ON PSEUDO SEMISIMPLE RINGS

A Thesis Submitted to
the Graduate School of Engineering and Sciences of
İzmir Institute of Technology
in Partial Fulfillment of the Requirements for the Degree of

MASTER OF SCIENCE

in Mathematics

by
Hatice Mutlu

June 2013
İZMİR

We approve the thesis of **Hatice Mutlu**

Examining Committee Members:

Assoc. Prof. Dr. Engin BÜYÜKAŞIK
Department of Mathematics,
İzmir Institute of Technology

Assoc. Prof. Dr. Engin MERMUT
Department of Mathematics,
Dokuz Eylül University

Assist. Prof. Dr. Başak AY
Department of Mathematics,
İzmir Institute of Technology

24 June 2013

Assoc. Prof. Engin BÜYÜKAŞIK
Supervisor, Department of
İzmir Institute of Technology

Prof. Dr. Oğuz YILMAZ
Head of the Department of
Mathematics

Prof. Dr. R. Tuğrul SENGER
Dean of the Graduate School of
Engineering and Sciences

ACKNOWLEDGMENTS

I would like to express my endless gratitude to my supervisor Assoc. Prof. Engin Büyükaşık for helping me in so many ways, for his patience and guidance throughout the preparation of this dissertation.

I am deeply grateful to Professor Saad Mohamed for his help and support during his visit to Izmir Institute of Technology.

Finally, I would like to thank my parents for their love and support.

ABSTRACT

ON PSEUDO SEMISIMPLE RINGS

In this thesis, we give a survey of right pseudo semisimple rings and prove some new results about these rings. Namely, we prove that a right pseudo semisimple ring is an internal exchange ring and a right pseudo semisimple ring is an SSP ring. We also give a complete characterization of right and left pseudo semisimple rings.

ÖZET

SÖZDE YARIBASİT HALKALAR ÜZERİNE

Bu tezde, sağ sözde yarıbasit halkarın incelemesi yapıldı ve bu halkalarla ilgili yeni sonuçlar ispatlandı. Şöyle ki bir sağ sözde yarıbasit halkanın iç değişim halka ve SSP halka olduğu ispatlandı. Ayrıca sağ ve sol yarıbasit halkaların tam karakterizasyonu verildi.

TABLE OF CONTENTS

LIST OF ABBREVIATIONS

R	an associative ring with unit unless otherwise stated
M_R	a unitary right R- module
$\oplus_{i\in I}M_i$	direct sum of R- modules M_i
$\mathbb{Z}$	the ring of integers
$\mathbb{Z}_{\ltimes}$	the cyclic group $\mathbb{Z}/\ltimes\mathbb{Z}$
${}^0X$	left annihilator of the set X
X^0	right annihilator of the set X
S	right socle of the ring R
S'	left socle of the ring R
J	Jacobson radical of the ring R
Z	right singular ideal
Z'	left singular ideal
$\subseteq$	submodule
$\subset$	proper submodule
$<$	proper ideal
$\leq$	ideal
$\ll$	small (or superfluous) submodule
$\leq^{\oplus}$	direct summand
$ker f$	the kernel of the map f
$im f$	the image of the map f
$End(M)$	the endomorphism ring of a module M

CHAPTER 1

INTRODUCTION

Throughout this thesis, the rings that we consider are associative with an identity element. A ring R is called a right pseudo semisimple if any right ideal of R is either semisimple or isomorphic to R. Trivial examples of these rings are principal right ideal domains and semisimple rings. Pseudo semisimple rings are investigated and studied by S. H. Mohamed and B. Muller in a series of papers (see, (Mohamed & Muller, 1982), (Mohamed & Muller, 1991), (Mohamed & Muller, 1990), , (Mohamed, 2010)). Besides proving some properties of these rings, they also characterized the structure of right pseudo semisimple rings under some particular conditions. The complete structure of right pseudo semisimple rings is still not known.

In this thesis, we investigate some further properties of pseudo semisimple rings. Also we characterize the right and left pseudo semisimple rings.

In chapter 2 we give some known results related with our work and used in the sequel. For the results in this chapter we refer to (Anderson & Fuller, 1992), (Bland, 2010), (Kasch, 1982), (Alizade & Pancar, 1999), (Goodearl, 1979), (Lambek, 1966), (Lam, 1991), (Lam, 1999), (Wisbauer, 1991), and (Tuganbaev, 2002).

In Chapter 3 we give a survey of some results on the structure of right pseudo semisimple rings from (Mohamed, 2010) and (Mohamed & Muller, 1991). In the case $S^2 = 0$, they proved that R is right pseudo semisimple if and only if R/S is a principal right ideal domain and S is torsion-free as a left R/S module. In the case S is maximal, R is right and left pseudo semisimple if and only if R is semiprime and has enough shifts. Also in the case $S^2 = 0$ and $S \neq 0$, R is a right and left pseudo semisimple ring if and only if R is a local ring with radical square 0. They also give an example in order to show that right pseudo semisimple rings are not left pseudo semisimple, in general.

In Chapter 4 we prove that a right pseudo semisimple ring is an internal exchange ring, and a right and left pseudo semisimple ring is an SSP ring. We obtain that if R is a right and left pseudo semisimple ring, then either S is maximal or $J = 0$. In both cases we have proved some equivalent conditions for a right pseudo semisimple ring to be left pseudo semisimple. As a consequence, a complete structure of right and left pseudo semisimple rings is obtained.

CHAPTER 2

PRELIMINARIES

In this chapter, we give some fundamental properties of rings and modules that will be used later.

2.1. The Radical and Socle

For an element $x \in R$ and a right ideal L of R, the set $\{r \in R : xr \in L\}$ will be denoted by $(L : x)$.

Definition 2.1 *A division ring is a ring whose non-zero elements are invertible.*

Definition 2.2 *An R- module M is simple if $M \neq 0$ and it has no non-trivial submodules.*

Proposition 2.1 *((Anderson & Fuller, 1992), Proposition 5.5) Let $M = K \oplus K'$, let p_K be the projection of M on K along K', and let L be a submodule of M. Then*

$$M = L \oplus K'$$

if and only if

$$(p_K|L) : L \to K$$

is an isomorphism.

Let M be an R- module. Then for each subset X of M, *the (left) annihilator of X in R* is

$${}^{0}X = \{r \in R \mid rx = 0 \ for \ all \ x \in X\},$$

and *the (right) annihilator of X in R* is

$$X^{0} = \{r \in R \mid xr = 0 \ for \ all \ x \in X)\}.$$

Proposition 2.2 *((Anderson & Fuller, 1992), Theorem 2.14) Let M be a R- module and X be a subset of M. Then ${}^{0}X$ is a right ideal of R. Moreover, if X is a submodule of M, then ${}^{0}X$ is an ideal of R.*

Proof Let $x, y \in {}^{0}X$ and $r \in R$. Then for each $a \in X$, we have

$$(rx - y)a = (rx)a - ya = r(xa) - ya = 0.$$

Thus $rx - y \in {}^0X$ and ${}^0X$ is a right ideal of R. Assume that X is a submodule of M. Then

$$a(xr - y) = a(xr) - ya = (ax)r - ya = 0.$$

It means that ${}^0X$ is a left ideal of R. $\square$

Proposition 2.3 *((Anderson & Fuller, 1992), Theorem 9.6.) A right R-module T is simple if and only if $T \cong R/I$ for some maximal right ideal I of R.*

Definition 2.3 *A right (left) ideal A of R is said to be a $minimal\ right\ (left)\ ideal$ if 0 and A are the only right (left) ideals of R that contained in A.*

Lemma 2.1 *Let R be a ring and a be a non-zero element of R. The right aR is a minimal right ideal of R if and only if $aR = abR$ for any $b \in R$ such that $ab \neq 0$.*

Definition 2.4 *An element e of a ring R is called idempotent if $e^2 = e$. An idempotent e of R is* central idempotent *in case it is in the center of R. A pair of idempotents e_1 and e_2 in a ring R is said to be* orthogonal *if $e_1e_2 = 0 = e_2e_1$. An idempotent $e \in R$ is said to be* indecomposable *if it is nonzero and it is not the sum of two nonzero orthogonal idempotents of R.*

Lemma 2.2 *The ideal I of R is a direct summand of R if and only if $I = eR$ for some $e^2 = e \in R$.*

Proof Assume that $I = eR$ for some $e = e^2 \in R$. Since $x = ex + (1 - e)x$ for all $x \in R$, $R_R = eR + (1 - e)R$. If $ex = (1 - e)y$ for some $x, y \in R$, then $ex = e^2x = e((1 - e)y) = 0$. Thus $R = eR \oplus (1 - e)R$. If I is a direct summand of R, $R = I \oplus J$ for some $J \subseteq R$. Then $1 = e + f$ for some $e \in I$ and $f \in J$.

$$e^2 = (1 - f)^2 = 1 - f - f + f^2 = e - f(1 - f) = e - fe = e.$$

Also if $x \in I$, then

$$x = (e + f)x = ex + fx = ex \in eR.$$

Thus $I = eR$ for some $e = e^2 \in R$. $\square$

The idempotents in a ring R represent idempotents in every factor ring of R. However, idempotent cosets in a factor ring of R need not have idempotent representatives in R. For example, $\mathbb{Z}$ has two idempotents, while $\mathbb{Z}_6$ has four.

