APPROXIMATION THEOREMS FOR KRULL DOMAINS

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ABSTRACT

APPROXIMATION THEOREMS FOR KRULL DOMAINS

Let R be an integrally closed domain, and denote by $I(R)$ the multiplicative group of all invertible fractional ideals of R. Let $\{V_i\}_{i\in I}$ be the family of valuation overrings of R, and denote by G_i the corresponding value group of the valuation domain V_i. We show that $R = \bigcap_{i\in I} V_i$, and there is a map from $I(R)$ *into* $\prod_{i\in I} G_i$, the cardinal product of the G_i's. Furthermore, it is well known when R is a Dedekind domain, this map becomes an isomorphism *onto* $\coprod_{i\in I} G_i$, the cardinal sum of the G_i's. In this case, $G_i \cong \mathbb{Z}$ for each i. It is shown, by J. Brewer and L. Klingler, that this map is also an isomorphism *onto* $\coprod_{i\in I} G_i$ when R is an h-local Prüfer domain. In this thesis, we investigate the existence of such a map, and whether it is injective when R is a Krull domain.

ÖZET

KRULL TAMLIK BÖLGELERİ İÇİN YAKLAŞIM TEOREMLERİ

R bir tamsayıca kapalı tamlık bölgesi olsun ve $I(R)$, R'nin terslenebilir kesirli ideallerinin çarpımsal grubunu belirtsin. $\{V_i\}_{i\in I}$, R'nin valüasyon üsthalkalarının ailesi olsun ve G_i, V_i valüasyon tamlık bölgesine karşılık gelen değer grubunu belirtsin. Biz $R = \bigcap_{i\in I} V_i$ eşitliğini ve $I(R)$'den G_i'lerin kardinal çarpımının içine, yani $\prod_{i\in I} G_i$'nin içine bir fonksiyon olduğunu gösterdik. R bir Dedekind tamlık bölgesi olduğunda bu fonksiyonun G_i'lerin kardinal toplamına, yani $\coprod_{i\in I} G_i$'ye örten bir izomorfizma olduğu bilinmektedir. Bu durumda her bir i için G_i tam sayılar grubuna izomorftur. R bir h-yerel Prüfer tamlık bölgesi olduğunda bu fonksiyonun G_i'lerin kardinal toplamına örten bir izomorfizma olduğu J. Brewer ve L. Klingler tarafından gösterilmiştir. Bu tezde, R bir Krull tamlık bölgesi olduğunda böyle bir fonksiyonun varlığı ve birebirliği araştırılmıştır.

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LIST OF SYMBOLS

R	a commutative domain with identity element 1
Q	quotient field of R
$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the ring of integers
$\mathbb{Q}$	the ring of rational numbers
$R^{\times}$	$R - \{0\}$
R_P	localization of R at a prime ideal P
$F(R)$	the set of all fractional ideals of R
$I(R)$	the set of all invertible fractional ideals of R
I_v	divisorial ideal in class of I
$\mathrm{div}(I)$	divisor represented by the ideal I
$D(R)$	the set of all divisors of R
$\prod$	cardinal product
$\coprod$	cardinal sum
$\bigoplus$	direct sum
$\leq$	submodule
G^+	positive elements of G
$\mathrm{Min}\, G^+$	minimal elements amongst strictly positive elements of G

CHAPTER 1

INTRODUCTION

Throughout this thesis, R is an integral domain with quotient field Q unless otherwise stated. We will denote by $F(R)$ the multiplicative monoid of all fractional ideals of R and by $I(R)$ the multiplicative group of all invertible fractional ideals of R.

Definition 1.1 *A Dedekind domain is a Noetherian, integrally closed, integral domain of Krull dimension 1, i.e., every non-zero prime ideal is maximal.*

If R is a Dedekind domain, then all fractional ideals of R are invertible, i.e. $F(R) = I(R)$. Furthermore, if $\{M_i\}_{i\in I}$ is the set of all maximal ideals of a Dedekind domain R, then each non-zero fractional ideal of R can be written uniquely in the form $A = M_{i_1}^{z_{i_1}} \cdots M_{i_n}^{z_{i_n}}$, and the mapping $A \to (z_{i_1}, \ldots, z_{i_n})$ is an order isomorphism from $I(R)$ *onto* the cardinal sum $\coprod_{i\in I} \mathbb{Z}_i$, where $\mathbb{Z}_i \cong \mathbb{Z}$ for each i.

If R is the ring of integers, then any ideal $m\mathbb{Z}$ of $\mathbb{Z}$ can be written in the form $m\mathbb{Z} = (p_1\mathbb{Z})^{z_1} \cdots (p_n\mathbb{Z})^{z_n}$, where $m = p_1^{z_1} \cdots p_n^{z_n}$, $z_1, \ldots, z_n \in \mathbb{Z}$, and $p_1, \ldots, p_n$ are different prime integers. This is a well-known example of the above fact about Dedekind domains.

One can drop both the Noetherian and the one-dimensional assumptions and consider $I(R)$ when R is a Prüfer domain of finite character. A similar fact will become true if any non-zero prime ideal of R contained in one maximal ideal of R. It was proven by James Brewer and Lee Klingler in the paper ”The Ordered Group of Invertible Ideals of a Prüfer Domain of Finite Character” by using two important approximation theorems for Prüfer domains.

In Chapter 2 we mention about fundamental properties of fractional ideals, integrally and completely integrally closed domains which will be useful for our work. We also give the definitions and some specific properties of totally ordered and lattice-ordered groups. For further information and proofs we refer to (L. Fuchs & L. Salce), (M.F. Atiyah & I.G. Macdonald), (R. Gilmer).

In Chapter 3 we give definitions and properties of valuations and valuation domains. In addition, we give the fact that every integrally closed domain is the intersection of its valuation overrings. This fact gives us an order-preserving isomorphism from $I(R)$ *into* $\prod_{i\in I} G_i$, the cardinal product of value groups of valuation overrings of an integrally closed domain R.

In Chapter 4 we give the definition and properties of Prüfer domains. Also, we give the ”Strong Approximation Theorem” and the ”Very Strong Approximation Theorem” for Prüfer domains which are used to prove that there is an isomorphism from the group of all invertible fractional ideals of R *onto* the cardinal direct sum of value groups of valuation overrings of R, where R is a Prüfer domain, if and only if every nonzero element of R is contained in but a finite number of maximal ideals and every nonzero prime ideal of R is contained in only one maximal ideal. For further information and proofs we refer to (J. Brewer & L. Klingler, 2005).

In Chapter 5 it is proven that a Krull domain R is the intersection of its discrete rank 1 valuation overrings, which are exactly the localizations of R at its minimal prime ideals. Furthermore, we have defined and proved the "Strong Approximation Theorem" and the "Very Strong Approximation Theorem" for Krull domains.

CHAPTER 2

PRELIMINARIES

This chapter consists of some preliminary information about fractional ideals and integral dependence. Also, totally ordered and lattice-ordered groups structures ,which are used in following chapters, are given.

2.1. Fractional Ideals

Let R be an integral domain with the quotient field Q.

Definition 2.1 *A fractional ideal of an integral domain R is an R-submodule J of Q such that $rJ \leq R$ for some non-zero $r \in R$.*

Remark 2.1 ***(1)*** *An R-submodule of Q is a fractional ideal if and only if it is isomorphic to an ideal of R.*

(1) *The ideals of R are clearly fractional ideals, and they are called integral ideals.*

(3) *A finitely generated submodule of Q is a fractional ideal.*

For R-submodules I and J of Q, we already have two binary operations; sum of I and J, $I+J$ and intersection of I and J, $I \cap J$. In addition, we define two more binary operations, which are called the product and the residual, respectively:

$$IJ = \{\sum_{i=1}^{n} a_i b_i | a_i \in I, b_i \in J, n < w\} \text{ and } I : J = \{a \in Q | aJ \leq I\}.$$

We list a few properties of the operations mentioned above. Let I, J and K be R-submodules of Q; then:

(i) $I(J+K) = IJ + IK$;

(ii) $I : (J+K) = (I : J) \cap (I : K)$;

(iii) $(I \cap J) : K = (I : K) \cap (J : K)$;

(iv) $(I : J) : K = I : JK = (I : K) : J$;

(v) $I(IJ : I) = IJ$;

(vi) $(I \cap J) + (I \cap K) \leq I \cap (J + K)$;

(vii) $I(J \cap K) \leq IJ \cap IK$;

(vii) $(I : K) + (J : K) \leq (I + J) : K$.

Furthermore, properties (i), (ii) and (iii) can be extended to infinite sums and infinite intersections:

(i)′ $I(\sum_{\lambda\in\Lambda} J_\lambda) = \sum_{\lambda\in\Lambda} IJ_\lambda$;

(ii)′ $I : (\sum_{\lambda\in\Lambda} J_\lambda) = \bigcap_{\lambda\in\Lambda}(I : J_\lambda)$;

(iii)′ $(\sum_{\lambda\in\Lambda} J_\lambda) : I = \bigcap_{\lambda\in\Lambda}(J_\lambda : I)$.

The set of all non-zero fractional ideals of R is denoted by $F(R)$, and it becomes a multiplicative monoid with the associative binary operation product and the identity element R.

