MODULES WITH COPRIMARY DECOMPOSITION

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ABSTRACT

MODULES WITH COPRIMARY DECOMPOSITION

This thesis presents the theory of coprimary decomposition of modules over a commutative noetherian ring and its coassociated prime ideals. This theory is first introduced in 1973 by I. G. Macdonald as a dual notion of an important tool of associated primes and primary decomposition in commutative algebra. In this thesis, we studied the basic properties of coassociated prime ideals to a module M and gathered some modules in the literature which have coprimary decomposition. For example, we showed that artinian modules over commutative rings are representable. Moreover if R is a commutative noetherian ring, then we showed that injective modules over R are representable. Finally, we discussed the uniqueness properties of coprimary decomposition.

ÖZET

EŞ DOĞAL ASALIMSI AYRIŞIMA SAHİP OLAN MODÜLLER

Bu tezde, değişmeli noether bir halka üzerindeki eş doğal asalımsı ayrışım kuramı ve onların eş ilişkili asal idealleri verilmiştir. Bu kuram, değişmeli cebirde önemli bir araç olan ilişkili asal idealler ve asalımsı ayrışım kavramının duali olarak ilk defa 1973'de I. G. Macdonald tarafından ortaya konulmuştur. Bu tezde, bir M modülünün eş ilişkili asal ideallerinin temel özelliklerini inceledik ve eş doğal asalımsı ayışımı olan literatürdeki bazı modülleri bir araya topladık. Örneğin, değişmeli halkalar üzerindeki artin modüllerin temsil edilebilir olduklarını gösterdik. Eğer R değişmeli noether bir halka ise injektif modüllerin R üzerinde temsil edilebilir olduklarını da gösterdik. Son olarak, eş doğal asalımsı ayrışımın teklik özelliklerini tartıştık.

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NOTATION

R	an associative ring with unit unless otherwise stated
R-Mod	the category of left R-modules
$\subseteq$	inclusion
$\subsetneq$	strict inclusion
Ω	the set of all maximal ideals of the corresponding ring
$\mathrm{Spec}(R)$	the prime spectrum of R : the set of all prime ideals of R
$\mathrm{Var}(\mathfrak{a})$	the variety of the ideal $\mathfrak{a}$: the set of all prime ideals containing $\mathfrak{a}$
$I \leq R$	I is an ideal of R
$N \subseteq_e M$	N is an essential submodule of M
$\mathrm{Ass}(M)$	the set of associated prime ideals of M
$\mathrm{Ann}(M)$	annihilator of M
$\mathrm{Att}(M)$	the set of attached prime ideals of M
$\mathrm{Coass}(M)$	the set of coassociated prime ideals of M
$\mathrm{Ker}\, f$	the kernel of the map f
$\mathrm{Im}\, f$	the image of the map f
$\mathrm{Soc}\, M$	the socle of the R-module M
$\mathrm{Rad}\, M$	the radical of the R-module M
$\prod$	direct product
$\coprod$	external direct sum
$\bigoplus$	internal direct sum
$\bigotimes$	tensor product
M^I	$\prod_{i \in I} M$ (direct product of I copies of M)
$M^{(I)}$	$\coprod_{i \in I} M$ (direct sum of I copies of M)
$\lvert A \rvert$	cardinality of the set A
$M[\mathfrak{a}]$	the submodule $\mathrm{Ann}_M(\mathfrak{a}) = \{m \in M \mid \mathfrak{a}m = 0\}$ of the R-module M where $\mathfrak{a}$ is an ideal of R

CHAPTER 1

INTRODUCTION

Throughout this thesis all rings are commutative and unless otherwise stated they are noetherian. By a module, we mean a unital left R-module.

In the study of modules over commutative noetherian rings, the set of *associated prime ideals*, $\mathrm{Ass}(M)$, has proved to be an important tool. For a module M, $\mathrm{Ass}(M)$ can be introduced as the set of prime ideals $\mathfrak{p}$ such that $\mathfrak{p} = \mathrm{Ann}(m)$ for some element m in M.

There have been four attempts in the literature to dualize the theory of associated prime ideals in (Macdonald 1973), in (Chambles 1981), in (Zöschinger 1983) and in (Yassemi 1997). In (Yassemi 1997), it is shown that when the ring is noetherian, all these definitions are equivalent. Here we shall follow Zöschinger's notation and terminology from (Zöschinger 1983). An aim of this thesis is to study a theory dual to that of associated primes by defining $\mathrm{Coass}(M)$ to be the set of prime ideals such that there exists an artinian homomorphic image M' of M with $\mathfrak{p} = \mathrm{Ann}(M')$. In Chapter 3, we shall give basic properties of coassociated prime ideals and give several examples.

There have been several accounts of the theory dual to the well-known theory of primary decomposition for modules over a commutative ring R, for example see (Kirby 1973), (Macdonald 1973) and (Zöschinger 1990). We shall follow again Zöschinger's terminology from (Zöschinger 1990). A module M over a commutative ring R is called *coprimary* if $M \neq 0$ and for every $x \in R$ either $xM = M$ or $x^k M = 0$ for some $k \geq 1$, i.e. for every $x \in R$ the R-endomorphism produced by multiplication by x is either surjective or nilpotent. A module M is called *representable* if M is the sum of finitely many coprimary submodules. A representation $M = U_1 + \cdots + U_n$ in which all U_i are coprimary is called a *coprimary decomposition* in (Kirby 1973), and also a *secondary representation* of M in (Macdonald 1973). Both authors investigated the existence and uniqueness of such a decomposition analogous to the classical Noether-Lasker theory of primary decomposition of noetherian modules. In particular, they showed that every artinian module is representable. By (Sharp 1976, Theorem 2.3), every injective module over a noetherian ring is also representable .

Another aim of this thesis is to gather a wide class of representable modules over a noetherian ring R. In particular, every artinian module is representable and every injective module over a noetherian ring is representable. With the help of the set $\mathrm{Att}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p}$ is the annihilator of a factor module of $M\}$, we obtain the following sufficient criterion in Theorem 4.4: If M is an R-module such that $\mathrm{Att}(M)$ is discrete (i.e. from $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ it always follows $\mathfrak{p}_1 = \mathfrak{p}_2$), then M is representable.

Finally, in Chapter 5, we discuss uniqueness properties of coprimary decompositions of modules.

CHAPTER 2

PRELIMINARIES

2.1. Fundamental Definitions and Facts

Throughout R is a commutative ring with identity, all modules are unital left R-modules. In this section we shall give some fundamental definitions and facts.

Lemma 2.1 *(Zorn's Lemma) Let* $(V, \leq)$ *be a non-empty partially ordered set which has the property that every (non-empty) totally ordered subset of* V *has an upper bound in* V*. Then* V *has at least one maximal element.*

Definition 2.1 *An ideal* M *of a commutative ring* R *is said to be maximal precisely when* M *is a maximal member, with respect to inclusion, of the set of proper ideals of* R*.*

In other words, the ideal M *of* R *is maximal if and only if* $M \subsetneq R$*, and there is no ideal* I *of* R *with* $M \subsetneq I \subsetneq R$*.*

Definition 2.2 *Let* R *be a commutative ring. The Jacobson radical of* R*, denoted by* $\mathrm{Jac}(R)$ *or sometimes* $J(R)$*, is defined to be the intersection of all the maximal ideals of* R*.*

Thus $\mathrm{Jac}(R)$ is an ideal of R. Even in the case when R is trivial, our convention concerning the intersection of the empty family of ideals of a commutative ring means that $\mathrm{Jac}(R) = R$.

Definition 2.3 *Let* $\mathfrak{p}$ *be an ideal in a commutative ring* R*. We say that* $\mathfrak{p}$ *is a prime ideal of* R *precisely when* $\mathfrak{p}$ *is a proper ideal of* R*, and whenever* $a, b \in R$ *with* $ab \in \mathfrak{p}$*, then either* $a \in \mathfrak{p}$ *or* $b \in \mathfrak{p}$*.*

Definition 2.4 *Let* R *be a commutative ring and let* I *be an ideal of* R*. Then*

$$\sqrt{I} := \{r \in R \mid \text{there exists } n \in \mathbb{N} \text{ with } r^n \in I\}$$

is an ideal of R which contains I, and is called the radical of I.

Lemma 2.2 *Let I,J be ideals of the commutative ring R. We define the ideal quotient* $(I : J)$ *by*

$$(I : J) = \{a \in R \mid aJ \subseteq I\}$$

clearly this is another ideal of R and $I \subseteq (I : J)$*. In the special case in which* $I = 0$*, the ideal quotient*

$$(0 : J) = \{a \in R \mid aJ = 0\} = \{a \in R \mid ab = 0 \textit{ for all } b \in J\}$$

is called the annihilator of J and is also denoted by $\operatorname{Ann} J$ *or* $\operatorname{Ann}_R J$*.*

Definition 2.5 *A commutative ring R which has exactly one maximal ideal, M say, is said to be quasi-local. A commutative Noetherian ring which is quasi-local is called a local ring.*

When R is quasi-local, $\operatorname{Jac}(R)$ is the unique maximal ideal of R.

Theorem 2.1 *(First Isomorphism Theorem) Let M and N be modules over a commutative ring R, and let* $f : M \to N$ *be an R-homomorphism. Then f induces an isomorphism* $f' : M/\operatorname{Ker} f \to \operatorname{Im} f$ *for which*

$$f'(m + \operatorname{Ker} f) = f(m) \textit{ for all } m \in M.$$

Theorem 2.2 *(Second Isomorphism Theorem) Let M be a module over a commutative ring R. Let N and K be submodules of M such that* $N \subseteq K$ *so that* K/N *is a submodule of the R-module* M/N*. Then there is an isomorphism*

$$f : (M/N)/(K/N) \to M/K$$

such that $f((m + N) + K/N) = m + K$ *for all* $m \in M$*.*

Theorem 2.3 *(Third Isomorphism Theorem) Let M be a module over a commutative ring R. Let N and K be submodules of M. Then there is an isomorphism*

$$f : N/(N \cap K) \to (N + K)/K$$

such that $f(n + N \cap K) = n + K$ *for all* $n \in N$

Definition 2.6 *A module M is called finitely generated (finitely cogenerated) in case for every set* $\mathcal{A}$ *of submodules of M*

$$\sum \mathcal{A} = M \ (\bigcap \mathcal{A} = 0) \textit{ implies } \sum \mathcal{F} = M \ (\bigcap \mathcal{F} = 0)$$

for some finite $\mathcal{F} \subseteq \mathcal{A}$.

Definition 2.7 *Let M be a module over the commutative ring R. We say that M is a noetherian R-module precisely when it satisfies the following conditions which are equivalent*

(i) *Whenever* $(G_i)_{i\in\mathbb{N}}$ *is a sequence of submodules of M such that*

$$G_1 \subseteq G_2 \subseteq \cdots \subseteq G_i \subseteq G_{i+1} \subseteq \cdots,$$

then there exists $k \in \mathbb{N}$ *such that* $G_k = G_{k+i}$ *for all* $i \in \mathbb{N}$. *This is called the ascending chain condition for submodules of M.*

(ii) *Every non-empty set of submodules of M contains a maximal element with respect to inclusion. This is called the maximal condition for submodules.*

(iii) *Every submodule of M is finitely generated.*

Definition 2.8 *We say that M is an artinian R-module precisely when it satisfies the following conditions which are equivalent*

(i) *Whenever* $(G_i)_{i\in\mathbb{N}}$ *is a sequence of submodules of M such that*

$$G_1 \supseteq G_2 \supseteq \cdots \supseteq G_i \supseteq G_{i+1} \supseteq \cdots,$$

then there exists $k \in \mathbb{N}$ *such that* $G_k = G_{k+i}$ *for all* $i \in \mathbb{N}$. *This is called the descending chain condition for submodules of M.*

(ii) *Every non-empty set of submodules of M contains a minimal element with respect to inclusion. This is called the minimal condition for submodules.*

(iii) *Every factor module of M is finitely cogenerated.*

Definition 2.9 *Let M be a module over the commutative ring R. A zero-divisor on M is an element $r \in R$ for which there exists $m \in M$ such that $m \neq 0$ but $rm = 0$. An element of R which is not a zero-divisor on M is often referred to as a non-zero-divisor on M.*

The set of all zero-divisors on M is denoted by $\mathrm{Zdv}(M)$.

Definition 2.10 *Let M be a module over the commutative ring R. A proper submodule Q of M is said to be a primary submodule of M precisely when $(M/Q) \neq 0$ and, for each $a \in \mathrm{Zdv}_R(M/Q)$, there exists $n \in \mathbb{N}$ such that $a^n(M/Q) = 0$, i.e. for all $a \in R$, $m \in M$ $am \in Q$ then $m \in Q$ or $a^n M \subseteq Q$.*

Let Q be a primary submodule of M. It is easy to show that $\mathfrak{p} := \sqrt{\mathrm{Ann}(M/Q)}$ is a prime ideal of R. In this case, we say that Q is a $\mathfrak{p}$-primary submodule of M, or that Q is $\mathfrak{p}$-primary in M. Also if $Q_1, \ldots, Q_n$ (where $n \in \mathbb{N}$) are $\mathfrak{p}$-primary submodules of M, then so too is $\bigcap_{i=1}^{n} Q_i$.

