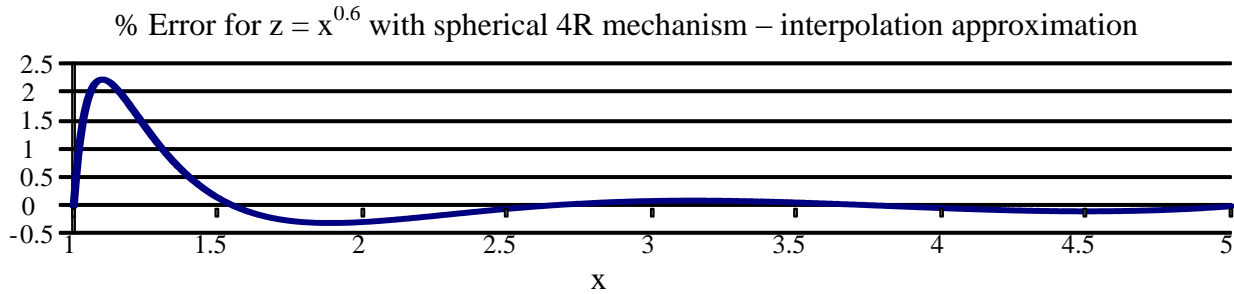
% Error for $z = x^{0.6}$ with spherical 4R mechanism – interpolation approximation

Fig. 6 Percentage error variation for $z = x^{0.6}$ for spherical four-bar mechanism with interpolation approximation

When we use Chebyshev approximation for the same function $z = x^{0.6}$ for $1 \leq x \leq 5$ with the spherical four-bar mechanism with same joint variable limits and five construction parameters, relatively lower error values are obtained as shown in Fig. 7. In this case, the maximum absolute error is 1.28%.
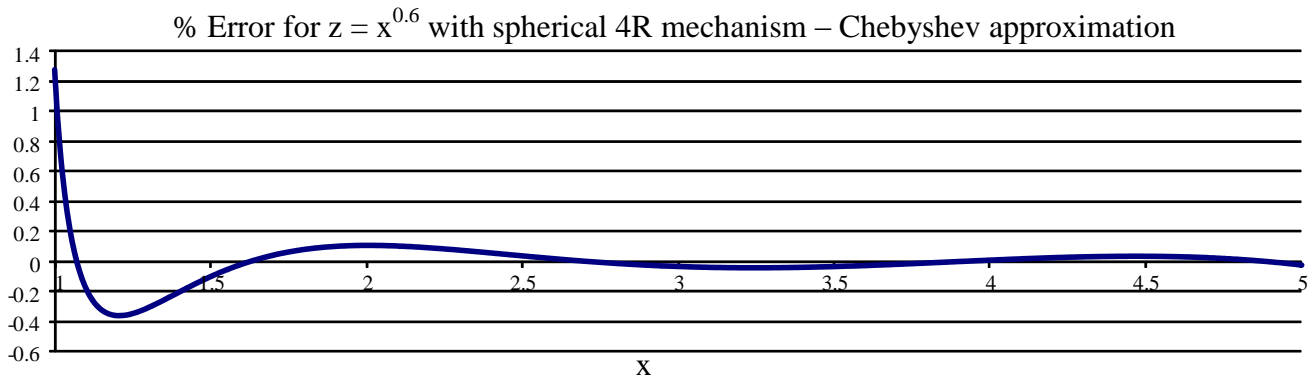
% Error for $z = x^{0.6}$ with spherical 4R mechanism – Chebyshev approximation

Fig. 7 Percentage error variation for $z = x^{0.6}$ for spherical four-bar mechanism with Chebyshev approximation

When we apply the Chebyshev approximation with the double spherical mechanism for the same function with the same input-output joint limits and choose $y = x^{0.75}$ and $75° \leq \psi \leq 160°$, the error variation in Fig. 8 is obtained. The maximum absolute error is 0.016%.