Definition 2.2 *A non-zero fractional ideal I of R is said to be invertible if it is invertible as an element of the multiplicative monoid $F(R)$. In other words, there exists a $J \in F(R)$ such that $IJ = JI = R$.*

The inverse of a fractional ideal I of R is unique and denoted by I^{-1}. The set of all invertible fractional ideals of R is denoted by $I(R)$, and it becomes a multiplicative group which is a submonoid of the monoid $F(R)$.

Definition 2.3 *A ring R is a local ring if it has only one maximal ideal, equivalently, if r or $1 - r$ is a unit for any $r \in R$. In addition, if R has finitely many maximal ideals, then it is called a semilocal ring.*

Proposition 2.1 *((L. Fuchs & L. Salce), Proposition I.2.5) Let I be an invertible fractional ideal of a domain R. Then:*

(a) *$I^{-1} = R : I$;*

(b) *I is finitely generated;*

(c) *if R is semilocal, then I is a principal ideal; moreover, if R is local, every generating set of I contains an element generating I;*

(d) *if I is an integral ideal and P is a minimal prime of I, then P is a minimal prime of some generator in any generating set of I;*

- ***(e)*** *If I is an integral ideal and there is an $a \in I$ contained in finitely many maximal ideals, then $I = aR + bR$ for some $b \in R$.*

Proposition 2.2 *((L. Fuchs & L. Salce), Proposition I.2.7) A finitely generated ideal I of an integral domain R is invertible if and only if IR_M is invertible for all maximal ideals M of R.*

Definition 2.4 *A non-zero fractional ideal I of an integral domain R is said to be divisorial if $I = R : (R : I)$.*

Proposition 2.3 *((L. Fuchs & L. Salce), Proposition I.2.9) Let R be a domain and $I \in F(R)$. Then*

- ***(a)*** *$R : I$ is divisorial for all I;*
- ***(b)*** *$R : (R : I) = \bigcap\{aR | I \leq aR\}$;*
- ***(c)*** *I is divisorial if and only if $I = \bigcap\{aR | I \leq aR\}$;*
- ***(d)*** *if I is divisorial, then $I : J$ is divisorial for all $J \in F(R)$.*

2.2. Integrally Closed Domains

Definition 2.5 *Let T be an integral domain and R a subring of T. An element $x \in T$ is said to be integral over R if there exists a monic polynomial $f \in R[x]$ such that $f(x) = 0$.*

Theorem 2.1 *((L. Fuchs & L. Salce), Theorem I.3.1) Let T be an integral domain, R a subring of T, and $x \in T$. The following are equivalent:*

- ***(a)*** *x is integral over R;*
- ***(b)*** *the subring $R[x]$ of T, which is generated by R and x, is a finitely generated R-module;*
- ***(c)*** *there is a subring S of T containing x, which is finitely generated as an R-module.*

Corollary 2.1 *((L. Fuchs & L. Salce), Corollary I.3.2) The elements of an integral domain T, which are integral over a subring R, form a subring containing R.*

The subring of T mentioned in Corollary 2.1 is called the **integral closure of R in T**. If each element of T is integral over R, we say that T is **integral over** R, and if R coincides with its integral closure in T, we say that R **is integrally closed in** T. Moreover, R is an **integrally closed domain** if it is integrally closed in its quotient field.

Lemma 2.1 *((L. Fuchs & L. Salce), Exercise I.3.1) Let R be a domain with quotient field Q. Suppose that $0 \neq x \in Q$. Then the element x^{-1} is integral over R if and only if $x^{-1} \in R[x]$.*

Proof Let $0 \neq x \in Q$. Suppose that x^{-1} is integral over R. Then we have $a_1 + a_2x^{-1} + \cdots + a_nx^{-n+1} + x^{-n} = 0$, $a_i \in R$ for $1 \leq i \leq n$. Now, if we multiply the previous equation by x^{n-1}, we will have $a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n + x^{-1} = 0$. This gives $x^{-1} = -(a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n) \in R[x]$. Conversely, suppose that $x^{-1} \in R[x]$. Then we can write $x^{-1} = a_0 + a_1x + \cdots + a_nx^n$, $a_i \in R$, $0 \leq i \leq n$. After multiplying both sides of the equation by $(x^{-1})^n$, we have $0 = a_n + a_{n-1}x^{-1} + a_{n-2}(x^{-1})^2 + \cdots + a_0(x^{-1})^n - (x^{-1})^{n+1}$. Hence, x^{-1} is integral over R. $\square$

Definition 2.6 *An integral domain R is called a GCD-domain if every pair a, b of elements has a greatest common divisor g.c.d$(a, b) = d$, i.e., $d|a, b$ and $c|a, b$ implies $c|d$.*

Proposition 2.4 *((L. Fuchs & L. Salce), Proposition I.3.4) GCD-domains are integrally closed.*

2.3. Completely Integrally Closed Domains

Definition 2.7 *Let T be an integral domain containing the subring R. An element $x \in T$ is called almost integral over R if there is a finitely generated R-submodule of T containing $R[x]$.*

Remark 2.2 *By Theorem 2.1(c), if $x \in T$ is integral over R, then x is almost integral over R as well.*

The set of elements of an integral domain T, which are almost integral over R, form a subring of T is called the **complete integral closure of** R in T. If R coincides with its complete integral closure in T, R is called **completely integrally closed in** T. Furthermore, R is a completely integrally closed domain if it is completely integrally closed in its quotient field.

Proposition 2.5 *((L. Fuchs & L. Salce), Proposition I.3.9) Let R be an integral domain and Q its quotient field. Then $x \in Q$ is almost integral over R if and only if there is an element $r \in R^{\times}$ such that $rx^n \in R$ for all $n \in \mathbb{N}$.*

There is a relation, which has an attractive role for completely integrally closed domains, for non-zero fractional ideals I, J in $F(R)$, defined by $I \sim J$ if and only if $R : I = R : J$. Using Proposition 2.3(a), it follows that:

(1) this is a congruence relation in $F(R)$;

(2) I is congruent to the divisorial ideal $I_v = R : (R : I)$;

(3) distinct divisorial ideals are incongruent.

The equivalence classes under $\sim$ are called **divisors**. The divisor containing the fractional ideal I is denoted by $\mathrm{div}(I)$, and the set of all divisors is denoted by $D(R)$. Also, we will use $\mathrm{div}(x)$ for the divisor containing principal fractional ideal xR, where $0 \neq x \in Q$.

The set of all divisors $D(R)$ becomes an additive monoid under the operation $\mathrm{div}(I)+\mathrm{div}(J) = \mathrm{div}(IJ)$ with the identity $\mathrm{div}(R)$. This operation is well-defined since $R : IJ = (R : I) : J = (R : I_v) : J = (R : J) : I_v = (R : J_v) : I_v = R : I_vJ_v$ implies that $\mathrm{div}(I) + \mathrm{div}(J) = \mathrm{div}(I_v) + \mathrm{div}(J_v)$. Moreover, $D(R)$ is a partially ordered monoid by defining $\mathrm{div}(I) \leq \mathrm{div}(J)$ if $R : I \leq R : J$.

Proposition 2.6 *((L. Fuchs & L. Salce), Proposition I.3.11) If I and J are non-zero fractional ideals of an integral domain R, the supremum of $\mathrm{div}(I)$ and $\mathrm{div}(J)$ is $\mathrm{div}(I \cap J)$, while their infimum is $\mathrm{div}(I + J)$.*

Proposition 2.6 shows that the monoid $D(R)$ is lattice-ordered. However, next theorem proves that $D(R)$ becomes a lattice-ordered group if R is an integrally closed domain.

Proposition 2.7 *((L. Fuchs & L. Salce), Proposition I.3.10) Each non-unit r of a completely integrally closed domain R satisfies $\bigcap_{n\in\mathbb{N}} r^nR = 0$.*

Theorem 2.2 *((L. Fuchs & L. Salce), Theorem I.3.12) The monoid $D(R)$ of the divisors of R is a lattice ordered group if and only if R is completely integrally closed.*

2.4. Totally Ordered and Lattice-Ordered Groups

Definition 2.8 *An abelian group G, which is a totally ordered set under a binary relation $\leq$, is called a totally ordered group if it satisfies that $a \leq b$ implies $a+c \leq b+c$ for all $a, b, c \in G$.*

An element a of a totally ordered group G is called **positive** or **strictly positive** if $a \geq 0$ or $a > 0$, respectively. The set of all the positive elements of G is called **positivity domain of** G and it is denoted by G^+.

Definition 2.9 *A subgroup H of a totally ordered group G is said to be convex or isolated if $a < g < b$ with $a, b \in H$, $g \in G$ implies that $g \in H$.*

The order type of the set of all proper convex subgroups of G is called the **rank of** G.

Definition 2.10 *An abelian group G, which is partially ordered set under a binary relation $\leq$, is called a lattice-ordered group if any two elements of G have a least upper bound, i.e., given $a, b \in G$, there is $g \in G$ such that $a \leq g$ and $b \leq g$.*

Let G be a partially ordered set. We say that G is **filtered** if for any $a, b \in G$, there exists $c \in G$ such that $a \leq c$ and $b \leq c$ (or, equivalently, $c \leq a$ and $c \leq b$).