Definition 2.11 *Let M be a module over the commutative ring R, and let G be a proper submodule of M. A primary decomposition of G in M is an expression for G as an intersection of finitely many primary submodules of M. Such a primary decomposition*

$$G = Q_1 \cap \ldots \cap Q_n \text{ with } Q_i \ \mathfrak{p}_i\text{-primary in } M \ (1 \leq i \leq n)$$

of G in M is said to be minimal precisely when

(i) $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ *are n distinct prime ideals of R; and*

(ii) for all $j = 1, \ldots, n$, we have

$$Q_j \not\supseteq \bigcap_{\substack{i=1 \\ i \neq j}}^{n} Q_i.$$

We say that G is a decomposable submodule of M precisely when it has a primary decomposition in M.

Lemma 2.3 *(First Uniqueness Theorem for Primary Decomposition) Let M be a module over the commutative ring R, and let G be a decomposable submodule of M. Let*

$$G = Q_1 \cap \ldots \cap Q_n \text{ with } Q_i \ \mathfrak{p}_i\text{- primary in } M \ (1 \leq i \leq n)$$

and

$$G = Q'_1 \cap \ldots \cap Q'_{n'} \textit{ with } Q'_i \; \mathfrak{p}'_i\textit{-primary in } M \; (1 \le i \le n')$$

be two minimal primary decomposition of G in M. Then $n = n'$ *and*

$$\{\mathfrak{p}_1, \ldots \mathfrak{p}_n\} = \{\mathfrak{p}'_1, \ldots \mathfrak{p}'_n\}.$$

Lemma 2.4 *(Second Uniqueness Theorem for Primary Decomposition) Let M be a module over the commutative ring R, and let G be a decomposable submodule of M. Let*

$$G = Q_1 \cap \ldots \cap Q_n \textit{ with } Q_i \; \mathfrak{p}_i\textit{-primary in } M \; (1 \le i \le n)$$

and

$$G = Q'_1 \cap \ldots \cap Q'_n \textit{ with } Q'_i \; \mathfrak{p}_i\textit{-primary in } M \; (1 \le i \le n)$$

be two minimal primary decompositions of G in M. (Here we use the first uniqueness theorem.) Suppose that $\mathfrak{p}_j$ *is a minimal member of* $\{\mathfrak{p}_1, \ldots \mathfrak{p}_n\}$ *with respect to inclusion. Then* $Q_j = Q'_j$.

Theorem 2.4 *(Krull's Intersection Theorem) Let* $\mathfrak{a}$ *be an ideal of the commutative Noetherian ring R such that* $\mathfrak{a} \subseteq \mathrm{Jac}(R)$. *Then*

$$\bigcap_{i=1}^{\infty} \mathfrak{a}^n = 0.$$

Definition 2.12 *Let R be a commutative ring, let G, M and N be R-modules, and let* $g : G \to M$ *and* $f : M \to N$ *be R-homomorphisms. We say that the sequence*

$$G \xrightarrow{g} M \xrightarrow{f} N$$

is exact precisely when $\mathrm{Im}\, g = \mathrm{Ker}\, f$.

Theorem 2.5 *Let* $\mathfrak{a}$ *be a proper ideal of the commutative ring R. Then*

$$\mathrm{Var}(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supseteq \mathfrak{a}\}$$

has at least one minimal member with respect to inclusion. Such a minimal member is called a minimal prime ideal of $\mathfrak{a}$ *or a minimal prime ideal containing* $\mathfrak{a}$.

Corollary 2.1 *Let* $\mathfrak{a}$ *be a proper ideal of the commutative ring R, and let* $\mathrm{Min}(\mathfrak{a})$ *denote the set of minimal prime ideals of* $\mathfrak{a}$. *Then*

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in \operatorname{Min}(\mathfrak{a})} \mathfrak{p}$$

Definition 2.13 *Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then the height of $\mathfrak{p}$, denoted by $\operatorname{ht} \mathfrak{p}$ is defined to be the supremum of lengths of chains*

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

of prime ideals of R for which $\mathfrak{p}_n = \mathfrak{p}$ if this supremum exists, and ∞ otherwise.

Definition 2.14 *A proper submodule A of M is called small if $A + B = M$, then B=M for all submodules B of M.*

Definition 2.15 *Let M, N be R-modules. A homomorphism $f : M \to N$ is called essential (small) if $\operatorname{Im} f$ is essential in N ($\operatorname{Ker} f$ is small in M).*

Definition 2.16 *Let M be a module over a commutative noetherian ring R, and let $\mathfrak{p} \in \operatorname{Spec}(R)$. We say that $\mathfrak{p}$ is an associated prime (ideal) of M precisely when there exists $m \in M$ with $(0 : m) = \operatorname{Ann}(m) = \mathfrak{p}$. Observe that, if $m \in M$ has $(0 : m) = \mathfrak{p}$ as above, then $m \neq 0$. The set of associated prime ideals of M is denoted by $\operatorname{Ass}(M)$.*

If M and M′ are isomorphic R-modules, then $\operatorname{Ass}(M) = \operatorname{Ass}(M')$.

Definition 2.17 *Let U be a submodule of an R-module M. If there exists a submodule V of M minimal with respect to the property $M = U + V$, then V is called a supplement of U in M. (This is equivalent of saying that $M = U + V$ and $U \cap V$ is small in V.)*

Proposition 2.1 ((Matlis 1960), Proposition 3) *Let M be an R-module. Then the following are equivalent:*

(i) *M is artinian.*

(ii) *M is a submodule of $E_1 \oplus \cdots \oplus E_n$ where $E_i = E(R/\mathfrak{m}_i)$ with $\mathfrak{m}_i$ a maximal ideal of R.*

(ii) *M has maximal orders and finitely generated socle.*

2.2. Lasker-Noether Theorem

The Lasker-Noether theorem is an extension of the fundamental theorem of arithmetic, and more generally the fundamental theorem of finitely generated abelian groups to all noetherian rings. The theorem was first proven by Emanuel Lasker (1905) for the special case of polynomial rings, and was proven in its full generality by Emmy Noether (1921).

Definition 2.18 *A submodule N of a module M is called irreducible if it is not an intersection of two strictly larger submodules.*

Theorem 2.6 *(***Lasker-Noether Theorem***) Every submodule of a finitely generated module over a noetherian ring is a finite intersection of primary submodules.*

The proof of Lasker-Noether theorem follows immediately from the following three facts:

(i) Any submodule of a finitely generated module over a noetherian ring is an intersection of a finite number of irreducible submodules.

(ii) If N is an irreducible submodule of a finitely generated module M over a noetherian ring then M/N has only one associated prime ideal.

(iii) A finitely generated module over a noetherian ring is primary if and only if it has at most one associated prime.

CHAPTER 3

COASSOCIATED AND ATTACHED PRIME IDEALS

3.1. Coassociated Prime Ideals

In this section, we shall derive basic results about coassociated primes without mentioning coprimary submodule or coprimary decomposition.

Definition 3.1 *An R-module M is called hollow if $M \neq 0$ and every proper submodule is small in M.*

Definition 3.2 *A module M is called indecomposable if $A \oplus B = M$ implies that either $A = 0$ or $B = 0$ for all submodules A,B of M.*

Clearly, if M is hollow, then M is indecomposable. But the converse is true if every non-zero factor of M is indecomposable (see 41.4 in (Wisbauer 1991)).

Lemma 3.1 *If M is hollow, then the set $\{x \in R \mid xM \neq M\}$ is a prime ideal of R which is denoted by $I(M)$.*

Proof Since $0 \in I(M)$, $I(M) \neq \emptyset$. Let $x, y \in I(M)$. Then $xM \neq M$ and $yM \neq M$. Since M is hollow $(x + y)M \subseteq xM + yM \neq M$, hence $(x + y)M \neq M$. Therefore $x + y \in I(M)$. Now for all $r \in R$, $x \in I(M)$, $rxM \subseteq xM \neq M$, so $rx \in I(M)$. Thus $I(M)$ is an ideal of R. Clearly $1 \notin I(M)$, so $I(M)$ is proper.

Let $x, y \notin I(M)$, then $xM = M$ and $yM = M$, so $x(yM) = M$ and $xy \notin I(M)$. Therefore $I(M)$ is prime ideal of R. □

Lemma 3.2 *If M is an artinian R-module, then it can be written as a finite sum of hollow submodules.*

Proof First, we will prove that an artinian R-module M has at least one hollow summand. If M is itself hollow, it is done. Suppose M is not hollow. Then $M = U_1 + X_1$ for some proper submodules U_1, X_1. Now if U_1 is hollow, it is done. If U_1 is not hollow, then again $U_1 = U_2 + X_2$ for some proper submodules U_2 and

X_2. Keep arguing that U_2 is hollow or not. Continuing this way, we obtain a descending sequence $M \supseteq U_1 \supseteq U_2 \supseteq \cdots$ of submodules of M. Since M is artinian this chain must be terminate at a hollow submodule $U_n (n \in \mathbb{N})$. Now $M = U_n + X$ where $X = X_n + X_{n-1} + \cdots + X_1$, so U_n is the desired hollow submodule.

Now we will prove that M is a finite sum of hollow submodules. If X is hollow, it is done, if not, since X is artinian, by above paragraph $X = U_{n+1} + Y_1$ for some submodules U_{n+1}, Y of M where U_{n+1} is hollow. Now $M = U_n + U_{n+1} + Y_1$, and if Y_1 is hollow, it is done, if not continue this way to obtain the descending sequence $X \supseteq Y_1 \supseteq Y_2 \supseteq \cdots$ of submodules. This chain must terminate at a hollow submodule Y_m. Hence $M = U_n + U_{n+1} + \cdots + U_{n+m} + Y_m$ indicates that M is a finite sum of hollow submodules. □

A module C in R-Mod is called a *cogenerator* provided that C cogenerates every left R-module, that is, M can be embedded in a product of copies of C

$$0 \longrightarrow M \longrightarrow C^A .$$

Proposition 3.1 ((Anderson and Fuller 1992), Proposition 18.15) *Let E be an injective left R-module, then E is a cogenarator if and only if E cogenerates every simple left R-module.*

Proof Necessity is clear. To prove the sufficiency, suppose E cogenerates every simple left R-module. Let M be a left R-module and let $0 \neq m \in M$. Consider the submodule Rm. Since Rm is finitely generated (cyclic) it contains a maximal submodule, say K. So $\mathrm{Hom}_R(Rm/K, E) \neq 0$. Otherwise $Rm/K = 0$ would give a contradiction. Here $Rm \xrightarrow{\pi} Rm/K \xrightarrow{\alpha_m} E$, say $\beta_m = \alpha_m o \pi$ where $0 \neq \alpha_m \in \mathrm{Hom}_R(Rm/K, E)$. Since E is injective, β_m can be extended to a homomorphism $\overline{\beta} : M \to E$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & Rm \arrow[r, "\iota"] \arrow[d, "\beta_m"'] & M \arrow[dl, dashed, "\overline{\beta}"] \\ & E & \end{tikzcd}$$

is commutative, i.e. $\overline{\beta}(m) = \beta_m(m) \neq 0$. Now define $\beta : M \to E^M$ by $\beta(a) = (\beta_m(a))$ where $a \in M$. Clearly, β is a monomorphism since $\beta_m(a) \neq 0$ for all $a \in M$. □

The following theorem will play a key role in the sequel.

Theorem 3.1 *If R is a noetherian ring, then every nonzero module over R has an artinian, hence also a hollow factor module.*

Proof Let $\Lambda_R := \{S_{\mathfrak{m}} \mid S_{\mathfrak{m}} \cong R/\mathfrak{m}$ where $\mathfrak{m}$ is a maximal ideal of R $\}$ and $\Lambda_R(E) := \{E(S_{\mathfrak{m}}) \mid S_{\mathfrak{m}} \in \Lambda\}$ where $E(S_{\mathfrak{m}})$ denotes the injective envelope of $S_{\mathfrak{m}}$. By Proposition 3.1, $C := \prod_{S_{\mathfrak{m}} \in \Lambda} E(S_{\mathfrak{m}})$ is a cogenerator for every R-module. So there exists a monomorphism $f : M \to C^I$ for some index set I. Note that $C^I = (\prod_{S_{\mathfrak{m}} \in \Lambda} E(S_{\mathfrak{m}}))^I \cong \prod_{S_{\mathfrak{m}} \in \Lambda} E(S_{\mathfrak{m}})^I$. Since f is a monomorphism there exists a $0 \neq m \in M$ such that $f(m) \neq 0$. Say $f(m) = (\ldots, s', \ldots)$ where $0 \neq s' \in E(S')$ for some $S' \in \Lambda_R$. Now the projection on $E(S')$ gives

$$0 \longrightarrow M \xrightarrow{f} \prod_{S_{\mathfrak{m}} \in \Lambda} E(S_{\mathfrak{m}})^{\pi_{E(S')}} \longrightarrow E(S').$$

Thus we have found a non-zero homomorphism $\alpha : M \to E(S')$ where $\alpha = \pi_{E(S')} \circ f$ and by the first isomorphism theorem $M/K \cong \operatorname{Im} \alpha \subseteq E(S')$ where $K = \operatorname{Ker} \alpha$. By Proposition 2.1 $E(S')$ is artinian. Hence M/K is the desired artinian factor of M.