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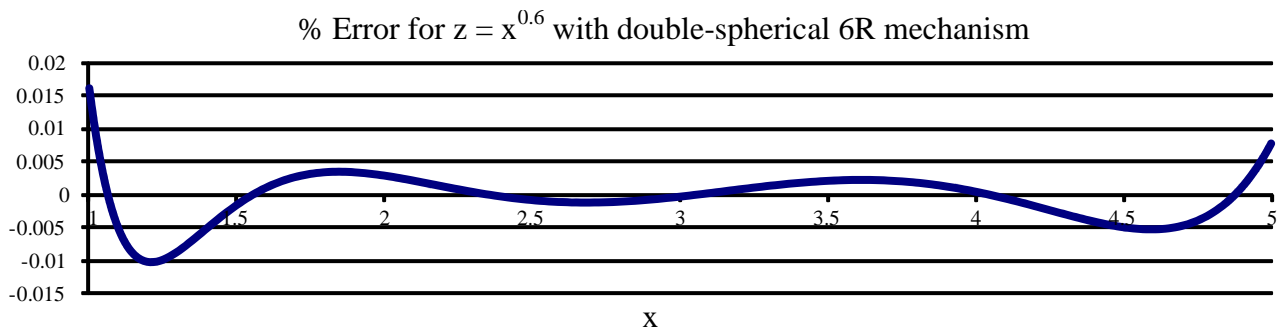% Error for $z = x^{0.6}$ with double-spherical 6R mechanism

Fig. 8 Percentage error variation for $z = x^{0.6}$ for double-spherical 6R mechanism with Chebyshev approximation

The designed construction parameters for the spherical four-bar and double-spherical 6R linkages are listed in Table 2.

Table 2. Designed construction parameters for generation of $z = x^{0.6}$

Linkage	Approximation	ϕ_0	θ_0	Link lengths	$|\%\text{Error}|_{max}$
Spherical 4R	Interpolation	-	11.03°	$\alpha_1 = 10.75°$, $\alpha_2 = 294.90°$, $\alpha_3 = 30.46°$, $\alpha_4 = 270.47°$	2.23%
Spherical 4R	Chebyshev	-	9.96°	$\alpha_1 = 271.92°$, $\alpha_2 = 87.00°$, $\alpha_3 = 34.01°$, $\alpha_4 = 142.30°$	1.28%
Double-spherical	Chebyshev	52°	–4.91°	$\alpha_1 = 10.75°, \alpha_2 = 114.9°, \alpha_3 = 28.34°, \alpha_4 = 93.17°,$ $\alpha_5 = 337.74°, \alpha_6 = 203.92°, \alpha_7 = 355.24°, \alpha_8 = 185.03°$	0.016%

As a final case study, we demonstrate design of the same type of linkage for three different types of functions: $z = x^a$, $z = a^x$ and $z = \sin(x)$. The worked-out functions are $z = x^{0.5}$ for $1 \le x \le 5$, $z = 2^x$ for $\pi/4 \le x \le 2\pi/3$, and $z = \sin(x)$ for $-1 \le x \le 1$. The variations of these functions in their respective ranges are illustrated in Fig. 9 in order to demonstrate the behavior of the functions to be generated.

$z = x^{0.5}$ for $1 \le x \le 5$

$z = 2^x$ for $-1 \le x \le 1$

a)

b)

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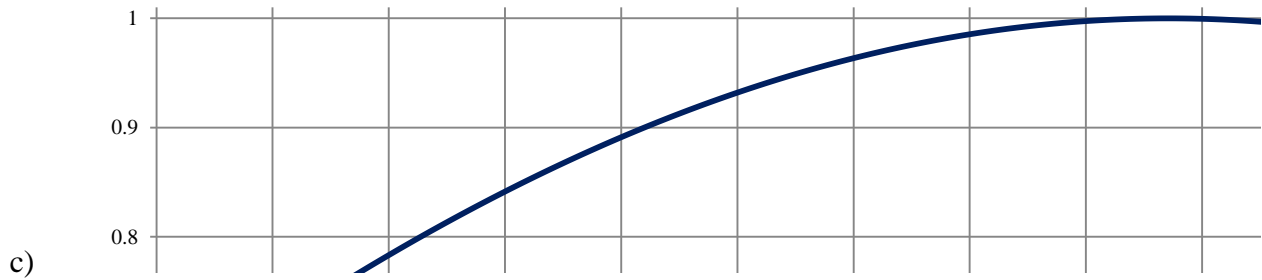
$$z = \sin(x) \text{ for } \pi/4 \le x \le 2\pi/3$$