Theorem 2.3 *((R. Gilmer), Theorem 15.4.(1)) Let G be a partially ordered and filtered abelian group. Then the following are equivalent:*

(1) *G is lattice-ordered;*

(2) *$\sup(a, b)$ exists for all $a, b \in G^+$;*

(3) *$\inf(a, b)$ exists for all $a, b \in G^+$.*

The set of minimal elements amongst the strictly positive elements of G is denoted by $\operatorname{Min} G^+$ and for $\gamma \in G$, $\gamma \in \operatorname{Min} G^+$ if and only if $\gamma > 0$ and there is no $\beta \in \operatorname{Min} G^+$ with $0 < \beta < \gamma$.

Lemma 2.2 *((L. Fuchs & L. Salce), Lemma III.4.8) Let G be a lattice-ordered group and $\gamma \in \operatorname{Min} G^+$. If $\gamma \leq \alpha_1 + \cdots + \alpha_n$ with $0 < \alpha_i$ for $1 \leq i \leq n$, then $\gamma \leq \alpha_i$ for some $i \in \{1, \ldots, n\}$.*

Theorem 2.4 *((L. Fuchs & L. Salce), Theorem III.4.9) Let G be a lattice-ordered abelian group such that every non-empty set of positive elements contains a minimal member. Then G is order-isomorphic to the free group $\bigoplus_{\gamma \in \operatorname{Min} G^+} \gamma\mathbb{Z}$ endowed by the pointwise ordering.*

Corollary 2.2 *((L. Fuchs & L. Salce), Corollary III.4.10) Let G be a lattice-ordered subgroup of a free abelian group F lattice-ordered by the pointwise ordering. Then $G \cong \bigoplus_{\gamma \in \operatorname{Min} G^+} \gamma\mathbb{Z}$.*

CHAPTER 3

VALUATION DOMAINS

In this chapter we review the most useful properties of valuation domains which play a distinguished role in our discussions in latter chapters. We will see for every valuation domain R, there exists a valuation from the quotient field Q of R to a totally ordered value group, and this valuation satisfies that elements of Q, which have non-negative values, are exactly from the domain R. Furthermore, in the last section of this chapter, we have mentioned that there is an isomorphism from the set of invertible ideals of an integrally closed domain R *into* the cardinal product of the value groups of valuation overrings of R.

3.1. Fundamental Properties of Valuation Domains

Proposition 3.1 *The following are equivalent for a ring R:*

(a) *for all ideals A,B of R, $A \subseteq B$ or $B \subseteq A$;*

(b) *for all elements a,b of R, $aR \subseteq bR$ or $bR \subseteq aR$.*

Proof $(a \Rightarrow b)$ It is clear.

$(b \Rightarrow a)$ Suppose $A \nsubseteq B$ for any given two ideals of R. Then there is an element $a \in A - B$. Now for all $b \in B$, $bR \subseteq aR$ since $aR \nsubseteq B$. So, $B \subseteq aR \subseteq A$.

□

Definition 3.1 *When a ring R satisfies the equivalent conditions in Proposition 3.1, it is called a valuation ring. A valuation ring which is an integral domain will be called a valuation domain.*

Proposition 3.2 *Let R be an integral domain with the quotient field Q. Then R is a valuation domain if and only if for all $0 \neq x \in Q, x \in R$ or $x^{-1} \in R$.*

Proof $(\Rightarrow)$ Let us take $x \in Q$. Then we can write $x = ab^{-1}$ for some $a, b \in R$ with $b \neq 0$. Since R is a valuation domain, we have $aR \subseteq bR$ or $bR \subseteq aR$, i.e., $a = br_1$, or $b = ar_2$ for some $r_1, r_2 \in R$. Therefore, $x = br_1b^{-1} = r_1$ or $x = aa^{-1}r_2^{-1} = r_2^{-1}$, that is, $x \in R$ or $x^{-1} \in R$.

$(\Leftarrow)$ Let us take any two elements $a, b \in R$, and set $x = ab^{-1} \in Q$. Then, by assumption, $x \in R$ or $x^{-1} \in R$. So, we have $a \in bR$ or $b \in aR$. Thus, R is a valuation domain. □

Example 3.1 *The localization $\mathbb{Z}_p = \{p^k \frac{a}{b} \in \mathbb{Q} : p \nmid b, p \nmid a, k \in \mathbb{Z}^+ \cup \{0\}\}$ of the ring of integers at a prime ideal $p\mathbb{Z}$ is a valuation domain:*

Let us take two different elements $q_1 = p^{k_1}\frac{a}{b}, q_2 = p^{k_2}\frac{c}{d} \in \mathbb{Z}_p$. If $k_1 \geq k_2$, then $q_1 = q_2 p^{k_1-k_2}\frac{ad}{bc}$ or if $k_2 \geq k_1$, then $q_2 = q_1 p^{k_2-k_1}\frac{cb}{da}$. So, for the principal ideals (q_1) and (q_2) of $\mathbb{Z}_p$, we have $(q_1) \subseteq (q_2)$ or $(q_2) \subseteq (q_1)$. Therefore, $\mathbb{Z}_p$ is a valuation domain.

Remark 3.1 ***(1)*** *A valuation ring R is local, i.e., R has a unique maximal ideal.*

(2) *If R is a valuation ring and $I \nsubseteq R$ is an ideal of R, then R/I is a valuation ring as well.*

(3) *If R is a valuation ring and $S \subsetneq R$ is a multiplicatively closed subset of R such that $0 \notin S$ and $1 \in S$, then $S^{-1}R$ is a valuation ring as well.*

Lemma 3.1 *((L. Fuchs & L. Salce), Lemma II.1.3) For a valuation domain R, we have:*

(a) *finitely generated ideals are principal;*

(b) *the only principal ideals which can possibly be primes are P and 0;*

(c) *for a proper ideal I of R, either $I^n = 0$ for some $n \in \mathbb{N}$ or the intersection $J = \bigcap_{n\in\mathbb{N}} I^n$ is a prime ideal of R.*

Before we prove a fact, we need to remember an important lemma which is called Nakayama's Lemma: Let M be a finitely generated R-module and I an ideal contained in the Jacobson radical of R, i.e., contained in the intersection of all maximal ideals of R. If $IM = M$, then $M = 0$.

Lemma 3.2 *Let R be a valuation domain with the maximal ideal M. Then:*

(a) *R is Noetherian, but not Artinian if and only if its non-zero ideals are: $R > M = pR > \ldots > p^nR > \ldots$ for $n \in \mathbb{N}$. In this case, R is a principal ideal domain;*

(b) *R is Artinian if and only if it has finitely many ideals which are all principal: $R > M = pR > \ldots > p^nR = 0$ for some $p \in R$ and $n \in \mathbb{N}$.*

Proof To prove (a), suppose R is Noetherian. Then each ideal of R is finitely generated. So, by Lemma 3.1(a), R is a principal ideal domain. Thus, the maximal ideal $M = pR$ for some $p \in R$. Now, we need to show that any non-zero proper ideal I of R is of the form p^kR. Also, since R is a principal ideal domain, there is an element $a \in R$ such that $I = aR$. Then $a \in M = pR$, i.e., $a = pa_1$ for some $a_1 \in R$. If a_1 is a unit of R, then $p = aa_1^{-1}$ which implies that $I = aR = pR = M$. If a_1 is not a unit of R, then $a_1 \in pR$, i.e., $a_1 = pa_2$ for some $a_2 \in R$. Furthermore, if we proceed this pattern, we will have a chain of ideals

$I = aR \subsetneq a_1R \subsetneq a_2R \subsetneq \ldots$. Ideals in the chain are not equal because if $a_nR = a_{n+1}R$ then $a_n = pa_{n+1}$. However, by Nakayama's Lemma, $a_nR = Ma_{n+1}R$ implies $a_n = 0$, i.e., $a_{n+1} = 0$. So, $I = 0$, which is impossible. Since R is Noetherian, this chain must terminate at an ideal a_nR, i.e., $a_nR = a_{n+1}R = \ldots$ for some $n \in \mathbb{N}$. This can only happen when a_n is a unit in R. Then we have $a = p^n a_n$. Hence, $I = aR = p^nR$. Conversely, the assumptions on R satisfies the maximum condition on ideals of R. Clearly, R is Noetherian.