Now by Lemma 3.2

$$(M/K) = (H_1/K) + (H_2/K) + \cdots + (H_n/K)$$

where $H_i/K (1 \leq i \leq n)$ are hollow. Now

$$(M/K)/((H_2 + \cdots + H_n)/K) \cong M/(H_2 + \cdots + H_n) \cong H_1/H_1 \cap (H_2 + \cdots + H_n).$$

Since H_1/K is hollow, and $H_1K \cong M/(H_2 + \cdots + H_n)$, the desired hollow factor is $M/(H_2 + \cdots + H_n)$. □

Definition 3.3 *Let R be a noetherian ring and M be an R-module. Then a prime ideal $\mathfrak{p}$ is called coassociated to M if there is a hollow factor module M' of M with $\mathfrak{p} = I(M')$. The set of all coassociated prime ideals to M is denoted by* $\operatorname{Coass}(M)$.

We shall give an equivalent statement to this definition after the following example.

Example 3.1 ((Zöschinger 1983), Example 1) *If M is an artinian module over a noetherian ring R, then* $\bigcap \operatorname{Coass}(M) = \sqrt{\operatorname{Ann}(M)}$.

Proof For every module M we have $\bigcap \mathrm{Coass}(M) \supseteq \sqrt{\mathrm{Ann}(M)}$ because from $\mathfrak{p} = I(M/M_0)$ with M/M_0 hollow it follows that $\mathfrak{p} \supseteq \mathrm{Ann}(M/M_0) \supseteq \mathrm{Ann}(M)$. But if M is artinian and $M = U_1 + \cdots + U_n$, thus that all U_i are hollow and none of them is redundant, then it follows with $\mathfrak{p}_i = I(U_i)$ that every $x \in \mathfrak{p}_i$ is already nilpotent with respect to U_i, i.e. $\mathfrak{p}_i = \sqrt{\mathrm{Ann}(U_i)}$. Hence $\bigcap \mathrm{Coass}(M) = \bigcap_{i=1}^{n} \mathfrak{p}_i = \sqrt{\mathrm{Ann}(M)}$. □

Lemma 3.3 ((Zöschinger 1986), Lemma 3.1) *Let R be a noetherian ring and M be an R-module. Then a prime ideal $\mathfrak{p}$ is coassociated to M if and only if there is an artinian factor module A of M with $\mathfrak{p} = \mathrm{Ann}(A)$.*

Proof For $\mathfrak{p} \in \mathrm{Coass}(M)$ there is a artinian hollow factor module M/M_0 where $\mathfrak{p} = I(M/M_0)$, and for every submodule U where $M_0 \subseteq U \subsetneq M$ we have $\mathfrak{p} = \sqrt{\mathrm{Ann}(M/U)}$. To eliminate the square root choose a maximal element $\mathfrak{a}_0 = \mathrm{Ann}(M/U_0)$ in the set $\{\mathrm{Ann}(M/U) \mid M_0 \subseteq U \subsetneq M\}$. Then $\mathfrak{a}_0$ is a prime ideal (see (Bourbaki 1967, chap.IV, §1 ,Prop.1)), hence $\mathfrak{p} = \mathfrak{a}$.

Conversely, if $\mathfrak{p} = \mathrm{Ann}(M/U)$ and $C = M/U$ is artinian then the canonical map $\gamma \in \mathrm{Hom}_R(M, C)$ since $\mathrm{Ann}(\gamma) = \mathfrak{p}$. If $f \in \mathrm{Hom}_R(M, C)$ with $\mathfrak{p} = \mathrm{Ann}(f)$, then $A = M/\operatorname{Ker} f$ is an artinian factor module of M with $\mathrm{Ann}(A) = \mathfrak{p}$. By Example 3.1 we have $\bigcap \mathrm{Coass}(A) = \sqrt{\mathrm{Ann}(A)}$, so $\mathfrak{p} \in \mathrm{Coass}(A)$, $\mathfrak{p} \in \mathrm{Coass}(M)$ as we wished. □

By Theorem 3.1, we can say that $\mathrm{Coass}(M) \neq \emptyset$ for every non-zero R-module M. If M is itself hollow and $I(M) = \{x \in R \mid xM \neq M\}$ as above, then $I(M) = I(M/M_0)$ for every submodule $M_0 \subsetneq M$, hence $\mathrm{Coass}(M) = \{I(M)\}$. More generally, we have the following lemma.

Lemma 3.4 *Let M be a module over a noetherian ring R. If $M = U_1 + \cdots + U_n$ such that all U_i are hollow and none of them can be omitted, then* $\mathrm{Coass}(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ *where* $\mathfrak{p}_i = I(U_i)$.

Proof Set $K_i = \sum_{\substack{j=1 \\ j \neq n}}^{n} U_j$, so K_i is proper.

$$M/K_i = (U_1 + \cdots + U_i + \cdots + U_n)/K_i \cong U_i/(U_i \cap K_i).$$

Hence $I(M/K_i) \cong I(U_i/U_i \cap K_i) = I(U_i) = \mathfrak{p}_i$ since U_i is hollow, so $\mathfrak{p}_i \in \mathrm{Coass}(M)$. Therefore $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \subseteq \mathrm{Coass}(M)$.

Conversely, let $\mathfrak{p} \in \mathrm{Coass}(M)$. Then $\mathfrak{p} = I(M/T)$ for some hollow factor M/T of M. Now

$$M/T = (U_1+\cdots+U_n)/T = (U_1+T)/T+\cdots+(U_n+T)/T \cong U_1/(U_1\cap T)+\cdots+U_n/(U_n\cap T).$$

Since M/T is hollow $M/T \cong U_i/(U_i \cap T)$ for some $i(1 \le i \le n)$. Therefore $I(M/T) \cong I(U_i/(U_i \cap T)) = \mathfrak{p}_i = I(U_i)$, since U_i is hollow. Thus $\mathfrak{p} = \mathfrak{p}_i$ for some i $(1 \le i \le n)$. Hence $\mathrm{Coass}(M) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. □

Example 3.2 ((Zöschinger 1983), Example 2) *For every ideal* $\mathfrak{a}$ *of a noetherian ring* R *we have*

$$\mathrm{Coass}(\coprod_{i=1}^{\infty} R/\mathfrak{a}^{\mathrm{i}}) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p}+\mathfrak{a} \neq R\}.$$

Proof Let $\mathfrak{a}$ be an ideal of a ring R. An R-module M is called $\mathfrak{a}$-torsion if $\sum_{i=1}^{\infty} \mathrm{Ann}_M(\mathfrak{a}^i) = M$. For every $\mathfrak{a}$-torsion module N it follows from $\mathfrak{p}+\mathfrak{a} = R$ that N is $\mathfrak{p}$-divisible. Then by (Lemma 3.6, i) $\mathfrak{p} \notin \mathrm{Coass}(N)$, i.e. for the particular module $N = \coprod_{i=1}^{\infty} R/\mathfrak{a}^i$ we have shown that $\mathrm{Coass}(N) \subseteq \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p}+\mathfrak{a} \neq R\}$. For the converse, let now $\mathfrak{p} \in \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p}+\mathfrak{a} \neq R\}$, hence $\mathfrak{p}+\mathfrak{a} \subseteq \mathfrak{m}$ for some $\mathfrak{m} \in \Omega$. The injective hull E of $R/\mathfrak{m}$ is artinian, hence also the submodule $M = \mathrm{Hom}_R(R/\mathfrak{p}, E)$ is, and from $\mathrm{Ann}(M) = \mathfrak{p}$ it follows by Example 3.1 that $\mathfrak{p} \in \mathrm{Coass}(M)$. Since in addition, M is $\mathfrak{a}$-torsion and countably generated, there is an epimorphism of $N = \coprod_{i=1}^{\infty} R/\mathfrak{a}^i$ on M such that $\mathfrak{p} \in \mathrm{Coass}(N)$. (Here we have also $\bigcap \mathrm{Coass}(N) = \sqrt{\mathrm{Ann}(N)}$ because with $\mathfrak{a}' = \{x \in R \mid x = ax \text{ for some } a \in \mathfrak{a}\}$ by Krull's Intersection Theorem we have $\mathrm{Ann}(N) = \mathfrak{a}'$. Now if $\mathfrak{p}$ is minimal over $\mathrm{Ann}(N)$, then it is shown in (Zöschinger 1982, Lemma 1.2) that $\mathfrak{p}+\mathfrak{a} \neq R$, hence $\mathfrak{p} \in \mathrm{Coass}(N)$.) □

Lemma 3.5 ((Zöschinger 1983), Lemma 2.1) *Let R be a noetherian ring and M be an R-module. Let U be a submodule of M.*

(i) We always have $\mathrm{Coass}(M) \subseteq \mathrm{Coass}(U) \cup \mathrm{Coass}(M/U)$.

(ii) If $\mathfrak{p} \in \mathrm{Coass}(U)$, *then there is* $\mathfrak{p}_0 \in \mathrm{Coass}(M)$ *with* $\mathfrak{p}_0 \subseteq \mathfrak{p}$.

(iii) If U is small in M, then $\mathrm{Coass}(M) = \mathrm{Coass}(M/U)$.

(iv) If U is coclosed in M, then $\mathrm{Coass}(M) = \mathrm{Coass}(U) \cup \mathrm{Coass}(M/U)$.

Proof (i) Let $\mathfrak{p} \in \mathrm{Coass}(M)$, $\mathfrak{p} \notin \mathrm{Coass}(U)$. In $\mathfrak{p} = I(M/M_0)$ with M/M_0 hollow, we must have $U + M_0 \neq M$. Otherwise $U + M_0 = M$ implies that $U/(U \cap M_0) = M/M_0$

and $\mathfrak{p} = I(M/M_0) \cong I(U/(U \cap M_0))$, hence $\mathfrak{p} \in \mathrm{Coass}(U)$ which is a contradiction. Therefore from $I(M/M_0) = I(M/(U + M_0))$ it follows that $\mathfrak{p} \in \mathrm{Coass}(M/U)$.

(ii) In $\mathfrak{p} = I(U/U_0)$ we can additionally assume that U/U_0 is artinian, and then for some maximal element V_0 in the set $\{U_0 \subseteq V \subseteq M \mid V \cap U = U_0\}$ we have that $U/U_0 \to M/V_0$ is an essential monomorphism, i.e. $\mathrm{Im}\,\varphi \subseteq_e V_0$, hence also M/V_0 is artinian. By Example 3.1 it follows $\bigcap \mathrm{Coass}(M/V_0) \subseteq \sqrt{\mathrm{Ann}(U/U_0)} \subseteq \mathfrak{p}$ (also $U/U_0 = U/V_0 \cap U \cong (U + V_0)/V_0 \subseteq M/V_0$), hence $\mathfrak{p}_0 \subseteq \mathfrak{p}$ for some $\mathfrak{p}_0 \in \mathrm{Coass}(M/V_0)$ and of course $\mathfrak{p}_0 \in \mathrm{Coass}(M)$.

(iii) Since every factor of M/U is a factor of M, it is clear that $\mathrm{Coass}(M/U) \subseteq \mathrm{Coass}(M)$. Conversely, if $\mathfrak{p} \in \mathrm{Coass}(M)$, then $\mathfrak{p} = I(M/L)$ where M/L is a non-zero hollow factor. Since U is small in M, $M/(U + L)$ is non-zero, so that $\mathfrak{p} = I(M/L) = I(M/(U + L)) \in \mathrm{Coass}(M/U)$.

(iv) A submodule U is called *coclosed* in M, if for every $X \subsetneq U$ we have that U/X is not small in M/X. Always $\mathrm{Coass}(M/U) \subseteq \mathrm{Coass}(M)$. By (i), we only need to prove that $\mathrm{Coass}(U) \subseteq \mathrm{Coass}(M)$. Let $\mathfrak{p} \in \mathrm{Coass}(U)$. Then $\mathfrak{p} = I(U/U_0)$ where U/U_0 is hollow. Hence U/U_0 is not small in M/U_0, i.e. $U + M_0 = M$ for some $M_0/U_0 \subsetneq M/U_0$. It follows that $\mathfrak{p} = I(U/U_0) \cong I((U/U_0)/(U \cap M_0)/U_0) \cong I(U/U \cap M_0) \cong I(U + M_0/M_0) \cong I(M/M_0)$. Therefore $\mathfrak{p} \in \mathrm{Coass}(M)$. □

Remark. All four parts of the Lemma (as well as Lemma 3.6) are well-known in the corresponding formulation for associated prime ideals. But for the fact $\mathrm{Ass}(\coprod_{\lambda\in\Lambda} M_\lambda) = \bigcup_{\lambda\in\Lambda} \mathrm{Ass}(M_\lambda)$ there is no analogy by coassociated prime ideals: For all $n \geq 1$, $\mathrm{Coass}(R^n) = \Omega$ while $\mathrm{Coass}(R^{(\mathbb{N})}) = \mathrm{Coass}(R^{\mathbb{N}}) = \mathrm{Spec}(R)$ by Example 3.2.