c)
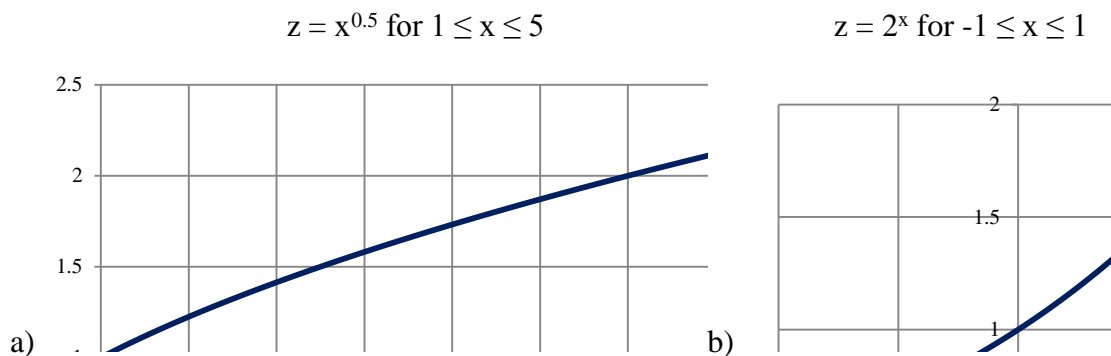
Fig. 9 a) $z = x^{0.5}$ for $1 \le x \le 5$, b) $z = 2^x$ for $-1 \le x \le 1$ and c) $z = \sin(x)$ for $\pi/4 \le x \le 2\pi/3$

A double-spherical 6R linkage with 8 construction parameters is designed for each of the three functions. For the function $z = x^{0.5}$ for $1 \le x \le 5$, we repeat the solution above with $y(x) = x^{0.8}$, $126° \ge \phi \ge 59°$, $193° \le \theta \le 260°$ and $94° \le \psi \le 199°$. The maximum absolute percentage error is as low as 0.0074%. For the function $z = 2^x$ for $-1 \le x \le 1$ we found a good design with $y = e^x$, $30° \le \phi \le 92°$, $34° \le \theta \le 82°$ and $210° \le \psi \le 249°$. In this case, the maximum absolute percentage error is about 0.65%. For the function $z = \sin(x)$ for $\pi/4 \le x \le 2\pi/3$ we chose $y = \tan(x/2)$, $81° \le \phi \le 165°$, $59° \le \theta \le 86.26°$, and $219° \le \psi \le 339°$. The maximum absolute percentage error is found as 1.4%. For all three functions we tried to minimize the maximum absolute percentage error. The results show that the performance of the function generator mechanism is highly dependent on the behavior of the function to be generated. The variation of the percentage errors in z for each of these functions are illustrated in Fig. 10. The designed construction parameters are given in Table 3.

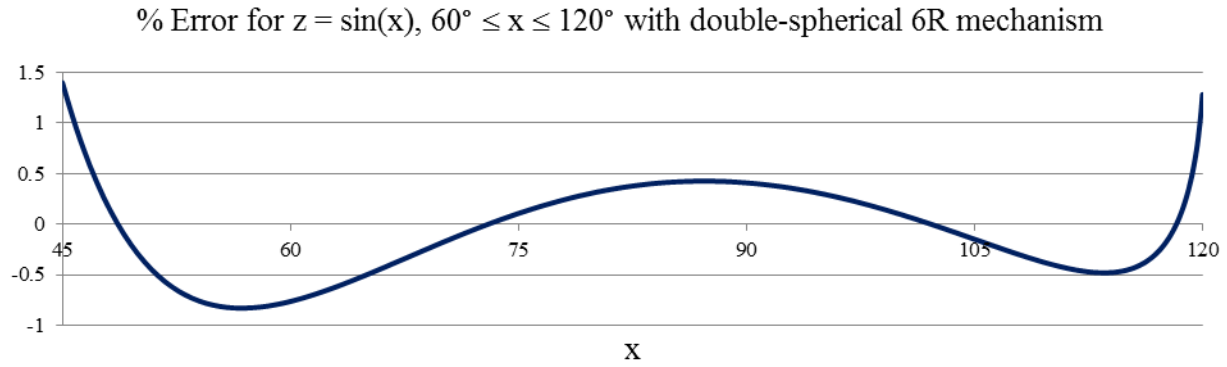% Error for $z = x^{0.5}$, $1 \le x \le 5$ with double-spherical 6R mechanism
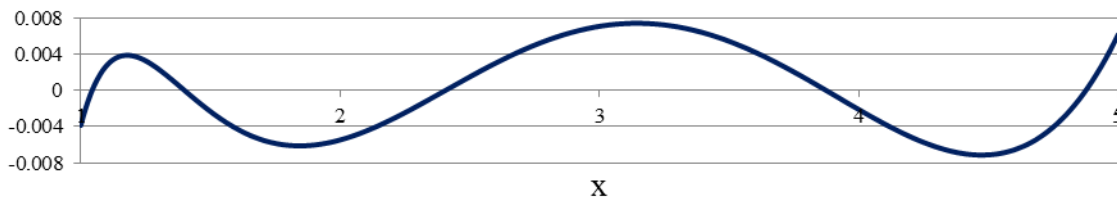
a)