To prove (b), suppose R is Artinian. Then, because of the fact that Artinian rings are Noetherian, by (a), R has ideals of the form $R > M = pR > \ldots > p^nR > \ldots$ for $n \in \mathbb{N}$. This chain must terminate at a finite step, i.e., $R > M = pR > \ldots > p^nR = p^{n+1} = \ldots$ for some $n \in \mathbb{N}$, since R is Artinian. $p^nR = p^{n+1}R$ gives the equality $p^n = p^{n+1}r$ for some $r \in R$. Then $p^n(1-pr) = 0$. Consequently, $p^n = 0$ since $1-pr$ is a unit by definition of local ring. Thus, R has finitely many ideals which are principal: $R > M = pR > \ldots > p^nR = 0$ for some $n \in \mathbb{N}$. Conversely, the assupmtions on R satisfies the minimum condition on ideals of R. Hence, R is Artinian. □

Definition 3.2 *An R-module M is called uniserial if all submodules of M are totally ordered under inclusion, in other words; for all $m_1, m_2 \in M$, either $m_1R \leq m_2R$ or $m_2R \leq m_1R$.*

Lemma 3.3 *((L. Fuchs & L. Salce), Lemma II.1.4) If R is a valuation domain, then*

(a) *its quotient field Q is a uniserial R-module;*

(b) *every proper submodule of Q is a fractional ideal of R.*

Definition 3.3 *An overring of R is a subring of the quotient field Q of R which contains R.*

Proposition 3.3 *((L. Fuchs & L. Salce), Proposition II.1.5) Let R be a valuation domain. A subring S of Q is an overring of R if and only if $S = R_P$ for some prime ideal P of R. It is necessarily a valuation domain.*

3.2. Valuations

Definition 3.4 *Let K be a field, G a totally ordered abelian group, and ∞ a symbol which is regarded to be larger than any element of G. We set for every $g \in G, g + \infty = \infty + \infty = \infty$. Then a map $v : K \to G \cup \infty$ is said to be a valuation if it satisfies:*

V1. $v(x) = \infty$ *for* $x \in K$ *if and only if* $x = 0$;

V2. $v(x.y) = v(x) + v(y)$ *for all* $x, y \in K$;

V3. $v(x+y) \geq \min\{v(x), v(y)\}$ *for all* $x, y \in K$.

The subset $R_v = \{x \in K | v(x) \geq 0\}$ of K is a ring, which is called the **valuation ring of** v, the maximal ideal of R_v is given by the set $P_v = \{x \in K | v(x) > 0\}$. Moreover, G is called the **value group** of R, and the rank of G is the **rank** of the valuation ring R.

If the value group G is isomorphic to the additive group of integers $\mathbb{Z}$, then the valuation v is called **discrete valuation**, and the valuation ring R_v is called **discrete valuation ring**.

Remark 3.2 *$a|b$ holds for two elements a, b of R_v if and only if $v(a) \leq v(b)$:*

Since $a|b$, $b = ar$ for some $r \in R$. Then $v(b) = v(a) + v(r)$, and hence $v(b) \geq v(a)$. Conversely, if $v(b) - v(a) \geq 0$, then $v(ba^{-1}) \geq 0$ implying that $ba^{-1} \in R$, and hence $ba^{-1} = r$ for some $r \in R$, that is $b = ar$ for some $r \in R$. Thus, $a|b$.

Theorem 3.1 *((L. Fuchs & L. Salce), Theorem II.3.1) Every valuation domain R is the valuation ring R_v of a valuation v of its quotient field.*

Proof Let R be an integral domain with the quotient field Q. The set of invertible elements of R, which is denoted by U, is a subgroup of the multiplicative group $Q^\times$. Setting $aU \leq bU$ for $a, b \in Q^\times$ if and only if $ba^{-1} \in R$ shows that the group $G = Q^\times/U$ becomes a partially ordered abelian group. The positive elements of G by the partial order "$\leq$" correspond to the cosets aU, where $a \in R$. Also, the canonical surjection $v : Q^\times \to G$ satisfies $v(a) \geq v(c)$ and $v(b) \geq v(c)$ implying that $v(a+b) \geq v(c)$ for $a, b, c, a+b \in Q^\times$, so $c|a$ and $c|b$ imply $c|(a+b)$. If we had taken the domain R as a valuation domain, G would have become a totally ordered group since for $aU, bU \in G$, where $a, b \in Q^\times$, $ab^{-1} \in R$, or its inverse $a^{-1}b \in R$, i.e., $aU \leq bU$ or $bU \leq aU$. Furthermore, we would rather view the operation on G as addition, and we could extend v to Q by defining $v(0) = \infty$. Then v clearly satisfies the properties **V1**, **V2** and **V3**, and the corresponding ring R_v coincides with the domain R. $\square$

Example 3.2 *The map $v : \mathbb{Q} \to \mathbb{Z}$ defined by $v(p^n \frac{a}{b}) = n$, where p is a prime integer and $a, b \in \mathbb{Z}$ such that gcd$(ab, p) = 1$, is a valuation on $\mathbb{Q}$. The valuation ring $R_v = \{x \in \mathbb{Q} | v(x) \geq 0\} = \{x \in \mathbb{Q} | x = \frac{a}{b}$ with gcd$(a, b) = 1$ and $p \nmid b\}$, where $a, b \in \mathbb{Z}$, coincides with the localization $\mathbb{Z}_p$ of the ring of integers at a prime ideal $p\mathbb{Z}$.*

Lemma 3.4 *((L. Fuchs & L. Salce), Exercise II.3.2) Let v be a valuation of a field K and R_v its valuation ring. Then:*

(a) $v(x) = 0$ *if and only if x is a unit in R_v;*

(b) *for $x, y \in K$ $v(x) = v(y)$ if and only if $xR_v = yR_v$;*

(c) *K is the quotient field of R_v.*

Proof To prove (a), firstly we need to see that $v(1) = v(1.1) = v(1) + v(1)$ implies $v(1) = 0$. Then $0 = v(1) = v(xx^{-1}) = v(x) + v(x^{-1})$ implies $v(x^{-1}) = -v(x)$. So, if $v(x) = 0$, then $-v(x) = v(x^{-1}) = 0$. Thus, both x and x^{-1} are elements of R_v, i.e. x is a unit in R_v. Conversely, by assumption, $x \in R_v$ and $x^{-1} \in R_v$. Then we have $v(x) \geq 0$ and $v(x^{-1}) \geq 0$. However, $-v(x) = v(x^{-1})$. This means that either $v(x) \leq 0$ or $v(x^{-1}) \leq 0$. Therefore, $v(x) = 0$.

By Remark 3.2, we have $v(x) = v(y)$ if and only if $x = ry$ and $y = sx$ for some $r, s \in R_v$, i.e., $xR_v \subseteq yR_v$ and $yR_v \subseteq xR_v$. This proves (b).

To prove (c), let us take a non-unit element $r \in R_v$. Then $v(r) > 0$. So, $-v(r) = v(r^{-1}) < 0$. Therefore, $r^{-1} \in K$. Also, for each non-zero $q \in K$, by definition of valuation, we have $v(q) \geq 0$ or $v(q) < 0$, i.e, $q \in R_v$ or $q^{-1} \in R_v$. Hence, K is the quotient field of R_v. $\square$

Theorem 3.2 *Let R be a valuation domain with the maximal ideal M. Then R is a discrete valuation domain if and only if it is Noetherian.*

Proof Let v be the valuation of R having value group $\mathbb{Z}$. There exists an element $m \in M$ such that $v(m) = 1$. For a non-zero element $x \in M$, $v(x)$ is a positive integer, say $v(x) = n$, $n \in \mathbb{Z}^+$. Then $v(x) - nv(m) = v(xm^{-n}) = 0$, i.e., $x = m^n u$ for some unit u of R. So, $M = mR$. Let I be a non-zero proper ideal of R. Then $\{v(a) | 0 \neq a \in I\}$ is the set of positive integers, and it has a smallest element, say k, $k > 0$. Then there exists an element $x \in I$ such that $v(x) = k$. Then $I = xR = m^k R$. Therefore, R is a principal ideal domain, so it is Noetherian. Conversely, by Lemma 3.2 (a), we can write the maximal ideal $M = mR$ fo some m and for every non-zero elment $a \in R$, there is $k \in \mathbb{Z}$ such that $aR = m^k R$, i.e., $a \in m^k R$, but $a \notin m^{k+1} R$. Then we can set $v(a) = k$ which implies that if $a, b, c, d \in R$ with $ab^{-1} = cd^{-1}$ then $v(a) - v(b) = v(c) - v(d)$. Therefore, setting $v(q) = v(a) - v(b)$, where $q = ab^{-1} \in Q$, gives a map $v : Q \to \mathbb{Z}$. It can be easily seen that this map v is a valuation of Q whose valuation ring is R with value group $\mathbb{Z}$. So, R is a discrete valuation ring. $\square$

3.3. More on Valuation Domains

In this section we give some connections between integrally closed domains and valuation domains.

First of all, a valuation domain is integrally closed since it is a GCD-domain by its definition. This fact gives us an important theorem on integrally closed domains.

