FOURIER ANALYSIS ON THE LORENTZ GROUP AND RELATIVISTIC QUANTUM MECHANICS

A Thesis Submitted to
the Graduate School of Engineering and Sciences of
İzmir Institute of Technology
in Partial Fulfillment of the Requirements for the Degree of

MASTER OF SCIENCE

in Mathematics

by
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June 2008
İZMİR

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ACKNOWLEDGEMENTS

I would like to thank my advisor Prof. Dr. Rıfat Mirkasım for his continual advice, supervision and understanding in research and writing of this thesis. I would also like to thank my family; Ok, Özsan, Aktaş and Tuna Bayrakdar for their love and infinite moral support.

ABSTRACT

FOURIER ANALYSIS ON THE LORENTZ GROUP AND RELATIVISTIC QUANTUM MECHANICS

The non-relativistic Schrödinger and Lippman-Schwinger equations are described. The expressions of these equations are investigated in momentum and configuration spaces, using Fourier transformation. The plane wave, which is generating function for the matrix elements of three dimensional Euclidean group in spherical basis, expanded in terms of Legendre polynomials and spherical Bessel functions. Also explicit calculation of Green's function is done.

The matrix elements of the unitary irreducible representations of Lorentz group are used to introduce Fourier expansion of plane waves. And the kernel of Gelfand-Graev transformation, which is the relativistic plane wave, is expanded in to these matrix elements. Then relativistic differential difference equation in configuration space is constructed.

Lippman-Schwinger equations are studied in Lobachevsky space (hyperbolic space). An analogous to the non-relativistic case, using the finite difference Schrödinger equation, one dimensional Green's function is analyzed for the relativistic case . Also the finite difference analogue of the Heavyside step function is investigated.

ÖZET

LORENTZ GRUBU ÜZERİNE FOURIER ANALİZİ VE RÖLATİVİSTİK KUANTUM MEKANİĞİ

Rölativistik olmayan Schrödinger ve Lippman-Schwinger denklemleri tanımlandı. Bu denklemlerin hem momentum hem de konfigürasyon uzayındaki ifadeleri Fourier dönüşümü kullanılarak incelendi. Üç boyutlu Öklid grubunun küresel bazlarca ifade edilen matris elemanlarının doğuranı olan ”düzlem dalga” nın seri açılımı, Legendre polinomları ve küresel Bessel fonksiyonları cinsinden yapıldı. Bunun yanısıra, dalga denkleminde kullanılmak üzere, Green fonksiyonunun hesabı gerçekleştirildi.

Lorentz grubunun birimsel indirgenemez temsillerinin matris elemanları tanıtıldı ve bu elemanlar kullanılarak Gelfand-Graev dönüşümlerinin çekirdeği olan rölativistik dalga fonksiyonunun seri açılımı gerçekleştirildi. Bu bakış açısıyla, konfigürasyon uzayında, sonlu fark Schrödinger denklemi yapılandırıldı.

Rölativistik olmayan denklemlerin benzeri olan rölativistik Lippman-Schwinger denklemlerini yapılandırmak üzere üç boyutlu Lobachevsky uzayında çalışıldı. Son olarak, sonlu fark Schrödinger denklemi kullanılarak, rölativistik dalga denkleminde kullanılmak üzere, bir boyutlu Green fonksiyonun hesabı gerçekleştirildi.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

The description of the two-particle relativistic system is one of the central problems of the Quantum Field Theory. Traditionally for studying this problem in the framework of the four-dimensional Feynman-Dyson's formalism the covariant equation was used known under the name "Bethe and Salpeter equation". This equation being advantageous in many senses is nevertheless not completely satisfactory. In particular, in the 4-dimenional Bethe-Salpeter approach the clear treatment of the wave function dependence of the relative time of two particles is absent.(Salpeter, et al. 1951, Landau and Lifshitz 1958, Davydov 1963, Kadyshevsky, et al. 1968, Kadyshevsky 1988, Kagramanov, et al. 1989, Mir-Kasimov , et al. 1990, Mir-Kasimov 1991, Thaller 1992, Nagiev 1995, Mir-Kasimov 2000, Mir-Kasimov 2002, Mir-Kasimov, et al. 2003)

The natural question arises: does in the framework of the covariant field theory such formalism exist, which is from the one hand side three dimensional and admits the probabilistic interpretation of the wave function and from the other hand side possesses all main advantages (renormalisibility, analyticity) of the completely covariant approach. There exist several formulations (unified under common title "quasi-potential approach") of the 3-dimensional relativistic 2-body problem which give a positive answer to this question. In the momentum representation the equations of quasi-potential approach are the direct relativistic generalizations of the standard (non-relativistic) Schrödinger and Lippmann-Schwinger equations. The goal of this Thesis is to consider and apply to solution of scattering theory the concept of the relativistic configurational space in the framework of the quasi-potential approach.

The concept of relativistic configurational space is based on the simple observation that a free motion of a relativistic particle can be described on the basis of the Gelfand-Graev transformation , i.e., expansion in relativistic spherical functions. (Kontorovich and Lebedev 1938, Bateman 1953, Gel'fand, et al. 1966, Vilenkin 1968, Biedenharn 1989). We start with the well-known fact that the equation describing the relativistic relation between energy and momentum of the particle (mass shell equation), describes at the same time the three-dimensional momentum space of constant negative curvature or the Lobachevsky space. The isometry group of this space is the Lorentz group. To introduce an adequate Fourier expansion, we must find the matrix elements of the unitary irreducible representations of this group. These matrix elements are the eigen-functions of the Casimir operator, or the Laplace-Beltrami operator in the Lobachevsky space

Later on it became clear that the geometry of relativistic configurational space carries the non-commutative geometry (its "Snyder version"). (Carlitz 1957, Witten 1986, Deser, et al. 1988 , Witten 1988, Kadyshevsky and Fursaev 1990, Wess and Zumino 1990, Dubois-Violette, et al.1990, Mir-Kasimov 1991, Connes 1994, Madore 1995, Dimakis and Müller 1998, Mir-Kasimov 2000, Güven, et al. 2001).

It is important to stress also the connection of given approach with q-deformations. (Kulish 1981, Manin 1988, Woronowich 1989, Kulish 1991, Floreanini, et al. 1991, Mir-Kasimov 1996). The oscillator model in relativistic configurational space is an explicit realization of the q-oscillator. (Macfarlane 1989, Mir-Kasimov 1991, Rajagopal 1993).

$$\hat{C} = \frac{1}{2} M^{\mu\nu} M_{\mu\nu} \qquad \mu, \nu = 0, 1, 2, 3 \tag{1.1}$$

Eigenfunctions of $\hat{C}$ are unitary irreducible representations of the Lorentz group and can be written in a form of the Gelfand - Graev kernels

$$< \mathbf{r} \mid \mathbf{p} >= \left(\frac{p_0 - \mathbf{p} \cdot \mathbf{n}}{mc} \right)^{-1-i\frac{r}{\lambda}} \tag{1.2}$$

$$\mathbf{r} = r\mathbf{n} \qquad \mathbf{n}^2 = 1 \qquad 0 \leq r < \infty \tag{1.3}$$

$$\lambda = \frac{\hbar}{mc} \quad \text{Compton wave length of the particle} \tag{1.4}$$

Note that $\mathbf{r}$ in (1.3) does not transform as the spatial component of a four vector.

There are several strong arguments for considering $\mathbf{r}$ as a relativistic position vector of the particle:

1. The range of variation of r (1.3) coincides with that of the standard non-relativistic coordinate vector.

2. The magnitude r of $\mathbf{r}$ is Lorentz invariant in full analogy with the non-relativistic relative distance, which is Galilean invariant.

3. ”Relativistic plane wave” $< \mathbf{r} \mid \mathbf{p} >$ in the non-relativistic limit

$$|\mathbf{p}| \ll mc \qquad r \gg \lambda \qquad p_0 \simeq mc + \frac{\mathbf{p}^2}{2mc} \tag{1.5}$$

goes over into the standard non-relativistic plane wave

$$< \mathbf{r} \mid \mathbf{p} > \simeq \exp\left(ir\frac{\mathbf{p}\cdot\mathbf{n}}{\hbar}\right) = \exp\left(i\frac{\mathbf{p}\cdot\mathbf{x}}{\hbar}\right) \tag{1.6}$$

In the last expression $\mathbf{x}$ denotes the standard non-relativistic position vector.

4. The most important physical argument is that there exists in r-space the differential – difference operator $\widehat{H_0}$ of free energy

$$\widehat{H_0} = mc^2\{\cosh i\lambda\frac{\partial}{\partial r} + i\frac{\lambda}{r}\sinh i\lambda\frac{\partial}{\partial r} - \frac{\lambda^2}{r^2}\,\Delta_{\vartheta,\phi}\, e^{\lambda\frac{\partial}{\partial r}}\} \tag{1.7}$$

such that

$$\left(\widehat{H_0} - E\right) < \mathbf{r} \mid \mathbf{p} >= 0 \tag{1.8}$$

The last two equations show that the plane wave $< \mathbf{r} \mid \mathbf{p} >$ is the wave function of the relativistic free particle, i.e. the state with a fixed value of the relativistic energy and momentum. It is worthwhile to stress that (1.8) can be considered as a solution of the problem of extracting the square root in the expression for the relativistic energy

$$E = \frac{p_0}{c} = \sqrt{\mathbf{p}^2c^2 + m^2c^4} \tag{1.9}$$

in a form of differential – difference operator (1.7)

The ”price” for this extraction is the presence of the exponential derivation which amounts to a finite-difference character of the Hamiltonian operator (1.7). The interaction can be described in terms of a potential function $V(r)$. On this basis, the quantum theory in the relativistic configurational space had been developed. This theory will be called hereafter the Relativistic Quantum Mechanics (RQM). It proved to be an efficient approach to solving problems in a wide range: from analytic properties of relativistic wave functions and amplitudes to relativistic confinement models of composite hadrons. We refer the reader further references therein where this theory is presented.

In chapter two the non-relativistic Schröndinger and Lippman-Schwinger equations are described in both momentum and configuration spaces, using Fourier transformation. The plane wave expansion and it's group theoretical meaning is described.

In chapter three the matrix elements of the unitary irreducible representations of Lorentz group are used to introduce Fourier expansion of plane waves. Then relativistic differential difference equation in configuration space is constructed.

Finally in chapter four relativistic two-body problem and general formalism of the scattering theory and related equations are described. Lippman-Schwinger equations are described in Lobachevsky (hyperbolic)space.

CHAPTER 2

A BRIEF REVIEW OF NON-RELATIVISTIC SCATTERING THEORY

2.1. Non-Relativistic Two Body Problem

This chapter in a sense is introductory. We analyze here the non-relativistic two-body problem giving the group-theoretical interpretation of a number of aspects of this well established theory keeping in mind to generalize these properties if possible from the non-relativistic (Galilean) case to the relativistic (Lorentz) case.

It is well known from classical mechanics that in the non-relativistic theory, the problem of the scattering of a particle of mass m_1 by a particle of mass m_2, when the interaction $V(\mathbf{r})$ between the particles depends on the relative coordinate, amounts to a problem of the scattering of a single effective particle with reduced mass in a potential field $V(\mathbf{r})$. This reduction of the problem of the elastic scattering of two particles to the motion of a fictitious particle with reduced mass m in the potential field $V(\mathbf{r})$ is realized by the simple change to a system of coordinates fixed in the center of mass of the colliding particles,(Davydov 1963). Now let us show this reduction. The Hamiltonian of two particles with masses m_1 and m_2 for the stationary state has the form;

$$\hat{\mathsf{H}} = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + \hat{V}(\mathbf{r}_1, \mathbf{r}_2) \tag{2.1}$$

We introduce new variables; the center of mass radius vector $\mathbf{R}$ and the relative position vector $\mathbf{r}$ defined by

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}, \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{2.2}$$

where $M = m_1 + m_2$ is total mass and

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_1, \quad \mathbf{p} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{m_1 + m_2} \tag{2.3}$$

$\mathbf{P}$ and $\mathbf{p}$ are total and relative momenta respectively.

Now considering the relation

$$-i\hbar\frac{\partial}{\partial \mathbf{r}_i} = \mathbf{p}_i \quad (i = 1, 2) \tag{2.4}$$

we can determine the equation which describes the motion of the center of mass. Let's write $\mathbf{r}_1$ and $\mathbf{r}_2$ as;

$$\begin{aligned}\mathbf{r}_1 &= \frac{m_2}{M}\mathbf{r} + \mathbf{R} \\ \mathbf{r}_2 &= -\frac{m_1}{M}\mathbf{r} + \mathbf{R}\end{aligned} \tag{2.5}$$

and determine the following

$$\begin{aligned}\frac{\partial}{\partial \mathbf{r}_1} &= \frac{\partial \mathbf{R}}{\partial \mathbf{r}_1}\frac{\partial}{\partial \mathbf{R}} + \frac{\partial \mathbf{r}}{\partial \mathbf{r}_1}\frac{\partial}{\partial \mathbf{r}} = \frac{m_1}{M}\frac{\partial}{\partial \mathbf{R}} + \frac{\partial}{\partial \mathbf{r}} \\ \frac{\partial}{\partial \mathbf{r}_2} &= \frac{\partial \mathbf{R}}{\partial \mathbf{r}_2}\frac{\partial}{\partial \mathbf{R}} + \frac{\partial \mathbf{r}}{\partial \mathbf{r}_2}\frac{\partial}{\partial \mathbf{r}} = \frac{m_2}{M}\frac{\partial}{\partial \mathbf{R}} - \frac{\partial}{\partial \mathbf{r}}.\end{aligned} \tag{2.6}$$

Now we insert these relations into Hamiltonian equation;

$$\hat{\mathsf{H}} = -\frac{\hbar^2}{2m_1}\left(\frac{m_1}{M}\frac{\partial}{\partial \mathbf{R}} + \frac{\partial}{\partial \mathbf{r}}\right)^2 - \frac{\hbar^2}{2m_2}\left(\frac{m_2}{M}\frac{\partial}{\partial \mathbf{R}} - \frac{\partial}{\partial \mathbf{r}}\right)^2 + \tilde{V}(\mathbf{R}, \mathbf{r}) \tag{2.7}$$

$$= -\frac{\hbar^2}{2M}\left(\frac{\partial}{\partial \mathbf{R}}\right)^2 - \frac{\hbar^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\left(\frac{\partial}{\partial \mathbf{r}}\right)^2 + \tilde{V}(\mathbf{R}, \mathbf{r}) \tag{2.8}$$

and substitute

$$\frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 + m_2}{m_1 m_2} = \frac{1}{m}, \qquad m = \frac{m_1 m_2}{M} \tag{2.9}$$

where m is the reduced mass and M is the total mass of the system. Then equation (2.8) can be written in the form

$$\hat{\mathsf{H}} = -\frac{\hbar^2}{2M}\nabla_R^2 - \frac{\hbar^2}{2m}\nabla_r^2 + \tilde{V}(\mathbf{R}, \mathbf{r}) \tag{2.10}$$

$$= \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}^2}{2m} + \tilde{V}(\mathbf{R}, \mathbf{r}) \tag{2.11}$$

$$= \hat{H}_{cm} + \hat{H} \tag{2.12}$$

This equation represents the separation of $\hat{\mathsf{H}}$ into the Hamiltonian of the center of mass, $\hat{H}_{cm}$, and the Hamiltonian of the, relative motion $\hat{H}$.

For an isolated system, relative coordinate $\mathbf{r}$ is invariant in respect to translations, time shifts and pure Galilean transformations. Therefore if the potential does not depend on the coordinate of center of mass, then the motion of center of mass and relative motion are separated i.e. in respect to full inhomogeneous Galilean group. So we can write,

$$\hat{H}_{cm} = -\frac{\hbar^2}{2M}\nabla_R^2 \tag{2.13}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla_r^2 + \tilde{V}(\mathbf{r}). \tag{2.14}$$

We stress that transfer to the center of mass reference system is an Galilean transformation. Separating the motion of the center of mass of two body system we can consider the relative motion as a motion of an effective particle in the field of potential $\tilde{V}(\mathbf{r})$.

2.2. General Formalism of Non-Relativistic Scattering Theory

Now we will concentrate on the time independent formulation of relative motion of free particle with momentum $\mathbf{q}$ and energy $E_q = \mathbf{q}^2/2m$, $(2/\hbar^2\tilde{V}(\mathbf{r}) = V(\mathbf{r}))$which is determined by Schrödinger equation

$$\left[-\frac{1}{m}\nabla_{\mathbf{r}}^2 + V(\mathbf{r}) - 2E_q\right]\Psi_q(\mathbf{r}) = 0 \tag{2.15}$$

where $\mathbf{r}$ is the relative distance of the particles with respect to each others. The solution of the free Schrödinger equation $\varphi_{o\mathbf{q}}$, corresponding to the case

$$V(\mathbf{r}) = 0 \tag{2.16}$$

$$\left[-\frac{1}{m}\nabla_{\mathbf{r}}^2 - 2E_q\right]\varphi_{o\mathbf{q}}(\mathbf{r}) = 0 \tag{2.17}$$

where

$$\varphi_{o\mathbf{q}}(\mathbf{r}) = e^{i\mathbf{q}.\mathbf{r}}. \tag{2.18}$$

We stress again that the plane waves $\varphi_{o\mathbf{q}}(\mathbf{r})$ are the free motion of the non-relativistic particle, i.e. the motion with definite momentum $\mathbf{q}$ and energy E_q. At the same time $\varphi_{o\mathbf{q}}(\mathbf{r})$ are generating functions for the unitary irreducible representations of the isometry group of the Euclidean space of relative (non-relativistic) momenta.