Definition 2.5 *Let I be an ideal of a ring R and let $g + I$ be an idempotent element of R/I. We say that this idempotent can be* lifted modulo I *in case there is an idempotent $e \in R$ such that $g + I = e + I$. We say that* idempotents lift modulo I *in case every idempotent in R/I can be lifted to an idempotent in R.*

Proposition 2.4 *((Anderson & Fuller, 1992), Theorem 27.1) If I is a nil ideal in a ring R, then idempotents lift modulo I.*

Proposition 2.5 *((Bland, 2010), Proposition 6.3.5) If A is a minimal right ideal of a ring R, then either $A^2 = 0$ or $A = eR$ for some idempotent e of R.*

Proof Let A be a minimal right ideal of R, and assume that $A^2 \neq 0$. Then there exists $a \in A$ such that $aA \neq 0$. Since aA is a nonzero right ideal of R contained in A, we must have $A = aA$. Let $e \in A$ be such that $a = ae$. If $B = a^0$, then B is a right ideal of R and $A \cap B \neq A$. Hence, $A \cap B = 0$. But $ae = ae^2$, so $a(e - e^2) = 0$. Therefore, $e - e^2 \in A \cap B$, so $e = e^2$. Hence, e is an idempotent of R, and $e \neq 0$ since $a \neq 0$. Thus, $0 \neq eR \subseteq A$ gives $eR = A$. $\square$

Proposition 2.6 *If R is a domain, then R does not contain any nonzero proper minimal right ideal.*

Corollary 2.1 *((Lambek, 1966), Corollary) If $e^2 = e \in R$ and $f^2 = f \in R$, then $eR \cong fR$ if and only if $Re \cong Rf$.*

Definition 2.6 *A submodule N of an R-module M is said to be an* essential (or a large) *submodule of M, written $N \leq_{ess} M$, if $N \cap N' \neq 0$ for each nonzero submodule N' of M. If N is an essential submodule of M, then M is referred to as an* essential extension *of N. It is easily seen that for a submodule N of M we have $N \ll_{ess} M$ if and only if for every $0 \neq m \in M$, there is an element $r \in R$ such that $rm \neq 0$ and $rm \in N$.*

Proposition 2.7 *((Anderson & Fuller, 1992), Proposition 5.16) Let M be an R- module with submodules $K \leq N \leq M$ and $H \leq M$. Then*

(1) *$K \ll_{ess} M$ if and only if $K \ll_{ess} N$ and $N \ll_{ess} M$.*

(2) *$H \cap K \ll_{ess} M$ if and only if $H \cap M$ and $K \ll_{ess} M$.*

Definition 2.7 *If N is a submodule of an R-module M, then a submodule C of M such that $N \oplus C$ is essential in M is said to be a* complement *of N in M.*

Proposition 2.8 *((Alizade & Pancar, 1999), Proposition 9.8) Every submodule of a module M has a complement in M.*

Definition 2.8 *A submodule K of an R-module M is called* superfluous *or* small *in M, written $K \ll M$, if, for every submodule $L \subseteq M$, the equality $K + L = M$ implies $L = M$.*

Lemma 2.3 *If N is an ideal of a ring R such that $N^0 \leq N$, then N is essential as a left ideal.*

Proof Let A be a left ideal such that $A \cap N = 0$. Since $NA \leq A \cap N = 0$, we obtain $NA = 0$. Then we have $A \leq N^0 \leq N$. Thus $0 = A \cap N = A$. □

Let M be a left R-module. The *radical* of M is defined by

$$Rad(M) = \bigcap\{K \subseteq M \mid K \ is \ a \ maximal \ submodule \ in \ M\}$$

$$= \sum\{L \subseteq M \mid L \ is \ a \ small \ submodule \ in \ M\}$$

and the *socle* of M is defined by

$$Soc(M) = \sum\{K \subseteq M \mid K \ is \ a \ minimal \ submodule \ in \ M\}$$

$$= \bigcap\{L \subseteq M \mid L \ is \ an \ essential \ submodule \ in \ M\}.$$

The right socle of a ring is $S = Soc(R_R)$ and left socle is $S' = Soc(_RR)$, and they are ideals of R. They need not to be equal for example; if R is the ring of 2×2 upper triangular matrices over a field, then $S \neq S'$.

Lemma 2.4 *((Lam, 1991), Lemma 4.1) For $y \in R$, the following statements are equivalent:*

(1) *$y \in J(R)$;*

(2) *$1 - yx$ is right invertible for any $x \in R$;*

(3) *$My = 0$ for any simple right R-module M.*

Proof

(1)⇒ (2) Assume $y \in J(R)$. If, for some x, $1 - yx$ is not right invertible, then $(1 - yx)R \subsetneqq R$ is contained in a maximal right ideal M of R. But $1 - yx \in M$ and $y \in M$ imply that $1 \in M$, a contradiction.

(2)⇒ (3) Assume $my \neq 0$ for some $m \in M$. Then since M is simple, we must have $myR = M$. In particular, $m = myx$ for some $x \in R$, so $m(1 - yx) = 0$. Using (2), we get, $m = 0$ a contradiction.

(3)⇒ (1) For any maximal right ideal M of R, R/M is a simple right R-module, so by (3), $(R/M)y = 0$ which implies that $y \in M$. By definition, we have $y \in J(R)$.

□

Corollary 2.2 *((Anderson & Fuller, 1992), Corollary 15.4) If R is a ring, then*

$$Rad({}_RR) = Rad(R_R).$$

The Jacobson radical of a ring is $J(R) = Rad(R_R)$ and it is an ideal.

Corollary 2.3 *((Anderson & Fuller, 1992), Corollary 15.5) If R is a ring, then $J(R)$ is the annihilator in R of the class of simple right (left) R-modules.*

Corollary 2.4 *((Anderson & Fuller, 1992), Corollary 15.6) If I is an ideal of a ring R, and if $J(R/I) = 0$, then $J(R) \subseteq I$.*

Proof If $J(R/I) = 0$, then the intersection of the maximal right ideals of R containing I is exactly R. It follows that $J(R)$, the intersection of the maximal right ideals of R , is contained in I. □

Corollary 2.5 *((Anderson & Fuller, 1992), Corollary 15.8) If R and R' are rings and if $\phi : R \to R'$ is a surjective ring homomorphism, then $\phi(J(R)) \subseteq J(R')$. Moreover, if $ker\phi \subseteq J(R)$, then $\phi(J(R)) = J(R')$. In particular, $J(R/J(R)) = 0$.*

Corollary 2.6 *((Anderson & Fuller, 1992), Corollary 15.11) If R is a ring, then $J(R)$ contains no non-zero idempotent.*

Proof If $e \in R$ is idempotent and if $e \in J(R)$, then eR is a small direct summand of R_R. Thus $e = 0$. □

Corollary 2.7 *((Lam, 1991), Lemma 11.4) If a right (left) ideal $U \leq R$ is nil, then $U \leq J(R)$.*

Proof Let $y \in U$. Then for any $x \in R$, $yx \in U$ is nilpotent. It follows that $1 - yx$ has an inverse given by $\Sigma_{i=0}^{\infty}(xy)^i$. Therefore, by Lemma 2.4, we have $y \in J(R)$. □

2.2. Polynomial Rings

Theorem 2.1 *Let $f(x)$, $g(x)$ be nonzero polynomials over a ring R. If $f(x)g(x) \neq 0$, then*

$$\deg[f(x)g(x)] \leq \deg[f(x)] + \deg[g(x)].$$

If R has no zero divisors, then we always have

$$\deg[f(x)g(x)] = \deg[f(x)] + \deg[g(x)].$$

In particular, this applies if R is an integral domain.

Proof Write $f(x) = a_0 + ... + a_n x^n$ and $g(x) = b_0 + ... + b_m x^m$, with $a_n, b_m \neq 0$. Then

$$f(x)g(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + ... + a_n b_m x^{n+m}.$$

Thus the largest power of x that can occur is x^{n+m}, so $\deg[f(x)g(x)] \leq \deg[f(x)] + \deg[g(x)]$.
If R has no zero divisors, then $a_n b_m \neq 0$ and $n+m$ is the degree, giving us $\deg[f(x)g(x)] = \deg[f(x)] + \deg[g(x)]$. □

Corollary 2.8 *If R is an integral domain, so is $R[x]$.*

Theorem 2.2 *Let R be an integral domain. The polynomial $f(x) \in R[x]$ is a unit if and only if $f(x)$ is a constant polynomial that is a unit in R.*

Proof Clearly such polynomials are units in $R[x]$. On the other hand, suppose that $f(x)g(x) = 1$. Then $\deg[f] + \deg[g] = \deg[1] = 0$, so $\deg[f] = \deg[g] = 0$. Hence they are in R, and units in R because their product is 1. □

Corollary 2.9 *If F is a field, then $f(x) \in F[x]$ is a unit if and only if it is a nonzero constant polynomial.*

Corollary 2.10 *The Jacobson radical of $R[x]$ is zero when R is a domain.*

Proof Let $f \in J(R[x])$. Then $1 + xf$ is a unit by Lemma 2.4, and so $1 + xf$ must be constant by Theorem 2.2. Thus $f = 0$. □

2.3. The Singular Ideal

Let M be an R- module. Consider the following set:

$$Z(M_R) = \{x \in M \,|\, xI = 0 \ for \ some \ I \ll_{ess} R_R\} = \{x \in M \,|\, x^0 \ll_{ess} R_R\}.$$

Lemma 2.5 *$Z(M)$ is a submodule of M.*

Proof Since R is an essential right ideal over itself, we get $0 \in Z(M)$. Given any $x,\, y \in Z(M)$, there are essential right ideals $I,\, J$ in R such that $xI = yJ = 0$. By Proposition 2.7 $I \cap J$ is an essential right ideal of R and so $x \mp y \in Z(M)$. Now for any $t \in R$ and $x \in Z(M)$, we will show that $xt \in Z(M)$. Consider the right ideal $K = \{r \in R \,|\, tr \in I\}$. It is essential by Lemma **??**, and we have $xtK \leq xI = 0$, whence $xt \in Z(M)$. Thus $Z(M)$ is a submodule of M. ☐

Proposition 2.9 *If $f : M \to N$ is a homomorphism of right R- modules, then $f(Z(M)) \subseteq Z(N)$.*

Proof Let $x \in Z(M)$. Then there exists an essential right ideal of R such that $Ix = 0$, that is for every $a \in I$, $xa = 0$. So $f(xa) = f(x)a = 0$ for every $a \in I$, that is $I \subseteq f(x)^0$. Since $I \ll_{ess} R$, by Proposition 2.7, $f(x)^0 \ll_{ess} R$, so we get $f(x) \in Z(N)$. ☐

Corollary 2.11 *If N is a submodule of a module M, then $Z(N) \subseteq Z(M)$.*

Proof This is clear from Proposition 2.9. ☐

Lemma 2.6 *If N is a submodule of a module M, then $Z(M) \cap N = Z(N)$.*

Proof By Corollary 2.11, $Z(N) \subseteq Z(M)$ and $Z(N) = Z(N) \cap N \subseteq Z(M) \cap N$. Conversely, let $x \in Z(M) \cap N$. Then there is an essential right ideal I such that $xI = 0$, on the other hand, $x \in N$, so $x \in Z(N)$. ☐

Definition 2.9 *The right singular ideal of a ring R is the ideal $Z = Z(R_R)$, and the left singular ideal of R is the ideal $Z^l = Z({}_RR)$.*

Lemma 2.7 *If R is a domain, then $Z = 0$.*

Proof Let $x \in Z$. Then $xI = 0$ for some essential right ideal of R. But then as R is a domain and $I \neq 0$, $x = 0$. ☐

Corollary 2.12 *The following statements hold for a ring R.*

(1) *Z is an ideal of R.*

(2) *If $R \neq 0$, then $Z \neq R$.*

(3) *Z does not contain any nonzero idempotent.*

2.4. Semisimple Modules

Let $(T_\alpha)_{\alpha \in A}$ be an indexed set of simple submodules of M. If M is the direct sum of this set, then

$$M = \bigoplus_A T_\alpha$$

is a semisimple decomposition of M. A module M is said to be *semisimple* in case it has a semisimple decomposition.