Theorem 3.3 *((L. Fuchs & L. Salce), Theorem I.3.6) An integral domain R is integrally closed if and only if R is the intersection of its valuation overrings.*

Proof Suppose R is integrally closed. To prove R is the intersection of its valuation overrings, it is enough to show that for each $x \in Q - R$ there is a valuation overring of R that fails to contain x. By Lemma 2.1, x is not integral over R implies $x \notin R[x^{-1}]$. Then owing to Zorn's lemma, we can have an overring $R^\star$ of $R[x^{-1}]$, which is maximal with respect to the exclusion of x. In order to prove that $R^\star$ is a valuation domain, we will show that for any non-zero $y \in Q$, either $y \in R^\star$, or $y^{-1} \in R^\star$, i.e., $x \in R^\star[y]$, or $x \in R^\star[y^{-1}]$. Assume to the contrary that $y, y^{-1} \notin R^\star$, i.e., both $x \in R^\star[y]$ and $x \in R^\star[y^{-1}]$. Then we have equations

$$x = a_0 + a_1 y + \cdots + a_n y^n, x = b_0 + b_1 y^{-1} + \cdots + b_m y^{-m},$$

where $a_i, b_j \in R^\star, 1 \leq i \leq n, 1 \leq j \leq m$, and n, m have been chosen as small as possible. By symmetry, we may suppose $n \geq m$. Also, since $x \notin R^\star$, neither $a_0 = x$ nor $b_0 = x$. Then if we multiply the second equation by x^{-1}, we get $1 = b'_0 + b'_1 y^{-1} + \cdots + b'_m y^{-m}$ with $b'_j = b_j x^{-1}$ and $b'_0 \neq 1$. Therefore, $(1 - b'_0)y^n = b'_1 y^{n-1} + \cdots + b'_m y^{n-m}$. Then

$$\begin{aligned} x &= a_0 + a_1 y + \cdots + a_n y^n \\ (1 - b_0 x^{-1})x &= (1 - b_0 x^{-1})(a_0 + a_1 y + \cdots + a_n y^n) \\ x - b_0 &= a_0(1 - b_0 x^{-1}) + \cdots + a_{n-1}(1 - b_0 x^{-1})y^{n-1} + a_n(1 - b_0 x^{-1})y^n. \end{aligned}$$

So, we can write $x = c_0 + c_1 y + \cdots c_k y^k$ for some $k < n$ since $a_n(1 - b_0 x^{-1})y^n = a_n(b'_1 y^{n-1} + \cdots + b'_m y^{n-m})$. This gives a contradiction, therefore; $R^\star$ is a valuation domain, and from the way of defining $R^\star$, R is the intersection of those valuation domains. $\square$

Definition 3.5 *A $*$-operation on R is a mapping $F \to F^*$ of $F(R)$ into $F(R)$ such that for each $q \in Q$ and all $A, B \in R$:*

1. $(q)^* = (q)$*;* $(qA)^* = aA^*$*,*
2. $A \subseteq A^*$*; if* $A \subseteq B$*, then* $A^* \subseteq B^*$*,*
3. $(A^*)^* = A^*$*.*

Moreover, an ideal A is called a $*$-ideal if $A = A^*$.

Theorem 3.4 *((R. Gilmer), Theorem 32.5) Let R be an integral domain with the quotient field Q, and assume that $\{V_i\}_{i \in I}$ is a family of overrings of R such that $R = \bigcap_{i \in I} V_i$. If F is a non-zero fractional ideal of R, we define F^* to be $\bigcap_{i \in I} FV_i$. Then the mapping $F \to F^*$ is a $*$-operation on R and $FV_i = F^* V_i$ for each non-zero fractional ideal F of R and for each i.*

Lemma 3.5 *((R. Gilmer), Lemma 32.17) If $F \to F^*$ is a $*$-operation on an integral domain R and if A is an invertible fractional ideal of R, then for each $B \in F(R)$, $(AB)^* = AB^*$. In particular, $A^* = (AD)^* = AD^* = A$; that is, A is a $*$-ideal.*

The group $I(R)$ of all invertible fractional ideals is partially ordered under the order $A \leq B$ if and only if $B \subseteq A$.

Proposition 3.4 *((J. Brewer & L. Klingler, 2005), Proposition 1) Let R be an integrally closed domain with $\{V_i\}_{i\in I}$ a collection of valuation overrings of R such that $R = \bigcap_{i\in I} V_i$. Denote by v_i the valuation associated with V_i, and by G_i the corresponding value group. Let $A = (a_1, \ldots, a_n)$ be an invertible fractional ideal of R. Then the mapping*

$$\Phi : I(R) \to \prod_{i\in I} G_i \text{ defined by}$$
$$\Phi(A) = (v_i(A))_{i\in I} = (\min\{v_i(a_j)\}_{1\leq j\leq n})_{i\in I}$$

is an order-preserving isomorphism from $I(R)$ into $\prod_{i\in I} G_i$, the cardinal product of the G_i's.

Proof We begin with by fixing notation. Let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_m)$ be invertible fractional ideals of R, where $n, m \in \mathbb{N}$. Then for $i \in I$, we have $v_i(A) = v_i(a_{j(i)})$, where $AV_i = a_{j(i)}V_i$ for $v_i(a_{j(i)}) = \min\{v_i(a_1), \ldots, v_i(a_n))\}$ and $v_i(B) = v_i(b_{k(i)})$, where $BV_i = b_{k(i)}V_i$ for $v_i(b_{k(i)}) = \min\{v_i(b_1), \ldots, v_i(b_m))\}$.

Firstly, we show that Φ is order-preserving. $A \geq B$ if and only if $A \subseteq B$ if and only if $A + B = B$ if and only if $(A+B)V_i = BV_i$ for all $i \in I$ if and only if $v_i(a_{j(i)}) \geq v_i(b_{k(i)})$ for all $i \in I$ if and only if $\Phi(A) \geq \Phi(B)$.

Next, for each $i \in I$, we have

$$\begin{aligned} v_i(AB) &= \min\{v_i(a_j b_k)\}_{1\leq j\leq n, 1\leq k\leq m} \\ &= \min\{v_i(a_j) + v_i(b_k)\}_{1\leq j\leq n, 1\leq k\leq m} \\ &= v_i(a_{j(i)}) + v_i(b_{k(i)}) = v_i(A) + v_i(B), \end{aligned}$$

then we also have

$$\begin{aligned} \Phi(AB) &= (v_i(a_{j(i)}) + v_i(b_{k(i)})_{i\in I} \\ &= (v_i(a_{j(i)}))_{i\in I} + (v_i(b_{k(i)})_{i\in I} \\ &= \Phi(A) + \Phi(B). \end{aligned}$$

Therefore, Φ is a group homomorphism.

To show Φ is one-to-one, we first need to say that if $F \in F(R)$, then by Theorem 3.4, the mapping $F \to \bigcap_{i \in I} FV_i$ is a $*$-operation. Let $A, B \in I(R)$. If $\Phi(A) = \Phi(B)$, then $v_i(A) = v_i(B)$ for each $i \in I$, i.e., $AV_i = BV_i$ for each $i \in I$. Then $\bigcap_{i \in I} AV_i = \bigcap_{i \in I} BV_i$. However, by Lemma 3.5, we have $A = \bigcap_{i \in I} AV_i$ and $B = \bigcap_{i \in I} BV_i$ since A and B are invertible. Hence, Φ is one-to-one.

$\square$

CHAPTER 4

PRÜFER DOMAINS

In this chapter we define Prüfer domains and review the most useful properties of Prüfer domains which are essential for our work. Additionally, let R be a Prüfer domain. Then we give two significant approximation theorems for R which give us an isomorphism from the group of all invertible ideals of R *onto* the cardinal direct sum of corresponding value groups of valuation overrings of R. For further information and proofs, we refer to (J. Brewer & L. Klingler, 2005).

4.1. Fundamental Properties of Prüfer Domains

Let R be an integral domain with the quotient field Q.

Definition 4.1 *R is called a Prüfer domain if its localization at a maximal ideal R_M is a valuation domain for all maximal ideals M of R.*

Theorem 4.1 *((L. Fuchs & L. Salce), Theorem III.1.1) The following are equivalent for an integral domain R:*

(a) *R is a Prüfer domain;*

(b) *every finitely generated non-zero fractional ideal is invertible;*

(c) *the lattice of the fractional ideals of R is distributive: for fractional ideals I, J, K of R,*

$$I \cap (J + K) = (I \cap J) + (I \cap K);$$

(d) *every overring of R is a Prüfer domain.*

Proof (a)⇒(b) Let I be a finitely generated and M a maximal ideal of R. Then IR_M is a finitely generated ideal of the valuation domain R_M, since $I = Ra_1 + \cdots + Ra_n$ implies that $S^{-1}I = S^{-1}Ra_1 + \cdots + S^{-1}Ra_n$ for a localization of R at any multiplicative subset S of R. Hence, IR_M is principal; so, it is invertible. Then by Proposition 2.2, I is invertible.

(b)⇒(a) Let M be a maximal ideal of R. We will show that given two elements $\frac{a}{b}, \frac{c}{d} \in R_M$, where $a, b, c, d \in R$ and with $b, d \notin M$, either $\frac{a}{b} \in (\frac{c}{d})R_M$ or $\frac{c}{d} \in (\frac{a}{b})R_M$. Clearly, it is enough to show that either $a \in cR_M$ or $c \in aR_M$. Then $aR_M + cR_M$ is an invertible ideal in the local domain R_M. Therefore, by Proposition 2.1(c), either $aR_M + cR_M = aR_M$ or $aR_M + cR_M = cR_M$. This proves R_M is a valuation domain.