Using the free solution (2.18) of equation (2.15), then with the following equation,

$$\left(\frac{1}{m}\nabla_{\mathbf{r}}^2 + 2E_q\right)\Psi_q(\mathbf{r}) = \int V(\mathbf{r}')\Psi_q(\mathbf{r}')d\mathbf{r}' \tag{2.19}$$

the Schrödinger equation can be written as

$$\Psi_q(\mathbf{r}) = \exp[i\mathbf{q}.\mathbf{r}] + \int G_q^+(\mathbf{r},\mathbf{r}')V(\mathbf{r}')\Psi_q(\mathbf{r}')d\mathbf{r}' \tag{2.20}$$

where the Green's function $G_q^+(\mathbf{r},\mathbf{r}')$ satisfying the equation

$$(\nabla^2 + q^2)G_q^+(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r}, \mathbf{r}') \tag{2.21}$$

is given by

$$G_q^+(\mathbf{r}, \mathbf{r}') = \langle \mathbf{r}|(2E_q + i\epsilon - H_0)^{-1}|\mathbf{r}'\rangle$$

$$= \frac{1}{(2\pi)^3}\int \exp[i\mathbf{k}.\mathbf{r}]\frac{d\mathbf{k}}{2E_q - 2E_k + i\epsilon}\exp[-i\mathbf{k}.\mathbf{r}']. \tag{2.22}$$

and $E_q = \mathbf{q}^2/2m$, $E_k = \mathbf{k}^2/2m$ defines the energy shell.

The scattering amplitude $A(\mathbf{p}, \mathbf{q})$ in momentum space is defined as the Fourier transform of the product $-m/4\pi V(\mathbf{r})\Psi_q(\mathbf{r}')$ which is

$$A(\mathbf{p}, \mathbf{q}) = \frac{-m}{4\pi}\int V(\mathbf{r})\Psi_q(\mathbf{r})\exp[-i\mathbf{p}.\mathbf{r}]d\mathbf{r} \tag{2.23}$$

$$= \frac{-m}{4\pi}\frac{1}{(2\pi)^3}\int \tilde{V}(\mathbf{p}-\mathbf{k})\tilde{\Psi}_q(\mathbf{k})d\mathbf{k} \tag{2.24}$$

where the Fourier transform of the potential is

$$\tilde{V}(\mathbf{p}, \mathbf{k}) = \int \exp[-i\mathbf{p}.\mathbf{r}]V(r, r')\exp[i\mathbf{k}.\mathbf{r}']d\mathbf{r}d\mathbf{r}'. \tag{2.25}$$

If the potential is local,

$$V(\mathbf{r}, \mathbf{r}') = V(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}') \tag{2.26}$$

then we can write

$$\tilde{V}(\mathbf{p}, \mathbf{k}) = \tilde{V}(\mathbf{p} - \mathbf{k}) = \int \exp[-i(\mathbf{p} - \mathbf{k}).\mathbf{r}]V(\mathbf{r})d\mathbf{r}. \tag{2.27}$$

To obtain the Lippmann-Schwinger equation for the off energy-shell scattering amplitude we multiply both sides of equation (2.20) by $(-m/4\pi)V(\mathbf{r})$ and take the Fourier transform, using (2.22) and (2.24) we get

$$A(\mathbf{p}, \mathbf{q}) = \frac{-m}{4\pi}\tilde{V}(\mathbf{p} - \mathbf{q}) + \frac{1}{(2\pi)^3}\int \tilde{V}(\mathbf{p} - \mathbf{k})\frac{d\mathbf{k}A(\mathbf{k},\mathbf{q})}{2E_q - 2E_k + i\epsilon}. \tag{2.28}$$

Its off-shell extrapolation is dictated by the boundary conditions of the Schrödinger equation. We note that on the energy shell, the denominator of the integrand in equation (2.28) vanishes. It can be proved that if V($\mathbf{r}$) decreases at infinity fast enough so that

$$\int |V(\mathbf{r})|d\mathbf{r} < \infty \tag{2.29}$$

then $\Psi_q(\mathbf{r})$ has the following asymptotic behavior:

$$\Psi_q(\mathbf{r}) = \exp[i\mathbf{q}.\mathbf{r}] + A_q(\theta, \varphi)\frac{e^{iqr}}{r} \tag{2.30}$$

where $A_q(\theta, \varphi)$ is the on energy-shell scattering amplitude in the spherical coordinates which are described by

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta \end{aligned} \tag{2.31}$$

with the range of variables $0 < \theta < \pi$ and $0 \leq \varphi < 2\pi$. The amplitude $A(\mathbf{p}, \mathbf{q})$ is normalized to the elastic-scattering $(E_q = E_p)$ differential cross-section by

$$\frac{d\sigma}{dw_k} = |A(\mathbf{p}, \mathbf{q})|^2 \tag{2.32}$$

where $dw_{\mathbf{k}} = \sin\theta d\theta d\varphi$

In the absence of absorption (ImV=0), the unitarity condition can be obtained from equation (2.28)

$$ImA(\mathbf{p}, \mathbf{q}) = \frac{|\mathbf{p}|}{4\pi}\int A(\mathbf{p}, \mathbf{k})A^*(\mathbf{k}, \mathbf{q})dw_{\mathbf{k}} \tag{2.33}$$

for $|\mathbf{p}| = |\mathbf{k}| = |\mathbf{q}|$.

Then if we take the Fourier transform of equation (2.20) directly, we get the Lippmann-Schwinger equation for the wave function of the continuous spectrum in momentum space

$$\tilde{\Psi}_q(\mathbf{p}) = (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}) - \frac{4\pi}{m}\frac{1}{2E_q - 2E_p + i\varepsilon}A(\mathbf{p}, \mathbf{q}) \tag{2.34}$$

which together with (2.24) gives the following Schrödinger equations in momentum space:

$$\tilde{\Psi}_q(\mathbf{p}) = (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}) + \frac{1}{(2\pi)^3}\frac{1}{2E_q - 2E_p + i\varepsilon}\int \tilde{V}(\mathbf{p} - \mathbf{k})\tilde{\Psi}_q(\mathbf{k})d\mathbf{k} \tag{2.35}$$

$$(2E_q - 2E_p)\tilde{\Psi}_q(\mathbf{p}) = \frac{1}{(2\pi)^3}\int \tilde{V}(\mathbf{p} - \mathbf{k})\tilde{\Psi}_q(\mathbf{k})d\mathbf{k}. \tag{2.36}$$

2.3. Partial Wave Analysis

2.3.1. The plane wave expansion and related relations

The solution of free Schrödinger equation in Cartesian coordinate system is

$$\varphi_{o\mathbf{q}}(\mathbf{r}) = \exp[i\mathbf{q}.\mathbf{r}] \tag{2.37}$$

whereas in spherical coordinate system it is possible to express the plane wave in the form

$$\varphi_{o\mathbf{q}}(\mathbf{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_l P_l^m(\cos\theta)\exp[im\varphi] \tag{2.38}$$

where $m = -l, -l+1, ..., l-1, l$. For simplicity we choose the z axis along **q**, then the free solution becomes symmetric with respect to the z axis and only $m = 0$ terms survive. Thus, the associated Legendre polynomials P_l^m reduced to the Legendre polynomials[1] $P_l(x)$ so we can write the solution in the form

$$\varphi_{o\mathbf{q}}(\mathbf{r}) = \sum_{l=0}^{\infty} a_l P_l(\frac{\mathbf{q}.\mathbf{r}}{qr}) \tag{2.39}$$

By substituting $x = \mathbf{q}.\mathbf{r}/qr$ and $y = qr$, we have

$$\exp[ixy] = \sum_{l=0}^{\infty} a_l(y)P_l(x). \tag{2.40}$$

To find the coefficients $a_l(y)$, let us multiply both sides of equation (2.40) by $P_{l'}$ and then integrate with respect to x

$$\int_{-1}^{1}\exp[ixy]P_{l'}(x)dx = \sum_{l=o}^{\infty} a_l(y)\int_{-1}^{1} P_{l'}(x)P_l(x)dx \tag{2.41}$$

Then by using the fact that Legendre polynomials form a complete set of orthogonal polynomials in the interval $x \in [-1, 1]$

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dt = \frac{2}{2l+1}\delta_{ll'} \tag{2.42}$$

we can write

$$a_l(y) = \frac{2l+1}{2}\int_{-1}^{1}\exp[ixy]P_l(x)dx. \tag{2.43}$$

[1] see Appendix B

If we substitute the Rodrigues formula (see Appendix B)

$$P_l(x) = \frac{1}{2^l}\frac{1}{l!}\frac{d^l}{dx^l}(x^2-1)^l, \tag{2.44}$$

for the Legendre Polynomial $P_l(x)$, we get

$$a_l(y) = \frac{2}{2l+1}\int_{-1}^{1} \exp[ixy]\frac{1}{2^l}\frac{1}{l!}\frac{d^l}{dx^l}(x^2-1)^l dx \tag{2.45}$$

and

$$a_l(y) = \frac{2}{2l+1}\int_{-1}^{1} \exp[ixy]\frac{1}{2^l}\frac{1}{l!}d\left[\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l\right]. \tag{2.46}$$

By applying the partial integration, we can write

$$a_l(y) = \frac{2}{2l+1}\frac{1}{2^l}\frac{1}{l!}$$

$$\left[\exp[ixy]\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l|_{-1}^{1} - \int_{-1}^{1} d\exp[ixy]\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l\right]. \tag{2.47}$$

The function $(x^2-1)^l$ has zeros of the lth order at $x=\pm 1$. If we differentiate $l-1$ times it still has zeros at $x=\pm 1$. Therefore, the integrated term (d^{l-1}/dx^{l-1}) vanishes. After $l-1$ times partial integration we can write

$$a_l(y) = \frac{2}{2l+1}\frac{1}{2^l}\frac{1}{l!}(-1)^l i^l y^l \int_{-1}^{1}(x^2-1)^l \exp[ixy]dx. \tag{2.48}$$

Recalling the integral representation of the Bessel functions $J_{l+\frac{1}{2}}(y)$ and the relation between Bessel and spherical Bessel functions $j_l(y)$: (see Appendix A)

$$J_{l+\frac{1}{2}}(y) = \frac{1}{\sqrt{\pi}l!}\left(\frac{y}{2}\right)^{l+\frac{1}{2}}\int_{-1}^{1}(1-x^2)^l\exp[ixy]dx \tag{2.49}$$

$$j_l(y) = \sqrt{\frac{\pi}{2y}}J_{l+1/2}(y). \tag{2.50}$$

Therefore, the coefficients $a_l(y)$ can be written as

$$a_l(y) = (2l+1)i^l j_l(y) \tag{2.51}$$

and we conclude that

$$e^{i\mathbf{q}.\mathbf{r}} = \sum_{l=o}^{\infty}(2l+1)i^l j_l(qr)P_l\left(\frac{\mathbf{q}.\mathbf{r}}{qr}\right). \qquad (2.52)$$

The expansion of the plane wave in terms of Legendre polynomials and the spherical Bessel functions has a clear group theoretical meaning. An incoming or outgoing plane wave is a generating function for the matrix elements of the three dimensional Euclidean group in spherical basis. This group can be defined in both the configuration space and the momentum space because of the fact that the expression of a plane wave is symmetric with respect to **r** and **q**. The group of motion of the three dimensional Euclidean space E_3 has some sub-group such as three dimensional rotation group O_3 and three dimensional translation group T_3. Since, the spherical harmonics Y_{lm} are the matrix elements of the $2l+1$ dimensional representation of the group O_3 and the spherical Bessel functions are the matrix elements of the the group T_3, the above expansion is clear. The plane wave can also be expanded in terms of other sub-groups of E_3 and it plays the role of kernel in Fourier transformation for the group E_3. (Gel'fand, et al. 1966)

Let's start from the free Schrödinger equation

$$(2E_q + \frac{\nabla^2}{m})\exp[i\mathbf{q}.\mathbf{r}] = 0 \qquad (2.53)$$

which is in spherical coordinates (2.31),

$$\left[2E_q + \frac{1}{mr^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{mr^2}\Delta_{\theta,\varphi}\right] exp[i\mathbf{q}.\mathbf{r}] = 0 \qquad (2.54)$$

where

$$\Delta_{\theta,\varphi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \qquad (2.55)$$

The wave functions then separates into radial part and angular part

$$\varphi_{o\mathbf{q}}(\mathbf{r}) = R(r)Y_{lm}(\theta\varphi) \qquad (2.56)$$

where $Y_{lm}(\theta\varphi)$ is a spherical harmonic, that is the eigenfunction of angular momentum operator.

Now it is possible to write a differential equation for the solution of the radial part

$$\frac{d^2 j_l(rq)}{dr^2} + \frac{2}{r}\frac{d}{dr}j_l(rq) - \frac{l(l+1)}{r^2}j_l(rq) + \mathbf{q}^2 j_l(rq) = 0 \qquad (2.57)$$

satisfied by the spherical Bessel functions. These functions also have following completeness and orthogonality relations:

$$\frac{2qq'}{\pi}\int_0^\infty r^2 dr j_l(rq) j_l(rq') = \delta(q-q'),$$

$$\frac{2rr'}{\pi}\int_0^\infty r^2 dr j_l(rq) j_l(r'q) = \delta(r-r'). \tag{2.58}$$

Using the relations (2.58), completeness and orthogonality relations for the spherical functions

$$\int d\Omega Y_{lm}(\Omega) Y^*_{l'm'}(\Omega) = \delta_{ll'}\delta_{mm'} \tag{2.59}$$

$(d\Omega = d\mathbf{n} = \sin\theta d\theta d\varphi)$

$$\sum_{lm} Y_{lm}(\theta\varphi) Y^*_{lm}(\theta'\varphi') = \frac{\delta(\theta-\theta')\delta(\varphi-\varphi')}{\sin\theta} \tag{2.60}$$

we obtain the standard orthogonality and completeness relations for the plane waves

$$\frac{1}{(2\pi)^3}\int e^{i\mathbf{q}.\mathbf{r}} e^{-i\mathbf{q}.\mathbf{r}'} d\mathbf{q} = \delta(\mathbf{r}-\mathbf{r}') \tag{2.61}$$

$$\frac{1}{(2\pi)^3}\int e^{i\mathbf{q}.\mathbf{r}} e^{-i\mathbf{q}'.\mathbf{r}} d\mathbf{r} = \delta(\mathbf{q}-\mathbf{q}'). \tag{2.62}$$

(See corresponding relativistic relations.)

2.3.2. Wave function expansion

If the potential of the field producing the scattering is spherically symmetric, we can expand the solution of the Schrödinger equation with the potential

$$\Psi_q(\mathbf{r}) = \sum_{l=0}^{\infty}(2l+1) i^l f_l(r,q) P_l\left(\frac{\mathbf{q}.\mathbf{r}}{qr}\right). \tag{2.63}$$

Hence, the functions $f_l(r,q)$ introduced here satisfies a similar differential equation with previous one

$$\frac{d^2 f_l(r,q)}{dr^2} + \frac{2}{r}\frac{d}{dr} f_l(r,q) - \frac{l(l+1)}{r^2} f_l(r,q) + \mathbf{q}^2 f_l(r,q) = mV(r) f_l(r,q). \tag{2.64}$$

which means that in the spherically symmetric potential case, the states corresponding to different values of angular momentum take part independently in the scattering. We can also write Lippmann-Schwinger type equation for $f_l(r,q)$

$$f_l(r,q) = j_l(rq) + \int_0^\infty G_{lq}(r,r')V(r')r'^2 dr' \tag{2.65}$$

where $G_{lq}(r,r')$ is the partial wave Green's function defined by

$$G_q(\mathbf{r},\mathbf{r'}) = \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)G_{lq}(r,r')P_l\left(\frac{\mathbf{r}.\mathbf{r'}}{rr'}\right), \tag{2.66}$$

$$G_{lq}(r,r') = \frac{2}{\pi}\int_0^\infty \frac{j_l(rk)j_l(r'k)}{2E_q - 2E_k + i\varepsilon}k^2 dk, \tag{2.67}$$

and satisfies the equation

$$\frac{1}{m}\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}\right]G_{lq}(r,r') + 2E_q G_{lq}(r,r') = \frac{1}{r^2}\delta(r-r'). \tag{2.68}$$

To prove this, let's insert $E_q = q^2/2m$ and $E_k = k^2/2m$ into equation (2.67) where $q = |\mathbf{q}|$ and $k = |\mathbf{k}|$. Then we get the integral

$$G_{lq}(r,r') = \frac{2m}{\pi}\int_0^\infty \frac{j_l(kr)j_l(kr')}{q^2 - k^2 + i\epsilon}k^2 dk \qquad (\epsilon = \varepsilon m). \tag{2.69}$$

Now we can insert this integral into the equation (2.68)

$$\frac{d^2}{dr^2}\frac{2}{\pi}\int_0^\infty \frac{j_l(kr)j_l(kr')}{q^2-k^2+i\epsilon}k^2 dk + \frac{2}{r}\frac{d}{dr}\frac{2}{\pi}\int_0^\infty \frac{j_l(kr)j_l(kr')}{q^2-k^2+i\epsilon}k^2 dk -$$

$$\frac{l(l+1)}{r^2}\frac{2}{\pi}\int_0^\infty \frac{j_l(kr)j_l(kr')}{q^2-k^2+i\epsilon}k^2 dk + q^2\frac{1}{\pi}\int_0^\infty \frac{j_l(kr)j_l(kr')}{q^2-k^2+i\epsilon}k^2 dk \tag{2.70}$$

and perform the differentiation

$$\frac{2}{\pi}\int_0^\infty \frac{j_l(r'k)}{q^2-k^2+i\epsilon}\left[\frac{d^2 j_l(kr)}{dr^2} + \frac{2}{r}\frac{d}{dr}j_l(kr) - \frac{l(l+1)}{r^2}j_l(kr)\right]k^2 dk +$$
$$+\frac{1}{\pi}\int_0^\infty \frac{j_l(kr')}{q^2-k^2+i\epsilon}q^2 j_l(kr)k^2 dk. \tag{2.71}$$

By using the equation (2.57)and orthogonality relation (2.58), we can write

$$\frac{2}{\pi}\int_0^\infty \frac{j_l(kr')}{q^2-k^2+i\epsilon}(-k^2+q^2)j_l(kr)dk = \frac{\delta(r-r')}{r^2} \tag{2.72}$$

and see that equation (2.68) holds.