Theorem 2.3 *((Anderson & Fuller, 1992), Theorem 9.6) For a right R-module M the following statements are equivalent;*

(1) *M is semisimple;*

(2) *M is generated by simple modules;*

(3) *M is the sum of some set of simple submodules;*

(4) *M is the sum of its simple submodules;*

(5) *Every submodule of M is a direct summand;*

(6) *Every short exact sequence*

$$0 \to K \to M \to N \to 0$$

of right R-modules splits.

Corollary 2.13 *((Kasch, 1982), Corollary 8.1.5) For a right R- module M, the following hold.*

(1) *Every submodule of a semisimple module M is semisimple.*

(2) *Every epimorphic image of a semisimple module M is semisimple.*

Corollary 2.14 *((Kasch, 1982), Corollary 8.2.2) For a ring R, the following are equivalent;*

(1) *R is semisimple;*

(2) *Every right and left R- module is semisimple.*

2.5. Local, Regular and Semiprime rings

A ring R is a *local* ring in case the set of non-invertible elements of R is closed under addition.

Proposition 2.10 *((Anderson & Fuller, 1992), Theorem 15.15) For a ring R, the following statements are equivalent;*

(1) *R is a local ring;*

(2) *R has a unique maximal left ideal;*

(3) *$J(R)$ is a maximal left ideal;*

(4) *The set of elements of R without left inverses is closed under addition;*

(5) $J(R) = \{x \in R \,|\, Rx \neq R\}$;

(6) *$R/J(R)$ is a division ring;*

(7) $J(R) = \{x \in R \,|\, x\, is\, not\, invertible\}$;

(8) *If $x \in R$, then either x or $1 - x$ is invertible.*

Lemma 2.8 *If R is a local ring with $J^2 = 0$, then $S = J$.*

Proof We know by Corollary 2.3 that $J(R)$ annihilates every simple R- module. Thus, $S \leq {}^0J$. But ${}^0JJ = 0$. Therefore, ${}^0J$ is an R/J- module. So by Corollary 2.14 ${}^0J$ is semisimple and ${}^0J \leq S$. Hence we have $S =^0 J$. If $J^2 = 0$, then $J \leq S$. On the other hand, since R is local, $S \leq J$. Thus $S = J$. □

A ring R is called *regular* provided that for every $x \in R$ there exists $y \in R$ such that $xyx = x$.

Theorem 2.4 *((Goodearl, 1979), Theorem 1.1) For a ring R, the following conditions are equivalent;*

(1) *R is regular;*

(2) *Every principal right (left) ideal of R is generated by an idempotent;*

(3) *Every finitely generated right (left) ideal of R is generated by an idempotent.*

Proof

(1)⇒(2) Given $x \in R$, there exists $y \in R$ such that $xyx = x$. Then xy is an idempotent in R such that $xyR = xR$.

(2)⇒(3) It suffices to show that $xR + yR$ is principal for any $x, y \in R$. Now, $xR = eR$ for some idempotent $e \in R$, and since $y - ey \in xR + yR$, we see that $xR + yR = eR + (y - ey)R$. There is an idempotent $f \in R$ such that $fR = (y - ey)R$, and we note that $ef = 0$. Consequently, $g = f - fe$ is an idempotent orthogonal to e. Observing that $fg = g$ and $gf = f$, we see that $gR = fR = (y - ey)R$, whence $xR + yR = eR + gR$. Since e and g are orthogonal, we conclude that $xR + yR = (e + g)R$.

(3)⇒(1) Given $x \in R$, there exists an idempotent $e \in R$ such that $xR = eR$. Then $e = xy$ for some $y \in R$, and $x = ex = xyx$.

□

A proper ideal A of R is said to be a *semiprime ideal* of R if whenever I is an ideal of R such that $I^2 \subseteq A$, then $I \subseteq A$. A ring R is said to be a *semiprime ring* if the zero ideal is a semiprime ideal of R.

Corollary 2.15 *((Goodearl, 1979), Corollary 1.2) Let R be a regular ring. Then the following hold statements are hold.*

(1) *All one-sided ideals of R are idempotent.*

(2) *All two-sided ideals of R are semiprime.*

(3) *The Jacobson radical of R is zero.*

Proof

(1) Let I be a right ideal of R. For $x \in I$, we have $xyx = y$ for some $y \in R$, and, consequently, $x = (xy)x \in I^2$. Thus $I^2 = I$.

(2) is clear from (1).

(3) Let $a \in J(R)$, there exists $y \in R$ such that $a = aya$. Then $a(1 - ya) = 0$, and since $(1 - ya)$ is invertible by Lemma 2.4, we get $a = 0$.

□

Proposition 2.11 *((Bland, 2010), Proposition 6.2.27) The following are equivalent for a ring R;*

(1) *R is semiprime;*

(2) *The zero ideal is the only nilpotent ideal of R;*

(3) *If A and B are right (left) ideals of R such that $AB = 0$, then $A \cap B = 0$.*

Proof

(1)⇒(3) If A and B are right ideals of R such that $AB = 0$, then $AB \subseteq P$ for every prime ideal P of R. Hence, $A \subseteq P$ or $B \subseteq P$, and so that $A \cap B \subseteq P$ for every prime P. Thus, $A \cap B \subseteq J(R) = 0$.

(3)⇒(2) Let I be a nilpotent ideal of R. If $I^n = 0$, then it follows from (3) that $I = I_1 \cap I_2 \cap ... \cap I_n = 0$, where $I_i = I$ for $i = 1, 2, ...n$.

(2)⇒(1) If $0 \neq a \in R$, let $a = a_0$. Then $Ra_0R \neq 0$, and so the ideal Ra_0R is not nilpotent. Hence, we can pick $a_1 \in Ra_0R$, $a_1 \neq 0$. For the same reasons, we can select a nonzero $a_2 \in Ra_1R$, and so on. Thus, a is not nilpotent, so a is not in $J(R)$. Hence, $J(R) = 0$ and R is therefore semiprime.

□

Remark 2.1 *((Tuganbaev, 2002), Remark 3.2.)If R is a semiprime ring then left socle coincides with right socle.*

2.6. Projective Modules

Definition 2.10 *An R- module M is said to be* projective *if each row exact diagram*

$$\begin{tikzcd} & M \arrow[dl, dashed, "g"'] \arrow[d, "f"] & \\ N_2 \arrow[r, "h"] & N_1 \arrow[r] & 0 \end{tikzcd}$$

of R-modules and R-module homomorphisms can be completed commutatively by an R-linear mapping $g : M \to N_2$.

Proposition 2.12 *((Bland, 2010), Proposition 5.2.3, Corollary 5.2.4, Lemma 5.2.5, Proposition 5.2.6) The following hold for a ring R.*

(1) *If $\{M_\alpha\}_\triangle$ is a family of R- modules, then $\bigoplus_\triangle M_\alpha$ is projective if and only if each M_α is projective.*

(2) *A direct summand of a projective R-module is projective.*

(3) *The ring R is a projective R-module.*

(4) *Every free R-module is projective.*

Proposition 2.13 *((Anderson & Fuller, 1992), Proposition 17.2) The following statements about a right R- module P are equivalent;*

(1) *P is projective ;*

(2) *Every epimorphism ${}_RM \to_R P \to 0$ splits;*

(3) *P is isomorphic to a direct summand of a free right R- module.*

Definition 2.11 *A ring R is* right hereditary *(respectively, right PP) if every right ideal (respectively, cyclic ideal) is projective.*

2.7. $(C_2), (C_3), SSP, and\, (C_4)$ Rings

Definition 2.12 *An R-module M has (C_2) if whenever a submodule N of M is isomorphic to a direct summand of M, then $N \leq^\oplus M$.*

Remark 2.2 *A ring R has (C_2) if R_R and ${}_RR$ has (C_2).*

Proposition 2.14 *Every regular ring has (C_2).*

Proof Let B be a right ideal of R and $B \cong A$ for some right ideal A of R such that $A \leq^{\oplus} B$. Then B is cyclic. By Theorem 2.4 B is generated by an idempotent and so B is a direct summand of R. $\square$

Definition 2.13 *An R-module M has (C_3) if whenever $N \leq^{\oplus} M$ and $K \leq^{\oplus} M$ such that $N \cap K = 0$ then, $N + K \leq^{\oplus} M$.*

Proposition 2.15 *(**?**, Proposition 2.2) If an R- module M has (C_2) then it has (C_3).*

Proof Write $M = M_1 \oplus M_1'$ and let ϕ denote the projection $M_1 \oplus M_1' \to M_1'$. Then $M_1 \oplus M_2 = M_1 \oplus \phi M_1'$. Since $\phi|_{M_2}$ is a monomorphism, we get $\phi M_2 \subset^{\oplus} M$ by C_2. As $\phi M_2 \leq M_1'$, $M_1 \oplus \phi M_2 \subset^{\oplus} M$. $\square$

Definition 2.14 *An R-module M has (C_4) if every submodule of M that contains an isomorphic copy of M, is itself isomorphic to M.*

Definition 2.15 *An R- module M has* the summand sum property (SSP) *if the sum of any two direct summands of M is a summand.*

Theorem 2.5 *((Shen, 2011), Theorem 2.4) The followings are equivalent for a ring R;*

(1) *R is right SSP;*

(2) *For any two idempotents e and f of R, $efR \leq^{\oplus} R_R$;*

(3) *R is left SSP;*

(4) *For any two idempotents e and f of R, $Ref \leq^{\oplus} {}_RR$.*

2.8. Exchange Rings

Definition 2.16 *A decomposition $M = \oplus_{i=1}^{n} M_i$ is exchangeable if for any summand N of M, $M = \oplus_{i=1}^{n} M_i' \oplus N$ with $M_i' \leq M_i$.*

Definition 2.17 *If every finite decomposition of M is exchangeable, then M is said to have the finite internal exchange property.*

Lemma 2.9 *((Mohamed & Muller, 2002), Lemma 5) Let $M = N \oplus K'$ where $K' \leq K \leq M$. If K has an exchangeable decomposition $K = \oplus_{i \in I} K_i$, then $M = N \oplus (\oplus_{i \in I} K'_i)$ with $K'_i \leq K_i$.*

Proof By the modular law, $K = (K \cap N) \oplus K'$. The hypothesis on $K = \oplus_{i \in} K_i$ implies $K = (K \cap N) \oplus (\oplus_{i \in I} K'_i)$ with $K'_i \leq K_i$. Write $L = \oplus_{i \in I} K'_i$. Then $M = N \oplus K' = N + K = N + N \cap K + L = N + L$, then $N \cap L = N \cap (K \cap L) = (N \cap K) \cap L = 0$. $\square$