(a)⇒(c) We always have $(I \cap J) + (I \cap K) \leq I \cap (J + K)$ for fractional ideals I, J, K of R. Conversely, let us take an element $x \in I \cap (J + K)$. Then $x \in I$ and $x \in J + K$ imply that $x \in IR_M$ and $x \in JR_M + KR_M$ for any maximal ideal M of R. Since R_M is a valuation ring, $JR_M \subseteq KR_M$ or $KR_M \subseteq JR_M$. So, $x \in (IR_M \cap JR_M) + (IR_M + KR_M)$. Also, since it is true for any maximal M, we have $x \in (I \cap J) + (I \cap K)$.

(c)⇒(a) We need to show that R_M is a valuation domain for any maximal ideal M of R. Since the assumptions on R hold for any localization of R, it is enough to show that a local domain R', which satisfies our assumptions, must be a valuation domain. For all $a, b \in R'$, we have $aR' = aR' \cap (bR' + (a-b)R') = (aR' \cap bR') + (aR' \cap (a-b)R')$. Thus, $a = x + y$ for some $x \in aR' \cap bR'$ and $y \in aR' \cap (a-b)R'$. Then we can write $y = r(a-b)$ for some $r \in R'$. If r is not contained in the maximal ideal of R', then $a - b = yr^{-1} \in aR'$, so $b \in aR'$. If r is contained in the maximal ideal of R', then $1 - r$ is a unit in R'. Therefore, $a = x + r(a-b)$ implies that $a(1-r) = x - rb \in bR'$, i.e. $a \in bR'$. Hence, R' is a valuation domain. This means that R_M is a valuation domain for all maximal ideals M of R.

(a)⇒(d) Let R' be an overring of R and M' a maximal ideal of R'. Then $P = M' \cap R$ is a prime ideal of R. Let M be the maximal ideal of R which contains P. Then, by Proposition 3.3, R_P is a valuation domain since R_P is an overring of R_M. We claim that $R'_{M'} = R_P$, so that $R'_{M'}$ is a valuation domain and R' is a Prüfer domain. It is clear that $R_P \subseteq R'_{M'}$. Also, since R_P is a valuation domain, by Proposition 3.3 again, $R'_{M'}$ is a localization of R_P, so that $R'_{M'} = R_L$ for some prime ideal L of R. Then, evidently, $L = M'R'_{M'} \cap R = M'R'_{M'} \cap R' \cap R = M' \cap R = P$. So, R' is a Prüfer domain.

(d)⇒(a) It is trivial since R is an overring of itself. □

Lemma 4.1 *((L. Fuchs & L. Salce), Lemma III.1.10) Let J be a finitely generated ideal of a Prüfer domain R. If I is an ideal contained in J, then $I = KJ$ with a unique ideal K of R.*

Proposition 4.1 *((L. Fuchs & L. Salce), Proposition III.1.11) If every finitely generated ideal of an integrally closed domain R can be generated by 2 elements, then R is a Prüfer domain.*

Remark 4.1 *Let D be an overring of a Prüfer domain R.*

(a) *An ideal J of D satisfies $J = (J \cap R)D$;*

(b) *If L is a prime ideal of D, then $P = L \cap R$ is a prime ideal of R, and $D_L = R_P$;*

(c) *A prime ideal P of R generates a proper ideal of D if and only if $D \leq R_P$;*

(d) *The prime ideals of D are exactly the ideals PD, where P is a prime ideal of R such that $D \leq R_P$;*

(e) *Any overring D satisfies $D = \bigcap_{PD<D} R_P$, where P is a prime ideal of R.*

Definition 4.2 *A domain R is called of finite character if every non-zero element, equivalently, every non-zero ideal of R is contained in but a finite number of maximal ideals.*

We specialize Proposition 3.4 to Prüfer domains and determine the embedding defined in Proposition 3.4 maps into the cardinal sum of the G_i's.

Theorem 4.2 *((J. Brewer & L. Klingler, 2005), Thoerem 2) Let R be a Prüfer domain with $\{M_i\}_{i\in I}$ the collection of all maximal ideals of R. Denote by v_i the valuation associated with the valuation ring R_{M_i}, and by G_i the associated value group. Let Φ be the same mapping which is defined in Proposition 3.4. Then:*

(1) *The group $I(R)$ is lattice-ordered;*

(2) *The mapping Φ is an order-preserving isomorphism from $I(R)$ into $\prod_{i\in I} G_i$, the cardinal product of the G_i's;*

(3) *The domain R is of finite character if and only if Φ maps $I(R)$ into $\coprod_{i\in I} G_i$, the cardinal sum of the G_i's.*

Proof (1) Let $A = a_1R + \cdots + a_nR$, $B = b_1R + \cdots b_mR \in I(R)$. Since R is a Prüfer domain, $A + B = a_1R + \cdots + a_nR + b_1R + \cdots + b_mR$ is invertible, and it is the infimum of A and B. Also, $AB = \sum_{1\leq i\leq n, 1\leq j\leq m} a_ib_jR$ is invertible and it is an upper bound for A and B. Thus, $I(R)$ is filtered. Then by Theorem 2.3, $I(R)$ is lattice-ordered.

(2) Since Prüfer domains are intersection of their valuation overrings, i.e., a Prüfer domain $R = \bigcap_{i\in I} R_{M_i}$, where M_i's are maximal ideals of R, Prüfer domains are integrally closed by Theorem 3.3. Therefore, the claim follows from Proposition 3.4.

(3) The Prüfer domain R is of finite character if and only if each non-zero finitely generated ideal $A = (a_1, \ldots, a_n)$ of R is contained in finitely many maximal ideals of R. However, A is also an invertible ideal since R is a Prüfer domain. Then the invertible ideal A of R is contained in only finitely many maximal ideals of R if and only if $\Phi(A) = (\min\{v_i(a_j)\}_{1\leq j\leq n})_{i\in I}$ has finitely many non-zero indices, i.e., $\Phi(A) \in \coprod_{i\in I} G_i$. $\square$

4.2. Approximation Theorems for Prüfer Domains

We determine the embedding defined in Proposition 3.4 maps *onto* the cardinal sum of the G_i's for a Prüfer domain R if and only if the "Strong Approximation Theorem" holds for R.

Definition 4.3 *Two valuation rings V and W with the same quotient field Q are said to be independent if and only if V and W generate the field Q, i.e., there does not exist a valuation ring $U \subseteq Q$ such that $V \subseteq U$ and $W \subseteq U$.*

Proposition 4.2 *((J. Brewer & L. Klingler, 2005), Proposition 3) Let R be a Prüfer domain with $\{M_i\}_{i\in I}$ the collection of all maximal ideals of R. Denote by v_i the valuation associated with the valuation ring R_{M_i} and by G_i the associated value group. Then the following are equivalent:*

(1) *The valuation rings $\{R_{M_i}\}_{i\in I}$ are pairwise independent;*

(2) *Each non-zero prime ideal P of R is contained in a unique maximal ideal of R;*

(3) *D/P is a valuation ring for each non-zero prime ideal P of R;*

(4) *The "Strong Approximation Theorem" holds for elements in R; that is, for every finite collection of maximal ideals $\{M_1, \ldots, M_n\}$ of R, and every choice of non-negative elements $g_i \in G_i$, there is an element $r \in R$ such that $v_i(r) = g_i$ for $1 \leq i \leq n$.*

Definition 4.4 *An integral domain R is an h-local domain if R is of finite character and each non-zero prime ideal of R contained in a unique maximal ideal of R.*

Now, we can claim a stronger version of Proposition 4.2.

Proposition 4.3 *((J. Brewer & L. Klingler, 2005), Proposition 4) Let R be a Prüfer domain with $\{M_i\}_{i\in I}$ the collection of all maximal ideals of R. Denote by v_i the valuation associated with the valuation ring R_{M_i} and by G_i the associated value group. Then the following are equivalent:*

(1) *R is h-local;*

(2) *The "Very Strong Approximation Theorem" holds for finitely generated ideals of R; that is, for every finite collection of maximal ideals $\{M_1, \ldots, M_n\}$ of R, and every choice of non-negative elements $g_i \in G_i$, there is a finitely generated ideal A of R such that $v_i(A) = g_i$ for $1 \leq i \leq n$, and $v_j(A) = 0$ for all other maximal ideals M_j of R.*

Moreover, when these equivalent conditions hold, the finitely generated ideal A in (2) can always be chosen to be 2-generated.

By Theorem 4.2, the mapping Φ defined in 3.4 is an isomorphism from the group of all invertible fractional ideals of a Prüfer domain R *into* the cardinal sum of G_i's if and only if R is of finite character, and by Proposition 4.3, Φ maps *onto* the cardinal sum of G_i's if and only if R is h-local. So, we can claim the next theorem.

Theorem 4.3 *((J. Brewer & L. Klingler, 2005), Theorem 5) Let R be a Prüfer domain with maximal ideals $\{M_i\}_{i\in I}$ and corresponding value groups $\{G_i\}_{i\in I}$, and let Φ be the mapping defined in Proposition 3.4. Then Φ is an isomorphism from the group $I(R)$ of invertible fractional ideals of R onto the cardinal direct sum $\coprod_{i\in I} G_i$ if and only if R is h-local.*

CHAPTER 5

KRULL DOMAINS

In this chapter we review fundamental properties of Krull domains and we investigate if we can prove a ”Strong Approximation Theorem” and a ”Very Strong Approximation Theorem” for Krull domains.