Now, to determine the integral of partial wave Green's function (2.69) we will use the spherical Hankel functions (see Appendix A) of first and second kind, which are defined respectively as

$$h_l^{(1)}(z) = j_l(z) + in_l(z)$$
$$h_l^{(2)}(z) = j_l(z) - in_l(z) \tag{2.73}$$

and also have a relation

$$h_l^{(1)}(-z) = (-1)^l h_l^{(2)}(z) \qquad h_l^{(2)}(-z) = (-1)^l h_l^{(1)}(z). \tag{2.74}$$

So we can write the integral (2.69) in terms of spherical Hankel functions as,

$$\frac{2m}{\pi}\frac{1}{4}\int_0^\infty \frac{[h_l^{(1)}(kr) + h_l^{(2)}(kr)][h_l^{(1)}(kr') + h_l^{(2)}(kr')]}{-k^2+q^2}k^2 dk. \tag{2.75}$$

By using equation (2.74) we get

$$\frac{m}{2\pi}\int_0^\infty \frac{[h_l^{(1)}(kr) - (-1)^l h_l^{(1)}(-kr)][h_l^{(1)}(kr') - (-1)^l h_l^{(1)}(-kr')]}{-k^2+q^2}k^2 dk. \tag{2.76}$$

After multiplying the terms and substituting $k \to -k$ we can separate the integral in the following form,

$$\frac{m}{2\pi}\int_{-\infty}^\infty \frac{h_l^{(1)}(kr)h_l^{(1)}(kr')}{-k^2+q^2}k^2 dk + (-1)^l\int_{-\infty}^\infty \frac{h_l^{(1)}(kr)h_l^{(1)}(-kr')}{-k^2+q^2}k^2 dk \tag{2.77}$$

Now we can apply the Jordan's lemma and Cauchy's residue theorem to the both integral by considering the behavior of the Hankel functions at $r \to \infty$ which are,

$$h_l^{(1)}(kr) \sim \frac{1}{kr}e^{i(kr - \frac{l+1}{2}\pi)} \tag{2.78}$$

$$h_l^{(1)}(kr)h_l^{(1)}(kr') \sim \frac{1}{k^2rr'}e^{i(k(r+r')-(l+1))\pi}$$
$$h_l^{(1)}(kr)h_l^{(1)}(-kr') \sim \frac{1}{k^2rr'}e^{i(k(r-r')-(l+1))\pi}.$$

In equation (2.77) we shift the poles of the integrand from the real axis as,

$$\frac{m}{2\pi}\int_{-\infty}^\infty \frac{h_l^{(1)}(kr)h_l^{(1)}(kr')}{-k^2+q^2+i\varepsilon}k^2 dk + (-1)^l\int_{-\infty}^\infty \frac{h_l^{(1)}(kr)h_l^{(1)}(-kr')}{-k^2+q^2+i\varepsilon}k^2 dk \tag{2.79}$$

and find the poles

$$k_1 = q + i\varepsilon$$

$$k_2 = -q - i\varepsilon.$$

In general Cauchy's residue theorem defined as,

$$\oint_C f(z)dz = 2\pi i \Sigma_j Res(z_j) \tag{2.80}$$

where $\oint_C$ defines the closed contour in the complex z plane and $Res(z_j)$ is the residues of the poles, z_j.

To determine the integral, as a first step we choose a closed contour. Then we apply the Cauchy's residue theorem with the help of the Jordan's lemma. Jordan's lemma states that; If C_R is a closed semicircle in the upper half of the complex plane, then the condition

$$\lim_{R\to\infty}\left(\int_{C_R} e^{iaz} f(z)dz\right) = 0 \tag{2.81}$$

satisfies. Where $\lim_{|z|\to\infty} f(z) = 0$.

Now we can turn back to integral (2.79). For the first integral, considering the asymptotic values, for $r + r' > 0$ and $k > 0$ we close the contour in the upper half plane. For the second integral,

if $r - r' > 0$ we close the contour in the upper half plane and,

if $r - r' < 0$ we close the contour in the lower half plane.

Then we apply the Cauchy's residue theorem for both cases.

For $r - r' > 0$, for the first integral,

$$\lim_{\epsilon\to 0} Res(q + i\epsilon) = \lim_{\epsilon\to 0}\left[\frac{m}{2\pi}.2\pi i\left(\frac{h_l^{(1)}(kr)h_l^{(1)}(kr')k^2}{-2k}\right)\Bigg|_{k=q+i\varepsilon}\right]$$

$$= -iqm\frac{h_l^{(1)}(qr)h_l^{(1)}(qr')}{2} \tag{2.82}$$

and for the second integral

$$\lim_{\epsilon\to 0} Res(q + i\epsilon) = \lim_{\epsilon\to 0}\left[\frac{m}{2\pi}.2\pi i(-1)^l\left(\frac{h_l^{(1)}(kr)h_l^{(1)}(-kr')k^2}{-2k}\right)\Bigg|_{k=+q+i\varepsilon}\right]$$

$$= -iqm(-1)^l\frac{h_l^{(1)}(qr)h_l^{(1)}(-qr')}{2}. \tag{2.83}$$

For $r - r' < 0$

$$\lim_{\epsilon\to 0} Res(-q - i\epsilon) = \lim_{\epsilon\to 0}\left[\frac{m}{2\pi}.2\pi i(-1)^l\frac{h_l^{(1)}(-qr)h_l^{(1)}(qr')}{2}\right]$$

$$= -iqm(-1)^l \frac{h_l^{(1)}(-qr)h_l^{(1)}(qr')}{2}. \tag{2.84}$$

With combining these results we can introduce here step function $\theta(r-r')$ which is ,

$$\theta(r-r') = \begin{cases} 0 & , r-r' < 0 \\ 1 & , r-r' > 0 \end{cases} \tag{2.85}$$

and arrange the equations as;

$$-iqm\Bigg[\frac{h_l^{(1)}(qr)h_l^{(1)}(qr')}{2} + (-1)^l\theta(r-r')\frac{h_l^{(1)}(qr)h_l^{(1)}(-qr')}{2} +$$
$$+(-1)^l\theta(r'-r)\frac{h_l^{(1)}(-qr)h_l^{(1)}(qr')}{2}\Bigg]. \tag{2.86}$$

As a conclusion by using the identities

$$h_l^*(z) = (-1)^l h_l(-z) \tag{2.87}$$
$$j_l(z) = \frac{1}{2}[h_l(z) + h_l^*(z)]$$

we determine the radial Green's function in the following form,

$$G_{lq}(r,r') = -iqm[j_l(qr)h_l^{(1)}(qr')\theta(r-r') - j_l(qr')h_l^{(1)}(qr)\theta(r'-r)]. \tag{2.88}$$

Let us substitute this function into equation (2.68)

$$-iq\frac{d}{dr}\left[j_l'(qr)h_l^{(1)}(qr')\theta(r-r') + j_l(qr')h_l^{(1)}(qr)\theta(r-r')\right] -$$

$$iq\frac{d}{dr}\left[j_l(qr)h_l^{(1)}(qr') - j_l(qr')h_l^{(1)}(qr)\right]\delta(r'-r) -$$

$$\frac{2iq}{r}\left[j_l'(qr)h_l^{(1)}(qr')\theta(r-r') + j_l(qr')h_l^{(1)}(qr)\theta(r-r')\right] -$$

$$\frac{2iq}{r}\left[j_l(qr)h_l^{(1)}(qr')\delta(r-r') - j_l(qr')h_l^{(1)}(qr)\delta(r'-r)\right] +$$

$$\frac{l(l+1)}{r^2}iq[j_l(qr)h_l^{(1)}(qr')\theta(r-r') - j_l(qr')h_l^{(1)}(qr)\theta(r-r')] -$$

$$iq^3[j_l(qr)h_l^{(1)}(qr')\theta(r-r') - j_l(qr')h_l^{(1)}(qr)\theta(r-r')]. \tag{2.89}$$

In this equality we will cancel the $\delta(r-r')$ terms, because of continuity of Green's Function.

Then;

$$-iqh_l^{(1)}(qr')\left[\frac{d^2 j_l(rq)}{dr^2}+\frac{2}{r}\frac{d}{dr}j_l(rq)-\frac{l(l+1)}{r^2}j_l(rq)+q^2 j_l(rq)\right]\theta(r-r')-$$

$$iqj_l(qr)\left[\frac{d^2 h_l(rq)}{dr^2}+\frac{2}{r}\frac{d}{dr}h_l(rq)-\frac{l(l+1)}{r^2}h_l(rq)+q^2 h_l(rq)\right]\theta(r-r')-$$

$$iq\left[j_l'(qr)h_l^{(1)}(qr')-j_l(qr')h_l'^{(1)}(qr)\right]\delta(r-r')$$

$$=\frac{\delta(r-r')}{r^2}. \tag{2.90}$$

We convinced that equation (2.88) also is a solution of equation (2.68).

CHAPTER 3

RELATIVISTIC Q.M. BASED ON THE CONCEPT OF THE RELATIVISTIC CONFIGURATION SPACE

3.1. Lorentz Group Representations and Relativistic Position Operator

In this section we expand the kernel of the Gelfand-Graev transformation in the matrix elements of the unitary representations of the Lorentz group[1]. The Gelfand-Graev transformation links the curved **p** space (Lobachevsky space) and a new relativistic configuration space to each others.

Let us consider the hyperboloid

$$p_0^2 - \mathbf{p}^2 = m^2 \tag{3.1}$$

which from the geometrical point of view realizes the three dimensional Lobachevsky space.

We consider the Gelfand-Graev transformation (which was introduce by Gelfand, Shapiro, Graev and other mathematicians) in a form

$$f(\mathbf{p}) = \frac{1}{(2\pi)^3}\int \xi(\mathbf{p};\mathbf{n},r)\tilde{f}(\mathbf{r})d\mathbf{r} \tag{3.2}$$

$$\tilde{f}(\mathbf{r}) = \frac{1}{(2\pi)^3}\int \xi^*(\mathbf{p};\mathbf{n},r)\tilde{f}(\mathbf{p})d\Omega_p \tag{3.3}$$

where $f(\mathbf{p})$ is the function determined on the hyperboloid (3.1), $\tilde{f}(\mathbf{r})$, is the function determined in the relativistic configurational **r**-space. (**r**=r**n**)

In hyperpolar coordinates

$$\begin{aligned} p_0 &= m\cosh\chi \\ \mathbf{p} &= m\sinh\chi\mathbf{n} \\ \mathbf{n} &= \frac{\mathbf{p}}{|\mathbf{p}|} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \end{aligned} \tag{3.4}$$

[1] See Appendix C, D

kernels of the Gelfand-Graev transformation on the hyperboloid (3.1) are

$$\xi(\mathbf{p};\mathbf{n},r)=\left(\frac{p_0-\mathbf{p}.\mathbf{n}}{m}\right)^{-1-irm}. \tag{3.5}$$

The parameter r appearing in the kernel of the Gelfand-Graev transformation plays the main role in describing the new **r** space. Therefore, before the expansion, we try to get more familiar to the parameter r. The Casimir operators of the Lorentz group are

$$\boldsymbol{x}^2-\frac{1}{m^2}\mathbf{L}^2$$

and

$$\mathbf{L}.\boldsymbol{x}$$

where **L** is the angular momentum operator, **x** is the coordinate operator in Lobachevsky space.

The second Casimir operator $\mathbf{L}.\boldsymbol{x}$ is identical to zero in the spinless case. Let the eigenvalue of the first one be defined by

$$(\boldsymbol{x}^2-\frac{1}{m^2}\mathbf{L}^2)\tilde{\Psi}_q(\mathbf{p})=(\frac{1}{m^2}+r^2)\tilde{\Psi}_q(\mathbf{p}),\qquad 0<r<\infty \tag{3.6}$$

where $\tilde{\Psi}_q(\mathbf{p})$ is the relativistic wave function.

As is well known, the relation (3.6) selects the so called principal series of the unitary representations of the Lorenz group. The functions $\xi(\mathbf{p};\mathbf{n},r)$ used in the Gelfand-Graev transformations are the matrix elements of these representations, thus, (3.6) connects the square of the coordinate operator $\boldsymbol{x}$ to the parameter r.

Since, we are dealing with the quantity r^2 instead of $\boldsymbol{x}^2$ in the relativistic case, we should specify the relations between these two. First, let us find the non-relativistic limit of the operator $\boldsymbol{x}^2-\frac{1}{m^2}\mathbf{L}^2$

$$\lim_{m\to\infty}\boldsymbol{x}^2-\frac{1}{m^2}\mathbf{L}^2=\lim_{m\to\infty}\boldsymbol{x}^2, \tag{3.7}$$

in virtue of (4.12)

$$\lim_{m\to\infty}\boldsymbol{x}^2=-\frac{\partial^2}{\partial\mathbf{p}^2}=\hat{\rho}^2 \tag{3.8}$$

where ρ^2 is the eigenvalue of the operator $\hat{\rho}^2$ which should be equal the non-relativistic limit of the eigenvalue of the first Casimir operator of the Lorentz group and ρ is the radial part of the non-relativistic coordinates. The non-relativistic limit of $1/m^2+r^2$ is

$$\lim_{m\to\infty}(1/m^2+r^2)=r^2=\rho^2 \tag{3.9}$$

Second, $\mathbf{r}^2$ with the eigenvalue r^2 is the Casimir operator of the group of motions of the curved $\mathbf{p}$-space, just as $\hat{\rho}^2$ is the Casimir operator of the group of motions of the flat $\mathbf{p}$-space. Therefore, the range of $\mathbf{r}^2$ is independent of the angular quantum number l (compare with the range of $\mathbf{x}^2$) and the entire relativistic formalism becomes very similar to the non-relativistic one. If the parameter r plays the role of a relativistic distance, is it possible to construct the vector $\mathbf{r}$? In this connection let us consider the Gelfand-Graev transformation, which is expansion in relativistic spherical functions with generating functions (relativistic plane waves) $\xi(\mathbf{p};\mathbf{n},r)$ satisfies the following completeness and orthogonality conditions:

$$\frac{1}{(2\pi)^3}\int \xi(\mathbf{p};\mathbf{n},r)\xi^*(\mathbf{p};\mathbf{n},r')d\Omega_p = \delta(\mathbf{r}-\mathbf{r}') \tag{3.10}$$

$$\mathbf{r} = r\mathbf{n}, \quad \mathbf{r}' = r\mathbf{n}, \quad d\Omega_p = \frac{d\mathbf{p}}{\sqrt{1+\frac{\mathbf{p}^2}{m^2}}}.$$

$$\frac{1}{(2\pi)^3}\int \xi(\mathbf{p};\mathbf{n},r)\xi^*(\mathbf{p}';\mathbf{n},r) = \delta(\mathbf{p}(-)\mathbf{p}') = \delta(\mathbf{p}-\mathbf{p}')\sqrt{1+\frac{\mathbf{p}^2}{m^2}}. \tag{3.11}$$

(Compare these formula with non-relativistic plane waves orthogonality and completeness equations (2.61) and (2.62)).

Let's find the non-relativistic limit ($rm \gg 1$ and $\chi \ll 1$) of Gelfand-Graev transformation. In this limit, one has

$$p = m\sinh\chi \cong m\chi$$

$$\chi \cong \frac{\mathbf{p}}{m}$$

$$\frac{p_0}{m} \cong \sqrt{1+\frac{\mathbf{p}^2}{m^2}} \cong 1+\frac{\mathbf{p}^2}{2m^2}$$

$$\left(\frac{p_0-\mathbf{p}.\mathbf{n}}{m}\right)^{-1-irm} \cong (1+\frac{\mathbf{p}^2}{2m^2}-\frac{\mathbf{p}.\mathbf{n}}{m})^{-1-irm}$$

$$\exp\left[\ln(1+\frac{\mathbf{p}^2}{2m^2}-\frac{\mathbf{p}.\mathbf{n}}{m})^{-1-irm})\right] \cong \exp\left[(-1-irm)\ln(1+\frac{\mathbf{p}^2}{2m^2}-\frac{\mathbf{p}.\mathbf{n}}{m})\right], \tag{3.12}$$

$$\ln(1+x) = x - \frac{x^2}{2} + ...$$

for$-1 < x \leq 1$,

$$\exp\left[(-1-irm)(\frac{\mathbf{p}^2}{2m^2}-\frac{\mathbf{p}.\mathbf{n}}{n})\right] \cong \exp\left[-(\frac{\mathbf{p}^2}{2m^2}+\frac{\mathbf{p}.\mathbf{n}}{m})-\frac{ir\mathbf{p}^2}{2m}+ir\mathbf{p}.\mathbf{n}\right],$$
$$\cong \exp[i\mathbf{p}.\mathbf{r}].$$

The vector $\mathbf{r} = r\mathbf{n}$ in $\xi(\mathbf{p}; \mathbf{n}, r)$ appears as a variable canonically conjugated to the momentum **p**. Therefore, the unit vector $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is exactly the angular part which we wanted to find. From the four dimensional point of view **n** is the space part of the isotropic 4-vector

$$n = (1, \mathbf{n}). \tag{3.13}$$

Therefore, the modulus of the radius vector r is a relativistic invariant, and its direction is transformed as the three-dimensional part of (3.13)[2].

3.2. Relativistic Plane Wave Partial Expansion

Now, we can determine the expansion. Since, the Lorentz group has the rotation group O_3 as a sub-group and since, the Legendre functions are the matrix elements of $2l+1$ dimensional unitary representation of the group O_3 in spherical coordinates, it is possible to expand the Gelfand-Graev functions in terms of the Legendre functions in the same coordinate system.