Definition 2.18 *M is said to have* n- exchange property *if whenever $M \leq^{\oplus} A = \oplus_{i=1}^{n} A_i$, then $A = (\oplus_{i=1}^{n} A'_i) \oplus M$ with $A'_i \leq A_i$.*

Definition 2.19 *M has the* finite exchange property *if M has the n-exchange property for every positive integer n.*

Definition 2.20 *A ring R is said to be an* exchange (internal exchange) ring *if R_R, equivalently $_RR$ has the exchange (internal exchange) property.*

Proposition 2.16 *Semisimple rings have exchange property.*

Proposition 2.17 *((Mohamed & Muller, 2002), Proposition 15) The 2-internal exchange property is inherited by summands.*

Proof Assume $M = A \oplus B$ has the 2- internal exchange property. Let $A = A_1 \oplus A_2$ and let X be a summand of A. Consider decomposition $M = A_1 \oplus (A_2 \oplus B)$. By the 2- internal exchange property for M, $M = (X \oplus B) \oplus A'_1 \oplus (A_2 \oplus B)' = X \oplus A'_1 \oplus (A_2 \oplus B)' \oplus B$, with $A'_1 \leq A_1$ and $(A_2 \oplus B)' \leq A_2 \oplus B$. Let π denote the projection $M \to A$ along B. Then $\pi((A_2 \oplus B)') \leq \pi(A_2 \oplus B) = A_2$. Write $A'_2 \oplus B = (A_2 \oplus B)' \oplus B$. Hence $M = X \oplus A'_1 \oplus A'_2$ with $A'_i \leq A_i$. $\square$

Proposition 2.18 *((Mohamed & Muller, 2002), Proposition 16) The 2-internal exchange property implies the finite exchange property.*

Proof Let $n > 2$ be an integer, and assume inductively that any module K with 2- internal exchange property has the (n- 1)- internal exchange property. Let $M = M_1 \oplus ... \oplus M_n$ be a module with the 2- internal exchange property, and let X be a summand of M. Write $K = M_2 \oplus ... \oplus M_n$. Then $M = M_1 \oplus K$. By the 2- internal exchange property for M, $M = X \oplus M'_1 \oplus K'$, with $M'_1 \leq M_1$ and $K' \leq K$. As K is a summand of M, K has the 2- internal exchange property by Proposition 2.17. Hence K has the (n- 1)- internal exchange property by induction. It then follows by Lemma 2.17 that $M = X \oplus M'_1 \oplus M'_2 ... \oplus M'_n$ with $M'_i \leq M_i$. $\square$

Corollary 2.16 *((Mohamed, 2006), Corollary 2.3) The following are equivalent for a module M;*

(1) *M has the finite internal exchange property;*

(2) *For any idempotents e and f of $End(M)$, there exists an idempotent $g \in End(M)$ such that $eEnd(M) = gEnd(M)$ and $gf(1-g) = 0$.*

(3) *For any idempotents e and f of $End(M)$, there exists an idempotent $\gamma \in End(M)fe$ such that $e - \gamma \in End(M)fe$ such that $e - \gamma \in End(M)(1-f)e$;*

(4) *$End(M)_{EndM}$ has the internal exchange property.*

Example 2.1 *((Mohamed, 2006), Example 2.5) The ring $R = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$ is an internal exchange ring.*

We show that (2) of Corollary 2.16 is satisfied for all possible choices of idempotents e and f of R. Let $k = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ be an idempotent of R. Then

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}^2 = \begin{pmatrix} a^2 & 0 \\ ab+cd & c^2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}.$$

implies $a^2 = a$, $b^2 = b$ and $ab + cd = b$. Now we have four cases:

(1) *If $a = 0$, $c = 0$ then $b = 0$ and $k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.*

(2) *If $a = 1$, $c = 0$ then $k = \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}$ for all $b \in \mathbb{Z}$.*

(3) *If $a = 0$, $c = 1$ then $k = \begin{pmatrix} 0 & 0 \\ b & 1 \end{pmatrix}$ for all $b \in \mathbb{Z}$.*

(4) *If $a = 1$, $c = 1$ then $k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.*

Thus all idempotents of R are: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ b & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where $b \in \mathbb{Z}$.

For $e = \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 1 & 0 \\ b' & 0 \end{pmatrix}$, take $g = e$.

For $e = \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}$ *and* $f = \begin{pmatrix} 0 & 0 \\ b' & 1 \end{pmatrix}$, *take* $g = e$.

For $e = \begin{pmatrix} 0 & 0 \\ b & 1 \end{pmatrix}$ *and* $f = \begin{pmatrix} 1 & 0 \\ b' & 0 \end{pmatrix}$, *take* $g = \begin{pmatrix} 0 & 0 \\ -b' & 1 \end{pmatrix}$. *For* $e = \begin{pmatrix} 0 & 0 \\ b & 1 \end{pmatrix}$ *and* $f = \begin{pmatrix} 0 & 0 \\ b' & 1 \end{pmatrix}$, *take* $g = f$.

Proposition 2.19 *((Nicholson, 1977), Proposition 2.9) The following conditions are equivalent for a projective module* P;

(1) P *has the finite exchange property;*

(2) *If* $P = M_1 + M_2 + ... + M_n$ *where* M_i *are submodules, then there is a decomposition* $P = P_1 \oplus P_2 \oplus ... \oplus P_n$ *with* $P_i \leq M_i$ *for each i;*

(3) *If* $P = M + N$ *where* M *and* N *are submodules, then there exists a summand* P_1 *of* P *such that* $P_1 \leq M$ *and* $P = P_1 + N$.

CHAPTER 3

NON-TRIVIAL RIGHT PSEUDO SEMISIMPLE RINGS

This chapter includes the definition of a right pseudo semisimple ring and fundamental facts of these rings.

Definition 3.1 *A right ideal P of R is called* right pseudo maximal *if P is maximal in the set of right ideals not isomorphic to R.*

Pseudo maximal ideals of a ring R need not to be a maximal ideal, for an example consider $\mathbb{Z}$. Socle of $\mathbb{Z}$ is 0, and every right and left ideal of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$. So 0 is a pseudo maximal ideal, but it is not maximal ideal.

Definition 3.2 *A ring R is called* right pseudo semisimple *if any right ideal of R is either semisimple or isomorphic to R_R.*

Trivial examples of such rings are semisimple rings ($S = R$) or principal right ideal domains ($S = 0$). So it is only interesting to study pseudo semisimple rings in which $0 < S < R$.

Proposition 3.1 *((Mohamed, 2010), Proposition 2.1.) The following hold in a right pseudo semisimple ring R.*

(1) *If $R = A \oplus B$ for non-zero right ideals A and B of R, then exactly one of them is semisimple and the other one is isomorphic to R; in particular, none of them is an ideal, and so any nontrivial idempotent of R is not central .*

(2) *S is the smallest essential right ideal of R and is right pseudo maximal.*

(3) $^0S = Z \leq S \cap J$.

(4) *$S = {}^0x$ for every $0 \neq x \in J$; in particular, if $J \neq 0$, then $S = {}^0J$.*

(5) *$Z \leq A$ for any right ideal A not contained in S.*

(6) *If $b^0 = 0$, then $(Z : b) = Z$.*

(7) *If a is not in S, then $(S : a) = S$ and $aZ = Z$.*

(8) *R/S is a principal right ideal domain.*

(9) *$SZ = 0$ and Z is torsion free divisible as a left R/S module.*

Proof

(1) If $A \nleq S$ then $A \cong R$, so that $A = A_1 \oplus B_1$ with $A_1 \cong A$ and $B_1 \cong B$. Then $R = A_1 \oplus B_1 \oplus B$. Iterating this process, we obtain $R = A_n \oplus B_n \oplus ... \oplus B_1 \oplus B$, with $B_i \cong B$, for every $n \in \mathbb{N}$. Thus, R contains the right ideal $\bigoplus_{i\in\mathbb{N}} B_i$, which is not finitely generated, hence not isomorphic to R. Therefore, $\bigoplus_{i\in\mathbb{N}} B_i$ is semisimple and consequently B is semisimple. Suppose $A \cong R$ and $B \leq S$. If B is an ideal, then $AB \leq A \cap B = 0$, so that $B \subseteq A^0$. On the other hand, since $A \cong R$, we have $A^0 = R^0 = 0$. Then $B = 0$ which is a contradiction.

(2) A nonzero right ideal A of R is either contained in S or is isomorphic to R, and hence contains a copy of S. In either case $S \cap A \neq 0$, and therefore, S is smallest essential right ideal in R_R. If S is contained in a right ideal I of R, then by the definition of right pseudo semisimple ring $I \cong R$. So, S is right pseudo maximal.

(3) $Z \subseteq {}^0S$ follows from (2) that x^0 contains S, for every $x \in Z$. Let $x \in {}^0S$. Since S is essential in R from (2), x^0 is essential in R, and so ${}^0S = Z$. This also proves that $Z \ncong R$, and we obtain $Z \leqslant S$. Therefore, $Z^2 = 0$, and consequently $Z \leqslant J$ by Corollary 2.7.

(4) We know that $SJ = 0$ by Corollary 2.3, so for every $x \in J$, we have $S \leq {}^0J \leq {}^0x$. For an element a of R which is not in S, $aR \cong R$. Then aR is projective by Proposition 2.12 . It is clear that

$$0 \to a^0 \to R \to aR \to 0$$

is exact. By Proposition 2.13, $R = a^0 \oplus B$ with $B \cong R$ and $a^0 \leq S$ by (1). It follows that $a^0 \cap J = 0$, and consequently $ax \neq 0$. Hence ${}^0x \leq S$, and so $S = {}^0J = ^0 x$.

(5) We know that $Z \leq S$ by (3). Then by Theorem 2.3, $Z = A \cap Z \oplus K$, for some right ideal K of R, and we get $A + Z = A + A \cap Z \oplus K$. Since A is not contained in S, $R \cong A \oplus K$ and $K \cong eR$ with $e^2 = e \in S$ by (1). As $KeR \leq ZS = 0$, $eR = 0$, and hence $K = 0$. It follows that $Z = A \cap Z$, and so $Z \leq A$.

(6) $Z \leq (Z : b)$ is clear. $br \in Z$ implies $brS = 0$ by (3), which in turn implies $rS = 0$, and so by (3) $r \in Z$.

(7) We have $aR \cong R$, and so by Proposition 2.12 and Proposition 2.13, $R = a^0 \oplus C$ for some right ideal C of R with $C \cong R$. Then $a^0 \leq S$ by (1). Assume $ar \in S$, and write $r = s + c$, where $s \in a^0$ and $c \in C$, and so $ar = as + ac = ac$. Then $S \geq arR = acR \cong cR$ because $a^0 \cap C = 0$. Hence, $c \in S$, and consequently $r \in S$. Now, $aR \cong R$ implies $aR = bR$ with $b^0 = 0$. Then applying (5) and (6), we get $aZ = aRZ = bRZ = bZ = bR \cap Z = Z$.