5.1. Fundamental Properties of Krull Domains

Definition 5.1 *An integral domain R is a Krull domain if*

K1 $R = \bigcap_{i\in\Lambda} V_i$*, where the V_i's are overrings of R which are discrete rank 1 valuation domains, and*

K2 *every non-zero element $a \in R$ is invertible in almost all of the V_i's.*

Lemma 5.1 *((L. Fuchs & L. Salce), Exercise II.1.12) A discrete rank 1 valuation domain is completely integrally closed.*

Proof Let R be a discrete rank 1 valuation domain. So, R is a Noetherian valuation domain as well. Let $0 \neq x \in Q$, which is the quotient field of R, be almost integral over R. Then there exists $r \in R$ such that $rx^n \in R$ for all $n \in \mathbb{N}$. Then the ideal $A = (rx, rx^2, \ldots, rx^n, \ldots)$ is a finitely generated ideal of R since R is Noetherian. Therefore, $A = (rx^{k_1}, \ldots, rx^{k_n})$, where $k_1, \ldots, k_n \in \mathbb{N}$. So, we can write rx^k as a combination of generators of A for any $k \in \mathbb{N}$. By choosing $k > k_i, 1 \leq i \leq n$, we have $rx^k = a_1 rx^{k_1} + \cdots + a_n rx^{k_n}$. Hence, $0 = -x^k + a_1 x^{k_1} + \cdots + a_n x^{k_n}$, i.e., x is integral over R. Since R is intersection of its valuation overrings, R is integrally closed, and consequently $x \in R$. Thus, R is completely integrally closed. □

Lemma 5.1 shows that a Krull domain is completely integrally closed. Therefore, by Theorem 2.2, the lattice-ordered monoid $D(R)$ of the divisors of R is a lattice-ordered group.

Since every overring V_i in (K1) is a discrete rank 1 valuation domain, there exists a valuation $v_i : Q \to \mathbb{Z} \cup \{0\}$ such that $V_i = R_{v_i} = \{x \in Q | v_i(x) \geq 0\}$. Furthermore, for a non-zero fractional ideal $I \in F(R)$, we will set $v_i(I) = \max\{v_i(q) | q \in Q, I \leq qR\}$.

Lemma 5.2 *((L. Fuchs & L. Salce), Lemma IV.1.2) For a non-zero fractional ideal I of a Krull domain R, $v_i(I) = 0$ holds for almost all indices $i \in \Lambda$.*

By Proposition 2.3(c), we have $v_i(I) = v_i(I_v)$. Using this property and Lemma 5.2, we have the following lemma.

Lemma 5.3 *((L. Fuchs & L. Salce), Lemma IV.1.3) If R is a Krull domain, then the mapping $\phi : D(R) \to \bigoplus_{i\in\Lambda} \mathbb{Z}_i$, $\mathbb{Z}_i \cong \mathbb{Z}$, defined by letting $v_i(I)$ be the i-th coordinate of $\phi(\mathrm{div}(I))$, is an order-isomorphism of the lattice-ordered group of divisorial ideals I of R with a subgroup of the pointwise ordered lattice-ordered group $\bigoplus_{i\in\Lambda} \mathbb{Z}_i$.*

Theorem 5.1 *((L. Fuchs & L. Salce), Theorem IV.1.4) The following conditions on a domain R are equivalent:*

(a) *R is a Krull domain;*

(b) *R is completely integrally closed and satisfies the ascending chain condition on divisorial ideals;*

(c) *the group $D(R)$ of divisors is a free abelian group with basis $\mathrm{Min}\, D(R)^+$, lattice-ordered by the pointwise ordering.*

Proof (a)⇒(b) A Krull domain is already completely integrally closed. By Lemma 5.3, an ascending chain $I_1 \leq I_2 \leq \ldots \leq I_n \leq \ldots$ of divisorial ideals of R corresponds bijectively to a decreasing chain of strictly positive elements in $\bigoplus_{i\in\Lambda} \mathbb{Z}_i$, i.e., $v_i(I_1) \geq v_i(I_2) \geq \ldots \geq v_i(I_n) \geq \ldots$ for each $i \in \Lambda$. Since $\bigoplus_{i\in\Lambda} \mathbb{Z}_i$ is lattice-ordered, the latter chain contains a minimal element. Thus, the chain of divisorial ideals has a maximal member.

(b)⇒ (c) Theorem 2.2 guarantees that $D(R)$ is a latticed-ordered abelian group. Then the maximum condition on divisorial ideals translates into the minimum condition on positive elements of $D(R)$ via the order reversing bijection $I \to \mathrm{div}(I)$ and followed by the mapping ϕ in Lemma 5.3. Therefore, by Theorem 2.4, $D(R) \cong \bigoplus_{\gamma\in \mathrm{Min}\, G^+} \gamma\mathbb{Z}$.

(c) ⇒ (a) By assumptions on $D(R)$, for every $0 \neq q \in Q$, we can write

$$\mathrm{div}(q) = \sum_{\gamma\in \mathrm{Min}\, D(R)^+} w_\gamma(q)\gamma,$$

where the $w_\gamma(q)$ are integers uniquely determined by q and almost all of $w_\gamma(q)$'s are zero since $D(R)$ is free. Then from the relations $\mathrm{div}(qq') = \mathrm{div}(q) + \mathrm{div}(q')$ and $\mathrm{div}(q + q') \geq \inf(\mathrm{div}(q), \mathrm{div}(q'))$ for non-zero $q, q' \in Q$, we conclude that each w_γ defines a discrete rank 1 valuation of Q. Furthermore, $q \in R$ if and only if $\mathrm{div}(q) \geq 0$ if and only if $w_\gamma(q) \geq 0$ for all $\gamma \in \mathrm{Min}\, D(R)^+$. This guarantees that $R = \bigcap_{\gamma\in \mathrm{Min}\, D(R)^+} W_\gamma$, where W_γ is the discrete rank 1 valuation domain defined by the discrete rank 1 valuation w_γ. Therefore, the property (K1) holds for R. Moreover, since $D(R)$ is a free abelian group, $w_\gamma(q) = 0$ for almost all

$\gamma \in \operatorname{Min} D(R)^+$, which shows that the property (K2) holds for R as well. Hence, R is a Krull domain. $\square$

The discrete rank 1 valuations w_γ and the discrete rank 1 valuation domains W_γ, which are defined in the proof of Theorem 5.1, are called **essential valuations** and **essential valuation overrings** of the Krull domain R, respectively.

Corollary 5.1 *((L. Fuchs & L. Salce), Corollary IV.1.5) A Noetherian domain is a Krull domain if and only if it is integrally closed.*

Proposition 5.1 *((L. Fuchs & L. Salce), Proposition IV.1.6) Let R be a Krull domain and $D(R)$ its group of divisors. For each $\gamma \in MinD(R)^+$, let P_γ be the maximal proper divisorial ideal, w_γ the essential valuation of R associated with γ, and W_γ the corresponding valuation ring. Then:*

(a) *P_γ is a prime ideal of R and $W_\gamma = R_{P_\gamma}$;*

(b) *$\{P_\gamma | \gamma \in MinD(R)^+\}$ is the set of minimal prime ideals of R.*

Proposition 5.2 *((L. Fuchs & L. Salce), Exercise IV.1.4) Let R be a Krull domain. Let S be a submonoid of $R^\times$. Then $S^{-1}R = \bigcap\{W_\gamma | \gamma \in \operatorname{Min} D(R)^+, w_\gamma(s) = 0$ for all $s \in S\}$, and $S^{-1}R$ is a Krull domain.*

Proof Let $rs^{-1} \in S^{-1}R$, where $r \in R$ and $s \in S$. Then $w_\gamma(rs^{-1}) = w_\gamma(r) - w_\gamma(s) = w_\gamma(r) \geq 0$ if $w_\gamma(s) = 0$. Therefore, $S^{-1}R \subseteq \bigcap\{W_\gamma | \gamma \in \operatorname{Min} D(R)^+, w_\gamma(s) = 0$ for all $s \in S\}$. Conversely, take an element $x \in \bigcap\{W_\gamma | \gamma \in \operatorname{Min} D(R)^+, w_\gamma(s) = 0$ for all $s \in S\}$. Now, define the set $\{W_\alpha | \alpha \in \operatorname{Min} D(R)^+, w_\alpha(s) > 0$ for some $s \in S\}$. Then there exist finitely many elemets $\alpha_1, \ldots, \alpha_n$ of $\operatorname{Min} D(R)^+$ such that $w_{\alpha_i}(x) < 0$ since R is a Krull domain, and $x = r_1 r_2^{-1}$ for some non-zero $r_1, r_2 \in R$. Also, for each α_i, there exists $s_i \in S$ such that $w_{\alpha_i}(s_i) > 0$. Choosing a positive integer k large enough such that $w_{\alpha_i}(s_i^k x) \geq 0$ and setting $s = (s_1 \cdots s_n)^k$, we have $w_\beta(sx) \geq 0$ for all $\beta \in \operatorname{Min} D(R)^+$. So, $sx \in R$ which implies $x \in R_S$ since $s \in S$. Hence, $S^{-1}R = \bigcap\{W_\gamma | \gamma \in \operatorname{Min} D(R)^+, w_\gamma(s) = 0$ for all $s \in S\}$. Furthermore, this ensures that K1 and K2 hold for $S^{-1}R$. So, $S^{-1}R$ is a Krull domain. $\square$

Since a Krull domain R is the intersection of its valuation overrings , R is also an integrally closed domain. Thus, we can specialize Proposition 3.4 for Krull domains.