The coefficients of the expansion can be found by a similar integration of the one in the non-relativistic quantum mechanics. In this case we substitute $x = \mathbf{p}.\mathbf{n}/pn$ where $n = 1$ into the Gelfand-Graev function and proceed as the following:

$$\left(\frac{p_0 - px}{m}\right)^{-1-irm} = \sum_{l=0}^{\infty} a_l(p_0, p) P_l(x) \tag{3.14}$$

$$\frac{2}{2l+1} a_l(p_0, p) = \int_{-1}^{1} \left(\frac{p_0 - px}{m}\right)^{-1-irm} P_l(x) dx \tag{3.15}$$

Thus;

$$a_l(p_o, p) = \frac{2l+1}{2}\int_{-1}^{1}\left(\frac{p_0 - px}{m}\right)^{-1-irm}\frac{l}{2^l.l!}\frac{d^l}{dx^l}(1-x^2)^l dx$$
$$= \frac{2l+1}{2}\int_{-1}^{1}\left(\frac{p_0 - px}{m}\right)^{-1-irm}\frac{l}{2^l.l!}d(\frac{d^{l-1}}{dx^{l-1}}(1-x^2)^l)$$

[2]Note that **r** does not transform as the spatial component of o four vector

$$= \frac{2l+1}{2}\frac{l}{2^l.l!}\left[\left(\frac{p_0 - px}{m}\right)^{-1-irm}\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l\right]_{-1}^{1}$$

$$-\frac{2l+1}{2}\frac{l}{2^l.l!}\left[\int_{-1}^{1}\left[\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l\right]d\left(\frac{p_0 - px}{m}\right)^{-1-irm}\right] \quad (3.16)$$

We get,

$$a_l(p_o, p) = -\frac{2l+1}{2}\frac{l}{2^l.l!}\int_{-1}^{1}(-1-irm)(-p/m)\left(\frac{p_0 - px}{m}\right)^{-1-irm}\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l dx$$

$$= \frac{2l+1}{2}\frac{l}{2^l.l!}(p/m)^l\frac{\Gamma(1+irm+l)}{\Gamma(1+irm)}\int_{-1}^{1}\left(\frac{p_0 - px}{m}\right)^{-1-irm-l}(x^2-1)^l dx. \quad (3.17)$$

By taking the following formulas into the consideration

$$\frac{\Gamma(z+n)}{\Gamma(z)} = z(z+1)(z+2)...(z+n-1), \quad (3.18)$$

$$P_\nu^\mu(z) = \frac{\pi^{-1/2}2^\mu(z^2-1)^{-\mu/2}}{\Gamma(1/2-\mu)}\int_0^\pi [z+(z^2-1)^{1/2}\cos t]^{\nu+\mu}(\sin t)^{-2\mu}dt, \quad (3.19)$$

and substituting the equalities $z = p_0/m$, $(z^2-1)^{1/2} = p/m$, $x = \cos t$, $\mu = -l-1/2$ and $\nu = -irm - 1/2$ into the equation (3.17), it can be written as

$$a_l(p_o, p) = (2l+1)\sqrt{\frac{\pi}{2\sinh\chi}}\frac{\Gamma(1+irm+l)}{\Gamma(1+irm)}P_{-irm-1/2}^{-l-1/2}(\cosh\chi) \quad (3.20)$$

where we've used the hyperpolar coordinates (3.4) for $p_0/m = \cosh\chi$ and $p/m = \sinh\chi$. So, the expansion of the Gelfand-Graev function in terms of the generalized associated Legendre function is

$$\xi(\mathbf{p}; \mathbf{n}, r) = \sqrt{\frac{\pi}{2\sinh\chi}}\sum_{l=o}^{\infty}(2l+1)\frac{\Gamma(1+irm+l)}{\Gamma(1+irm)}P_{-irm-1/2}^{-l-1/2}(\cosh\chi)P_l\left(\frac{\mathbf{p}.\mathbf{n}}{pn}\right)$$

$$= \sum_{l=o}^{\infty} i^l(2l+1)p_l(\cosh\chi, r)P_l\left(\frac{\mathbf{p}.\mathbf{n}}{pn}\right). \quad (3.21)$$

$$\xi^*(\mathbf{p}; \mathbf{n}, r) = \sum_{l=o}^{\infty}(-i)^l(2l+1)p_l^*(\cosh\chi, r)P_l\left(\frac{\mathbf{p}.\mathbf{n}}{pn}\right). \quad (3.22)$$

or by other words the Kernel of the Gelfand-Graev transformation (the relativistic plane wave) $\xi(\mathbf{p}; \mathbf{n}, r)$ is the generating function for the relativistic spherical functions $p_l(\cosh\chi, r)$.

In the non-relativistic limit *rm* $\gg 1$ and $\chi \ll 1$,

$$p_l(\cosh\chi, r) = \sum_{l=o}^{\infty}(-i)^l\sqrt{\frac{\pi}{2\sinh\chi}}\frac{\Gamma(1+irm+l)}{\Gamma(1+irm)}P^{-l-1/2}_{-irm-1/2}(\cosh\chi) \to j_l(qr) \quad (3.23)$$

$$p_l^*(\cosh\chi, r) = \sum_{l=o}^{\infty}(i)^l\sqrt{\frac{\pi}{2\sinh\chi}}\frac{\Gamma(1-irm+l)}{\Gamma(1-irm)}P^{-l-1/2}_{-irm-1/2}(\cosh\chi) \to j_l(qr) \quad (3.24)$$

The functions $p_l(\cosh\chi, r)$, like the spherical Bessel functions $j_l(qr)$, can be expressed in terms of elementary functions:

$$p_0(\cosh\chi, r) = \frac{\sin(rm\chi)}{rm\sinh\chi},$$

$$p_l(\cosh\chi, r) = \frac{i^l\Gamma(-irm+l)}{\Gamma(-irm+l+1)}(\sinh\chi)^l\left(\frac{d}{d\cosh\chi}\right)^l p_0(\cosh\chi, r). \quad (3.25)$$

The analogues of the relations (2.58) can be obtained as

$$\frac{2\sinh\chi\sinh\chi'}{\pi}m^3\int_0^\infty r^2 dr p_l(\cosh\chi, r)p_l^*(\cosh\chi', r) = \delta(\chi-\chi')$$

$$\frac{2rr'}{\pi}m^3\int_0^\infty \sinh\chi^2 d\chi p_l(\cosh\chi, r)p_l^*(\cosh\chi, r') = \delta(r-r'). \quad (3.26)$$

Further, in complete analogy with (2.66) and (2.67) we have

$$G_p = \frac{1}{4\pi}\Sigma_{l=0}^{\infty}(2l+1)G_{lp}(r, r')P_l\left(\frac{\mathbf{r}, \mathbf{r'}}{r, r'}\right),$$

$$G_{lp}(r, r') = \frac{2m^2}{\pi}\int_0^\infty \frac{p_l(\cosh\chi, r)p_l^*(\cosh\chi, r')\sinh\chi^2 d\chi}{2\cosh\chi_p - 2\cosh\chi + i\epsilon}. \quad (3.27)$$

3.3. Relativistic Differential Difference Schrödinger Equation

Now, we are ready to write the relativistic relations in configuration space. Defining the Green' s function $G_p(\mathbf{r};\mathbf{r}')$

$$G_p(\mathbf{r};\mathbf{r}') = \frac{1}{(2\pi)^3}\int \xi(\mathbf{k};\mathbf{n},r)\frac{d\Omega_k}{2E_p - 2E_k + i\epsilon}\xi^*(\mathbf{k};\mathbf{n}',r') \tag{3.28}$$

we can write an integral equation for the wave function $\Psi_p(\mathbf{r})$ in relativistic configuration space:

$$\Psi_p(\mathbf{r}) = \xi(\mathbf{p};\mathbf{n},r) + \int G_p(\mathbf{r};\mathbf{r}')V(\mathbf{r}';E_p)\Psi_p(\mathbf{r}')d\mathbf{r}'. \tag{3.29}$$

When we want to write the corresponding differential equation, we can use the following recursion formulae for the generalized associated Legendre functions :

$$(2\nu+1)zP^{\mu}_{\nu}(z) = (\nu-\mu+1)P^{\mu}_{\nu+1}(z) + (\nu+\mu)P^{\mu}_{\nu-1}(z). \tag{3.30}$$

For values

$$\mu \to -l - 1/2$$

$$\nu \to -irm - 1/2$$

we construct the $P^{\mu}_{\nu+1}(z)$ and $P^{\mu}_{-\nu-1}(z)$ as;

$$P^{\mu}_{\nu+1}(z) = P^{-l-1/2}_{irm-1/2+1}(z) = P^{\mu}_{i(rm-i)-1/2}(z) = e^{-i\frac{d}{dr}}P^{-(l+1/2)}_{irm+1/2}(z) = e^{-i\frac{d}{dr}}P^{\mu}_{\nu}(z) \tag{3.31}$$

$$P^{\mu}_{\nu-1}(z) = P^{-l-1/2}_{irm-1/2-1}(z) = P^{\mu}_{i(rm+i)-1/2}(z) = e^{-i\frac{d}{dr}}P^{-(l+1/2)}_{irm-1/2}(z) = e^{-i\frac{d}{dr}}P^{\mu}_{\nu}(z). \tag{3.32}$$

In equations (3.31) and (3.32) the terms $e^{\pm i\frac{d}{dr}}$ are the finite difference operators(see chapter 4.3). To see how these operators act to any function $f(r)$, we use the Taylor series expansion of $e^{\pm i\frac{d}{dr}}$ which is

$$e^{i\frac{d}{dr}} = \left(1 + i\frac{d}{dr} + \frac{i^2}{2!}\frac{d^2}{dr^2} + ... + \frac{i^n}{n!}\frac{d^n}{dr^n} + ...\right) = \sum_{n=0}^{\infty}\frac{1}{n!}i^n\left(\frac{d}{dr}\right)^n. \tag{3.33}$$

For the function $f(r)$ this expression becomes

$$e^{i\frac{d}{dr}}f(r) = \left(f(r) + i\frac{df(r)}{dr} + \frac{i^2}{2!}\frac{d^2f(r)}{dr^2} + ... + \frac{i^n}{n!}\frac{d^nf(r)}{dr^n} + ...\right) \tag{3.34}$$

$$e^{i\frac{d}{dr}}f(r)=f(r+i). \tag{3.35}$$

Correspondingly

$$\cosh i\frac{d}{dr}f(r)=\frac{e^{i\frac{d}{dr}}+e^{-i\frac{d}{dr}}}{2}f(r) \tag{3.36}$$

$$=\frac{f(r+i)+f(r-i)}{2} \tag{3.37}$$

$$\sinh i\frac{d}{dr}f(r)=\frac{e^{i\frac{d}{dr}}-e^{-i\frac{d}{dr}}}{2}f(r) \tag{3.38}$$

$$=\frac{f(r+i)-f(r-i)}{2}. \tag{3.39}$$

Then with the equation (3.31) and (3.32), recursion relation (3.30) can be written in the form

$$-2ir.\cosh\chi.P^{-l-1/2}_{-irm-1/2}(\cosh\chi)=$$

$$=(-irm+l+1)e^{-i\frac{d}{dr}}P^{-l-1/2}_{-irm-1/2}(\cosh\chi)+(-irm-l-1)e^{i\frac{d}{dr}}P^{-l-1/2}_{-irm-1/2}(\cosh\chi). \tag{3.40}$$

From equation (3.23) we can also construct the following relations

$$e^{-i\frac{d}{dr}}P^{-l-1/2}_{-irm-1/2}(\cosh\chi)=(-i)^{-l}\sqrt{\frac{2\sinh\chi}{\pi}}\frac{\Gamma(irm+2)}{\Gamma(irm+l+2)}e^{-i\frac{d}{dr}}p_l(\cosh\chi,r) \tag{3.41}$$

$$e^{i\frac{d}{dr}}P^{-l-1/2}_{-irm-1/2}(\cosh\chi)=(-i)^{-l}\sqrt{\frac{2\sinh\chi}{\pi}}\frac{\Gamma(irm)}{\Gamma(irm+l+1)}e^{i\frac{d}{dr}}p_l(\cosh\chi,r). \tag{3.42}$$

And if we insert them into formulae (3.40), we get

$$\Bigg[2irm\cosh\chi\frac{\Gamma(irm+1)}{\Gamma(irm+l+1)}-(irm+l+1)\frac{\Gamma(irm+2)}{\Gamma(irm+l+2)}e^{-i\frac{d}{dr}}+$$

$$+(irm-l-1)\frac{\Gamma(irm)}{\Gamma(irm+l)}e^{-i\frac{d}{dr}}\Bigg]p_l(\cosh\chi,r). \tag{3.43}$$

With these constructions, it is easy to see that the function $p_l(\cosh\chi,r)$ introduced in (3.23) satisfy the following differential equation :

$$\left[2m\cosh\chi_q-2m\cosh\left(i\frac{1}{m}\frac{d}{dr}\right)-\frac{2i}{r}\sinh\left(i\frac{1}{m}\frac{d}{dr}\right)-\frac{l(l+1)}{mr^2}\exp\left[i\frac{1}{m}\frac{d}{dr}\right]\right]\times$$

$$\times\ p_l(\cosh\chi,r)=0 \tag{3.44}$$

From here, taking into account (3.21), we obtain the finite-difference analogue of the free Schröndinger equation in the relativistic domain

$$(2E_p-H_0)\xi(\mathbf{p};\mathbf{n},r)=$$

$$= \left[2E_p - 2m\cosh\left(i\frac{1}{m}\frac{\partial}{\partial r}\right) - \frac{2i}{r}\sinh\left(i\frac{1}{m}\frac{\partial}{\partial r}\right) + \frac{\Delta_{\theta,\varphi}}{mr^2}\exp\left[i\frac{1}{m}\frac{\partial}{\partial r}\right]\right].\xi(\mathbf{p};\mathbf{n},r) = 0. \quad (3.45)$$

In the non-relativistic limit, using the Taylor expansion of the functions

$$\cosh\left(i\frac{1}{m}\frac{\partial}{\partial r}\right) = \frac{e^{i\frac{1}{m}\frac{\partial}{\partial r}} + e^{-i\frac{1}{m}\frac{\partial}{\partial r}}}{2} \approx \frac{(2 - \frac{1}{m^2}(\frac{\partial}{\partial r})^2 + ...)}{2}$$

$$\sinh\left(i\frac{1}{m}\frac{\partial}{\partial r}\right) = \frac{e^{i\frac{1}{m}\frac{\partial}{\partial r}} - e^{-i\frac{1}{m}\frac{\partial}{\partial r}}}{2} \approx \frac{(2i\frac{1}{m}\frac{\partial}{\partial r} + ...)}{2} \quad (3.46)$$

and considering the non-relativistic analogue of energy term,

$$\frac{E}{c} = p_0 = \sqrt{m^2c^2 + p^2} = mc(1 + \frac{1}{2}\frac{p^2}{m^2c^2} + ...) \quad (3.47)$$

equation (3.45) obviously reduces to (2.54). From the derivation of (3.44) and (3.45) it is clear that these equations are recursion relations for the functions $p_l(\cosh\chi, r)$ and $\xi(\mathbf{q};\mathbf{n},r)$. For instance, (3.45) may be written in the form

$$H_0\xi(\mathbf{p};\mathbf{n},r) =$$

$$m\left[((1 + \frac{1}{mr})\xi(\mathbf{p};\mathbf{n},r + \frac{i}{m}) + (1 - \frac{1}{mr})\xi(\mathbf{p};\mathbf{n},r - \frac{i}{m}) - \frac{\Delta_{\theta,\varphi}}{mr^2}\xi(\mathbf{p};\mathbf{n},r + \frac{i}{m})\right]$$

$$= E_p\xi(\mathbf{p};\mathbf{n},r). \quad (3.48)$$

The function $\xi^*(\mathbf{p};\mathbf{n},r)$ also satisfies (3.44) and(3.45) . We shall write down two more equations, without explanations since their analogy with the non-relativistic formalism is quite evident.

a) The Schröndinger equation with quasi-potential

$$\left[2E_p - 2m\cosh\left(i\frac{1}{m}\frac{\partial}{\partial r}\right) - \frac{2i}{r}\sinh\left(i\frac{1}{m}\frac{\partial}{\partial r}\right) + \frac{\Delta_{\theta,\varphi}}{mr^2}\exp\left[i\frac{1}{m}\frac{\partial}{\partial r}\right]\right]\Psi_p(\mathbf{r})$$

$$= V(\mathbf{r};E_p)\Psi_p(\mathbf{r}), \quad (3.49)$$

b) The equation for the partial wave Green' s function (3.27)

$$\left[-2m\cosh\left(i\frac{1}{m}\frac{d}{dr}\right) - \frac{2i}{r}\sinh\left(i\frac{1}{m}\frac{d}{dr}\right) - \frac{l(l+1)}{mr^2}\exp\left[i\frac{1}{m}\frac{d}{dr}\right]\right].G_{lp}(r,r') +$$

$$+2E_pG_{lp}(r,r') = \frac{1}{r^2}\delta(r - r'). \quad (3.50)$$

The recursion form of these relations must not be discouraging since the recursion formulae are often as the differential equations for defining the functions.

We continue our consideration to the case of spherically symmetric real potential. Equation for the radial wave function is;

$$\left[H_0^r - 2E + \tilde{V}(r)\right] \psi_{ql}(r) = 0, \tag{3.51}$$

where

$$H_0^r = 2\cosh(i\frac{d}{dr}) + \frac{l(l+1)}{r(r+1)} e^{i\frac{d}{dr}}. \tag{3.52}$$

The free solutions of equation (3.51) are analogues of spherical Bessel, Neumann and Hankel functions (see Appendix A) which appear in the non-relativistic partial-wave analysis. They defined as;[3]

$$s_l(r,\chi) = \sqrt{\frac{\pi \sinh \chi}{2}} (-1)^{l+1} (-r)^{(l+1)} P_{ir-\frac{1}{2}}^{-l-\frac{1}{2}}(\cosh \chi) \tag{3.53}$$

$$c_l(r,\chi) = s_{-l-1}(r,\chi) = \sqrt{\frac{\pi \sinh \chi}{2}} (-r)^{(-l)} P_{ir-\frac{1}{2}}^{l+\frac{1}{2}}(\cosh \chi) \tag{3.54}$$

$$e_l^{(1,2)}(r,\chi) = \sqrt{\frac{2 \sinh \chi}{\pi}} (-1)^{l+1} (-r)^{(l+1)} Q_{ir-\frac{1}{2}}^{-l-\frac{1}{2}}(\cosh \chi) \tag{3.55}$$

$$e_l^{(1,2)} = c_l \pm i s_l$$

where $E = \cosh \chi$, $r^{(\lambda)}$ is the "generalized degree"which is defined as;

$$r^{(\lambda)} = i^{\lambda} \frac{\Gamma(-ir + \lambda)}{\Gamma(-ir)} \tag{3.56}$$

and $P_\nu^\mu(\cosh \chi)$ and $Q_\nu^\mu(\cosh \chi)$are the Legendre functions of first and second kinds.