Lemma 3.6 ((Zöschinger 1983), Lemma 2.2) *Let M be a module over a noetherian ring R, $\mathfrak{a}$ be an ideal of R and Ω be the set of all maximal ideals of R.*

(i) M is $\mathfrak{a}$-divisible if and only if $\mathfrak{a}$ lies in none of the coassociated prime ideals to M.

(ii) $\mathfrak{a}M$ is small in M if and only if $\mathfrak{a}$ lies in all of the coassociated prime ideals to M.

(iii) M is radical if and only if $\mathrm{Coass}(M)$ *contains no maximal ideals, (i.e.* $\mathrm{Coass}(M) \cap \Omega = \emptyset$*).*

(iv) M is coatomic (i.e. every submodule is contained in a maximal submodule) if and only if $\mathrm{Coass}(M)$ *consists only of maximal ideals, (i.e.* $\mathrm{Coass}(M) = \Omega$*).*

Proof (i) We will first show more generally that $\mathrm{Coass}(M/\mathfrak{a}M) = \mathrm{Coass}(M) \cap \mathrm{Var}(\mathfrak{a})$ where $\mathrm{Var}(\mathfrak{a})$ denotes the variety of $\mathfrak{a}$. Always $\mathrm{Coass}(M/\mathfrak{a}M) \subseteq \mathrm{Coass}(M)$ and clearly for every $\mathfrak{p} \in \mathrm{Coass}(M/\mathfrak{a}M)$ we have that $\mathfrak{a} \subseteq \mathrm{Ann}(M/\mathfrak{a}M) \subseteq \mathfrak{p}$, so $\mathfrak{p} \in \mathrm{Var}(\mathfrak{a})$. Conversely, let $\mathfrak{p} \in \mathrm{Coass}(M) \cap \mathrm{Var}(\mathfrak{a})$, i.e. $\mathfrak{a} \subseteq \mathfrak{p} = I(M')$ for some hollow factor M' of M: If M' were $\mathfrak{a}$-divisible , then $xM' = M'$ for some $x \in \mathfrak{a} \subseteq \mathfrak{p}$. Hence $x \notin I(M') = \mathfrak{p}$, a contradiction. So M' is not $\mathfrak{a}$-divisible, i.e. $yM' \neq M'$ for some $y \in \mathfrak{a}$. Therefore $\mathfrak{p} = I(M') = I(M'/yM) = I((M'/yM)/(\mathfrak{a}M/yM)) \cong I(M'/\mathfrak{a}M')$. Therefore $M'/\mathfrak{a}M'$ is a factor module of $(M/\mathfrak{a}M)$. Therefore $\mathfrak{p} \in \mathrm{Coass}(M/\mathfrak{a}M)$.

Now we prove the particular statement of part (i) in the lemma: M is not $\mathfrak{a}$-divisible if and only if $\mathfrak{a}M \neq M$. In that case $\emptyset \neq \mathrm{Coass}(M/\mathfrak{a}M) = \mathrm{Coass}(M) \cap \mathrm{Var}(\mathfrak{a})$. Therefore there exists $\mathfrak{p} \in \mathrm{Coass}(M) \cap \mathrm{Var}(\mathfrak{a})$, i.e. $\mathfrak{a} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Coass}(M)$.

(ii) If $\mathfrak{a}M$ is small in M, then it follows by Lemma 3.5(iii) and newly proved above equality that $\mathrm{Coass}(M) \subseteq \mathrm{Var}(\mathfrak{a})$, i.e. $\mathfrak{a} \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Coass}(M)$; but if $\mathfrak{a}M$ is not small in M then there is a hollow $\mathfrak{a}$-divisible factor module M' of M, and $\mathfrak{p} = I(M')$ does not lie over $\mathfrak{a}$ by (i).

(iii) This follows immediately from (i) with $\mathfrak{a} = \mathfrak{m}$ for all $\mathfrak{m} \in \Omega$.

(iv) Let $\mathfrak{p} \in \mathrm{Coass}(M)$. Then $\mathfrak{p} = I(M/M_0) = I((M/M_0)/(K/M_0)) \cong I(M/K)$ for some submodule M_0 and for some maximal submodule K containing M_0. Since $M/K \cong R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, we have that $\mathfrak{p} \cong I(R/\mathfrak{m}) = \{x \in R \mid xR + \mathfrak{m} \neq R\}$. If $x \in \mathfrak{m}$, then $xR + \mathfrak{m} = \mathfrak{m} \neq R$, so $x \in \mathfrak{p}$. Thus the inclusion $\mathfrak{m} \subseteq \mathfrak{p}$ implies that $\mathfrak{m} = \mathfrak{p}$. Conversely, suppose $\mathrm{Coass}(M) \subseteq \Omega$. Then for every $X \subsetneq M$, we have that M/X is not radical by (iii), i.e. X lies in a maximal submodule of M. □

Corollary 3.1 ((Zöschinger 1983), Corollary 1) *For every module M over a noetherian ring R, we have $\bigcup \mathrm{Coass}(M) = \{x \in R \mid xM \neq M\}$ and $\bigcap \mathrm{Coass}(M)$ is the largest ideal $\mathfrak{a}$ of R such that $\mathfrak{a}M$ is small in M.*

Part (ii) also yields a generalization of the Krull's Intersection Theorem. If J is the Jacobson radical of the ring R and M is a finitely generated R-module, then it is well-known that JM is small in M and $\bigcap_{i=1}^{\infty} J^i M = 0$. The generalization says:

Corollary 3.2 ((Zöschinger 1983), Corollary 2) *Let M be a module over a noetherian* *ring R. Let $\mathfrak{a}$ and M be arbitrary and assume that $\mathfrak{a}M$ is small in M, then*

$$\bigcap_{i=1}^{\infty} \mathfrak{a}^i M = 0.$$

Proof If M is artinian, then it follows from $\mathfrak{a} \subseteq \bigcap \mathrm{Coass}(M)$ by Example 3.1 that $\mathfrak{a}^e M = 0$ for some $e \geq 1$. But if M is arbitrary, then there is a family $(U_\lambda \mid \lambda \in \Lambda)$ of submodules such that all M/U_λ are artinian and $\bigcap_{\lambda \in \Lambda} U_\lambda = 0$. Since every M/U_λ is annihilated by a (dependent on λ) power of $\mathfrak{a}$, we have $\bigcap_{i=1}^{\infty} \mathfrak{a}^i M \subseteq U_\lambda$ for all $\lambda \in \Lambda$, hence the claim follows. □

3.2. Attached Prime Ideals

Now we will give a generalization of the concept "coassociated". A prime ideal $\mathfrak{p}$ of R is called *attached* to the R-module M if $\mathfrak{p} = \mathrm{Ann}_R(M/U)$ for some submodule U of M. In this case, since $\mathfrak{p} = \mathrm{Ann}_R(M/\mathfrak{p}M)$, the set $\mathrm{Att}(M)$ of all attached prime ideals behaves very simply under the direct product of modules: $\mathrm{Att}(M^{(I)}) = \mathrm{Att}(M^I) = \mathrm{Att}(M)$ for every non-empty index set I. Also in difference to $\mathrm{Coass}(M)$, an element of $\mathrm{Att}(M)$ is easily given: By (Zöschinger 1987, p.592) , every minimal prime ideal of $\mathrm{Ann}_R(M)$ belongs to $\mathrm{Att}(M)$. In particular, $\bigcap \mathrm{Att}(M) = \sqrt{\mathrm{Ann}_R(M)}$.

Example 3.3 ((Zöschinger 1988), Example 1) *If M is a finitely generated module over a noetherian ring R, then*

$$\mathrm{Att}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supseteq \mathrm{Ann}_R(M)\}.$$

Proof For every module M and every $\mathfrak{p} \in \mathrm{Att}(M)$, we have $\mathfrak{p} \supseteq \mathrm{Ann}_R(M)$. But if M is finitely generated, then by (Bourbaki 1967, Chapter II §4 Proposition 18, Cor.) for every prime ideal $\mathfrak{p}$ we have $\sqrt{\mathrm{Ann}_R(M/\mathfrak{p}M)} = \sqrt{\mathfrak{p} + \mathrm{Ann}_R(M)}$. If $\mathfrak{p} \supseteq \mathrm{Ann}_R(M)$, then $\mathrm{Ann}_R(M/\mathfrak{p}M) \subseteq \mathfrak{p}$, i.e. $\mathfrak{p} \in \mathrm{Att}(M)$. –Of course, if M is finitely generated, then $\mathrm{Coass}(M) = \{\mathfrak{m} \in \Omega \mid \mathfrak{m} \supseteq \mathrm{Ann}_R(M)\}$, so $\mathrm{Coass}(M) = \mathrm{Att}(M)$ only if M is of finite length. □

Example 3.4 ((Zöschinger 1988), Example 2) *If M is a flat module over a noetherian ring R, then* $\mathrm{Att}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M/\mathfrak{p}M \neq 0\}$.

Proof We only need to show " $\supseteq$ ". Since $M/\mathfrak{p}M$ is a torsion-free nonzero module over the integral domain $R/\mathfrak{p}$, we have $\mathrm{Ann}_{R/\mathfrak{p}}(M/\mathfrak{p}M) = 0$, i.e. $\mathrm{Ann}_R(M/\mathfrak{p}M) = \mathfrak{p}$. We do not know that under what extra conditions on a flat module M, really $\mathrm{Coass}(M) = \mathrm{Att}(M)$. □

Example 3.5 ((Zöschinger 1988), Example 3) *If M is an injective module over a noetherian ring R, then* $\mathrm{Att}(M) = \{\mathfrak{p} \in \mathrm{Ass}(R) \mid M[\mathfrak{p}] \neq 0\} = \mathrm{Coass}(M)$.

Proof For every finitely generated R-module A it is shown in (Zöschinger 1986, Corollary 3.3) that

$$\mathrm{Att}(\mathrm{Hom}_R(A, M)) \subseteq \{\mathfrak{p} \in \mathrm{Ass}(A) \mid M[\mathfrak{p}] \neq 0\} \subseteq \mathrm{Coass}(\mathrm{Hom}_R(A, M)),$$

so that the claim yields $A = R$. □

Example 3.6 ((Zöschinger 1988), Example 4) *If R is local and M is radical, then every attached prime ideal is an intersection of coassociates.*

Proof For every $\mathfrak{p} \in \mathrm{Att}(M)$ the $R/\mathfrak{p}$-module $\overline{M} = M/\mathfrak{p}M$ is faithful, so by (Zöschinger 1988, Corollary 1.3) we have $\bigcap \mathrm{Coass}_{R/\mathfrak{p}}(\overline{M}) = 0$ so that $\mathfrak{p} = \bigcap\limits_{\lambda \in \Lambda} \mathfrak{p}_\lambda$ where $\mathrm{Coass}(M/\mathfrak{p}M) = \{\mathfrak{p}_\lambda \mid \lambda \in \Lambda\}$. In addition, if $\mathrm{Coass}(M)$ were countable, the same proof and (Zöschinger 1988, Corollary 1.5) shows that we even have $\mathrm{Coass}(M) = \mathrm{Att}(M)$. □

CHAPTER 4

EXISTENCE OF COPRIMARY DECOMPOSITION

Let M be a non-zero R-module. M is called *coprimary* if for every $r \in R$, the R-endomorphism of M produced by multiplication by r is either surjective or nilpotent, i.e. for every $r \in R$ either $rM = M$ or $r^k M = 0$ for some $k \geq 1$. It is said that M is *representable* if M is the sum of finitely many coprimary submodules.

Lemma 4.1 *If an R-module M is coprimary, then $\mathfrak{p} := \sqrt{\mathrm{Ann}(M)}$ is a prime ideal.*

Proof Let $r, s \in R$ and $rs \in \sqrt{\mathrm{Ann}(M)}$ but $s \notin \sqrt{\mathrm{Ann}(M)}$. Then $(rs)^n M = 0$ for some positive integer n, and $s^k M \neq 0$ for all positive integers k. Since M is coprimary, we have $sM = M$, and hence $s^n M = M$. Therefore $0 = (rs)^n M = r^n s^n M = r^n M$ implies that $r \in \sqrt{\mathrm{Ann}(M)}$. □

Following above lemma, M is called $\mathfrak{p}$-*coprimary*. A representation $M = U_1 + \cdots + U_n$ in which all U_i are coprimary is called a *coprimary decomposition* in (Kirby 1973) , and also *a secondary representation* of M in (Macdonald 1973). Both authors investigated the existence and uniqueness of such a decomposition analogous to the classical Noether-Lasker theory of primary decomposition of noetherian modules. In particular, they showed that every artinian module is representable. Later in (Sharp 1976, (Theorem 2.3)) it is shown that every injective module over a noetherian ring is representable. We shall give their proofs in the following sections.

If M has a coprimary decomposition, then we say that M has a *minimal coprimary decomposition* if it has the smallest possible number of coprimary modules, that is, there exist a positive integer n, distinct prime ideals $\mathfrak{p}_i (1 \leq i \leq n)$ of R, and $\mathfrak{p}_i$ −coprimary submodules $M_i (1 \leq i \leq n)$ of M such that

(i) $M = M_1 + \cdots + M_n$, and

(ii) $M_j \not\subseteq \sum\limits_{\substack{i=1 \\ i \neq j}}^{n} M_i$ for all j where $1 \leq j \leq n$.