% Error for $z = 2^x$, $-1 \le x \le 1$ with double-spherical 6R mechanism

b)

% Error for z = sin(x), 60° ≤ x ≤ 120° with double-spherical 6R mechanism
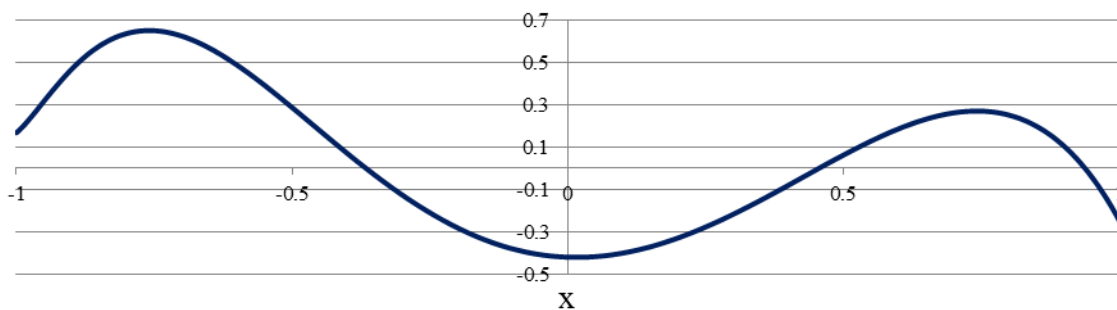
c)

Fig. 10 Percentage error variation for generation of a) $z = x^{0.5}$, $1 \leq x \leq 5$, b) $z = 2^x$, $-1 \leq x \leq 1$ and c) $z = \sin(x)$, $45° \leq x \leq 120°$

Table 3. Designed construction parameters for generation of $z = x^{0.5}$, $z = 2^x$ and $z = \sin(x)$

Function	Link lengths	$\lvert\%\,\text{Error}\rvert_{max}$
$z = x^{0.5}$, $1 \leq x \leq 5$	$\alpha_1 = 10.75°, \alpha_2 = 114.9°, \alpha_3 = 28.34°, \alpha_4 = 93.17°,$ $\alpha_5 = 337.74°, \alpha_6 = 203.92°, \alpha_7 = 355.24°, \alpha_8 = 185.03°$	0.0074%
$z = 2^x$, $-1 \leq x \leq 1$	$\alpha_1 = 150.51°, \alpha_2 = 145.82°, \alpha_3 = 127.57°, \alpha_4 = 14.50°,$ $\alpha_5 = 81.62°, \alpha_6 = 170.89°, \alpha_7 = 75.02°, \alpha_8 = 161.65°$	0.65%
$z = \sin(x)$, $45° \leq x \leq 120°$	$\alpha_1 = 61.49°, \alpha_2 = 62.51°, \alpha_3 = 64.63°, \alpha_4 = 57.60°,$ $\alpha_5 = 178.82°, \alpha_6 = 76.03°, \alpha_7 = 77.34°, \alpha_8 = 2.21°$	1.4%

The construction parameter values in Tables 1-2-3 are used to construct virtual models of 6R linkages and these models are tested to satisfy the input/output joint values.

6. Discussions and Conclusions

The single-loop Bennett 6R mechanisms possess much more construction parameters than the single-loop planar four-bar, spherical four-bar or planar slider-crank mechanisms and hence, they may be used for function approximation purposes with a relatively better accuracy. The method of decomposition makes it possible for us to analytically formulate the function synthesis problem for mechanisms with so many construction parameters. The relatively easy computer implementation of the formulation enables the designer to quickly work on several alternative designs and come up with an accurate function generator mechanism.

The case studies we presented in Section 5 illustrate how a designer can compare the three types of Bennett 6R mechanisms for the same function. Also the comparison of the spherical four-bar with the double-spherical 6R mechanism clearly shows that the accuracy is improved when the 6R linkage is used for generating the same function. The last case study shows how the performance of the same type of linkage may be affected for different functions to be generated.

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