Remembering that the equation (3.23) we see that we can write the following relation

$$s_l(r,\chi) = r \sinh \chi p_l(\cosh \chi, r). \tag{3.57}$$

Then, as an analogue of the equation (3.26), function $s_l(r,\chi)$ satisfies the following completeness and orthogonality relations

$$\frac{2}{\pi} \int dr s_l(r,\chi) s_l^*(r,\chi') = \delta(\chi - \chi') \tag{3.58}$$

[3]we will use the unit system in which $\hbar = m = c = 1$

$$\frac{2}{\pi}\int dr s_l(r,\chi) s_l(r',\chi) = \delta(r-r'). \tag{3.59}$$

Now let us insert the function $s_l(r,\chi)$ into equation (3.52)

$$H_0^r s_l(r,\chi) = \left(2\cosh(i\frac{d}{dr}) + \frac{l(l+1)}{r^{(2)}} e^{i\frac{d}{dr}}\right) s_l(r,\chi) \tag{3.60}$$

By using the definition of $\cosh(i\frac{d}{dr})$ we can write the equation as;

$$H_0^r s_l(r,\chi) = \left[e^{i\frac{d}{dr}} s_l(r,\chi) + e^{-i\frac{d}{dr}} s_l(r,\chi) + \frac{l(l+1)}{r^{(2)}} e^{i\frac{d}{dr}} s_l(r,\chi)\right]$$

and inserting the definition of the function $s_l(r,\chi)$ into previous equation,

$$\sqrt{\frac{\pi\sinh\chi}{2}}(-1)^{l+1} i^{(l+1)}\left[e^{i\frac{d}{dr}}\frac{\Gamma(ir+l+1)}{\Gamma(ir)} P_{ir-\frac{1}{2}}^{-l-\frac{1}{2}}(\cosh\chi) + e^{-i\frac{d}{dr}}\frac{\Gamma(ir+l+1)}{\Gamma(ir)} P_{ir-\frac{1}{2}}^{-l-\frac{1}{2}}(\cosh\chi) + \right.$$

$$\left. + \frac{l(l+1)}{r(r+i)} e^{i\frac{d}{dr}}\frac{\Gamma(ir+l+1)}{\Gamma(ir)} P_{ir-\frac{1}{2}}^{-l-\frac{1}{2}}(\cosh\chi)\right]$$

$$= \sqrt{\frac{\pi\sinh\chi}{2}}(-1)^{l+1}\ i^{(l+1)}\left[\frac{\Gamma(ir+l)}{\Gamma(ir-1)} P_{ir-\frac{3}{2}}^{-l-\frac{1}{2}}(\cosh\chi) + \frac{\Gamma(ir+l+2)}{\Gamma(ir+1)} P_{ir+\frac{1}{2}}^{-l-\frac{1}{2}}(\cosh\chi) + \right.$$

$$\left. + \frac{l(l+1)}{r(r+i)}\frac{\Gamma(ir+l)}{\Gamma(ir-1)} P_{ir-\frac{3}{2}}^{-l-\frac{1}{2}}(\cosh\chi)\right]$$

we get;

$$\sqrt{\frac{\pi\sinh\chi}{2}}(-1)^{l+1} i^{(l+1)}\frac{1}{ir}\left[(ir-l-1) P_{ir-\frac{3}{2}}^{-l-\frac{1}{2}}(\cosh\chi) + (ir+l+1) P_{ir+\frac{1}{2}}^{-l-\frac{1}{2}}(\cosh\chi)\right] \tag{3.61}$$

Now recall the recursion formulae (3.30) for the Legendre functions which can be written,

$$(2ir)\cosh\chi P_{ir-\frac{1}{2}}^{-l-\frac{1}{2}}(\cosh\chi) = (ir+l+1) P_{ir+\frac{1}{2}}^{-l-\frac{1}{2}}(\cosh\chi) + (ir-l-1) P_{ir-\frac{3}{2}}^{-l-\frac{1}{2}}(\cosh\chi) \tag{3.62}$$

then we see that with (3.61) following equation holds

$$H_0^r s_l(r,\chi) \equiv \left(2\cosh(i\frac{d}{dr}) + \frac{l(l+1)}{r^{(2)}} e^{i\frac{d}{dr}}\right) s_l(r,\chi) = 2E s_l(r,\chi). \tag{3.63}$$

We can also consider the asymptotic behavior of the functions (3.53), (3.54) and (3.55) when $r \to \infty$ which are given by

$$s_l(r,\chi) \sim \sin(r\chi - \frac{1}{2}l\pi) \tag{3.64}$$

$$c_l(r,\chi) \sim \cos(r\chi - \frac{1}{2}l\pi) \tag{3.65}$$

$$e_l^{(1,2)}(r,\chi) \sim e^{\pm i(r\chi - \frac{1}{2}l\pi)}. \tag{3.66}$$

The functions s_l, c_l, $e_l^{(1)}$ and $e_l^{(2)}$ are all two-by-two linearly independent. It is known that two solutions of a second order finite difference equation are linearly independent[4] iff

$$W(\phi_1,\phi_2) \neq 0, \tag{3.67}$$

where $W(\phi_1,\phi_2)$ is the "Wronskian"

$$W(\phi_1,\phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \Delta\phi_1 & \Delta\phi_2 \end{vmatrix} \tag{3.68}$$

and

$$\Delta = \frac{e^{-id/dr} - 1}{-i} \tag{3.69}$$

is the finite difference "derivative" operator.

It can be shown that

$$W[s_l(r,\chi), c_l(r,\chi)] = \frac{1}{2i} W[e_l^{(1)}(r,\chi), e_l^{(2)}(r,\chi)] \tag{3.70}$$

$$= \sinh\chi(-1)^l \frac{(-r)^{(l+1)}}{(r)^{(l+1)}}. \tag{3.71}$$

We can also consider the non-relativistic limit of these functions. First let us give the definition of Legendre function in terms of hypergeometric series.

$$P_\nu^\mu(z) = \frac{(z+1)^{\frac{-1}{2}\mu}(z-1)^{\frac{1}{2}\mu}}{\Gamma(1-\mu)} F(-\nu, 1+\nu; 1-\mu; \frac{1-z}{2}) \tag{3.72}$$

which is with suitable indices takes the form,

$$P_{ir+\frac{1}{2}}^{-l-\frac{1}{2}}(cosh\chi) = \frac{(\cosh\chi+1)^{\frac{-1}{2}\mu}(\cosh\chi-1)^{\frac{1}{2}\mu}}{\Gamma(l+\frac{3}{2})} \sum_{n=0}^{\infty} \frac{(-ir+\frac{1}{2})_n(ir+\frac{1}{2})_n}{(l+\frac{3}{2})_n} \frac{1}{n!} \left(\frac{1-\cosh\chi}{2}\right)^n. \tag{3.73}$$

where $(z)_n = \Gamma(z+n)/\Gamma(z)$.

[4]see Appendix A

Now we insert the relation (3.73) into function $s_l(r,\chi)$ and get,

$$s_l(r,\chi) = \sqrt{\frac{\pi \sinh\chi}{2}}(-1)^{l+1}(i)^{l+1}\frac{\Gamma(ir+l+1)}{\Gamma(ir)}\times$$

$$\times\frac{(\cosh\chi+1)^{\frac{-1}{2}\mu}(\cosh\chi-1)^{\frac{1}{2}\mu}}{\Gamma(-l+\frac{1}{2})}\sum_{n=0}^{\infty}\frac{(-ir-\frac{1}{2})_n(ir+\frac{3}{2})_n}{(-l+\frac{1}{2})}\frac{1}{n!}\left(\frac{1-\cosh\chi}{2}\right)^n.$$

So in the non-relativistic limit, for $\chi \ll 1$, $r \gg 1$;

$$s_l(r,\chi) \longrightarrow \sqrt{\frac{\pi pr}{2}}J_{l+\frac{1}{2}}(pr). \tag{3.74}$$

If we do the same calculation for the function $c_l(r,\chi)$ and $e_l^{(1,2)}(r,\chi)$ we find the following relations

$$c_l(r,\chi) \longrightarrow -\sqrt{\frac{\pi pr}{2}}N_{l+\frac{1}{2}}(pr) \tag{3.75}$$

$$e_l^{(1,2)}(r,\chi) \longrightarrow \pm i\sqrt{\frac{\pi rq}{2}}H^{(1,2)}_{l+\frac{1}{2}}(pr) \tag{3.76}$$

where $J_{l+\frac{1}{2}}(pr)$, $N_{l+\frac{1}{2}}(pr)$ and $H^{(1,2)}_{l+\frac{1}{2}}(pr)$ are spherical Bessel, Neumann and Hankel functions respectively.

Arbitrary Green's function of equation (3.51) can be obtained if a solution of the homogeneous equation, i.e., linear combination of (3.53), (3.54) with coefficients in general depending on r', is added to $G_l(r,r';E_q)$. We introduce now a Green's function

$$G_l(r,r';E_q) = \frac{s_l(r,\chi)c_l(r',\chi) - c_l(r,\chi)s_l(r',\chi)}{W[s_l(r',\chi),c_l(r',\chi)]}\hat{\theta}(r-r') \tag{3.77}$$

which satisfies the equation

$$\begin{aligned}(H_0 - 2E_q)G_l(r,r';E_q) &= \left[2\cosh\chi - 2\cosh(i\frac{d}{dr}) - \frac{l(l+1)}{r^{(2)}}e^{i\frac{d}{dr}}\right]G_l(r,r';E_q)\\ &= \delta(r-r'). \end{aligned}\tag{3.78}$$

To show that the previous equality holds, let's write the terms explicitly. By using the finite difference derivative operator (3.69) we can write

$$\cosh(i\frac{d}{dr}) = \frac{1}{2i}(\Delta - \Delta^*) + 1 \tag{3.79}$$

and inserting into equation (3.78) we get

$$\frac{c_l(r',\chi)}{W[s_l(r',\chi),c_l(r',\chi)]}\left[2\cosh\chi+(\Delta-\Delta^*)-2-\frac{l(l+1)}{r^{(2)}}e^{i\frac{d}{dr}}\right]s_l(r,\chi)\hat{\theta}(r-r')$$

$$+\frac{s_l(r',\chi)}{W[s_l(r',\chi),c_l(r',\chi)]}\left[2\cosh\chi+(\Delta-\Delta^*)-2-\frac{l(l+1)}{r^{(2)}}e^{i\frac{d}{dr}}\right]c_l(r,\chi)\hat{\theta}(r-r'). \tag{3.80}$$

Now we need to determine the relation when the finite difference derivative operator (3.69) acts to the multiplication of two functions. For given functions $f(r)$ and $g(r)$ we have

$$\Delta(f(r)g(r))=i(e^{i\frac{d}{dr}}-1)(f(r)g(r))=i(e^{i\frac{d}{dr}}f(r)g(r)-f(r)g(r)) \tag{3.81}$$

$$=i(f(r-i)g(r-i)-f(r)g(r)). \tag{3.82}$$

When we write the expression $f(r-i)$ as

$$f(r-i)=f(r)-i\Delta f(r) \tag{3.83}$$

and the same for $g(r-i)$, then we have the following relation

$$\Delta(f(r)g(r))=f(r)\Delta g(r)-g(r)\Delta f(r)-i(\Delta f(r))(\Delta g(r)). \tag{3.84}$$

We can also determine the similar relation for $\Delta^*(f(r)g(r))$ which is

$$\Delta^*(f(r)g(r))=f(r)\Delta^* g(r)-g(r)\Delta^* f(r)+i(\Delta^* f(r))(\Delta^* g(r)). \tag{3.85}$$

By using the equations (3.84), (3.85) and the following relations (see chapter 4.3)

$$\Delta\theta(r-r') \;=\; \delta(r-r') \quad \text{and} \quad e^{i\frac{d}{dr}}\Delta=\Delta^* \tag{3.86}$$

(3.87)

we can arrange our equation as,

$$\frac{1}{W[s_l(r',\chi),c_l(r',\chi)]}\left[c_l(r',\chi)e^{i\frac{d}{dr}}s_l(r,\chi)-is_l(r',\chi)e^{i\frac{d}{dr}}c_l(r,\chi)\right]\delta(r-r')+$$

$$-\left[\frac{ic_l(r',\chi)}{W[s_l(r',\chi),c_l(r',\chi)]}(e^{i\frac{d}{dr}}+\frac{l(l+1)}{r^{(2)}}e^{i\frac{d}{dr}})s_l(r,\chi)\right]e^{i\frac{d}{dr}}\delta(r-r')+$$

$$+\left[\frac{is_l(r',\chi)}{W[s_l(r',\chi),c_l(r',\chi)]}(e^{i\frac{d}{dr}}+\frac{l(l+1)}{r^{(2)}}e^{i\frac{d}{dr}})c_l(r,\chi)\right]e^{i\frac{d}{dr}}\delta(r-r'). \tag{3.88}$$

From definition of $\delta(r-r')$ function we consider the case $r-r'=0$. So the the second and the third terms give no input in integration over real r's and must be omitted. Then from equation (3.68) and (3.69) we have,

$$W[s_l(r',\chi),c_l(r',\chi)] = s_l(r',\chi)\Delta c_l(r',\chi) - c_l(r',\chi)\Delta s_l(r',\chi)$$
$$= is_l(r',\chi)e^{-i\frac{d}{dr}}c_l(r',\chi) - ic_l(r',\chi)e^{-i\frac{d}{dr}}s_l(r',\chi). \tag{3.89}$$

which is the relation in the first bracket in equation (3.88). So we get

$$\frac{1}{W[s_l(r',\chi),c_l(r',\chi)]}\left[c_l(r',\chi)e^{i\frac{d}{dr}}s_l(r,\chi) - is_l(r',\chi)e^{i\frac{d}{dr}}c_l(r,\chi)\right]\delta(r-r')$$
$$= \delta(r-r'). \tag{3.90}$$

and convinced that equality (3.78) holds.

Then, it is possible to write the following integral equation

$$\psi_{ql}(r) = A(r)s_l(r,\chi) + B(r)c_l(r,\chi) - \int_0^\infty G_l(r,r';E_q)\tilde{V(r')}\psi_{ql}(r')dr', \tag{3.91}$$

which is a direct generalization of the corresponding non-relativistic equation (2.20). In equation (3.91) $A(r)$ and $B(r)$ are i periodic functions

$$e^{\pm i\frac{d}{dr}}A(r) = A(r+i) = A(r) \tag{3.92}$$

$$e^{\pm i\frac{d}{dr}}B(r) = B(r+i) = B(r) \tag{3.93}$$

which play the role of constants in the finite-difference calculus.

CHAPTER 4

RELATIVISTIC 2-BODY PROBLEM AND RELATIVISTIC SCATTERING THEORY

4.1. General Theory

Although in the relativistic quantum mechanics, as in the non-relativistic one, the two body problem can be reduced to the problem of the behavior of one relativistic particle in a quasi-potential field within the framework of quasi potential approach.

The quasi-potential approach is a diagram technique in the old fashioned (non-covariant) perturbation theory. It is a rearrangement of the ordinary Feynman diagrams. This technique allows us to write a Lippmann-Schwinger type, on mass-shell equation with a quasi potential field for the scattering of two relativistic particles, in the center of mass frame of these particles. The specific features of such a quasi-potential equation are:

a) it is a three-dimensional equation,

b) all quantities in it are defined in the Lobachevsky space modelled by the hyperboloid $p_0^2 - \mathbf{p}^2 = m^2$ (the mass shell of the particle of mass m)

c) it is satisfied order by order by the Feynman perturbation expansion of the scattering amplitude,

d) for a hermitian potential it implies the elastic unitarity condition.

The derivation of these features of the quasi-potential equations is not the theme of this thesis.

In this thesis, we shall use the following quasi-potential equation to develop our own considerations (we recall that **p**, **q**, **k**, ... denote the spatial part of the 4-vectors belonging to the hyperboloid (3.1)):

$$A(\mathbf{p}, \mathbf{q}) = -\frac{m}{4\pi}\tilde{V}(\mathbf{p}, \mathbf{q}; E_q) + \frac{1}{(2\pi)^3}\int \tilde{V}(\mathbf{p}, \mathbf{k}; E_q)\frac{d\mathbf{k}}{\sqrt{1+\mathbf{k}^2/m^2}}\frac{A(\mathbf{k}, \mathbf{q})}{2E_q - 2E_k + i\epsilon}. \tag{4.1}$$

All variables in this equation are in the center of mass frame and the masses of the relativistic particles are considered to be identical and equal to m. The quantity

$\tilde{V}(\mathbf{p}, \mathbf{q}; E_q)$ called quasi-potential is in general a complex function of momenta and of energy and in the weak coupling case it can be constructed with the help of perturbation theory.

Each of the **p** and **k** denotes the relative three-momenta of the particles with respect to each others in the center of mass frame. $A(\mathbf{p}, \mathbf{q})$ is the relativistic scattering amplitude and E_q and E_k are the relativistic energies $E_q = \sqrt{\mathbf{q}^2 + m^2}$, $E_k = \sqrt{\mathbf{k}^2 + m^2}$.

When we compare the Lippmann-Schwinger equation (4.1) and (2.28) of the relativistic and non-relativistic quantum mechanics, we can see that if we take the geometrical relations (the volume elements and the energies), which are different in relativistic and non-relativistic cases, out of these equations they become indistinguishable by their forms. This means that their dynamical features are the same but their geometrical relations are different. Therefore, to get the analogues of the relations listed in the section 2, it is enough to study on the geometry of the space of the relativistic Lippman-Schwinger equation which is the Lobachevsky space.