The following proposition gives a test for prime-coprimary relationship:

Proposition 4.1 ((Kirby 1973), Proposition 3) *Let $M \neq 0$ be an R-module, and let $\mathfrak{p}$ be a prime ideal of R. Then M is a $\mathfrak{p}$-coprimary module if and only if,*

(i) $r \in R$ and $rM \neq M$ imply $r \in \mathfrak{p}$, and

(ii) $\mathfrak{p} \subseteq \sqrt{\mathrm{Ann}(M)}$.

Proof When M is $\mathfrak{p}$-coprimary, $\mathfrak{p} = \sqrt{\mathrm{Ann}(M)}$ and $r \in R$, $r \notin \mathfrak{p}$ imply $rM = M$. So (i) and (ii) are immediate.

Conversely, suppose (i) and (ii) hold. Consider $r \in R$ such that $r \notin \sqrt{\mathrm{Ann}(M)}$; then $r \notin \mathfrak{p}$ by (ii) and $rM = M$ by (i). So M is coprimary. Next consider $r \in R$ such that $r \notin \mathfrak{p}$. Hence $rM = M$ by (i), and $r^k M = M \neq 0$ for all k, i.e. $r \notin \sqrt{\mathrm{Ann}(M)}$. Therefore $\mathfrak{p} \supseteq \sqrt{\mathrm{Ann}(M)}$ which with (ii) shows that M is $\mathfrak{p}$-coprimary. □

4.1. Basic Facts and Examples of Coprimary and Representable Modules

Proposition 4.2 ((Muslim Baig 2009), Proposition 3.2.15) *If R is an integral domain, then its quotient field K is a 0-coprimary R-module.*

Proof For all $r \in R$ and $a/b \in K$, $a/b = r(a/rb)$ then $rK = K$. Hence K is coprimary Let $a/b \in K$ with $a, b \in R$ and $b \neq 0$. Without loss of generality, let $a \neq 0$ and for any $x \in R$, $xa/b = 0$ then $x = 0$. Therefore $\sqrt{\mathrm{Ann}(K)} = 0$. □

Example 4.1 ((Muslim Baig 2009), Example 3.2.16) *$\mathbb{Q}$ is a 0-coprimary $\mathbb{Z}$-module, and so is $\mathbb{Q}/\mathbb{Z}$.*

Proof Let $n \in \mathbb{Z}$, $n \neq 0$ and $a/b \in \mathbb{Q}$, $b \neq 0$. $n(a/nb) = a/b$ and $n\mathbb{Q} = \mathbb{Q}$. Without loss of generality, we may also assume $a \neq 0$, then $na/b = 0$ so $n = 0$.Therefore $\sqrt{\mathrm{Ann}\,\mathbb{Q}} = 0$. Then $\mathbb{Q}$ is a 0-coprimary. By Lemma 4.2, $\mathbb{Q}/\mathbb{Z}$ is also 0-coprimary. □

Proposition 4.3 ((Muslim Baig 2009), Proposition 3.2.17) *If $\mathfrak{m}$ is a maximal ideal of R, then $R/\mathfrak{m}^n$ is an $\mathfrak{m}$-coprimary R-module for every $n \geq 1$.*

Proof Let $x \in R$. If $x \in \mathfrak{m}$, then for any $\bar{r} = r + \mathfrak{m}^n \in R/\mathfrak{m}^n$, we have $x^n \bar{r} = x^n(r + \mathfrak{m}^n) = x^{^n} r + \mathfrak{m}^n = \mathfrak{m}^n$, where $r \in R$. Then $x^n(R/\mathfrak{m}^n) = 0$. Otherwise, if $x \notin \mathfrak{m}$, then $(x) + \mathfrak{m} = R$ and hence there exists $u \in R$ such that $ux + a = 1$ for some $a \in \mathfrak{m}$.

Moreover, $1 = 1^n = (ux + a)^n = vx + a^n$ for some $v \in R$. Now for any $\bar{r} \in R/\mathfrak{m}^n$ we have $\bar{r} = r.1 + \mathfrak{m}^n = r(vx + a^n) + \mathfrak{m}^n = xvr + ra^n + \mathfrak{m}^n = xvr + \mathfrak{m}^n = x(vr + \mathfrak{m}^n) = x(\overline{vr})$. Thus, $R/\mathfrak{m}^n$ is coprimary. Finally, note that $\sqrt{\mathrm{Ann}(R/\mathfrak{m}^n)} = \mathfrak{m}$ which gives $R/\mathfrak{m}^n$ is $\mathfrak{m}$-coprimary. $\square$

Example 4.2 ((Muslim Baig 2009), Example 3.2.18) *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}/\ \mathfrak{p}^n\ \mathbb{Z}$. If $x \in \mathbb{Z}$ and $(x, p) = 1$ then $xM = M$. Otherwise, if $p|x$ then $x^n M = 0$. That is, M is a $\mathfrak{p}$-coprimary $\mathbb{Z}$-module where $\mathfrak{p} = p\mathbb{Z}$ and p is a prime number.*

Proof Let $x \in \mathbb{Z}$. If $p \mid x$ then $x = pk$. Let $\overline{m} \in M$, then $x^n\overline{m} = (pk)^n\overline{m} = p^n k^n \overline{m} = p^n k^n m + p^n\mathbb{Z} = p^n\mathbb{Z}$ then $x^n M = 0$. If $(x, p) = 1$ then $p \nmid x$ and $(x, p^n) = 1$. Now given $\overline{m} \in M$, we have $(x, p^n) = 1$, hence $1 = xa + p^n b$ for some $a, b \in \mathbb{Z}$. Thus $\overline{m} = \overline{m}.1 = (m + p^n\mathbb{Z})(xa + p^n b) = xma + p^n bm + p^n\mathbb{Z} = \overline{xma}$. Hence $\overline{m} = x(\overline{ma})$ gives that M is $\mathfrak{p}$-coprimary. $\square$

Lemma 4.2 *Let $\mathfrak{p}$ be a prime ideal of R and let M be a $\mathfrak{p}$-coprimary module. Then M/K is a $\mathfrak{p}$-coprimary R-module for each proper submodule K of M.*

Proof Let $r \in R$. Suppose M is $\mathfrak{p}$-coprimary and $r(M/K) \neq M/K$. In this case $rM + K \neq M$, and this gives that $rM \neq M$. Otherwise, $rM = M$ would give the contradiction $M \neq M$. Now, since M is coprimary $r^k M = 0$ for some $k \geq 1$. But $r^k M = 0 \subseteq K$ gives that $r^k(M/K) = 0$. Hence M/K is coprimary.

Now let $s \in \mathrm{Ann}(M/K)$. Then $sM \subseteq K$ and K is proper together gives that $sM \neq M$. Since M is $\mathfrak{p}$-coprimary $s^t M = 0$ for some $t \geq 1$, and hence $s^t \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime we have $s \in \mathfrak{p}$. Therefore $\sqrt{\mathrm{Ann}(M/K)} = \mathrm{Ann}(M/K) = \mathfrak{p}$ $\square$

Corollary 4.1 *If M is representable, then M/K is representable for every proper submodule K of M.*

Proof There exist a positive integer n and coprimary submodules $M_i (1 \leq i \leq n)$ of M such that $M = M_1 + \cdots + M_n$. Then $M/K = ((M_1 + K)/K) + \cdots + ((M_n + K)/K)$. Then,for each $1 \leq i \leq n$, $(M_i + K)/K \cong M_i/(M_i \cap K)$ so that $(M_i + K)/K = 0$ or $(M_i + K)/K$ is coprimary by Lemma 4.2. $\square$

Lemma 4.3 *Let $\mathfrak{p}$ be a prime ideal of R, let n be a positive integer, and let $M_i (1 \leq i \leq n)$ be non-zero left R-modules. Then the R-module $M_1 \oplus \cdots \oplus M_n$ is $\mathfrak{p}$-coprimary if and only if M_i is $\mathfrak{p}$-coprimary for each $1 \leq i \leq n$.*

Proof Necessity follows from Lemma 4.2. Now let $M = M_1 \oplus \cdots \oplus M_n$. Suppose M_i are $\mathfrak{p}$-coprimary for every $i \in \{1, \ldots, n\}$. Let $r \in R$ and assume that $rM \neq M$. Then there exists $j \in \{1, \ldots, n\}$ such that $rM_j \neq M_j$. Since M_j is $\mathfrak{p}$-coprimary $r^{k_j}M_j = 0$ for some $k_j \geq 1$ and hence $r \in \sqrt{\operatorname{Ann} M_j} = \mathfrak{p}$. But $\mathfrak{p} = \sqrt{\operatorname{Ann} M_i}$ for every $i \in \{1, \ldots, n\}$. Thus $r^{k_i}M_i = 0$ for some $k_i \geq 1$ for each $i \in \{1, \ldots, n\}$. Say $k = max\{k_1, \ldots, k_n\}$. Then $r^k M = 0$. Since $r \in \sqrt{\operatorname{Ann} M} = \mathfrak{p}$, M is $\mathfrak{p}$-coprimary. □

Corollary 4.2 *Let $\mathfrak{p}$ be a prime ideal of R, let n be a positive integer, and let $M_i (1 \leq i \leq n)$ be $\mathfrak{p}$-coprimary submodules of M. Then the submodule $M_1 + \cdots + M_n$ of M is a $\mathfrak{p}$-coprimary R-module.*

Corollary 4.3 *If M has a coprimary decomposition, then M has a minimal coprimary decomposition.*

Proof Follows immediately from Corollary 4.2. □

4.2. Artinian Modules are Representable

Throughout this section R is a commutative ring which is not necessarily noetherian. The proofs of this section closely follows the ones from (Kirby 1973).

Lemma 4.4 *Let M be an Artinian R-module. If M is not coprimary, then there exist proper submodules N_1, N_2 of M such that $M = N_1 + N_2$.*

Proof Suppose that M is not coprimary, i.e. there exists $r \in R$ such that $r \notin \sqrt{\operatorname{Ann}(M)}$ and $rM \subsetneq M$. Thus $r^k M \neq 0$ for all $k \geq 1$, and so $M[Rr^k] \subsetneq M$ for all k. Consider the descending sequence

$$M \supsetneq rM \supseteq r^2M \supseteq \cdots,$$

and suppose $r^tM = r^{t+1}M$. Put $N_1 = rM$ and $N_2 = M[Rr^t]$; so N_1, N_2 are both proper submodules of M. Let $m \in M$; then $r^tm \in r^tM = r^{t+1}M$, i.e.

$$m \in rM + M[Rr^t] = N_1 + N_2.$$

Hence $M = N_1 + N_2$ as required. □

Theorem 4.1 *Every artinian R-module is the sum of a finite number of coprimary R-modules.*

Proof Let $\mathcal{F}$ denote the family of submodules of the artinian module M which cannot be written as a finite sum of coprimary modules. Suppose that $\mathcal{F}$ is non-empty; so $\mathcal{F}$ contains a minimal element, say $\overline{M}$. Note that $\overline{M}$ is not coprimary, so, by Lemma 4.4, $\overline{M} = N_1 + N_2$ where $\overline{M} \supsetneq N_1$ and $\overline{M} \supsetneq N_2$. By the minimality of $\overline{M}$, both N_1 and N_2 are finite sums of coprimary modules, and so, therefore, is $\overline{M}$. This contradiction shows that $\mathcal{F}$ is empty, and so M itself is the sum of a finite number of coprimary R-modules. □

4.3. Injective Modules over Noetherian Rings are Representable

The results in this section are due to (Sharp 1976).

Lemma 4.5 *Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal of R, and E be an injective R-module. Then $E[\mathfrak{q}] = \{x \in E \mid \mathfrak{q}\, x = 0\}$, if non-zero, is $\mathfrak{p}$-coprimary.*

Proof Let $r \in R$. If $r \in \mathfrak{p}$, then $r^n \in \mathfrak{q}$ for some $n \geq 1$, so that r^n annihilates $E[\mathfrak{q}]$. On the other hand, if $r \notin \mathfrak{p}$, then we can see that $E[\mathfrak{q}] = rE[\mathfrak{q}]$ as follows. Let $x \in E[\mathfrak{q}]$. Using the bar notation to denote the natural homomorphism from R to $R/\mathfrak{q}$, there is a homomorphism $\phi : R/\mathfrak{q} \to E$ for which $\phi(\overline{b}) = bx$ for all $\overline{b} \in R/\mathfrak{q}$.