(8) Let $\overline{x}$, $\overline{y} \in R/S$. Suppose that x is not in S and $\overline{xy} = \overline{0}$. Then $xy \in S$ and by (7) $y \in S$. That is R/S is a domain. Let A/S be a nonzero right ideal of R/S. Then by the definition of right pseudo semisimple ring, $A \cong R$, and so A is cyclic, that is A/S is cyclic.

(9) The result is trivial for $Z = 0$, assume that $Z \neq 0$. By (3) $Z \leq J$, and $SZ = 0$ by (4). Thus Z is an R/S module. Let x be a nonzero element of Z and $\overline{r} \in R/S$ such that $\overline{r}x = 0$. Then $rx = 0$ and by (3) and (4) $r \in S$. That is R/S is torsion free as an R/S module. Let $\overline{a}$ be a nonzero element of R/S. Then by (7) $aZ = Z$, and so $\overline{a}Z = Z$. Thus Z is divisible as R/S module.

□

Proposition 3.2 *Let R be any ring with idempotent g. Then the following are equivalent.*

(1) $R \cong (1-g)R$;

(2) *There exists $t, t^* \in R$ such that $t^*t = 1$, $tt^* = 1 - g$;*

(3) $R \cong R(1-g)$.

Proof

(1)⇒(2) Let ϕ be the isomorphism between R and $(1-g)R$. Let $\phi(1) = t$ for some $t \in (1-g)R$. Then for some $r \in R$, we have

$$t = (1-g)r = (1-g)^2 r = (1-g)t.$$

Since ϕ is onto, $(1-g)R = \phi(R) = tR$. So there exists $t^* \in R$ such that $tt^* = 1-g$. Now

$$\phi(1 - t^*t) = t(1 - t^*t) = t - tt^*t = t - (1-g)t = t - t = 0.$$

Since ϕ is monomorphism, $1 - t^*t = 0$ and so $t^*t = 1$.

(2)⇒(1) We have $(1-g)t = tt^*t = t$. Define $\phi : R \to (1-g)R$ such that $\phi(r) = tr$. Clearly, ϕ is well defined. Suppose that $\phi(r) = tr = 0$ for some $r \in R$. Then $t^*tr = 0$, and so $r = 0$. Therefore, ϕ is one to one. Also,

$$(1-g)R = tt^* \leq tR = (1-g)tR \leq (1-g)R.$$

Hence ϕ is onto. That is $R \cong (1-g)R$.

(2)⇔ (3) Proof is similar to $(1) \Leftrightarrow (2)$.

□

Proposition 3.3 *Let R be a ring with idempotent g which is in S. Then the following are equivalent.*

(1) $R \cong (1-g)R$;

(2) *There exist $t, t^* \in R$ such that $t^*t = 1,\ tt^* = 1-g$;*

(3) $R \cong R(1-g)$;

(4) $R \oplus gR \cong R$;

(5) $R \oplus Rg \cong R$.

Proof

(1)⇔ (2)⇔ (3) is by Proposition 3.2

(1)⇒ (4) Suppose $R \cong (1-g)R$. Then

$$R \oplus gR \cong (1-g)R \oplus gR = R.$$

Thus $R \oplus gR \cong R$.

(4)⇒ (1) Suppose $R \oplus gR \cong R$. Then $R \oplus gR \cong (1-g)R \oplus gR$. Since $g \in S$, gR is semisimple, and so gR has exchange property by Proposition 2.16. Therefore, $R \cong (1-g)R$.

(3)⇔ (5) Proof is similar to (1)⇔ (4).

□

Definition 3.3 *We call t in the Proposition 3.3 a shift for g. We say that R has* enough shifts *if for every isomorphism type of indecomposable idempotents in S, there is a representative f which has a shift.*

Corollary 3.1 *Let R be a right pseudo semisimple ring and $e^2 = e \in S$. Then $(1-e)R \cong R_R$ and $R(1-e) \cong {}_RR$.*

Proof $(1-e)R \cong R_R$ follows by Proposition 3.1(1). Then $R(1-e) \cong {}_RR$ by Corollary 2.1. □

Corollary 3.2 *Assume that R has enough shifts, and let $R = A \oplus B$ for some left ideals A and B. If $A \leq S$, then $B \cong {}_RR$.*

Proposition 3.4 *((Mohamed, 2010), Proposition 2.2.) Let R be a ring with $S \neq 0$. Then R is non-trivial right pseudo semisimple if and only if S is right pseudo maximal and R has enough shifts.*

Proof Suppose that R is right pseudo semisimple ring. Then S is right pseudo maximal by Proposition 3.1 (2). Since $R \cong eR \oplus (1-e)R$, by Proposition 3.1 (1) $e \in S$ or $1-e \in S$. Thus R has enough shifts by Proposition 3.3. Conversely, let A be a right ideal of R which is not semisimple. Write $S = (A \cap S) \oplus K$ for some right ideal K of R. Then $S < A + S = A \oplus K$, and hence $A \oplus K \cong R$. Therefore, $R = (1-e)R \oplus eR$ with $(1-e)R \cong A$ and $eR \cong K \leq S$. As R has enough shifts, by Proposition 3.3, $A \cong (1-e)R \cong R$. □

3.1. Right Pseudo Semisimple Rings with $S^2 = 0$

Lemma 3.1 *((Mohamed, 2010), Lemma 2.4.) Let R be a non-trivial right pseudo semisimple ring. Then:*

(1) *$S^2 = 0$ if and only if $Z = S \leq J \leq S^0$;*

(2) *$S^2 \neq 0$ if and only if $Z \leq S^0 = J < S$, and S contains a countable set of non-zero orthogonal idempotents.*

Proof

(1) If $S^2 = 0$, then $S \leq J$ by Corollary 2.3, and $S = Z$ by Proposition 3.1 (3). Therefore, $Z = S \leq J \leq S^0$. Conversely, if $Z = S \leq J \leq S^0$, then $S^2 = 0$ by Corollary 2.3.

(2) Suppose that $S^2 \neq 0$, and let A be a minimal right ideal of R. Then by Proposition 2.5, either $A^2 = 0$ or $A = e_1R$ for some idempotent e_1 of R. If $A^2 = 0$, then $A \leq J$, by Corollary 2.7. It follows $S \leq J$ which contradicts to the assumption, by Corollary 2.3. So, $A = e_1R$ for some idempotent $e_1 \in S$. Then by Proposition 3.3, $R = e_1R \oplus (1 - e_1)R$ and $(1 - e_1)R \cong R$. Again $(1 - e_1)R \cong e_2R \oplus (1 - e_2)R$ with $(1 - e_2)R \cong R$. Iterating this process, we get a countable set of orthogonal idempotents $\{e_i\} \in S$. Write $S = S \cap J \oplus K$ for some right ideal K of R. Now we claim that the projections of the e_i into K are still non-zero orthogonal idempotents. Let $e \in S$ such that $e^2 = e$. Then $e = j + k$ for some $j \in J \cap S$ and $k \in K$. Then

$$j + k = e = e^2 = (j + k)e = je + ke.$$

Since $(S \cap J) \cap K = 0$, $j = je$ and $k = ke$. Since $e \in S$ and $j \in J$, we have $ej = 0$ by Corollary 2.3. Then

$$e = e^2 = e(j + k) = ej + ek = ek.$$

It follows $k^2 = keke = kee = ke = k$. Let $e_1,\, e_2 \in S$ such that $e_1^2 = e_1$ and $e_2^2 = e_2$. Then $e_1 = j_1 + k_1$ and $e_2 = j_2 + k_2$ for some $j_1,\, j_2 \in J$ and $k_1,\, k_2 \in K$. It follows that

$$0 = e_1e_2 = (j_1 + k_1)(j_2 + k_2) = k_1k_2.$$

That is $k_1k_2 = 0$. Hence K is not finitely generated. Suppose that J is not in S, then $R \cong J + S = J \oplus K$, and it follows that K is finitely generated. This contradiction proves that $J \leqslant S$. Then by Corollary 2.3, $S^2 \neq 0$ implies $J < S$. In the same manner, one can prove that $S^0 < S$. This implies $(S^0)^2 = 0$, hence $S^0 \leqslant J$. As $J \leqslant S^0$, we get $J = S^0$.

□

Theorem 3.1 *((Mohamed, 2010), Theorem 2.5.) Let R be a ring with $S^2 = 0$. The following are equivalent:*

(1) *R is right pseudo semisimple;*

(2) *R/S is a right principal ideal domain, and S is torsion-free divisible as a left R/S module;*

(3) *S is right pseudo maximal.*

Proof $S^2 = 0$ implies $S \neq R$, and the result is trivial if $S = 0$. So we may assume $0 < S < R$.

(1)⇒(2) As $S = Z$ by Lemma 3.1, the result follows by (8) and (9) of Proposition 3.1.

(2)⇒(3) Let A be a right ideal of R with $S < A$. Then R/S is a principal right ideal ring implies that $\overline{A} = \overline{a}\overline{R}$ with $a^0 \leq S$. As S is torsion free as left R/S module, $a^0 = 0$. Then $A = aR + S = aR + aS = aR$ because S is divisible as R/S module. Hence $A \cong R$.

(3)⇒(1) As S has no non-zero idempotents, the result follows by Proposition 3.4.

□

Lemma 3.2 *((Mohamed & Muller, 1991), Lemma 2.6.) If R is right pseudo semisimple, then the left socle of R is contained in S.*

Proof The result is obvious in the trivial cases. So assume $0 < S < R$, and consider the two cases in Lemma 3.1.

If $S^2 = 0$, then by Proposition 3.1(1), R contains no non-trivial idempotents. Thus, every minimal left ideal A of R satisfies $A^2 = 0$ by Proposition 2.5, and so $\overline{A}^2 = \overline{0}$ in $\overline{R} = R/S$. Since $\overline{R}$ is a domain by Proposition 3.1(8), we get $A \leq S$.

If $S^2 \neq 0$, then by Lemma 3.1, $S^0 < S$, and hence S is essential as a left ideal of R by Lemma 2.3, and therefore contains the left socle. So, the left socle of R is contained in S. □

Corollary 3.3 *((Mohamed, 2010), Corollary 2.7.) Let R be a ring with $S \neq 0$ and $S^2 = 0$. Then R is right and left pseudo semisimple if and only if R is a local ring with radical square 0.*

Proof Suppose that R is local ring with $J^2 = 0$. By Corollary 2.3, J annihilates every simple R module. Thus, $S \leqslant {}^0J$. But ${}^0JJ = 0$. Therefore, ${}^0J$ is an R/J module. Then we have by Corollary 2.14 that ${}^0J$ is semisimple and ${}^0J = S$. This gives us $J \leqslant S$ because $J^2 = 0$. Since R is local ring $S \leqslant J$. Thus, $S = J$. If A is a right (left) ideal which is not isomorphic to R, then $A \leqslant J = S$. Thus, R is a right (left) pseudo semisimple ring.