To study the Lobachevsky geometry we should parametrize it. To parametrize the Lobachevsky space, it is useful to project the hyperboloid (3.1) on some Euclidean hyperplane and to assign to the points (3.1) the Cartesian coordinate of their projections. If we project the hyperboloid from the point $(\infty, 0)$ onto the hyperplane $k_0 = 0$, the entire three dimensional **k**-space will be a model of the Lobachevsky space with metric

$$ds^2 = \frac{(\mathbf{k}.d\mathbf{k})^2}{m^2 + \mathbf{k}^2} - d\mathbf{k}^2 = g_{ij} dk_i dk_j \tag{4.2}$$

and volume element

$$d\Omega_{\mathbf{k}} = \sqrt{g(k)} d\mathbf{k} = \frac{d\mathbf{k}}{\sqrt{1 + \mathbf{k}^2/m^2}} \tag{4.3}$$

where $g(k)$ is the determinant of the metric g_{ij}. Thus, we can study the Lobachevsky space as the momentum space of a spinless relativistic particle with the metric (4.2)

Now, we want to define the operators of the angular momentum **L** and the coordinate **x** in Lobachevsky space. Let $\Psi(\mathbf{p})$ be the wave function of a particle with spin 0, mass m and momentum **p**, which is a vector in Lobachevsky space. The group of motions of Lobachevsky space-the Lorentz group-contains O_3 as a sub-group and the operator **L** has the ordinary form

$$L_k = \frac{1}{i}\varepsilon_{klm}p_l\frac{\partial}{\partial p_m} \tag{4.4}$$

In the non-relativistic theory, the coordinates x are generators of translations in the Euclidean **p**-space. In the curved **p**-space the boosts $\Lambda_{\mathbf{k}}$ play the role of translations and we could try to introduce the operators **x** as generators of these transformations. A boost $\Lambda(\chi)$ in the direction p_l is

$$\Lambda(\chi) = \begin{pmatrix} \cosh\chi & \sinh\chi & 0 & 0 \\ \sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.5}$$

It contains a single real parameter χ and value of χ in the range $-\infty < \chi < \infty$ corresponds to a different transformation $\Lambda(\chi)$. The transformed vector p' is given by

$$p' = \Lambda(\chi)p = (p_0\cosh\chi + p_1\sinh\chi, p_1\cosh\chi + p_0\sinh\chi, p_2, p_3) \tag{4.6}$$

We may clearly define a similar boost transformation in an arbitrary direction defined by a unit three-vector $\hat{\mathbf{k}}$:

$$\mathbf{p}' = \mathbf{p} + \hat{\mathbf{k}}[(\cosh\chi - 1)\mathbf{p}.\hat{\mathbf{k}} + p_0\sinh\chi], \tag{4.7}$$

$$p_0' = p_0\cosh\chi + \hat{\mathbf{k}}.\mathbf{p}\sinh\chi. \tag{4.8}$$

If we use the following " spherical " co-ordinates for the curved momentum space

$$\begin{aligned} k_0 &= m\cosh\chi \\ k_1 &= m\sinh\chi\sin\theta\cos\varphi \\ k_2 &= m\sinh\chi\sin\theta\sin\varphi \\ k_3 &= m\sinh\chi\cos\theta \end{aligned} \tag{4.9}$$

where the range of the variables are $-\infty < \chi < \infty$, $0 < \theta < \pi$ and $0 < \varphi < 2\pi$, the equation (4.7) can be written as

$$\mathbf{p}' = \Lambda_{\mathbf{k}}\mathbf{p} = \mathbf{p}(+)\mathbf{k} = \mathbf{p} + \mathbf{k}\left[\sqrt{1+\mathbf{p}^2/m^2} + \frac{\mathbf{p}.\mathbf{k}}{m^2[1+\sqrt{1+\mathbf{k}^2/m^2}]}\right] \quad (4.10)$$

For the infinitesimal momentum $d\mathbf{k}$ the transformation (4.10) becomes

$$\mathbf{p}' = \mathbf{p} + \sqrt{1+\mathbf{p}^2/m^2}d\mathbf{k} \quad (4.11)$$

and we can introduce the operator x as

$$\mathbf{x} = i\sqrt{1+\mathbf{p}^2/m^2}\frac{\partial}{\partial\mathbf{p}}. \quad (4.12)$$

Although the operator (4.12) is hermitian in the metric

$$(\Psi, \Phi) = \int \Psi^*(\mathbf{p})\Phi(\mathbf{p})\frac{d\mathbf{p}}{\sqrt{1+\mathbf{p}^2/m^2}}, \quad (4.13)$$

since it components do not commute amongst themselves and cannot be reduced to diagonal form simultaneously, the transformations (4.10) do not form a group, in contrast to the Euclidean shifts. However, being elements of the Lorentz group they have certain group theoretical properties. In particular,

$$\mathbf{p}(+)0 = \mathbf{p} \quad \text{and} \quad \mathbf{p}(-)\mathbf{p} = 0 \quad (4.14)$$

$$0(+)\mathbf{k} = \mathbf{k} \quad \text{and} \quad [\mathbf{p}(+)\mathbf{k}](-)\mathbf{k} = \mathbf{p} \quad (4.15)$$

where

$$\mathbf{p}(-)\mathbf{k} = \mathbf{p} - \mathbf{k}\left[\sqrt{1+\mathbf{p}^2/m^2} - \frac{\mathbf{p}.\mathbf{k}}{m^2[1+\sqrt{1+\mathbf{k}^2/m^2}]}\right]. \quad (4.16)$$

Evidently the volume element (4.3) is invariant with respect to the transformation (4.10). Thus, we have

$$d\Omega_{\mathbf{k}(+)\mathbf{q}} = d\Omega_{\mathbf{p}}. \quad (4.17)$$

This property of $d\Omega_{\mathbf{k}}$ allows a convolution to be defined for functions on Lobachevsky space:

$$\Psi_1(\mathbf{p}) * \Psi_2(\mathbf{p}) = \int d\Omega_{\mathbf{k}}\Psi_1(\mathbf{k})\Psi_2(-\mathbf{k}(+)\mathbf{p}). \quad (4.18)$$

It is evident that

$$\Psi_1(\mathbf{p}) * \Psi_2(\mathbf{p}) = \Psi_2(\mathbf{p}) * \Psi_1(\mathbf{p}) \tag{4.19}$$

Putting $\Psi_1(\mathbf{p}) = \delta^{(3)}(\mathbf{p})$ and $\Psi_2(\mathbf{p}) = \Psi(\mathbf{p})$ in (4.18) and taking into account (4.14) and (4.19), we obtain

$$\delta(\mathbf{p}) = \Psi(\mathbf{p}) = \sqrt{g(0)}\Psi(\mathbf{p}) = \int \delta^{(3)}(\mathbf{k}(-)\mathbf{p})\Psi(\mathbf{k})d\Omega_{\mathbf{k}} \tag{4.20}$$

Hence,

$$\delta^{(3)}(\mathbf{k}(-)\mathbf{p}) = \frac{\delta^{(3)}(\mathbf{k}\text{ - }\mathbf{p})}{\sqrt{g(k)}} = \sqrt{1 + \mathbf{k}^2/m^2}\delta^{(3)}(\mathbf{k}\text{ - }\mathbf{p}). \tag{4.21}$$

Now, we are ready to write the analogues of the corresponding non-relativistic relations. First, in the case of a real quasi-potential, from (4.1) follows a relation which coincides exactly with the non-relativistic unitarity condition (2.33):

$$ImA(\mathbf{p}, \mathbf{q}) = \frac{|\mathbf{q}|}{4\pi} \int A(\mathbf{p}, \mathbf{k})A^*(\mathbf{k}, \mathbf{q})dw_{\mathbf{k}} \tag{4.22}$$

where $E_p = E_q = E_k$. The relativistic wave function of the continuous spectrum (See the equations (2.34) and (4.21)) can be introduced as

$$\tilde{\Psi}_q(\mathbf{p}) = (2\pi)^3\delta^{(3)}(\mathbf{p}(\text{-})\mathbf{q}) - \frac{4\pi}{m}\frac{A(\mathbf{p}, \mathbf{q})}{E_q - 2E_p + i\epsilon}. \tag{4.23}$$

Using the Gelfand-Graev transformations(3.2) and (3.3) for the quasi-potential $V(\mathbf{p}; E_q)$ and applying the addition theorem for $\xi(\mathbf{p}; \mathbf{n}, r)$ function

$$\int \xi^*(\mathbf{p}(-)\mathbf{k}; \mathbf{n}, r)dw_{\mathbf{n}} = \int \xi^*(\mathbf{p}; \mathbf{n}, r)\xi(\mathbf{k}; \mathbf{n}, r)dw_{\mathbf{n}} \tag{4.24}$$

$$dw_{\mathbf{n}} = \sin\theta d\theta d\varphi, \tag{4.25}$$

we can generalize the equation (2.23) in an evident way:

$$A(\mathbf{p}, \mathbf{q}) = -\frac{m}{4\pi} \int \xi^*(\mathbf{p}; \mathbf{n}, r)V(\mathbf{r}; E_q)\Psi_q(\mathbf{r})d\mathbf{r} \tag{4.26}$$

$$= -\frac{m}{4\pi}\frac{1}{(2\pi)^3} \int \tilde{V}(\mathbf{p}(-)\mathbf{k}; E_q)\tilde{\Psi}_q(\mathbf{k})d\Omega_{\mathbf{k}}. \tag{4.27}$$

Then, we can obtain the following relativistic Schrödinger equation for the quantity $\tilde{\Psi}_q(\mathbf{p})$(compare with (2.35) and (2.36)):

$$\tilde{\Psi}_q(\mathbf{p}) = (2\pi)^3\delta^{(3)}(\mathbf{p}(-)\mathbf{q}) + \frac{1}{(2\pi)^3}\frac{1}{2E_q - 2E_p + i\epsilon}\int \tilde{V}(\mathbf{p}(-)\mathbf{k}; E_q)\tilde{\Psi}_q(\mathbf{k})d\Omega_{\mathbf{k}}, \quad (4.28)$$

$$(2E_q - 2E_p)\tilde{\Psi}_q(\mathbf{p}) = \frac{1}{(2\pi)^3}\int \tilde{V}(\mathbf{p}(-)\mathbf{k}; E_q)\tilde{\Psi}_q(\mathbf{k})d\Omega_{\mathbf{k}}, \quad (4.29)$$

In the spherically symmetric potential case we have, in addition

$$\begin{aligned}
\tilde{V}(\mathbf{p}(-)\mathbf{k}; E_q) &= \tilde{V}((\mathbf{p}(-)\mathbf{k})^2; E_q) && (4.30)\\
&= \int \xi^*(\mathbf{p}; \mathbf{n}, r)V(r, E_q)\xi * (\mathbf{k};\, \mathbf{n}, r)d\mathbf{r} \\
&= \int r^2 dr V(r, E_q)\int \xi^*(\mathbf{p};\, \mathbf{n}\,, r)\xi(\mathbf{k};\, \mathbf{n}, r)dw_{\mathbf{n}} && (4.31)
\end{aligned}$$

4.2. Evaluation of Green Function

In this chapter we will analyze the one dimensional Green's function for the relativistic case by using finite difference Schrödinger equation. With the potential $V(x)$, finite difference Schröndinger equation defined as;

$$\left[\cosh i\frac{d}{dx} - \cosh\chi\right]\psi(x) = -V(x)\psi(x). \quad (4.32)$$

and Green's function is defined as a solution of the equation

$$\left[\cosh i\frac{d}{dx} - \cosh\chi\right]G(x - x', \chi) = -\delta(x - x'). \quad (4.33)$$

So, if we determine solution of the previous equation, then we can write the Schrödinger equation in a form of integral equation as,

$$\psi(x) = e^{ix\chi} + \int_{-\infty}^{\infty} G(x - x', \chi)V(x')\psi(x')dx'. \quad (4.34)$$

Let's define δ function in relativistic configuration space in the following form;

$$\delta(x - x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha(x-x')}d\alpha \quad (4.35)$$

then the Fourier integral representation of Green's function can be written as

$$G(x - x', \chi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\alpha, \chi)e^{i\alpha(x-x')}d\alpha. \quad (4.36)$$

Inserting the equation (4.36) into equation (4.32)

$$\left(e^{i\frac{d}{dx}} - e^{-i\frac{d}{dx}} - \cosh\chi\right) G(x-x',\chi) = -\delta(x-x') \tag{4.37}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(e^{i\frac{d}{dx}} - e^{-i\frac{d}{dx}} - \cosh\chi\right) g(\alpha,\chi)e^{i\alpha(x-x')}d\alpha = -\delta(x-x') \tag{4.38}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\alpha,\chi)(2\cosh\alpha - \cosh\chi)e^{i\alpha(x-x')}d\alpha = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha(x-x')}d\alpha \tag{4.39}$$

then we get

$$g(\alpha,\chi) = \frac{1}{\cosh\chi - \cosh\alpha}. \tag{4.40}$$

Substituting equation (4.40) into equation (4.36) we can find the integral representation of Green's function, which is

$$G(x-x',\chi) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\alpha(x-x')}}{\cosh\chi - \cosh\alpha}d\alpha \tag{4.41}$$

In analogy with the non-relativistic scattering theory formalism we can shift the poles as

$$\chi \rightarrow \chi + i\varepsilon \tag{4.42}$$

and by using $x - x' = x$ for simplification, the integral representation of Green's function becomes

$$G(x,\chi) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\alpha x}}{\cosh(\chi + i\varepsilon) - \cosh\alpha}d\alpha. \tag{4.43}$$

We can find the poles of the integrand from the equation

$$\cosh(\chi + i\epsilon) - \cosh\alpha = 0. \tag{4.44}$$

Considering the periodicity of the hyperbolic functions with respect to imaginary axis, the poles are find as

$$\begin{aligned} \alpha &= \chi + i\varepsilon + 2\pi i n \\ \alpha &= -\chi - i\varepsilon - 2\pi i n, \end{aligned} \tag{4.45}$$

$n =$ integer

Now we can apply the Cauchy's residue theorem and Jordan's lemma by recalling the equations (2.80) and (2.81) from non-relativistic theory.

Then we turn back to the integral equation (4.43). If we close the contour in the upper half plane,

for $x > 0$ and large imaginary values of α

$$e^{i\alpha x} \to 0$$

and if we close the contour in the lower half plane,

for $x < 0$ and large imaginary values of α

$$e^{i\alpha x} \to 0$$

then we can apply the Cauchy's residue theorem for both cases.

For $x > 0$ Green's function is

$$G(x,\chi)_{x>0} = \frac{1}{2\pi} 2\pi i \lim_{\epsilon \to 0} \left[\sum_{n=0}^{\infty} \frac{e^{i\alpha\chi}}{-\sinh\alpha} \bigg|_{\alpha=\chi+i\varepsilon+2\pi in} + \sum_{n=1}^{\infty} \frac{e^{i\alpha\chi}}{-\sinh\alpha} \bigg|_{\alpha=-\chi+i\varepsilon+2\pi in} \right]$$

$$= -i \left[\sum_{n=0}^{\infty} e^{ix\chi} \frac{e^{-2n\pi x}}{\sinh(\chi + 2\pi in)} + \sum_{n=1}^{\infty} e^{-ix\chi} \frac{e^{-2n\pi x}}{\sinh(-\chi + 2\pi in)} \right]. \qquad (4.46)$$

Take into account that periodicity of the hyperbolic functions with respect to imaginary axis, we can write

$$\sinh(\chi + 2\pi in) = \sinh\chi \qquad \sinh(-\chi + 2\pi in) = -\sinh\chi \qquad (4.47)$$

If we insert these expression into equation (4.46) we get

$$G(x,\chi)_{x>0} = \frac{-i}{\sinh\chi} \left[e^{ix\chi} \sum_{n=0}^{\infty} e^{-2n\pi x} - e^{-ix\chi} \sum_{n=1}^{\infty} e^{-2n\pi x} \right]. \qquad (4.48)$$

The series expansion

$$\sum_{n=0}^{\infty} e^{-2n\pi x} = 1 + e^{-2\pi x} + e^{-4\pi x} + e^{-6\pi x} + ... \qquad (4.49)$$

is a geometric series which is in the form; $S_n = \sum_{n=0}^{\infty} ar^n$. And if $|r| < 1$ then the series is convergent and $S_n = a/(1-r)$.

In our case $a = 1$ and $|e^{-2n\pi x}| < 1$ for $n > 0$, so series is convergent also. Then the summation of the first series in equation (4.48) is

$$\sum_{n=0}^{\infty} e^{-2n\pi x} = \frac{1}{1-e^{-2\pi x}} \tag{4.50}$$

and the summation of the second series in equation (4.48) is

$$\sum_{n=1}^{\infty} e^{-2n\pi x} = \frac{e^{-2\pi x}}{1-e^{-2\pi x}}. \tag{4.51}$$

If we combine these two results and insert into equation (4.48), we get

$$G(x,\chi)_{x>0} = \frac{-i}{\sinh\chi}\left[e^{ix\chi}\frac{1}{1-e^{-2\pi x}} - e^{ix\chi}\frac{e^{-2\pi x}}{1-e^{-2\pi x}}\right]. \tag{4.52}$$

For $x<0$ Green's function is

$$G(x,\chi)_{x<0} = -\frac{1}{2\pi}2\pi i\sum_{n=1}^{\infty}\left.\frac{e^{i\alpha\chi}}{-\sinh\alpha}\right|_{\alpha=\chi+i\varepsilon+2\pi in} + \sum_{n=1}^{\infty}\left.\frac{e^{i\alpha\chi}}{-\sinh\alpha}\right|_{\alpha=-\chi-i\varepsilon-2\pi in} \tag{4.53}$$

$$= -i\left[\sum_{n=1}^{\infty} e^{ix\chi}\frac{e^{-2n\pi}}{-\sinh(\chi-2\pi in)} + \sum_{n=0}^{\infty} e^{-i\chi x}\frac{e^{-2n\pi x}}{\sinh(-\chi-2\pi in)}\right]. \tag{4.54}$$

using the equalities

$$\sinh(\chi-2\pi in) = \sinh\chi \qquad \sinh(-\chi-2\pi in) = -\sinh\chi \tag{4.55}$$

we can write

$$G(x,\chi)_{x<0} = \frac{i}{\sinh\chi}\left[\sum_{n=0}^{\infty} e^{i\chi x}e^{2n\pi} - \sum_{n=1}^{\infty} e^{-i\chi x}e^{2n\pi x}\right]. \tag{4.56}$$

And for the summation of the two geometric series in previous equation we can write respectively

$$\sum_{n=0}^{\infty} e^{2n\pi x} = \frac{1}{1-e^{2\pi x}} = \frac{e^{-2\pi x}}{e^{-2\pi x}-1}. \tag{4.57}$$

$$\sum_{n=1}^{\infty} e^{2n\pi x} = \frac{1}{1-e^{2\pi x}} = \frac{e^{2\pi x}}{1-e^{2\pi x}}. \tag{4.58}$$

And substituting these results into equation (4.56) we get

$$G(x,\chi)_{x<0} = \frac{-i}{\sinh\chi}\left[e^{ix\chi}\frac{1}{1-e^{-2\pi x}} - e^{-ix\chi}\frac{e^{-2\pi x}}{1-e^{-2\pi x}}\right]. \tag{4.59}$$

Then with the equations (4.52) and (4.56) relativistic Green's function can be defined as

$$G(x,\chi) = \frac{-2i}{\sinh\chi}\left[e^{ix\chi}\frac{1}{1-e^{-2\pi x}} - e^{-ix\chi}\frac{e^{-2\pi x}}{1-e^{-2\pi x}}\right]. \tag{4.60}$$

In the last equation corresponding to Green's function, the expressions $\frac{1}{1-e^{-2\pi x}}$ and $-\frac{e^{-2\pi x}}{1-e^{-2\pi x}}$ are called finite difference step operators that will be defined in the following chapter. These operators have different property from step functions which are used in non-relativistic formalism. This property is continuity for the following definition

$$\hat{\theta}(x) = \frac{1}{1-e^{-2\pi x}} \tag{4.61}$$

$$\hat{\theta}(-x) = -\frac{e^{-2\pi x}}{1-e^{-2\pi x}} = 1 - \frac{1}{1-e^{-2\pi x}}. \tag{4.62}$$

So with these equalities relativistic Green's function can be expressed by;

$$G(x,\chi) = \frac{-2i}{\sinh\chi}\left[e^{ix\chi}\hat{\theta}(x) - e^{-ix\chi}\hat{\theta}(-x)\right]. \tag{4.63}$$

4.3. Finite Difference Analogue of the Heavyside Step Function and Bernoulli Numbers

Now, we can analyze the step function and finite difference step function together. Let us start with the step function which is defined in equation (2.85). Derivative of this function defined as

$$\frac{d}{dx}\theta(x) = \delta(x). \tag{4.64}$$

From this relation we can find the integral representation of θ function. So let's define the θ function by using its Fourier image

$$\theta(x) = \int_{-\infty}^{\infty} g(q)e^{iq(x)}dq \tag{4.65}$$

and insert this relation into equation (4.64)

$$\frac{d}{dx}\theta(x) = \int_{-\infty}^{\infty} g(q).iqe^{iqx}dq = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iqx}dq \tag{4.66}$$

where the relation on the right hand side is the Fourier image of the δ function.