As the diagram

$$\begin{tikzcd} 0 \arrow[r] & R/\mathfrak{q} \arrow[r, "\alpha"] \arrow[d, "\phi"] & R/\mathfrak{q} \arrow[dl, dashed, "\psi"] \\ & E & \end{tikzcd}$$

has exact row, there exists a homomorphism $\psi : R/\mathfrak{q} \to E$ making the triangle commutative. Thus $x = \phi(\overline{1}) = \psi(r\overline{1}) = r\psi(\overline{1})$. Hence (since $\psi(\overline{1}) \in E[\mathfrak{q}]$) we have $E[\mathfrak{q}] = rE[\mathfrak{q}]$, and the result follows. □

Lemma 4.6 *Let $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ be ideals of R and E be an injective R-module. Then*

$$\sum_{i=1}^{n} E[\mathfrak{a}_i] = E[\bigcap_{i=1}^{n} \mathfrak{a}_i].$$

Proof Let $x \in E[\bigcap_{i=1}^{n} \mathfrak{a}_i]$. Let $\pi : R \to R/\bigcap_{i=1}^{n} \mathfrak{a}_i$ and, for each $i = 1, \ldots, n$, $\pi_i : R \to R/\mathfrak{a}_i$, be the natural homomorphisms. There is a monomorphism

$$\xi : R/\bigcap_{i=1}^{n} \mathfrak{a}_i \to \bigoplus_{i=1}^{n}(R/\mathfrak{a}_i)$$

for which $\xi(\pi(r)) = (\pi_1(r), \pi_2(r), \ldots, \pi_n(r))$ for all $r \in R$. Also, there is a homomorphism $\eta : R/\bigcap_{i=1}^{n} \mathfrak{a}_i \to E$ for which $\eta(\pi(r)) = rx$ for all $r \in R$.

Since E is injective, we may extend the following diagram

$$\begin{tikzcd} 0 \arrow[r] & R/\bigcap_{i=1}^{n} \mathfrak{a}_i \arrow[r, "\xi"] \arrow[d, "\eta"] & \bigoplus_{i=1}^{n}(R/\mathfrak{a}_i) \arrow[dl, dashed, "\zeta"] \\ & E & \end{tikzcd}$$

(which has exact row) by a homomorphism $\zeta : \bigoplus_{i=1}^{n}(R/\mathfrak{a}_i) \to E$ which makes the extended diagram commute. Now $x = \eta(\pi(1)) \in \mathrm{Im}(\zeta)$, and it is clear that $\mathrm{Im}(\zeta) \subseteq \sum_{i=1}^{n} E[\mathfrak{a}_i]$. It follows that $E[\bigcap_{i=1}^{n} \mathfrak{a}_i] \subseteq \sum_{i=1}^{n} E[\mathfrak{a}_i]$. Since the reverse inclusion is clear, we have that $\sum_{i=1}^{n} E[\mathfrak{a}_i] = E[\bigcap_{i=1}^{n} \mathfrak{a}_i]$. □

Before we state the main theorem of this section, recall that an injective R-module E is called an *injective cogenerator of R* if, for every R-module M and for every non-zero $m \in M$, there is a homomorphism $\varphi : M \to E$ such that $\varphi(m) \neq 0$.

Theorem 4.2 *Assume R is noetherian, and denote by* $\mathrm{Ass}(R)$ *the set of prime ideals of R which belong to the zero ideal (for primary decomposition). Let E be an injective R-module. Then E has a coprimary decomposition, and* $\mathrm{Coass}(E) \subseteq \mathrm{Ass}(R)$.

More precisely, let $0 = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n$ *be a minimal primary decomposition for the zero ideal of R, with (for i=1,. . . ,n)* $\mathfrak{q}_i$ *a* $\mathfrak{p}_i$*-primary ideal. Then*

$$E = E[\mathfrak{q}_1] + E[\mathfrak{q}_2] + \cdots + E[\mathfrak{q}_n], \tag{4.1}$$

and (for i=1,. . . ,n) $E[\mathfrak{q}_i]$ *is either zero or* $\mathfrak{p}_i$*-coprimary.*

Moreover, if j is an integer such that $1 \leq j \leq n$*, and* $J = \{1, \ldots, j-1, j+1, \ldots, n\}$*, then* $E = \sum_{i\in J} E[\mathfrak{q}_i]$ *if and only if* $\bigcap_{i\in J} \mathfrak{q}_i$ *annihilates E; consequently, if E is an injective cogenerator of R, then equation (4.1) is a minimal coprimary decomposition for E, and* $\mathrm{Coass}(E) = \mathrm{Ass}(R)$.

Proof Lemma 4.5 shows that $E[\mathfrak{q}_i]$ is either zero or $\mathfrak{p}_i$-coprimary, and Lemma 4.6 shows that $E = E[0] = E[\bigcap_{i=1}^{n} \mathfrak{q}_i] = \sum_{i=1}^{n} E[\mathfrak{q}_i]$. The same lemma also provides the information that if the integer j satisfies $1 \leq j \leq n$, then

$$\sum_{i\in J} E[\mathfrak{q}_i] = E[\bigcap_{i\in J} \mathfrak{q}_i];$$

the latter module is clearly equal to E if and only if $\bigcap_{i\in J} \mathfrak{q}_i$ annihilates E.

Now assume E is an injective cogenerator of R. To prove the final assertions of the theorem, it is enough to show that for each $j = 1, \ldots, n$, the ideal $\bigcap_{i\in J} \mathfrak{q}_i$ does not annihilate E; it is therefore sufficient to show that if $\mathfrak{b}$ is an arbitrary non-zero ideal of R, then $\mathfrak{b}$ does not annihilate E.

To this end, let y be a non-zero element of $\mathfrak{b}$. Since E is an injective cogenerator of R, there exists a homomorphism $\phi : R \to E$ such that $\phi(y) \neq 0$. Then $y\phi(1) = \phi(y) \neq 0$, so $\phi(1)$ is an element of E which is not annihilated by y, and so not annihilated by $\mathfrak{b}$. This completes the proof. □

4.4. More General Facts about Coprimary and Representable Modules

For every R-module M we have $\mathrm{Coass}(M) \subseteq \mathrm{Att}(M)$, and we show in Lemma 4.10(i) that these sets coincide for representable modules. Since every minimal prime ideal of the ideal $\mathrm{Ann}(M)$ is an element of $\mathrm{Att}(M)$, we have $\bigcap \mathrm{Att}(M) = \sqrt{\mathrm{Ann}(M)}$ while for $\mathfrak{a} = \bigcap \mathrm{Coass}(M)$ in general we only have $\bigcap_{i=1}^{\infty} \mathfrak{a}^i M = 0$. With the help of the set $\mathrm{Att}(M)$ we are able to describe coprimary R-modules very easily:

Lemma 4.7 ((Zöschinger 1990), Lemma 1.1) *Let R be a noetherian ring. For an R-module M and a prime ideal $\mathfrak{p}$ of R the following are equivalent:*

(i) *M is $\mathfrak{p}$-coprimary.*

(ii) $\mathrm{Att}(M) = \{\mathfrak{p}\}$.

(iii) $\mathrm{Coass}(M) = \{\mathfrak{p}\}$ *and* $\mathfrak{p}^e M = 0$ *for some* $e \geq 1$.

Proof (*i*) ⇒ (*ii*) For every coprimary module M, $\sqrt{\mathrm{Ann}(M)}$ is prime ideal, say $\mathfrak{p}$, and then M is called $\mathfrak{p}$-coprimary. Now, if $\mathfrak{q} \in \mathrm{Att}\, M$, $\mathfrak{q} = \mathrm{Ann}(M/U)$, it follows $\mathrm{Ann}(M) \subseteq \mathfrak{q}$. For all $x \in \mathfrak{q}$, $xM \subseteq U \subsetneq M$. Suppose $U = M$ then $\mathfrak{q} = \mathrm{Ann}(M/M) = R$. This is a contradiction, so $xM \neq M$. Then $x^k M = 0$, $x^k \in \mathrm{Ann}(M)$, so $x \in \sqrt{\mathrm{Ann}\, M} = \mathfrak{p}$. Therefore $\mathfrak{q} \subseteq \mathfrak{p}$. On the other hand $\mathfrak{p} = \sqrt{\mathrm{Ann}(M)} = \bigcap_{\mathrm{Ann}(M)\subseteq \mathfrak{p}} \mathfrak{p} \subseteq \mathfrak{q}$, $\mathfrak{p} \subseteq \mathfrak{q}$. Hence $\mathrm{Att}(M) = \{\mathfrak{p}\}$

$(ii) \Rightarrow (iii)$ $\mathrm{Att}(M) = \{\mathfrak{p}\}$, $\mathfrak{p} = \sqrt{\mathrm{Ann}(M/U)}$ for some $U \subseteq M$. $\mathrm{Coass}(M) = \{\mathfrak{p}\}$, $\mathfrak{p} = I(M/K) = \{x \in R \mid xM + K \neq M\}$ for some $K \subseteq M$ where M/K is hollow. Since for all $x \in R$ $xM = M$ or $x^k M = 0 (k \geq 1)$ we have $x^k M = 0$ for some $k \geq 1$. Then $x \in \mathfrak{p}$, $x^k M = 0$, and hence $x \in \sqrt{\mathrm{Ann}(M)} \subseteq \mathfrak{p}$.

$(iii) \Rightarrow (i)$ For every R-module N we have $\bigcup \mathrm{Coass}(N) = \{x \in R \mid xN \neq N\}$, so that here $x^k M = 0$ always follows from $xM \neq M$, hence M is coprimary and $\sqrt{\mathrm{Ann}(M)} = \mathfrak{p}$. □

We will give two corollaries of the above lemma. But first we need the following result from (Zöschinger 1986).

Lemma 4.8 ((Bourbaki 1967), p.280, Corollary 1) *If R is noetherian and $\mathfrak{p} \in \mathrm{Ass}_R(E \otimes_R F)$, then $\mathfrak{p} \in \mathrm{Ass}_R(E)$ and $\mathfrak{p}$ is the only prime ideal $\mathfrak{p}$ of R such that $\mathfrak{p} \in \mathrm{Ass}_R(F/\mathfrak{p}F)$.*

Lemma 4.9 ((Zöschinger 1986), Corollary 3.3) *Let R be a noetherian ring. If A is a finitely generated and M is an injective R-module, then*

$$\mathrm{Coass}(\mathrm{Hom}_R(A, M)) = \{\mathfrak{p} \in \mathrm{Ass}(A) \mid \mathrm{Ann}_M(\mathfrak{p}) \neq 0\}.$$

Proof Let $\mathfrak{p} \in \mathrm{Coass}(\mathrm{Hom}_R(A, M))$. Then by Lemma 3.3, $\mathfrak{p} = \mathrm{Ann}(\mathrm{Hom}_R(A, M)/U)$, where $U \subseteq \mathrm{Hom}_R(A, M)$, and an injective module Q and a homomorphism $f : \mathrm{Hom}_R(A, M) \to Q$ with $\mathrm{Ker} f = U$. Now $f \in \mathrm{Hom}(\mathrm{Hom}_R(A, M), Q)$ with $\mathrm{Ann}(f) = \mathfrak{p}$. For

$$\begin{aligned} \mathrm{Ann}(f) &= \{r \in R \mid rf = 0\} \\ &= \{r \mid (rf)(\alpha) = 0 \text{ for all } \alpha \in \mathrm{Hom}_R(A, M)\} \\ &= \{r \in R \mid r\alpha \in U \text{ for all } \alpha \in \mathrm{Hom}_R(A, M)\} \\ &= \mathfrak{p}. \end{aligned}$$

Note that $\mathrm{Hom}(\mathrm{Hom}_R(A, M), Q) \cong A \otimes \mathrm{Hom}(M, Q) = A \otimes_R F$. Therefore $\mathfrak{p} \in \mathrm{Ass}(A \otimes_R F)$ since $\mathfrak{p}$ is the annihilator of an element in $A \otimes_R F$ with the flat module $F = \mathrm{Hom}(M, Q)$. By Lemma 4.8 $\mathfrak{p} \in \mathrm{Ass}_R(A)$ and since $R/\mathfrak{p} \otimes_R F \cong F/\mathfrak{p}F$, we have $(R/\mathfrak{p}) \otimes_R F \neq 0$. Now $0 \neq (R/\mathfrak{p}) \otimes_R F = (R/\mathfrak{p}) \otimes \mathrm{Hom}(M, Q) \cong \mathrm{Hom}(\mathrm{Hom}_R(R/\mathfrak{p}, M), Q) \cong \mathrm{Hom}(\mathrm{Ann}_M(\mathfrak{p}), Q)$. Therefore $\mathrm{Ann}_M(\mathfrak{p}) \neq 0$.

Conversely, let A be an R-module and $\mathfrak{p} \in \operatorname{Ass}(A)$, M injective and $\operatorname{Ann}_M(\mathfrak{p}) \neq 0$. The monomorphism $R/\mathfrak{p} \to A$ yields an epimorphism $\operatorname{Hom}_R(A, M) \to \operatorname{Hom}_R((R/\mathfrak{p}), M)$, and any $\mathfrak{q} \in \operatorname{Coass}(\operatorname{Hom}_R((R/\mathfrak{p}), M))$ gives $\mathfrak{q} \in \operatorname{Ass}(R/\mathfrak{p})$, i.e. $\mathfrak{q} = \mathfrak{p}$, so that $\mathfrak{p} \in \operatorname{Coass}(\operatorname{Hom}_R(A, M))$. □

Corollary 4.4 ((Zöschinger 1990), Corollary 1.2) *Let M be a module over a noetherian ring R and $\mathfrak{p}$ be a prime ideal of R. Then we have:*

(i) If $\mathfrak{p}$ is a maximal element of $\operatorname{Att}(M)$, then the factor modules $M/\mathfrak{p}^i M$ (i=1,2,3,...) are all $\mathfrak{p}$-coprimary.