Proposition 5.3 *Let R be a Krull domain with $\{P_i\}_{i \in I}$ the collection of all minimal ideals of R. Denote by v_i the associated valuation with the valuation ring R_{P_i} and by $\mathbb{Z}_i$ the associated* *value group. Let $A = (a_1, \ldots, a_n)$ be an invertible fractional ideal of R. Then the mapping*

$$\Phi : I(R) \to \coprod_{i \in I} \mathbb{Z}_i \textit{ defined by}$$

$$\Phi(A) = (v_i(A))_{i \in I} = (\min\{v_i(a_j)\}_{1 \leq j \leq n})_{i \in I}$$

is an order-preserving isomorphism from $I(R)$ into $\coprod_{i \in I} \mathbb{Z}_i$, the cardinal sum of the $\mathbb{Z}_i$'s.

Proof By Proposition 3.4, Φ is already an order-preserving isomorphism *into* $\prod_{i \in I} \mathbb{Z}_i$. Let $A \in I(R)$. Then $A = (a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in A$. By (K2), $v_i(A) = \min\{v_i(a_j)\}_{1 \leq j \leq n} = 0$ in almost all of the V_i's. This means $\Phi(A)$ has finitely many non-zero indices. So, $\Phi(A) \in \coprod_{i \in I} \mathbb{Z}_i$. Therefore, Φ maps *into* $\coprod_{i \in I} \mathbb{Z}_i$. $\square$

5.2. Approximation Theorems for Krull Domains

There is already an approximation theorem for a Krull domain R. This approximation theorem gives an element in the quotient field Q which satisfies the required assumptions on R. However, we have proved that the "Strong Approximation Theorem" holds for elements in R. It gives us a progression to define another approximation theorem for finitely generated fractional ideals of R.

Proposition 5.4 *((L. Fuchs & L. Salce), Proposition IV.1.7) Let $w_1, \cdots, w_n$ be different essential valuations of a Krull domain R, and $z_1, \ldots, z_n \in \mathbb{Z}$. There exists an element $q \in Q$ such that $w_i(q) = z_i$ for all $i = 1, \ldots, n$, and $w(q) \geq 0$ for all essential valuations $w \neq w_1, \ldots, w_n$.*

Proposition 5.5 *Let R be a Krull domain with $\{P_i\}_{i \in I}$ the collection of minimal prime ideals of R. Denote by v_i the valuation associated with the valuation ring R_{P_i} and by $\mathbb{Z}_i$ the associated value group. Then the "Strong Approximation Theorem" holds for elements in R; that is, for every finite collection of minimal prime ideals $\{P_1, \ldots, P_n\}$ of R, and every choice of non-negative elements $z_i \in \mathbb{Z}_i$, there is an element $r \in R$ such that $v_i(r) = z_i$ for $1 \leq i \leq n$.*

Proof Firstly, we claim that given distinct minimal prime ideals $P, P_1, \ldots, P_n$ of R with corresponding valuations $v, v_1, \ldots, v_n$ and given non-negative value z of v, there is an element $r \in R$ such that $v(r) > z$ and $v_i(r) = 0$, $1 \leq i \leq n$.

Let z be a non-negative value of v. We know that $P - \bigcup_{i=1}^n P_i \neq \varnothing$ since $P, P_1, ..., P_n$ are distinct minimal prime ideals. Therefore, we can choose $c \in P - \bigcup_{i=1}^n P_i$. Now, since

$v(c) > 0$, there exists a positive integer k such that $v(c^k) > z$. Moreover, since $c \notin P_i$, $1 \leq i \leq n$, we also have $v_i(c^k) = k.v_i(c) = 0$. Hence, the required r is c^k for the claim.

Now, let $z_1, \ldots, z_n$ be non-negative elements of $\mathbb{Z}_1, \ldots, \mathbb{Z}_n$, respectively. By the claim, for each i, we can choose $r_i \in R$ such that $v_i(r_i) > z_i$ and for all $j \neq i, v_j(r_i) = 0$. Let $c_1, \ldots, c_n \in R$ be such that $v_i(c_i) = z_i$, $1 \leq i \leq n$, and set

$$b_i = c_i(r_1 \cdots r_{i-1} r_{i+1} \cdots r_n).$$

Then for $1 \leq i \leq n$, we get

$$v_i(b_i) = v_i(c_i) + v_i(r_1) + \cdots + v_i(r_{i-1}) + v_i(r_{i+1}) + \cdots + v_i(r_n) = z_i$$

since $v_i(r_j) = 0$ for $j \neq i$. Also, for $j \neq i$, we have

$$\begin{aligned} v_j(b_i) &= v_j(c_i) + \cdots + v_j(r_{i-1}) + v_j(r_{i+1}) + \cdots + v_j(r_n) \\ &= v_j(c_i) + v_j(r_j) \geq v_j(r_j) > z_j. \end{aligned}$$

Finally, if we set $b = b_1 + \cdots + b_n$ we get $v_i(b) = v_i(b_i) = z_i$, $v_i(b_j) > z_i$ for $j \neq i$.

□

Corollary 5.2 *Let R be a Krull domain with $\{P_i\}_{i \in I}$ the collection of minimal prime ideals of R. Denote by v_i the valuation associated with the valuation ring R_{P_i} and by $\mathbb{Z}_i$ the associated value group. Then the valuation rings $\{R_{P_i}\}_{i \in I}$ are pairwise independent.*

Proof Let P_1 and P_2 be distinct minimal ideals of R and q a non-zero element of Q. If $v_1(q) \geq 0$ or $v_2(q) \geq 0$, then $q \in R_{P_1}$ or $q \in R_{P_2}$. So, suppose that $v_1(q) < 0$ and $v_2(q) < 0$. By the Strong Approximation Theorem, there exists an element $r \in R$ such that $v_1(r) = -v_1(q)$ and $v_2(r) = 0$. Then we can write $q = (qr)r^{-1}$. Since $v_1(qr) = v_1(q) + v_1(r) = 0$, $qr \in R_{P_1}$. Also, $v_2(r^{-1}) = -v_2(r)$ implies that $r^{-1} \in R_{P_2}$. □

Furthermore, we have proved that the ”Very Strong Approximation Theorem” holds for finitely generated fractional ideals of a Krull domain R.

Proposition 5.6 *Let R be a Krull domain with $\{P_i\}_{i \in I}$ the collection of minimal prime ideals of R. Denote by v_i the valuation associated with the valuation ring R_{P_i} and by $\mathbb{Z}_i$ the associated value group. Then the ”Very Strong Approximation Theorem” holds for finitely generated ideals of R; that is, for every finite collection of minimal prime ideals $\{P_1, \ldots, P_n\}$*

of R, and every choice of non-negative elements $z_i \in \mathbb{Z}_i$, there is a finitely generated ideal A of R such that $v_i(A) = z_i$ for $1 \leq i \leq n$ and and $v_j(A) = 0$ for all other minimal prime ideals P_j of R. Moreover, A is two-generated.

Proof By Proposition 5.5, the Strong Approximation Theorem holds for Krull domains, so we can find an element $r \in R$ such that $v_i(r) = z_i$ for $1 \leq i \leq n$. By the definition of Krull domain, we have finitely many other minimal prime ideals $P_1, \ldots, P_m$ of R with corresponding valuations $w_1, \ldots, w_m$ such that $w_j(r) > 0$ for $1 \leq j \leq m$. By the Strong Approximation Theorem again, we can find an element $r' \in R$ such that $v_i(r') = z_i$ and $w_j(r') = 0$. Then the ideal (r, r') is the required ideal. $\square$

CHAPTER 6

CONCLUSION

Let R be an integrally closed domain, and denote by $I(R)$ the multiplicative group of all invertible fractional ideals of R. Let $\{V_i\}_{i\in I}$ be the family of valuation overrings of R, and denote by G_i the corresponding value group of the valuation domain V_i. We showed that if $R = \bigcap_{i\in I} V_i$ then is a map from $I(R)$ *into* $\prod_{i\in I} G_i$, the cardinal product of the G_i's. Furthermore, it is well known when R is a Dedekind domain, this map becomes an isomorphism *onto* $\coprod_{i\in I} G_i$, the cardinal sum of the G_i's. In this case, $G_i \cong \mathbb{Z}$ for each i.

It is shown, by J. Brewer and L. Klingler, that this map is also an isomorphism *onto* $\coprod_{i\in I} G_i$ when R is an h-local Prüfer domain by using two approximation theorems. In this thesis, we showed that such a map exists and that it is injective when R is a Krull domain. Furthermore, we reformed these approximation theorems for Krull domanins, which helped us to gain further insight about Krull domains.

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