Then we find

$$g(q) = \frac{1}{2\pi i}\frac{1}{q}. \tag{4.67}$$

If we insert this relation into equation (4.65) then we get

$$\theta(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{iqx}}{q - i\epsilon} \tag{4.68}$$

which is the integral representation of θ function. And the derivative of the equation (4.79) is the δ function for $x' = 0$.

The integral equation (4.79) has a single pole at $q = i\epsilon$. And for $x > 0$ the conditions of the Jordan's lemma are fulfilled in the upper half plane. So we can apply the Cauchy's residue theorem. Then the θ function is determined as

$$\theta(x) = \frac{1}{2\pi i}2\pi i \lim_{\epsilon \to 0}[Res(i\epsilon)] = \lim_{\epsilon \to 0}\left.\frac{e^{i}qx}{1}\right|_{q=i\epsilon} = 1 \tag{4.69}$$

for $x < 0$ the contour in the lower half plane doesn't enclose the pole so integral is zero. So, as in the definition, we convinced that the θ function is a partial function. Now let's turn back to the relativistic case.

Finite difference step operator can be expressed as

$$\Delta\hat{\theta}(x) = \delta(x) \tag{4.70}$$

which is the analogue of the derivative operator in the non-relativistic form. Then if we introduce the integral representation of finite difference step operator as

$$\hat{\theta}(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\alpha x}}{e^{\alpha - i\epsilon} - 1}d\alpha \tag{4.71}$$

and insert this into equation (4.70), then together with equation (4.35) we can see that

$$\Delta\hat{\theta}(x) = \lim_{\epsilon \to 0}\left[i\frac{1}{2\pi i}\left(e^{-i\frac{d}{dx}} - 1\right)\int_{-\infty}^{\infty}\frac{e^{i\alpha x}}{e^{\alpha - i\epsilon} - 1}d\alpha\right]$$

$$= \lim_{\epsilon \to 0}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\alpha x}(e^{\alpha} - 1)}{e^{\alpha - i\epsilon} - 1}d\alpha\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\alpha x}d\alpha = \delta(x). \tag{4.72}$$

In the same way as in the usual analysis, $\hat{\theta}(x)$ is a constant at $r \neq 0$, i.e., a periodic function

$$\hat{\theta}(x) = \frac{1}{1 - e^{-2\pi x \cdot}}. \tag{4.73}$$

The singularity at $r = 0$ is connected with the basic property (4.70). The relation

$$\hat{\theta}(x) + \hat{\theta}(-x) = 1 \tag{4.74}$$

is also satisfied.

From equation (4.71) we can write

$$\lim_{x\to\infty} \hat{\theta}(x) = 1$$

$$\lim_{x\to-\infty} \hat{\theta}(x) = 0. \tag{4.75}$$

For the equation (4.71) we find the pole $\alpha = i\epsilon$ and the contour is the semicircle in the upper half plane. For $x > 0$

$$\hat{\theta}(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\alpha(x)}}{e^{\alpha - i\epsilon} - 1} d\alpha = \frac{1}{2\pi i} 2\pi i \lim_{\epsilon\to 0}[Res(i\epsilon)] = \lim_{\epsilon\to 0}\left(\left.\frac{e^{i\alpha x}}{e^{\alpha}}\right|_{\alpha=i\epsilon} \right) = 1 \tag{4.76}$$

for $x < 0$ in the lower half plane pole is outside the contour so integral is zero.

Now if we insert the equation (4.63) into equation (4.34) and substitute $x \to x - x'$ then we get the wave equation as

$$\psi(x) = e^{ix\chi} - \frac{2i}{\sinh\chi}\int_{-\infty}^{\infty} (e^{i(x-x')\chi}\hat{\theta}(x-x') + e^{-i(x-x')\chi}\hat{\theta}(x'-x))V(x')\psi(x')dx'. \tag{4.77}$$

We can also define the finite difference step function (4.71) in different way that is in terms of Bernoulli numbers.

If B_ν denotes the Bernoulli numbers then that can be defined by the generating function as;

$$t(e^t - 1)^{-1} = \sum_{\nu=o}^{\infty} B_\nu \frac{t^\nu}{\nu!}. \tag{4.78}$$

Now, we introduce expression of $\hat{\theta}(r - r')$ such as

$$\hat{\theta}(r - r') = \left[\theta(r - r') - \sum_{\nu=1}^{\infty} \frac{B_\nu}{\nu!}(i^\nu)\frac{d^{\nu-1}}{dr^{\nu-1}}\delta(r - r') \right] \tag{4.79}$$

If we substitute here representations for $\theta(r - r')$ and $\delta(r - r')$ as an analogue of one dimensional case,

$$\theta(r - r') = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{iz(r-r')}}{z - i\varepsilon} dz \tag{4.80}$$

$$\delta(r-r') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iz(r-r')}dz \tag{4.81}$$

then the right side of equation (4.79) becomes

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{iz(r-r')}\left[\frac{1}{z-i\varepsilon} + \sum_{\nu=1}^{\infty}\frac{B_\nu}{\nu!}(-1)^{\nu-1}z^{\nu-1}\right]dz. \tag{4.82}$$

From relation (4.78) we can write,

$$z(e^z-1)^{-1} - 1 + \frac{z}{2} = \sum_{\nu=2}^{\infty} B_\nu \frac{z^\nu}{\nu!} \tag{4.83}$$

where $B_0 = 1$ and $B_1 = 1/2$. Then if we arrange the terms as,

$$(e^z-1)^{-1} - \frac{1}{z} + \frac{1}{2} = \sum_{\nu=1}^{\infty} B_{\nu+1}\frac{z^\nu}{(\nu+1)!}. \tag{4.84}$$

with $\nu \to \nu - 1$ we get,

$$(e^z-1)^{-1} - \frac{1}{z} = \sum_{\nu=1}^{\infty}\frac{B_\nu}{\nu!}z^{\nu-1}. \tag{4.85}$$

Finally, substituting this relation into equation (4.82) we obtain

$$\hat{\theta}(r-r') = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{iz(r-r')}}{e^{z-i\varepsilon}}dz \tag{4.86}$$

which is the integral representation of finite difference step operator.

4.4. Scattering Theory

In this section section we collect the formulae related to the relativistic two-body scattering theory based on the differential difference Schrödinger equation .

We stress the remarkable fact that all essential formulae (scattering phase shifts from the asymptotics of the wave function, unitary condition-in integral and partial form, etc.) can be obtained in the framework of the relativistic Quantum Mechanics. Even relativistic analogues of the Jost functions using which the problem of study of analytic properties of the scattering matrix can be reduced to study these properties of the wave function, exists in the relativistic r space approach.

The differential cross section is expressed through the amplitude $A(\mathbf{p},\mathbf{q})$, in a non-relativistic form

$$\frac{d\sigma}{d\Omega} = |A(\mathbf{p},\mathbf{q})|^2. \tag{4.87}$$

The wave function is connected to the amplitude by

$$A(\mathbf{p},\mathbf{q}) = -\frac{1}{4\pi}\int \xi^*(\mathbf{p},\mathbf{r})\tilde{V}(r)\psi_{\mathbf{q}}(\mathbf{r})d\mathbf{r}. \tag{4.88}$$

Using the partial-wave expansion of the functions $\xi(\mathbf{p},\mathbf{r})$ and the wave function $\psi_{\mathbf{q}}(\mathbf{r})$,

$$\xi^*(\mathbf{p},\mathbf{r}) = \frac{1}{r\sinh\chi_p}\sum_{l=0}^{\infty}(2l+1)(-i)^l s_l^*(r,\chi_q)P_l\left(\frac{\mathbf{p}.\mathbf{r}}{pr}\right), \tag{4.89}$$

$$\psi_{\mathbf{q}}(\mathbf{r}) = \frac{1}{r\sinh\chi_q}\sum_{l=0}^{\infty}(2l+1)(i)^l\psi_{lq}(r)P_l\left(\frac{\mathbf{q}.\mathbf{r}}{qr}\right) \tag{4.90}$$

and the well-known equality for Legendre polynomials,

$$\int d\Omega_r P_l\left(\frac{\mathbf{p}.\mathbf{r}}{pr}\right)P_{l'}\left(\frac{\mathbf{q}.\mathbf{r}}{qr}\right) = \frac{4\pi}{2l+1}P_l\left(\frac{\mathbf{p}.\mathbf{q}}{pq}\right)\delta_{ll'}, \tag{4.91}$$

we obtain the decomposition of the amplitude

$$A(\mathbf{p},\mathbf{q}) = \frac{1}{\sinh\chi_p\sinh\chi_q}\sum_{l=0}^{\infty}(2l+1)A_l(p,q)P_l\left(\frac{\mathbf{p}.\mathbf{q}}{pq}\right) \tag{4.92}$$

where the partial amplitudes are defined by the formula

$$A_l(p,q) = -\int_0^{\infty} dr s_l^*(r,\chi_q)\tilde{V}(r)\psi_{lq}(r). \tag{4.93}$$

The wave function $\psi_{\mathbf{q}}^{+}(\mathbf{r})$ of continuous spectrum, describing the scattering on a potential $V(r)$ satisfies the equation

$$\psi_{\mathbf{q}}^{+}(\mathbf{r}) = \xi(\mathbf{q},\mathbf{r}) + \int d\mathbf{r}' G^{(+)}(\mathbf{r},\mathbf{r}';E_q)\tilde{V}(r')\psi_q^{(+)}(\mathbf{r}'), \tag{4.94}$$

where $G^{(+)}(\mathbf{r},\mathbf{r}';E_q)$ is connected with the partial wave Green's functions

$$G_l^{(+)}(r,r';E_q) = \frac{1}{\pi}\int_0^{\infty}\frac{s_l(r,\chi_k)s_l^*(r',\chi_k)d\chi_k}{\cosh\chi_q - \cosh\chi_k + i\epsilon} \tag{4.95}$$

by the decomposition

$$G^{(+)}(\mathbf{r},\mathbf{r}';E_q) = \frac{1}{4\pi rr'}\sum_{l=0}^{\infty}(2l+1)G_l^{(+)}(r,r';E_q)P_l\left(\frac{\mathbf{r}.\mathbf{r}'}{rr'}\right). \tag{4.96}$$

Combining equation (4.89), (4.90), (4.93), (4.94) and (4.96), we obtain the following expression for $\psi_{ql}^{(+)}(r)$:

$$\psi_{ql}^{(+)}(r) = s_l(r,q) - \frac{1}{\pi}\int_0^{\infty} d\chi_k\frac{s_l(r,\chi_k)A_l(k,q)}{\cosh\chi_q - \cosh\chi_k + i\epsilon} \tag{4.97}$$

The asymptotic form of equation (4.97) at $r \to \infty$ can be found using equation (3.64):

$$\psi_{ql}^{(+)as}(r) = \sin(r\chi - \frac{\pi l}{2}) - \frac{1}{\pi}\int_0^\infty d\chi_k \frac{\sin(r\chi_k - \frac{\pi l}{2})A_l(k,q)}{\cosh\chi_q - \cosh\chi_k + i\epsilon} \tag{4.98}$$

Then using the symbolic equality

$$\frac{1}{2\pi i}\frac{e^{ir(\chi_q-\chi_k)}}{\cosh\chi_q - \cosh\chi_k - i\epsilon} = \begin{cases} \delta(\cosh\chi_q - \cosh\chi_k) & \text{at} \quad r \to \infty \\ 0 & \text{at} \quad r \to -\infty, \end{cases} \tag{4.99}$$

one gets

$$\psi_{ql}^{(+)as}(r) = \sin(r\chi_q - \frac{\pi l}{2}) + e^{i(r\chi_q - \frac{\pi l}{2})}\frac{A_l(q,q)}{\sinh\chi_q}. \tag{4.100}$$

Introducing now, in analogy with the non-relativistic theory, the phase shifts δ_l

$$\psi_{ql}^{(+)as}(r) = a_l(q)s_l(r\chi - \frac{\pi l}{2} + \delta_l) \tag{4.101}$$

we have

$$a_l(q) = e^{i\delta_l}, \tag{4.102}$$

$$e^{2i\delta_l} = 1 + \frac{2iA_l(q,q)}{\sinh\chi_q} \equiv S_l(q). \tag{4.103}$$

Equality (4.103) is the definition of the scattering matrix $S_l(q)$. Using equations (4.102), (4.103) and (4.92), we obtain the expression for $\psi_{\mathbf{q}}^{(+)}(\mathbf{r})$ in the form

$$\psi_{\mathbf{q}}^{(+)as}(\mathbf{r}) = \xi^{as}(\mathbf{q},\mathbf{r}) + \frac{e^{ir\chi_q}}{r}A(\mathbf{q},\mathbf{p})\Big|_{E_p=E_q} \tag{4.104}$$

where

$$\begin{aligned} A(\mathbf{p},\mathbf{q})\Big|_{E_p=E_q} &= \frac{1}{\sinh^2\chi_q}\sum_{l=0}^{\infty}(2l+1)A_l(q,q)P_l\left(\frac{\mathbf{p}.\mathbf{q}}{pq}\right) \\ &= \frac{1}{2i\sinh\chi_q}\sum_{l=0}^{\infty}(2l+1)[S_l(q)-1]P_l\left(\frac{\mathbf{p}.\mathbf{q}}{pq}\right). \end{aligned} \tag{4.105}$$

The asymptotic partial-wave expansion of the wave function in terms of the scattering matrix is

$$\psi_q^{(+)as}(\mathbf{r}) = \frac{1}{2ir\sinh\chi_q}\sum_{l=0}^{\infty}(2l+1)[S_l(q)e^{ir\chi_q} + (-1)^{l+1}e^{-ir\chi_q}]P_l\left(\frac{\mathbf{r}.\mathbf{q}}{rq}\right). \tag{4.106}$$

To conclude this section, let us note that in the scheme under consideration the relation between the total cross section and phase shifts is valid in a non-relativistic form

$$\sigma = \frac{4\pi}{\sinh^2\chi_q}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l. \tag{4.107}$$

The optical theorem also holds:

$$ImA(\mathbf{q},\mathbf{q}) = \frac{\sinh\chi_q}{4\pi}\sigma. \tag{4.108}$$

4.5. The Jost Functions

Now we shall construct solutions, satisfying different boundary conditions. Let us consider, together with equation (4.95), the Green's function

$$G_l^{(-)}(r,r';E_q) = \frac{1}{\pi}\int_0^\infty \frac{s_l(r,\chi_k)s_l^*(r',\chi_k)d\chi_k}{\cosh\chi_q - \cosh\chi_k + i\epsilon}. \quad (4.109)$$

Let us also introduce the functions $G_l^{(0)}(r,r';E_q)$ and $G_l^{(J)}(r,r';E_q)$ with the help of the non-relativistic relations

$$G_l^{(0)}(r,r';E_q) = \frac{1}{2}[G_l^{(+)}(r,r';E_q) + G_l^{(-)}(r,r';E_q)] - \frac{s_l(r,\chi_q c_l(r',\chi_q)}{W[s_l(r',\chi_q c_l(r',\chi_q)]}, \quad (4.110)$$

$$G_l^{(J)}(r,r';E_q) = G_l^{(0)}(r,r';E_q) - \frac{e_l^1(r',\chi_q)e_l^2(r,\chi_q) - e_l^2(r',\chi_q)e_l^1(r,\chi_q)}{W[e_l^1(r',\chi_q),e_l^2(r',\chi_q)]} \quad (4.111)$$

The corresponding wave functions satisfy the equations

$$\psi_{ql}^{(\pm,0)}(r) = s_l(r,\chi_q) + \int_0^\infty G_l^{(\pm,0)}(r,r';E_q)\tilde{V}(r')\psi_{ql}^{(\pm,0)}(r')dr', \quad (4.112)$$

(solutions regular at the origin)

$$\psi_{ql}^{(1,2)}(r) = \pm\frac{1}{2i}e_l^{(1,2)}(r,\chi_q) + \int_0^\infty G_l^{(J)}(r,r';E_q)\tilde{V}(r')\psi_{ql}^{(1,2)}(r')dr', \quad (4.113)$$

(Jost solutions). Insofar as $G_l^{(J)}(r,r';E_q) \to 0$ when $r \to \infty$, the asymptotics of the solutions $\psi_{ql}^{(1,2)}(r)$ coincide with the asymptotic of the free partial spherical waves (3.66).