(ii) If U is a submodule of M such that U and M/U are $\mathfrak{p}$-coprimary, then also M is $\mathfrak{p}$-coprimary.

(iii) If M is the injective hull of $R/\mathfrak{p}$ and $\mathfrak{a}$ is an ideal of R, then for $U = M[\mathfrak{a}] = \operatorname{Ann}_M(\mathfrak{a})$ we have: U is $\mathfrak{p}$-coprimary if and only if $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{a}$.

Proof (i) immediately follows with Lemma 4.7(ii) as well as (ii) since $\operatorname{Att}(M) \subseteq \operatorname{Att}(U) \bigcup \operatorname{Att}(M/U)$.

(iii) $U \cong \operatorname{Hom}_R(R/\mathfrak{a}, M)$, so by the proof of Lemma 4.9 we have $\operatorname{Att}(U) = \{\mathfrak{q} \in \operatorname{Ass}(R/\mathfrak{a}) \mid \mathfrak{q} \subseteq \mathfrak{p}\}$ so that $\operatorname{Att}(U) = \{\mathfrak{p}\}$ is equivalent with $\mathfrak{a} \subseteq \mathfrak{p}$ and $\mathfrak{p}$ is the minimal prime ideal of $\mathfrak{a}$. □

The next corollary was proved in Theorem 4.2 for the particular case $A = R$.

Corollary 4.5 ((Zöschinger 1990), Corollary 1.3) *If M is injective and A is finitely generated module over a noetherian ring R, then $\operatorname{Hom}_R(A, M)$ is representable.*

Proof By $A \neq 0$ one can choose irreducible factors A/A_i such that $\bigcap_{i=1}^{n} A_i = 0$, and the monomorphism $A \to \prod_{i=1}^{n}(A/A_i)$ induces an epimorphism $\prod_{i=1}^{n} \operatorname{Hom}_R(A/A_i, M) \to \operatorname{Hom}_R(A, M)$. For every $H_i = \operatorname{Hom}_R(A/A_i, M)$ again by Lemma 4.9 $\operatorname{Att}(H_i) = \{\mathfrak{q} \in \operatorname{Ass}(A/A_i) \mid M[\mathfrak{q}] \neq 0\}$, since $\mid \operatorname{Ass}(A/A_i) \mid = 1$, hence $\mid \operatorname{Att}(H_i) \mid \leq 1$, i.e. by the lemma H_i is zero or coprimary. Therefore $\prod_{i=1}^{n} H_i$ is representable, hence also the factor module $\operatorname{Hom}_R(A, M)$ is. □

Proposition 4.4 *Let R be a noetherian ring and M be an R-module. Let $M_1 + \cdots + M_n$ be a minimal coprimary decomposition of M where M_i is $\mathfrak{p}_i$-coprimary, $\mathfrak{p}_i$ are prime ideals. Then*

$$\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \subseteq \operatorname{Coass}(M)$$

Proof Let $M = M_1 + \cdots + M_n$ be a minimal coprimary decomposition where $M_i (1 \leq i \leq n)$ is $\mathfrak{p}_i$-coprimary. By the second isomorphism theorem we have

$$M/\sum_{\substack{j=1 \\ i\neq j}}^{n} M_j \cong M_i/(M_i \cap \sum_{\substack{j=1 \\ i\neq j}}^{n} M_j).$$

Since M_i is $\mathfrak{p}_i$-coprimary, by Lemma 4.2 we have that $M_i/(M_i \cap \sum_{\substack{j=1 \\ i\neq j}}^{n} M_j)$ is $\mathfrak{p}_i$-coprimary for every $i(1 \leq i \leq n)$. By Lemma 4.7 we have $\{\mathfrak{p}_i\} = \mathrm{Coass}(M/\sum_{\substack{j=1 \\ i\neq j}}^{n} M_j) \subseteq \mathrm{Coass}(M)$ for every $i(1 \leq i \leq n)$. Therefore $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \subseteq \mathrm{Coass}(M)$. □

Lemma 4.10 ((Zöschinger 1990), Lemma 1.4) *For every representable module M over a noetherian ring R we have:*

(i) $\mathrm{Coass}(M)$ *is finite and coincides with* $\mathrm{Att}(M)$.

(ii) For every decomposition $\mathrm{Coass}(M) = X \cup Y$ *there is a representable submodule U of M where* $\mathrm{Coass}(U) = X$ *and* $\mathrm{Coass}(M/U) = Y$.

(iii) For every ideal $\mathfrak{a}$ of R there is $e \geq 1$ where $M[\mathfrak{a}^e] + \mathfrak{a}M = M$.

(iv) The radical part $P(M)$ is again representable and the reduced part $M/P(M)$ is coatomic and semi-artinian. Besides, $P(M)$ is coclosed in M.

Proof For $M = 0$, all statements are clear, so let $M \neq 0$ and $M = U_1 + \cdots + U_n$ be a coprimary decomposition of M in which none of U_i is superfluous.

(i) With $\mathfrak{p}_i = \sqrt{\mathrm{Ann}(U_i)}$ we claim that $\mathrm{Coass}(M) = \mathrm{Att}(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. The epimorphism $U_1 \times \cdots \times U_n \longrightarrow M$ yields for every $\mathfrak{q} \in \mathrm{Att}(M)$ that $\mathfrak{q} \in \mathrm{Att}(U_j) = \{\mathfrak{p}_j\}$ for some $j \in \{1, \ldots, n\}$, hence $\mathrm{Coass}(M) \subseteq \mathrm{Att}(M) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. At $n = 1$ it is finished. At $n \geq 2$ for every $i \in \{1, \ldots, n\}$ where $A_i = U_1 + \cdots + \widehat{U_i} + \cdots + U_n$ we have $M/A_i \neq 0$, so as factor module of U_i it is $\mathfrak{p}_i$-coprimary, and it follows that $\{\mathfrak{p}_i\} = \mathrm{Coass}(M/A_i) \subseteq \mathrm{Coass}(M)$.

(ii) Write $\mathrm{Coass}(M) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_k\}$ with pairwise distinct $\mathfrak{q}_i$, hence for every $j \in \{1, \ldots, k\}$, the submodule $V_j = \sum\{U_i \mid \mathfrak{p}_i = \mathfrak{q}_j, 1 \leq i \leq n\}$ is similarly $\mathfrak{q}_j$-coprimary and $M = V_1 + \cdots + V_k$. By the given decomposition of $\mathrm{Coass}(M)$ we can assume $X = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_s\}$ and $Y = \{\mathfrak{q}_{s+1}, \ldots, \mathfrak{q}_k\}$, and then achieve $U = V_1 + \cdots + V_s$ as desired: The epimorphisms $V_1 \times \cdots \times V_s \to U$ and $V_{s+1} \times \cdots \times V_k \to M/U$ show that $\operatorname{Coass}(U) \subseteq X$ and $\operatorname{Coass}(M/U) \subseteq Y$, and therefore we have the equality in both cases (since $\operatorname{Coass}(M) \subseteq \operatorname{Coass}(U) \bigcup \operatorname{Coass}(M/U)$).

(iii) One can assume $\mathfrak{a}M = M$ and then the U_i's are numbered in such a way that $\mathfrak{a}U_i \neq U_i$ for $i \in \{1, \ldots, s\}$ and $\mathfrak{a}U_i = U_i$ for rest of i. It follows that $\mathfrak{a} \subseteq \sqrt{\operatorname{Ann}(U_i)}$, hence $U_i \subset M[\mathfrak{a}^e]$ for some common $e \geq 1$ and all $i \leq s$, and therefore $M[\mathfrak{a}^e] + \mathfrak{a}M = M$.

(iv) Let $P(M) \neq M$ and suppose $U_1, \ldots, U_s$ are not radical, U_i are radical for all $i > s$. For every $i \leq s$ there is a maximal ideal $\mathfrak{m}_i$ and an $e_i \geq 1$ with $\mathfrak{m}_i^{e_i} U_i = 0$, with $\mathfrak{b} = \mathfrak{m}_1^{e_1} \ldots \mathfrak{m}_s^{e_s}$ follows $U_i \subseteq M[\mathfrak{b}]$ for all $i \leq s$, with $B = \sum_{i>s} U_i$ finally $M[\mathfrak{b}] + B = M$. Since $R/\mathfrak{b}$ is artinian and $P(M)/B$ is annihilated by $\mathfrak{b}$, it follows $B = P(M)$, so does the first claim, and as factor module of $M[\mathfrak{b}]$ also $M/P(M)$ is coatomic and semi-artinian. Finally, if A is a submodule of $P(M)$ and $P(M)/A$ is small in M/A, as small cover of $M/P(M)$ similarly M/A becomes coatomic, so by (Zöschinger 1980, (Lemma 1.1)) it has no radical submodules and $P(M)/A = 0$ follows, i.e. $P(M)$ is coclosed in M. □

Corollary 4.6 ((Zöschinger 1990), Corollary 1.5) *A module M over a noetherian ring R is representable if and only if $P(M)$ is representable and there is an ideal $\mathfrak{b}$ of R such that $R/\mathfrak{b}$ is artinian and $M[\mathfrak{b}] + P(M) = M$.*

Proof If M is representable, then if $P(M) \neq M$ in the last part of (iv) we have constructed such an ideal $\mathfrak{b}$, and if $P(M) = M$ we choose $\mathfrak{b} = R$. For the converse it remains to show that $N = M[\mathfrak{b}]$ is representable: For every $\mathfrak{m} \in \Omega$, $L_{\mathfrak{m}}(N) = \sum_{i=1}^{\infty} N[\mathfrak{m}^i]$ is at most non-zero, if $\mathfrak{b} \subseteq \mathfrak{m}$, and from $\mathfrak{m}^e + \mathfrak{b} = \mathfrak{m}^{e+1} + \mathfrak{b}$ it follows that $\mathfrak{m}^e \cdot L_{\mathfrak{m}}(N)$ is radical, so is zero. Hence in the decomposition $N = \bigoplus_{\mathfrak{m} \in \Omega} L_{\mathfrak{m}}(N)$ almost all summands are zero and the others are coprimary. □

In particular, if M is reduced, i.e. $P(M) = 0$ one obtains:

Corollary 4.7 ((Zöschinger 1990), Corollary 1.6) *Let R be a noetherian ring. A reduced R-module M is representable if and only if $R/\operatorname{Ann}(M)$ is artinian.*

Remark to Lemma. From part (ii) it follows for an arbitrary R-module M: If Y is a finite subset of $\operatorname{Coass}(M)$, then there is a submodule U of M where $\operatorname{Coass}(M/U) = Y$. (To prove it choose an artinian factor module M/M_0 where $Y \subseteq \operatorname{Coass}(M/M_0)$ and apply it on (ii).) But for infinite Y it is no longer valid. For example, if $(R, \mathfrak{m})$ is a local integral domain with $\dim(R) > 1$ and $M = \coprod_{i=1}^{\infty} R/\mathfrak{m}^i$, then there are infinitely many pairwise distinct prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \ldots$ of height 1 such that $Y = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \ldots\}$ is a subset of $\mathrm{Coass}(M) = \mathrm{Spec}(R)$, but by (Zöschinger 1988, Corollary 1.6) there exists no R-module N with $\mathrm{Coass}(N) = Y$ in general. Finally, if X is a non-empty subset of $\mathrm{Coass}(M)\backslash\{\mathfrak{m}\}$, there is no submodule U of M where $\mathrm{Coass}(U) = X$ since M is reduced.

Lemma 4.11 ((Zöschinger 1990), Lemma 1.7) *Let M be a module over a noetherian ring R and S be a multiplicative subset of R such that all elements of S act bijectively on M. M is coprimary (representable) as R-module if and only if M_S is coprimary (representable) as R_S-module.*

Proof For an arbitrary S, the map $\mathrm{Att}_{R_S}(M_S) \ni \mathcal{P} \mapsto \mathcal{P} \bigcap R \in \mathrm{Att}_R(M)$ is well-defined and injective. Hence if $M_S \neq 0$ and M is $\mathfrak{p}$-coprimary, we must have $\mathfrak{p} \bigcap S = \emptyset$ by Lemma 4.7 and M_S must be $\mathfrak{p}R_S$-coprimary as R_S-module (see also(Macdonald 1973, p.27). But if M has a representation $M = U_1 + \cdots + U_n$ with coprimary U_i, then in $M_S = U_{1S} + \cdots + U_{nS}$ all U_{iS} are as above zero or coprimary, so that M_S is also representable as R_S-module.