Conversely, assume that R is a right and left pseudo semisimple. Then, S is the left socle by the right-left symmetry of Lemma 3.2. Consider a minimal left ideal A of R. Since S has no nontrivial idempotents, by Proposition 2.5, $A^2 = 0$. Hence, by Lemma 2.7, $A \leq J$. Therefore $A = Rx$, with $x \in J$. Also, we have $A \cong R/{}^0x$, and so ${}^0x$ is a maximal left ideal. Since $S = {}^0x$ by Proposition 3.1(4), S is a maximal left ideal. As $S \leq J$, $S = J$ and the result follows. □

Definition 3.4 The split extension of a ring R by an R-R bimodule M, *denoted by* $R \rtimes M$, *is the ring of all matrices of the form* $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, *with* $r \in R$ *and* $m \in M$.

Now we give an example of a right pseudo semisimple ring with $S^2 = 0$, which is not left pseudo semisimple.

Example 3.1 *((Mohamed, 2010), Example 2.8.) Let* $A = F[X]$, *the ring of polynomials over a field* F, *and* $M = F(X)$, *the quotient field of* A. *We make* M *as an* A*-bimodule by natural multiplication on the left and multiplying by constant coefficient on the right. Define* $R = A \rtimes M$. *Let* $N = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} = 0 \rtimes F(X)$. *If* $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in N$, *then* $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}^2 = 0$ *and so by Corollary 2.7* $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in J$. *That is* $N \leq J$. *It is clear that* $R/N \cong \begin{pmatrix} F[X] & 0 \\ 0 & F[X] \end{pmatrix} \cong F[X]$. *Since* $J(F[X]) = 0$ *by Corollary 2.10,* $J(R/N) = 0$. *Therefore* $J \leq N$. *Now, we have* $N = J$.

Claim : $S = 0 \rtimes F(X) = N$.

Let $a = \begin{pmatrix} 0 & m_1 \\ 0 & 0 \end{pmatrix} \in 0 \rtimes F(X)$. *Take an element* $b = \begin{pmatrix} a_2 & m_2 \\ 0 & a_2 \end{pmatrix} \in R$ *such that* $ab \neq 0$. *That is* $m_1 a_2' \neq 0$, *where* a_2' *is constant term of* a_2 *which means* a_2' *has an inverse* $(a_2')^{-1}$. *Let* c *be the matrix* $\begin{pmatrix} (a_2')^{-1} & 0 \\ 0 & (a_2')^{-1} \end{pmatrix}$. *Then,* $abc = a$ *which means* aR *is a minimal right ideal of* R *by Lemma 2.1. Hence* $0 \rtimes F(X) \leq S$. *Since* $R/N \cong \begin{pmatrix} F[X] & 0 \\ 0 & F[X] \end{pmatrix} \cong F[X]$ *and* $Soc(F[X]) = 0$, *we get* $Soc(R/N) = 0$ *and so* $S \leq N$. *Therefore,* $J = S = 0 \rtimes F(X)$. *Then for every element* $a \in S$, $a^2 = 0$, *and so* $S^2 = 0$. *One can see that* $R/S \cong F[X]$. *On the other hand, since* $F[X]$ *is a principal ideal domain,* R/S *is a principal ideal domain. Now we prove that* S *is torsion-free divisible as a left* R/S *module. Firstly let* $\begin{pmatrix} a_1 & m \\ 0 & a_1 \end{pmatrix}$ *be an element of* R *such that* $a_1 \neq 0$. *Now we get,*

$$\begin{pmatrix} a_1 & m \\ 0 & a_1 \end{pmatrix} + S = \begin{pmatrix} a_1 & 0 \\ 0 & a_1 \end{pmatrix} + \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} + S = \begin{pmatrix} a_1 & 0 \\ 0 & a_1 \end{pmatrix} + S \neq S$$

because $\begin{pmatrix} a_1 & 0 \\ 0 & a_1 \end{pmatrix}$ *is not is* S. *Let* $\begin{pmatrix} a_1 & m \\ 0 & a_1 \end{pmatrix}$ *be a nonzero element of* R/S *so* $a_1 \neq 0$ *and* $0 \neq \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in S$. *Then, we get*

$$(\begin{pmatrix} a_1 & m \\ 0 & a_1 \end{pmatrix} + S)\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1 m \\ 0 & 0 \end{pmatrix} \neq 0.$$

Thus, S is torsion free as a left R/S module. Let $0 \neq \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in S$*, and let* $\begin{pmatrix} a_1 & m \\ 0 & a_1 \end{pmatrix} +$ *S be a nonzero element of R/S. Since $F(X)$ is divisible, there exists $m_2 \in F(X)$ such that $m = am_2$. Now we get,*

$$\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & am_2 \\ 0 & 0 \end{pmatrix} = (\begin{pmatrix} a_1 & 0 \\ 0 & a_1 \end{pmatrix} + S)\begin{pmatrix} 0 & m_2 \\ 0 & 0 \end{pmatrix} =$$
$$(\begin{pmatrix} a_1 & m \\ 0 & a_1 \end{pmatrix} + S)\begin{pmatrix} 0 & m_2 \\ 0 & 0 \end{pmatrix}.$$

Thus, S is divisible as a left R/S module. Then, R is right pseudo semisimple by Theorem 3.1.

Claim :$S^0 = XF[X] \rtimes F(X) = \begin{pmatrix} XF[X] & F(X) \\ 0 & XF[X] \end{pmatrix}$.

Let $\begin{pmatrix} Xk & a \\ 0 & Xk \end{pmatrix} \in XF[X] \rtimes F(X)$ *and* $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in S$*. Then,*

$$\begin{pmatrix} Xk & a \\ 0 & Xk \end{pmatrix}\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $\begin{pmatrix} Xk & a \\ 0 & Xk \end{pmatrix} \in S^0$*. Let* $\begin{pmatrix} a & k \\ 0 & a \end{pmatrix} \in S^0$*. Then, for every $m \in M$ with constant term m',*

$$\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & k \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & m'a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So $m'a = 0$ and $a = Xk$ for every $k \in F[X]$. It follows $\begin{pmatrix} a & k \\ 0 & a \end{pmatrix} \in XF[X] \rtimes F(X)$*. Hence $S^0 = XF[X] \rtimes F(X)$. Also, we have $J < S^0$, and this means R is not local. So, by Corollary 3.3, R is not left pseudo semisimple ring.*

3.2. Right Pseudo Semisimple Rings with S Maximal

We know that S is a right pseudo maximal ideal of a ring R, which is a right pseudo semisimple ring by Proposition 3.1(2). Here, we study the pseudo semisimple property for rings with S maximal. So, we assume that $S \neq 0$, and is a maximal right ideal. Such a ring R is local if and only if $S^2 = 0$, and hence R is right and left pseudo semisimple by Corollary 3.3. So, we will consider non-local rings with maximal right socle.

Theorem 3.2 *((Mohamed & Muller, 1991), Theorem 2.2) Let R be a non-local ring with S maximal. Then R is right pseudo semisimple if and only if R has enough shifts.*

Proof It follows from Proposition 3.4. □

Theorem 3.3 *((Mohamed, 2010), Theorem 2.10) Let R be a ring with maximal non-zero right socle. The following are equivalent*

(1) *R is right and left pseudo semisimple and regular;*

(2) *R is right pseudo semisimple, and $J = 0$;*

(3) *R is semiprime and has enough shifts.*

Proof

(1)⇒ (2) Since R is a regular ring, $J = 0$ by Corollary 2.15.

(2)⇒ (3) Let I be an ideal of R such that $I^n = 0$ for some positive integer n. Then $I \leq J$, by Corollary 2.7, and so $I = 0$. Therefore, R is semiprime ring by Proposition 2.11. On the other hand, R has enough shifts by Proposition 3.4.

(3)⇒ (1) Since R is semiprime, $S^2 \neq 0$ by Proposition 2.11, hence R is non-local. Also, S is the left socle of R by Remark 2.1. Then, R is right and left pseudo semisimple by Theorem 3.2, and its right-left symmetry. For an element $a \in R$, let K be a complement of aR. Then, S is a maximal right ideal implying that $aR \oplus K = S$ or $aR \oplus K = R$. Since R is semiprime, we get aR is a summand of R in both cases. Hence R is regular.

□

Theorem 3.4 *((Mohamed, 2010), Theorem 2.11) Let R be a ring with $S^2 \neq 0$. If R is non-trivial right pseudo semisimple, then R/Z is non-trivial right pseudo semisimple with $Z(R/Z) = 0$. The converse holds if S is maximal.*

Proof Assume that R is a non-trivial right pseudo semisimple ring. We prove that $\overline{R} = R/Z$ is not semisimple. We first claim that R/Z contains no non-trivial central idempotents. Let u be a central idempotent in $\overline{R}$. We have $Z^2 = 0$ by Proposition 3.1 (3), and so we can lift idempotents modulo Z by Proposition 2.4. Then $u = \overline{e}$ for some idempotent $e \in R$. Now u is central in $\overline{R}$ implying that $eR(1-e) \leq Z$.On the other hand we may assume that $e \in S$ by Proposition 3.1(1). But, then by Proposition 3.1 (9),

$$eR(1-e) \leq Z \cap eR = eZ = 0.$$

Thus $e = 0$ or $e = 1$. This proves our claim, as required. Now if $\overline{R}$ is semisimple, then $\overline{R}$ got to be simple ring, and therefore Z is a maximal ideal in R. As $Z \leq S$ by Proposition 3.1 (3), we get $Z = S$, and hence $S^2 = 0$, a contradiction. Thus, $\overline{R}$ is not semisimple. Let $\overline{L}$ be the right socle of $\overline{R}$. Since S is semisimple, S/Z is semisimple, and so $S/Z \leq L/Z$. That is $S \leq L$. We have $L/S \cong (L/Z)/(S/Z)$. But, then L/S is semisimple because L/Z is semisimple. So $L/S \leq Soc(R/S)$. As R/S is a domain by Proposition 3.1 (8), Proposition 2.6 implying that $L/S = 0$ or $L/S = R/S$. This means $L = S$ or $L = R$. As $\overline{R}$ is not semisimple, $L \neq R$. Hence $Soc(\overline{R}) = S/Z$. Now let $\overline{A}$ be a right ideal in $\overline{R}$ such that $\overline{A} \neq \overline{S}$. Then $A \neq S$ and $A \cong R$. Let $f : R \to A$ be the isomorphism and $f(1) = a \in A$. We claim that $A = aR$ with $a^0 = 0$ and a is not in S. If $b \in A$, then since f is an epimorphism, there exists $x \in A$ such that $b = f(x) = f(1)x = ax \in aR$. Let $y \in a^o$, then $0 = ay = f(1)y = f(y)$, and so $y = 0$ because f is monomorphism. This proves our claim. On the other hand, by Proposition 3.1 (7), $Za = Z$, and so