Let us introduce then the Jost functions $f_l^{(\pm)}(q)$ as the coefficient of the expansion of $\psi_{ql}^{(0)}(r)$ in terms of Just solutions $\psi_{ql}^{(0)}(r)$:

$$\psi_{ql}^{(0)}(r) = f_l^{(-)}(q)\psi_{ql}^{(1)}(r) + f_l^{(+)}(q)\psi_{ql}^{(0)}(r). \quad (4.114)$$

A simple calculations gives

$$f_l^{(\pm)}(r) = 1 - \int_0^\infty \frac{e_l^{(1,2)}\tilde{V}(r')\psi_{ql}^{(0)}(r')}{W[s_l(r',\chi_q)c_l(r',\chi_q)]}dr' \quad (4.115)$$

From here it follows that if one defines the partial S-matrix as

$$S_l(q) = \frac{f_l^{(-)}(q)}{f_l^{(+)}(q)}(-1)^{l+1}, \quad (4.116)$$

then in the case of a real potential the unitary condition holds:

$$S_l^*(q)S_l(q) = 1. \tag{4.117}$$

Moreover, passing in equation (4.114) to the asymtotic form of $\psi_{ql}^{(0)}(r)$ and $\psi_{ql}^{(1,2)}(r)$, it is possible to obtain $S_l(q)$ as a function of the phase shifts δ_l, which coincides with equation (4.103).

Just as in the case of non-relativistic potential scattering, it is possible to express the Jost function $f_l^{(\pm)}(q)$ in terms of Wronskian of the solutions $\psi_{ql}^{(0,1,2)}(r)$. Indeed, calculating $W[\psi_{ql}^{(0)}(r), \psi_{ql}^{(1)}(r)]$ and $W[\psi_{ql}^{(0)}(0), \psi_{ql}^{(2)}(r)]$ we have

$$f_l^{(+)}(q) = \frac{W[\psi_{ql}^{(0)}(r), \psi_{ql}^{(1)}(r)]}{W[\psi_{ql}^{(2)}(r), \psi_{ql}^{(1)}(r)]}, \tag{4.118}$$

$$f_l^{(-)}(q) = -\frac{W[\psi_{ql}^{(0)}(r), \psi_{ql}^{(2)}(r)]}{W[\psi_{ql}^{(2)}(r), \psi_{ql}^{(1)}(r)]}. \tag{4.119}$$

Formulae (4.118) and (4.119) allow us to reduce the problem of investigating the analytical properties of the wave functions.

CHAPTER 5

CONCLUSION

In this thesis, we describe the general formalism of the non-relativistic scattering theory as a brief review. Non-relativistic Schröndinger and Lippman-Schwinger equations are described and the expressions of these equations are investigated in momentum and configuration spaces, using Fourier transformation. The plane wave, which is generating function for the matrix elements of three dimensional Euclidean group in spherical basis, is expanded in terms of Legendre polynomials and spherical Bessel functions.

Then, to get the analogues of the relations in non-relativistic theory we study on the geometry of the space of the relativistic Lippman-Schwinger equation which is Lobachevsky space. We start with the well-known fact that the equation describing the relativistic relation between energy and momentum of the particle , describes at the same time the three-dimensional momentum space of constant negative curvature or the Lobachevsky space. The isometry group of this space is the Lorentz group. We find the matrix elements of the unitary irreducible representations of this group which are the eigen-functions of the Casimir operator, or the Laplace-Beltrami operator in the Lobachevsky space.

And we restricted our attention to the most important physical argument is that there exists in r-space the differential difference operator of free energy. On this basis, the quantum theory in the relativistic configurational space in the framework of the quasi-potential approach had been developed.

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APPENDIX A

BESSEL FUNCTIONS

A.1. Bessel Differential Equation

Bessel functions are solutions of the Bessel's differential equation which is given as

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2)w = 0 \tag{A.1}$$

where ν, z are unrestricted (apart from condition that, for the present ν is not an integer. For brevity the differential operator which occurs in (A.1) will be called ∇_ν so that

$$\nabla_\nu \equiv z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + z^2 - \nu^2. \tag{A.2}$$

The standard method of obtaining solutions of a linear differential equation in the neighborhood of a regular singularity (z=0)lead to the solution

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m(\frac{1}{2z})^{2m+\nu}}{[m!\Gamma(m+\nu+1)]} \tag{A.3}$$

and $J_{-\nu}(z)$.

The first solution $J_\nu(z)$ is called the Bessel function of first kind. The linear combinations

$$\begin{aligned} N_\nu(z) &= (\sin\nu\pi){-}1[J_\nu(z)\cos\nu\pi - J_{-\nu}(z)], \\ H_\nu^{(1)}(z) &= J_\nu(z) + iN_\nu(z) = (i\sin\nu\pi)^{-1}[J_{-\nu}(z) - J_\nu(z)e^{-i\nu\pi}], \\ H_\nu^{(2)}(z) &= J_\nu(z) - iN_\nu(z) = (i\sin\nu\pi)^{-1}[J_\nu(z)e^{i\nu\pi} - J_{-\nu}(z)] \end{aligned} \tag{A.4}$$

are likewise solution of (A.1). N_ν is called the Neumann's function, $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are called first and second kind Hankel functions respectively.

A.2. Spherical Bessel Functions

The Bessel functions reduced to combinations of elementary functions if and only if ν is half of an odd integer. When $l = 0, 1, 2...$ and $\nu = l + 1/2$ we can write;

$$\begin{aligned} j_l(z) &= (\frac{\pi}{2z})^{1/2} J_{l+\frac{1}{2}}(z), \\ n_l(z) &= (\frac{\pi}{2z})^{1/2} N_{l+\frac{1}{2}}(z), \\ h_l^{(1)}(z) &= j_l(z) + in_l(z) = (\frac{\pi}{2z})^{1/2} H_{l+\frac{1}{2}}^{(1)}(z) \\ h_l^{(2)}(z) &= j_l(z) - in_l(z) = (\frac{\pi}{2z})^{1/2} H_{l+\frac{1}{2}}^{(2)}(z) \end{aligned} \tag{A.5}$$

where $j_l(z)$, $n_l(z)$, $h_l^{(1,2)}(z)$ are called spherical Bessel, Neumann and Hankel functions respectively.

A.3. A Fundamental System of Solutions of Bessel's Equation

It is well known that, if y_1 and y_2 are two solutions of a linear differential equation of the second order, and if y_1' and y_2' denote their derivatives with respect to the independent variable, then the solution are linearly independent if the Wronskian determinant does not vanish identically; and if the Wronskian does vanish identically then either one of the two solutions vanishes identically, or else the ratio of the two solutions is a constant.

If the Wronskian does not vanish identically, then any solution of the differential equation is expressible in the form $c_1 y_1 + c_2 y_2$ where c_1, c_2 are constants depending on the particular solution under consideration; then solutions y_1 and y_2 are then said to form a fundamental system.

Now we proceed to evaluate

$$W[J\nu, J_{-\nu}]. \tag{A.6}$$

If we multiply the equations

$$\nabla_\nu J\nu(z) = 0, \quad \nabla_\nu J_{-\nu}(z) = 0 \tag{A.7}$$

by $J\nu(z)$ and $J_{-\nu}(z)$ respectively and subtract these results, we obtain an equation which may be written in the form

$$\frac{d}{dz}\{zW[J\nu, J_{-\nu}]\} = 0, \tag{A.8}$$

and hence, on integration

$$W[J\nu, J_{-\nu}] = \frac{C}{z}, \tag{A.9}$$

C is a determinate constant. To evaluate C, we observe that when ν is not an integer, and $|z|$ is small, we have

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\{1+0(z^2)\}J'\nu(z) = \frac{(\frac{1}{2}z)^{\nu-1}}{2\Gamma(\nu)}\{1+0(z^2)\} \tag{A.10}$$

with similar expressions for $J_{-\nu}(z)$ and $J'_{-\nu}(z)$ and hence

$$J_\nu(z)J'_{-\nu}(z) - J_{-\nu}(z)J'\nu(z) = \frac{1}{z}\left[\frac{1}{\Gamma(\nu+1)\Gamma(-\nu)} - \frac{1}{\Gamma(\nu)\Gamma(-\nu+1)}\right] \tag{A.11}$$

$$= -\frac{2\sin\nu\pi}{\pi z} + 0(z). \tag{A.12}$$

Since $\sin\nu\pi$ is not zero the functions $J_\nu(z)$ and $J_{-\nu}(z)$ form a fundamental system of solutions of equation (A.1).

A variety of Wronskian formulas can be developed:

$$\begin{aligned}
J_\nu(z)J_{-\nu+1}(z) + J_{-\nu}(z)J_{\nu-1}(z) &= \frac{2\sin\nu\pi}{\pi z}, \\
J_\nu(z)N'_\nu(z) + J'_\nu(z)N_\nu(z) &= \frac{2}{\pi z}, \\
J_\nu(z)N_{\nu+1}(z) + J_{\nu+1}(z)N_\nu(z) &= -\frac{2}{\pi z}, \\
H_\nu(z)^{(2)}H^{(1)}_{\nu+1}(z) + H^{(1)}_\nu(z)H^{(2)}_{\nu+1}(z) &= \frac{4}{i\pi z}, \\
J_{\nu-1}(z)H^{(1)}_\nu(z) - J_\nu(z)H^{(1)}_{\nu-1}(z) &= \frac{2}{i\pi z}, \\
J_{\nu-1}(z)H^{(2)}_{\nu-1}(z) - J_{\nu-1}(z)H^{(2)}_\nu(z) &= \frac{2}{i\pi z}.
\end{aligned} \tag{A.13}$$

APPENDIX B

LEGENDRE FUNCTIONS

B.1. Legendre Differential Equation

Associated Legendre differential equation is given by

$$(1-z^2)\frac{d^2}{dz^2}w - 2z\frac{dw}{dz}w + [\nu(\nu+1) - \mu^2(1-z^2)^{-1}]w = 0. \tag{B.1}$$

where z, ν and μ are unrestricted. Solutions of this equation

$$w = P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}(z^2-1)^{\frac{-1}{2}\mu}F(1-\mu+\nu, -\mu-\nu; 1-\mu; \frac{1}{2}-\frac{1}{2}z), \quad |1-z|<2 \tag{B.2}$$

$$\begin{aligned} w = Q_\nu^\mu(z) &= e^{\mu i\pi}2^{-\nu-1}\pi^{1/2}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)}z^{-1-\nu+\mu}(z^2-1)^{\frac{-1}{2}\mu} \\ &\times F(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu, 1+\frac{1}{2}\nu-\frac{1}{2}\mu; \nu+\frac{3}{2}; z^{-2}) \end{aligned} \tag{B.3}$$

are called associated Legendre functions or associated Legendre functions of first and second kind respectively.

Associated Legendre's differential equation remains unchanged if μ is replaced by $-\mu$ and z by $-z$, and ν by $-\nu-1$. Therefore $P_\nu^{\pm\mu}(\pm z)$, $Q_\nu^{\pm\mu}(\pm z)$, $P_{-\nu-1}^{\pm\mu}(\pm z)$, $Q_{-\nu-1}^{\pm\mu}(\pm z)$ are solutions of equation (B.1).

For the case $\mu = 0$ and ν is an integer we have Legendre's differential equation, which is

$$(1-z^2)\frac{d^2s}{dz^2} - 2z\frac{ds}{dz} + \nu(\nu+1)s = 0. \tag{B.4}$$

Solutions of this equation are $P_\nu(z)$ and $Q_\nu(z)$ which are called Legendre functions. From equation (B.2) and (B.3) we have,

$$s = P_\nu(z) = F(1+\nu, -\nu; 1; \frac{1}{2}-\frac{1}{2}z). \tag{B.5}$$

$$s = Q_\nu(z) = 2^{-\nu-1}\pi^{1/2}\frac{\Gamma(\nu+1)}{\Gamma(\nu+3/2)}z^{-1-\nu} \times F(\frac{1}{2}+\frac{1}{2}\nu, 1+\frac{1}{2}\nu; \nu+\frac{3}{2}; z^{-2}). \tag{B.6}$$

If we differentiate (B.5) m times with respect to z (m=1,2,...)it follows that

$$P_\nu^m(z) = (z^2-1)^{\frac{1}{2}m}\frac{d^m P_\nu(z)}{dz^m} \tag{B.7}$$

and also

$$Q_\nu^m(z) = (z^2-1)^{\frac{1}{2}m}\frac{d^m Q_\nu(z)}{dz^m}. \tag{B.8}$$

B.2. Rodrigues' Formula

In particular case; $\mu = 0$ and ν is an integer, $P_\nu^\mu(z)$ becomes

$$P_{2n}(z) = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2} F(-n, n+1/2; 1/2; z^2)$$

$$P_{2n+1}(z) = \frac{(-1)^n(2n+1)!}{2^{2n}(n!)^2} z.F(-n, n+3/2; 3/2; z^2) \tag{B.9}$$

or together which can be written as

$$P_n(z) = \frac{(2n)!}{2^{2n}(n!)^2} \times \left[z^n - \frac{n(n-1)}{2(2n-1)} z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} z^{n-4} + ... \right]. \tag{B.10}$$

It can be written also in compact form

$$P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} (z^2 - 1)^n. \tag{B.11}$$

which is Rodrigues' formula.

APPENDIX C

LORENTZ GROUP

An "event" is something that happens at a definite time at a definite place. Events in space-time are described with respect to an "inertial frame" by coordinates[1] $x = (x^0, x)^{\mathbf{T}} \in \mathbb{R}^4$, where $x = (x^1, x^2, x^3)^{\mathbf{T}}$ are the space coordinates, and $x^0 = ct$ is the time coordinate of event. The factor c denotes the velocity of light. It gives x^0 the dimension of a length. The principals of relativity states that all inertial frames are equivalent for description of nature. The coordinate transformations $I \to I'$ between all possible inertial frames are called $Poincaré$ transformations.

In the vector space $\mathbb{R}^4$ we define the "$Lorentz\ metric$"

$$< y, x >= y^0x^0 - y^1x^1 - y^2x^2 - y^3x^3, \qquad x, y \in \mathbb{R}^4 \tag{C.1}$$

The bilinear form $< .,. >$ is symmetric and nondegenerate[2], but not positive definite. With 4×4 matrix

$$\mathrm{g} = (g_{\mu\nu}) = (g^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{C.2}$$

we can write (C.1) in the form

$$< y, x >= y^{\mathbf{T}}\mathbf{g}\, x = \sum_{\mu,\nu=0}^{4} g_{\mu\nu} y^\mu x^\nu \tag{C.3}$$

The vector space $\mathbb{R}^4$ endowed with the Lorentz metric is called "$Minkowsky$" "$space$".

Definition C.1 *A (homogeneous) Lorentz transformation of* $\mathbb{R}^4$ *is a linear map* $\Lambda : \mathbb{R}^4 \to \mathbb{R}^4$ *with*

$$< \Lambda y, \Lambda x >=< y, x >, \qquad \forall x, y \in \mathbb{R}^4. \tag{C.4}$$

The elements of the matrix of Λ *are denoted by* Λ^μ_ν*.*

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[1] "T" denotes the transposed of a vector or a matrix. We would like to consider x a column vector.

[2] i.e., for all $y \neq 0$ there exists an x such that $< y, x > \neq 0$

Eq. (C.4) is equivalent to

$$\Lambda^{\mathbf{T}} \mathrm{g}\, \Lambda = \mathrm{g} \qquad \text{or} \qquad \Lambda^{\mu}_{\rho} g_{\mu\nu} \Lambda^{\nu}_{\tau} = g_{\rho\tau}, \tag{C.5}$$

which consists of ten independent quadratic equations for the components of Λ, e.g.,

$$(\Lambda^0_0)^2 - (\Lambda^1_0)^2 - (\Lambda^2_0)^2 - (\Lambda^3_0)^2 = 1. \tag{C.6}$$

The composition of two Lorentz transformation Λ is again a Lorentz transformation. We find

$$1 = -\det \mathrm{g} = -\det\Lambda^{\mathbf{T}} \mathrm{g}\, \Lambda = -\det\Lambda^{\mathbf{T}} \det \mathrm{g}\, \det\Lambda = (\det\Lambda)^2 \tag{C.7}$$

and hence det $\Lambda = \pm 1$ for any Lorentz transformation Λ. Therefore Λ is invertible and set of all Lorentz transformations forms a group which is called "$Lorentz\ group$".

APPENDIX D

GROUP REPRESENTATIONS

Definition D.1 An m-dimensional matrix group $D(G)$ is called a $representation$ of the given group G, if G is homomorphic to $D(G)$. The element $D(R)$ in $D(G)$, which is nonsingular, is called the representation matrix of the group element $R \in G$ in the representation $D(G)$.

Definition D.2 If all matrices $D(R)$ in $D(G)$ are unitary, $D(G)$ is called $unitary$ $representation$.

Definition D.3 Two representations $D(G)$ and $\bar{D}(G)$ with the same dimension are called equivalent to each other if there is a similarity transformation X relating two representation matrices for each element R in the group G, $\bar{D}(R)$ = $X^{-1}D(R)X$.

The concept of equivalence of representations is reflexive, symmetric and transitive: every representation is equivalent to itself; if a representation $D(G)$ is equivalent to a representation $\bar{D}(G)$, then $\bar{D}(G)$ is equivalent to $D(G)$; if $D(G)$ is equivalent to $\bar{D}(G)$ and $\bar{D}(G)$ is equivalent to $\hat{D}(G)$, then $D(G)$ is equivalent to $\hat{D}(G)$. Therefore the set off all representations of the group G can be divided into equivalence classes of representations. Henceforth we shall not distinguish between equivalent representations, i.e. we shall study the properties of equivalence classes of representations.

Definition D.4 A representation $D(G)$ of the group G is said to be $reducible$ if the representation matrix $D(R)$ of any element$R$ of G in $D(G)$ can be transformed into the same form of the echelon matrix by a common similarity transformation X,

$$X^{-1}D(R)X = \begin{pmatrix} D^{(1)}(R) & T(R) \\ 0 & D^{(2)}R \end{pmatrix}. \tag{D.1}$$

Otherwise, it is called an irreducible representation.

The necessary and sufficient condition for a reducible representation is that there is a nontrivial invariant subspace with respect to $D(G)$ in its representation space. That is, if a representation possesses only trivial invariant subspaces it is called $irreducible$. A representation with nontrivial invariant subspaces is $reducible$.