Now if all $s \in S$ act bijectively on M, then the above map $\mathrm{Att}_{R_S}(M_S) \ni \mathcal{P} \mapsto \mathcal{P} \bigcap R \in \mathrm{Att}_R(M)$ becomes also surjective, and again with Lemma 4.7 the claim follows over "coprimary". But if M_S has a representation $M_S = X_1 + \cdots + X_n$ with coprimary R_S-submodules X_i, with $U_i = \{a \in M \mid \frac{a}{1} \in X_i\}$ it follows that $M = U_1 + \cdots + U_n$ and every U_i is an S-divisible submodule of M where $U_{iS} \cong X_i$, hence as above U_i is a coprimary R-module and therefore M is representable. □

Corollary 4.8 ((Zöschinger 1990), Corollary 1.8) *Let R be a noetherian ring. Every radical hollow R-module is coprimary.*

Proof As it is well-known a module M is called *hollow* if $M \neq 0$ and $M = U_1 + U_2$ always implies $U_1 = M$ or $U_2 = M$. In this case $\mathfrak{p} = \{x \in R \mid xM \neq M\}$ is a prime ideal and $\mathrm{Coass}(M) = \{\mathfrak{p}\}$, furthermore there is $\mathfrak{m} \in \Omega$ in such a way that for all $0 \neq a \in M$ the ring $R/\mathrm{Ann}(a)$ is local with the unique maximal ideal $\overline{\mathfrak{m}}$ (see (Zöschinger 1986, p.3)). Therefore $S = R\backslash\mathfrak{m}$ satisfies the assumptions in the lemma, because for every $a \in M$ and $s \in S$, $< s > + \mathrm{Ann}(a) = R$, i.e. $a = rsa$ for some $r \in R$. In addition, if M is radical, the last formula shows that M_S is also radical and hollow as R_S-module, now $\mathrm{Coass}_{R_S}(M_S) = \{\mathcal{P}\}$ implies, since R_S is local, by (Zöschinger 1988, Corollary 1.3) $\mathcal{P}^e \cdot M_S = 0$, so that M_S is coprimary as R_S-module, so also M is as R-module. □

Corollary 4.9 ((Zöschinger 1990), Corollary 1.9) *Let R be a noetherian ring. Let M be an R-module of finite Goldie dimension, such that* $\mathrm{Ass}(M)$ *is discrete and every non-zero-divisor acts bijectively on M. Then for every finitely generated R-module A we have that* $\mathrm{Hom}_R(A, M)$ *is representable.*

Proof Every element of $S = R \setminus \bigcup \mathrm{Ass}(M)$ acts bijectively on M, hence also does on $H = \mathrm{Hom}_R(A, M)$. By the lemma it suffices to show that H_S is artinian as R_S-module: Clearly, M_S is also finite dimensional as R_S-module, and since $\mathrm{Ass}(M)$ is finite and discrete, every $\mathcal{P} \in \mathrm{Ass}_{R_S}(M_S)$ is a maximal ideal in the ring R_S, i.e. M_S is semi-artinian. Thus by Matlis, M_S is even artinian, so also $\mathrm{Hom}_{R_S}(A_S, M_S) \cong H_S$ □

Lemma 4.12 ((Zöschinger 1990), Lemma 2.3) *Let M be a module over a noetherian ring R. If U is an artinian submodule of M and M/U is representable, then M is also representable.*

Proof First let M/U be $\mathfrak{q}$-coprimary. For a supplement V_0 of U in M, i.e. a minimal element in the set $\{V \subseteq M \mid V + U = M\}$ we have $\mathrm{Coass}(V_0) = \mathrm{Coass}(M/U) = \{\mathfrak{q}\}$, in particular, $\bigcap_{i=1}^{\infty} \mathfrak{q}^i V_0 = 0$. From $\mathfrak{q}^e(M/U) = 0$ but also $\mathfrak{q}^e V_0 \subseteq U$ follows, hence $\mathfrak{q}^f V_0 = 0$ for some $f \geq e$, and by Lemma 4.7 it implies that V_0 is $\mathfrak{q}$-coprimary, so with U also $V_0 + U = M$ is representable.

If M/U is only representable, it follows with a coprimary decomposition $M/U = (M_1/U) + \cdots + (M_n/U)$ that by the first step, all M_i are representable, so also $M = M_1 + \cdots + M_n$ is. □

4.5. Modules those Att(M) is Discrete are Representable

Theorem 4.3 ((Zöschinger 1990), Lemma 3.1) *Let M be a module over a noetherian ring R and $\mathfrak{p} \in$* $\mathrm{Spec}(R)$ *be simultaneously a minimal and a maximal element of* $\mathrm{Att}(M)$*. Then for $V = \bigcap\{sM \mid s \in R \setminus \mathfrak{p}\}$ we have:*

(i) V is the largest $\mathfrak{p}$-coprimary submodule of M.

(ii) V is the unique supplement of $\mathfrak{p}M$ in M.

(iii) There is an $e \geq 1$ *where* $\mathfrak{p}^e M = \mathfrak{p}^{e+1} M$*, and hence* $V = \mathrm{Ann}(\mathfrak{p}^e M) \cdot M$*.*

Proof Since R is noetherian, there is an $e \geq 1$ where $\mathrm{Ann}(\mathfrak{p}^e M) = \mathrm{Ann}(\mathfrak{p}^{e+1} M)$, and we claim that $\mathfrak{p}$ can not be a minimal prime ideal over $\mathfrak{c} = \mathrm{Ann}(\mathfrak{p}^e M)$: Otherwise, we would have $\mathfrak{p} = \mathrm{Ann}(\bar{r})$ for some $\bar{r} \in R/\mathfrak{c}$, $r \notin \mathrm{Ann}(\mathfrak{p}^{e+1} M)$ results from $\bar{r} \neq 0$, i.e. $rt \notin \mathfrak{c}$ for some $t \in \mathfrak{p}$, and this is impossible.

Now if $\mathfrak{p}$ is a minimal and maximal element of $\mathrm{Att}(M)$, we have $\mathfrak{c}M + \mathfrak{p}M = M$: Otherwise we would have $\mathfrak{q} \in \mathrm{Att}(M)$ where $\mathfrak{c} + \mathfrak{p} \subseteq \mathfrak{q}$, and $\mathfrak{c} \subseteq \mathfrak{p}$ follows from $\mathfrak{p} = \mathfrak{q}$, so that $\mathfrak{p}$ is also minimal over $\mathfrak{c}$ which contradicts the preliminary note. In particular, we obtain from $\mathfrak{c}\,\mathfrak{p}^e M = 0$ that $\mathfrak{p}^e M = \mathfrak{p}^{e+1} M$. Moreover, $\mathfrak{c}M$ is a supplement of $\mathfrak{p}M$, because from $X \subseteq \mathfrak{c}M$, $X + \mathfrak{p}M = M$ we have $\mathfrak{c}X = \mathfrak{c}M$, hence $X = \mathfrak{c}M$, and the same proof shows that $\mathfrak{c}M$ is the unique supplement of $\mathfrak{p}M$. Since $\mathrm{Coass}(\mathfrak{c}M) = \mathrm{Coass}(M/\mathfrak{p}M) = \{\mathfrak{p}\}$, by Lemma 4.7, $\mathfrak{c}M$ is $\mathfrak{p}$-coprimary, in particular $\mathfrak{c}M \subseteq V$, and as a result of $\mathfrak{c} \nsubseteq \mathfrak{p}$ we have an $s_0 \in \mathfrak{c} \cap R \setminus \mathfrak{p}$ such that $V = \mathfrak{c}M$ follows from $V \subseteq s_0 M \subseteq \mathfrak{c}M$ and all of the three parts are proved. □

Theorem 4.4 ((Zöschinger 1990), Theorem 3.2) *Let* M *be a module over a noetherian ring* R*. If* $\mathrm{Att}(M)$ *is discrete, then* M *is representable. By* $M \neq 0$ *we have: If* $\mathfrak{q}_1, \ldots, \mathfrak{q}_k$ *are the pairwise distinct elements of* $\mathrm{Att}(M)$*, then* $V_i = \bigcap\{sM \mid s \in R\setminus\mathfrak{q}_i\}$ *is the largest* $\mathfrak{q}_i$*-coprimary submodule of* M $(1 \leq i \leq k)$ *and* $M = V_1 + \cdots + V_k$*.*

Proof By Lemma 4.3(i) it is only the last claim to show, and for this let $M' = V_1 + \cdots + V_k$. By Lemma 4.3(ii), $V_i + \mathfrak{q}_i M = M$ holds for all i, with $\mathfrak{a} = \mathfrak{q}_1 \ldots \mathfrak{q}_k$ hence $M' + \mathfrak{a}M = M$. But $\mathfrak{a}^n M = 0$ for some $n \geq 1$ follows from $\mathfrak{a} \subseteq \bigcap \mathrm{Att}(M) = \sqrt{\mathrm{Ann}(M)}$, so $M = M'$. □

CHAPTER 5

UNIQUENESS OF COPRIMARY DECOMPOSITION

Throughout this chapter R is a commutative ring which is not necessarily noetherian. Now we want to study the extent to which the minimal decomposition on page 19 is unique. In order to do that for every multiplicatively closed subset S of R and every R-module M, we introduce a module ${}^S M = \bigcap_{s \in S} sM$. Note that when S is empty ${}^S M = M$.

Proposition 5.1 ((Kirby 1973), Proposition 5) *Let $M = M_1 + \cdots + M_k$ be an R-module, where M_i is a $\mathfrak{p}_i$-coprimary module, and let S be a multiplicatively closed subset of R. If S has empty intersection with $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$ and non-empty intersection with $\mathfrak{p}_{l+1}, \ldots, \mathfrak{p}_k$, then*

$${}^S M = M_1 + \cdots + M_l.$$

Proof Suppose that $s \in S$; so $s \notin \mathfrak{p}_i (i = 1, \ldots, l)$ and

$$sM = sM_1 + \cdots + sM_k \supseteq M_1 + \cdots + M_l.$$

Therefore ${}^S M \supseteq M_1 + \cdots + M_l$.

Conversely, as $\mathfrak{p}_i \cap S$ is non-empty ($i = l+1, \ldots, k$), there exists $s_i \in \mathfrak{p}_i \cap S$ and $s = \prod_{l+1}^{k} s_i \in \mathfrak{p}_i \cap S$ for $i = l+1, \ldots, k$. But $\mathfrak{p}_i = \sqrt{\operatorname{Ann}(M_i)}$; so there exists an integer t such that $s^t M_i = 0$ ($i = l+1, \ldots, k$). However $s^t \in S$, so $s^t \notin \mathfrak{p}_i$ ($i = 1, \ldots, l$). Therefore $s^t M_i = M_i$ ($i = 1, \ldots, l$), and

$${}^S M \subseteq s^t M = s^t M_1 + \cdots + s^t M_k = M_1 + \cdots + M_l,$$

which completes the proof. □

Theorem 5.1 ((Kirby 1973), Theorem 2) *Let $M = M_1 + \cdots + M_k$ and $M = M'_1 + \cdots + M'_l$ be two minimal coprimary decompositions of M. Let M_i be $\mathfrak{p}_i$-coprimary and M'_j be $\mathfrak{p}'_j$-coprimary. Then $k = l$ and the sets $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ and $\{\mathfrak{p}'_1, \ldots, \mathfrak{p}'_l\}$ coincide.*

Proof Let $\mathfrak{p}$ be any one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$. It suffices to prove that $\mathfrak{p}$ is contained in $\{\mathfrak{p}'_1, \ldots, \mathfrak{p}'_l\}$. We first renumber the M_i, M'_j such that $\mathfrak{p} \supsetneq \mathfrak{p}_i$ for $1 \leq i < m$, $\mathfrak{p} = \mathfrak{p}_m$,

$\mathfrak{p} \not\supseteq \mathfrak{p}_i$ for $m < i \leq k$, $\mathfrak{p} \supseteq \mathfrak{p}'_j$ for $1 \leq j \leq n$, and $\mathfrak{p} \not\supseteq \mathfrak{p}'_j$ for $n < j \leq l$. Put $S = R \setminus \mathfrak{p}$; then, by Proposition 5.1,

$$M_1 + \cdots + M_m = {}^S M = M'_1 + \cdots + M'_n.$$

Suppose $\mathfrak{p} \supsetneq \mathfrak{p}'_j$ for $1 \leq j \leq n$; so there exists $r \in \mathfrak{p}$ such that $r \notin \mathfrak{p}_i$ $(1 \leq i < m)$ and $r \notin \mathfrak{p}'_j$ $(1 \leq j \leq n)$ (see, for example,(Northcott 1968, p.81,Proposition 5)). Now

$$r \in \mathfrak{p} = \sqrt{\mathrm{Ann}(M_m)} \text{ implies } r^t M_m = 0 \text{ for some integer } t,$$
$$r \notin \mathfrak{p}_i = \sqrt{\mathrm{Ann}\, M_i} \text{ implies } rM_i = M_i (1 \leq i < m) \text{ and}$$
$$r \notin \mathfrak{p}'_j = \sqrt{\mathrm{Ann}\, M'_j} \text{ implies } rM'_j = M'_j.$$

Therefore

$$M_1 + \cdots + M_{m-1} = r^t({}^S M) = M'_1 + \cdots + M'_n,$$

and

$$M_1 + \cdots + M_{m-1} = M_1 + \cdots + M_m,$$

which contradicts the minimality of the decomposition $M = M_1 + \cdots + M_k$. Hence $\mathfrak{p} = \mathfrak{p}'_j$ for some j satisfying $1 \leq j \leq n$, and the theorem is proved. □

CHAPTER 6

CONCLUSION

In this thesis we studied about existence and uniqueness of coprimary decomposition of modules. To do this, we searched the literature and for the case of the modules over commutative noetherian rings we studied (Zöschinger 1990). To investigate coassociated prime ideals as dual notion of associated prime ideals we mainly studied (Zöschinger 1983) and (Zöschinger 1988).

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