$$\overline{A} = A/Z = aR/aZ \cong R/Z = \overline{R}.$$

Now, we prove that $Z(R/Z) = 0$. If $Z = S$, then R/Z is a domain by Proposition 3.1 (8) and so by Lemma 2.7 $Z(R/Z) = 0$. So, assume that $Z \neq S$. By Proposition 3.1 (3), $Z \leq S$ and hence by Proposition 2.3, $S = Z \oplus K$ for some right ideal K of R. Consider $\overline{x} \in Z(R/Z)$. Since R/Z is right pseudo semisimple, $(x + Z)S/Z = 0$ by Proposition 3.1 (9) and so $xS \leq Z$. Also, we have $xS = xZ \oplus xK$, and so $xK \leq xS \leq Z$. Hence, $xK \leq Z \cap K = 0$. On the other hand, if $xR \cong R$, then there exists a nonzero isomorphism $f : R \to xR$. Let $f(1) = xy$ for some $y \in R$. Then, $f(K) = f(1)K = xyK = 0$. But f is monomorphism $K = 0$, a contradiction, and so $xR \leqslant S$. Now, we have $xZ = 0$ by Proposition 3.1 (9). Therefore, $xS = xZ \oplus xK = 0$. So, by Proposition 3.1 (3), we get $x \in Z$.

For the converse, consider a right ideal $A \nleq S$, and let C be a complement of A. As S is maximal, $A \oplus C = R$. This proves that any right ideal is either contained in S or is a summand, hence projective. Now, let $x \in Z$, then either $xR \leqslant S$ or xR is a summand of R. If $xR \leqslant \oplus R$ the $xR = eR$ for some idempotent $e \in R$ by Lemma 2.2, and so $1 = e + f$ for some $f = f^2 \in R$ and $e = e^2 \in Z$. But, Z does not contain any nontrivial idempotent so $xR = 0$. Thus, $x \in S$, and so $Z \leq S$. Then by Proposition 2.3, $Z = A \cap Z \oplus K$ for some right ideal K of R. Now we have $A + Z = A + A \cap Z \oplus K = A \oplus K = R$. So $K = eR$ for some idempotent $e \in R$ by Lemma 2.2. But Z does not contain any nontrivial idempotent so $K = 0$. This implies $Z \leq A$. As $\overline{R} = R/Z$ is a non-trivial right pseudo semisimple ring, then the right socle of $\overline{R}$ is $\overline{S}$. Now $\overline{A} \nleqslant \overline{S}$ implies $A = aR$ with $a^0 \leqslant Z$. However, by Proposition 2.12, A is projective implying that a^0 is a summand of R . So that $a^0 = 0$, and hence $A \cong R$. $\square$

CHAPTER 4

RIGHT- LEFT PSEUDO SEMISIMPLE RINGS

In this chapter we find relations between right pseudo semisimple rings and the other classes of rings. Furthermore, from the Example 3.1, we see that pseudo semisimple rings are not left- right symmetric, and we give some characterizations of left-right pseudo semisimple rings.

Lemma 4.1 *A ring R with S maximal is an exchange ring.*

Proof Let $R = A + B$ for right ideals A and B of R. As S is maximal, we may assume $A \nleq S$. Let C be a complement of A. Then maximality of S implies $R = A \oplus C$. Hence, $R = A + B$ with $A \leq^{\oplus} R$. Now the result follows Proposition 2.12 and by Proposition 2.19. □

Proposition 4.1 *A right pseudo semisimple ring R is an internal exchange ring.*

Proof We only need to show that R_R has the 2-internal exchange property by Proposition 2.18. Let $R = A \oplus B$ for right ideals A and B, and let C be a summand of R. By Proposition 3.1(1), we may assume that B is semisimple. Hence, by Theorem 2.3, $B = ((A + C) \cap B) \oplus B'$ for some $B' \leq B$. Then

$$A + B = [A + ((A + C) \cap B)] \oplus B',$$

and therefore by modular law

$$R = [(A + C) \cap (A + B)] \oplus B' = (A + C) \oplus B'.$$

Let $f : A \oplus B \to B$ be the natural projection, and let f' denote the restriction of f to C. Again by Theorem 2.3, B is semisimple implying that $f'(C)$ is a summand of B, so it is projective by Proposition 2.12. It follows by Proposition 2.13 that $C = Kerf' \oplus D$ with $D \cong f'(C)$. Therefore, $A \cap C = Kerf' \leq^{\oplus} C \leq^{\oplus} R$, and so write $R = A \cap C \oplus K$, for some right ideal K of R. Then, by modular law, $A = (A \cap C) \oplus (A \cap K)$. Hence, $A + C = (A \cap K) \oplus C$. Consequently, we obtain that $R = (A \cap K) \oplus C \oplus B'$. □

Theorem 4.1 *A right pseudo semisimple ring R with $Z = J$ has SSP.*

Proof Let A and B be summands of R. We consider two cases.

(i) $B \leq S$: Then $B = \oplus_{i=1}^{n} B_i$ where B_i is a minimal right ideal and $B_i \leq^{\oplus} R$. Using induction, we may assume that B is minimal. If $B \leq A$, then we have nothing to prove. So, assume that $B \nleq A$. Since B is minimal right ideal, we have $A + B = A \oplus B$. Also, $R = A \oplus C$ for some right ideal C of R. So we get by modular law

$$A \oplus B = A + (A \oplus B) \cap C.$$

Then, by Proposition 2.1, $(A \oplus B) \cap C = X \cong B$. This implies $X \cap Z = 0$, and consequently $X \cap J = 0$. It follows that $X^2 \neq 0$, and so by Proposition 2.5 $X \leq^{\oplus} R$. Then, $R = X \oplus K$ for some right ideal K of R. As $X \leq C$, we obtain by modular law $C = X \oplus (C \cap K)$. Thus,

$$R = A \oplus C = A \oplus X \oplus (C \cap K) = A \oplus B \oplus (C \oplus K).$$

(ii) $B \nleq S$: Write $R = B \oplus D$ for some right ideal D of R. Then $D \leq S$ by Proposition 3.1(1). We get by modular law

$$A + B = B \oplus (A + B) \cap D.$$

As D is semisimple, by Theorem 2.3, we have $D = ((A + B) \cap D) \oplus K$ for some right ideal K of R. Hence,

$$R = B \oplus D = B \oplus ((A + B) \cap D) \oplus K = (A + B) \oplus K.$$

□

Remark 4.1 *The above theorem shows that R_R has SSP. By Theorem 2.5, ${}_RR$ also has SSP.*

Lemma 4.2 *For a right pseudo semisimple ring R, we have:*

(1) *If S is maximal, then $J \leq S'$,*

(2) *Either $J \cap S' = 0$ or S is maximal and $0 < J \leq S' \leq S$.*

Proof

(1) For a nonzero $x \in J$, we have ${}^0x = S$ by Proposition 3.1(4). Now $Rx \cong R/{}^0x = R/S$, and so Rx is a minimal left ideal of R because R/S is a division ring. This implies that $x \in S'$.

(2) Assume $J \cap S' \neq 0$, and consider a minimal left ideal $Rx \leq J$. Since $Rx \cong R/{}^0x$, ${}^0x$ is a maximal left ideal. As ${}^0x = S$ by Proposition 3.1(4), we have R/S is a division ring. Hence, S is a maximal right (left) ideal. Now the result follows by (1) and Lemma 3.2.

□

Proposition 4.2 *Let R be a right pseudo semisimple ring with $S^2 = 0$. Then either $S' = 0$ or $S' = J = S$, and R is a local ring with $J^2 = 0$.*

Proof As $S^2 = 0$, we get $S' \leq J$ by Corollary 2.3 and by Lemma 3.2. Hence, $J \cap S' = S'$. Then, it follows by Lemma 4.2 that either $S' = 0$ or S is maximal, and $0 < J \leq S' \leq S$. If $S' \neq 0$, then by Lemma 4.2 $J \leq S'$ and so we get $S' = J = S$. Therefore, R/J is a division ring, and so R is local. Also, $J^2 = SJ = 0$. □

Proposition 4.3 *Let R be a right and left pseudo semisimple ring. Then the following hold.*

(1) $S' = S$.

(2) $Z = J = Z'$.

(3) *S is maximal or $J = 0$.*

Proof

(1) is obvious by Lemma 3.2, and its left-right symmetry.

(2) By Corollary 2.3, $JS' = 0$, and so by (1) $JS = 0$. Hence, $J \leq Z$ by Proposition 3.1(3). However $Z \leq J$ by Proposition 3.1(3), hence $Z = J$. Similarly, $Z' = J$.

(3) Suppose S is not maximal, then by Lemma 4.2, $J \cap S' = 0$. We get, by (1), $J \cap S = 0$. It follows from Proposition 3.1(3) and (2) that $J = 0$.

□

Corollary 4.1 *A right and left pseudo semisimple ring is an SSP ring.*

Proof If R is a right and left pseudo semisimple ring, then $Z = J$ by Proposition 4.3. Thus, R is an SSP ring by Theorem 4.1. □

A right and left pseudo semisimple ring has S maximal or $J = 0$. Theorem 3.3 deals with the case S maximal and $J = 0$. In the following we will separate the two cases. First we note that this corollary may be rephrased as follows:

Theorem 4.2 *The following are equivalent for a ring R with $0 < S < R$.*

(1) *R is right pseudo semisimple and regular,*

(2) *R is semiprime, right and left pseudo semisimple with R/S division ring,*

(3) *R is left pseudo semisimple and regular.*

Proof

(1)⇒ (2) R is semiprime by Corollary 2.15. Suppose that $S < A$ for some right ideal of R. Since R is right pseudo semisimple $A \cong R$, and so A is finitely generated. Then, A is generated by an idempotent by Theorem 2.4. It follows that $R = A \oplus B$ for some right ideal B. Also, $B \leqslant S$ by Proposition 3.1 (1). Thus $B = 0$, that is, $R = A$, and S is maximal right ideal. Since R is regular ring, we get by Corollary 2.15, $J = 0$. Consequently, by Theorem 3.3 R is right and left pseudo semisimple ring.

(2)⇒ (1) Since R is right pseudo semisimple ring R has enough shifts by Proposition 3.4. Thus, by Theorem 3.3, R is regular ring.

(2)⇔(3) The proof is similar to (1)⇔(2).

□

Note that in a right pseudo semisimple ring, $J = 0$ if and only if R is semiprime. Suppose $J = 0$. Let A be a nilpotent ideal of R, then it is a nil ideal. It follows by Corollary 2.7 that $A = 0$. Thus, by Proposition 2.11, R is semiprime ring. Indeed, R being semiprime implies $S^2 \neq 0$, and hence $J < S$ by Lemma 3.1. So that $J^2 \leq SJ = 0$, and consequently $J = 0$.

We generalize Theorem 4.2 by dropping the semiprimeness condition in (2) and replacing regularity by the weaker condition (C_2).

Theorem 4.3 *The following are equivalent for a ring R with $0 < S < R$.*

(1) *R is right pseudo semisimple with (C_2),*

(2) *R is right and left pseudo semisimple with S maximal,*

(3) *R is left pseudo semisimple with (C_2).*

Proof

(1)⇒(2) Let A be right ideal of R such that $S < A$. Then, $A \cong R$. By (C_2), $R = A \oplus B$ for some right ideal B of R. Therefore, by Proposition 3.1 (1), $B \leq S$. So, $B = 0$ and S is maximal. Then, $J \leq S$. Now, by Theorem 2.3, we have $S = J \oplus K$ for some right ideal K of R. We prove that $K \leq S'$. Consider a minimal right ideal $E \leq K$. If $E^2 = 0$, then, by Corollary 2.7, $E \leqslant J$. It follows that $E \leqslant K \cap J$, which is a contradiction. Thus $E^2 \neq 0$, and so, by Proposition 2.5, $E = eR$ for some $e^2 = e \in R$. We prove that Re is a minimal left ideal. Consider a nonzero element $re \in Re$. As $reR \cong eR$, we get by (C_2) that $reR \leq^{\oplus} R$. Hence, $reRreR \neq 0$, and therefore $eRre \neq 0$. Since eRe is a division ring, $eReRre = eRe$. Hence,

$$Re = ReRe = ReReRre \leq Rre \leq Re.$$

So that Re is a minimal left ideal of R. It follows that $e \in Re \leq S'$. As S' is an ideal, we get $eR \leq S'$. This proves that $K \leq S'$. Hence, $S \leq S'$, and so $S = S'$. As R contains enough shifts, we get R is left pseudo semisimple by the left-handed version of Proposition 3.4.

(2)⇒(1) Let A be a right ideal of R such that $A \cong eR$ for some $e^2 = e \in R$. By Proposition 3.1, we may assume $eR \in S$. Let B be a complement of A then $S \leq A \oplus B$. Since S is maximal, $A \oplus B = S$ or $A \oplus B = R$. In the second case, we have nothing to prove. In the first case, we have $A \leq S$. Since eR is semisimple $Z \cap eR = 0$. Now $A \cong eR$ implies $Z \cap A = 0$. Since $Z = J$ by Proposition 4.3, $J \cap A = 0$, and so each simple right ideal contained in A is a direct summand of A. Using induction, we get $A = gR$ for some $g^2 = g \in R$.

(3)⇔(2) Follows by symmetry.

□

Corollary 4.2 *If R is a right pseudo semisimple ring with (C_2), then R/J is a regular right and left pseudo semisimple ring.*

Proof By Theorem 4.3, R is right and left pseudo semisimple with S maximal. Then $S' = S$ and $Z = J$ by Proposition 4.3. If $S^2 = 0$, then R is local by Proposition 4.2.

Hence R/J is a division ring. On the other hand, assume $S^2 \neq 0$. Then by the right-left symmetry of Theorem 3.4, R/J is right and left pseudo semisimple with $Soc(R/J) = S/J$. Thus, R/J is a semiprime right and left pseudo semisimple with maximal socle. Hence, R/J is regular by Theorem 4.2. $\square$

Next we deal with the case $J = 0$.

Theorem 4.4 *The following are equivalent for a ring R with $0 < S < R$.*

(1) *R is right pseudo semisimple, left PP and left (C_4),*

(2) *R is right and left pseudo semisimple with $J = 0$,*

(3) *R is left pseudo semisimple, right PP and right (C_4).*

Proof

(2)⇒(1) Suppose that R is right and left pseudo semisimple ring with $J = 0$. Let I be a left ideal of R which contains an isomorphic of R then by Proposition 3.1(1) $I \nleq S$. Again by Proposition 3.1(1) $I \cong R$. So R is left (C_4). Thus, it remains to show that R is left PP. We have $S' = S$ by Proposition 4.3. Since $J = 0$, by Proposition 2.5 every minimal right ideal is of the form eR with $e^2 = e$. Then we get by Proposition 2.12 every minimal right ideal of R is projective. Again by Proposition 2.12 S_R is projective. Now let A be a left ideal of R. If $A \leq S$, then by Theorem 2.3 $A \leq^{\oplus} S$, and hence projective by Proposition 2.12. On the other hand, by Proposition 3.1 $A \nleq S$ implies $A \cong {}_RR$, and hence free. (This proves that R is left hereditary).

(1)⇒(2) Consider an element $a \in R$ such that a is not in S. As R/S is a domain, by Proposition 3.1(8), ${}^0a \leq S$. Now R is left PP implies Ra is projective, and hence by Proposition 2.13 $R = {}^0a \oplus B$, with $B \cong Ra$. As R has enough shifts, we get by Corollary 3.2 that $Ra \cong B \cong {}_RR$. Now applying (C_4), we get $A \cong {}_RR$ for any left ideal A that is not contained in S. It remains to show that $J = 0$ (this also proves $S' = S$). To the contrary, let $0 \neq x \in J$. Then ${}^0x = S$ by Proposition 3.1(4). As Rx is projective, we get by Proposition 2.13 $R = S \oplus C$, with $C \cong Rx$. Again R has enough shifts implies $C \cong {}_RR$. Now $SC \leq S \cap C = 0$ implies $SR = 0$, and hence $S = 0$, a contradiction. Therefore, $J = 0$.

(3)⇔(2) follows by symmetry.

$\square$

Corollary 4.3 *Let R be any ring with $0 < S < R$. Then R is right and left pseudo semisimple if and only if*

(1) *R is a local ring with $J^2 = 0$, or*

(2) *R has enough shifts, $S' = S$ and S is maximal, or*

(3) *R has enough shifts, $J = 0$, R/S is a domain, R is hereditary with (C_4).*

Proof Suppose that R is right and left pseudo semisimple ring. We consider two cases:

(i) $S^2 = 0$: We obtain R is local ring with $J^2 = 0$ by Proposition 4.2.

(ii) $S^2 \neq 0$: We know that R has enough shifts by Proposition 3.4. Since R is right and left pseudo semisimple ring S is maximal or $J = 0$ by Proposition 4.3. If S is maximal, then there is nothing to prove. Suppose $J = 0$. R/S is a domain by Proposition 3.1 (8). Since $J = 0$, by Proposition 2.5 every minimal right ideal is of the form eR with $e^2 = e \in R$. Hence, ${}_RS$ is projective (also ${}_RS$ is projective) by Proposition2.12. Let A be a right ideal of R. If $A \leq S$, then A is projective by Theorem 2.3 and Proposition 2.12. If $A \nleq S$, then $A \cong R$ and A is projective by Proposition 2.12. That is R hereditary ring.

For the converse we consider three cases:

(i) Assume that R is local ring with $J^2 = 0$. Since R is local ring we have $S = J$. Then R is right and left pseudo semisimple ring by Corollary 3.3.

(ii) Assume that R has enough shifts, $S' = S$ and S is maximal. Then R is right and left pseudo semisimple ring by Proposition 3.4.

(iii) Assume that R has enough shifts, $J = 0$, R/S is a domain, R is hereditary ring with (C_4). Let A be a right ideal of R such that $S < A$. If $x \in A/S$, then by Proposition 2.12 $R \cong xR \oplus x^0$ because R is a hereditary ring. On the other hand, since R/S is a domain, $x^0 \leq S$. As R has enough shifts, $xR \cong R$ by Corollary 3.2. Thus, $A \cong R$ by (C_4). Now S is a right pseudo maximal ideal and so by Proposition 3.4 R is right pseudo semisimple ring. Consequently, by Theorem 4.4 R is right and left pseudo semisimple ring.

□

REFERENCES

Anderson, F. W. & Fuller, K. R. eds. (1992) *Rings and Categories of Modules* Springer-Verlag, New York.

Alizade, R. & Pancar, A. *Homoloji Cebire Giriş* 19 Mayıs Üniversitesi, Samsun.

Bland, Paul E. (2010) *Rings and Their Modules* De Gruyter, Berlin- New York.

Büyükaşık, E., Mohamed, S. H., and Mutlu, H. (2013) On pseudo-semisimple rings. *Journal Of Algebra and Its Appl.,* 12(2).

Goodearl, K. R. (1979) *Von Neumann Regular Rings* Pitman Publishing Limited, London.

Kasch, F. (1982) *Modules and Rings* London Mathematical Society Monographs 17, Academic Press, London-New York.

Lam, T. Y. (1999) *Lectures on Modules and Rings* Springer-Verlag, New York.

Lam, T. Y. (1991) *A First Course in Noncommutative Rings* Springer-Verlag, New York.

Lambek, J. (1966) *Lectures on Rings and Modules* Blaisdel Publishing Company.

Mohamed, S. H. & Muller, B. J. (2002) Ojective Modules. *Comm. Algebra ,* 30(4), 1817- 1827.

Mohamed, S. H. & Muller, B. J. (1982) Pseudo-semisimple rings. *Proc. Amer. Math. Soc,*85,157-160.

Mohamed, S. H. (2010) Pseudo Semisimple Rings. *AIP Conference Proc. ,* 1309, 693-701.

Mohamed, S. H. & Muller, B. J. (1991) Structure of pseudo-semisimple rings. *J. Austral. Math. Soc.,*50-A,53-66.

Mohamed, S. H. (2006) Internal exchange rings. *Contemp. Math. ,* 419.

Mohamed, S. H. & Muller, B. J. (1990) *Continuous and Discrete Modules* Cambridge University Press, New York.

Nicholson, W. K. (1977) Lifting İdempotents and Exchange Rings. *Trans. Amer. Math.*

Soc , 29, 269- 278.

Shen, L. (2011) A Note on Rings with The Summand Sum Property. , http://arxiv.org/abs/1107.0384.

Tuganbaev, A. (2002) *Rings Close to Regular Rings* Kluwer Academic Publishers, Dordrecht.

Wisbauer, R. (1991) *Foundations of Module and Ring Theory* Gordon and Breach, Reading, Philadelphia.