BORN - INFELD - RIEMANN GRAVITY

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ABSTRACT

BORN - INFELD - RIEMANN GRAVITY

The Universe we live in has started with a bang, a very big bang. Its evolution and global structure are controlled by gravitation and its matter and radiation content. Gravitation, curving of spacetime, formulated in minimal or extended versions, provides different layers of understanding about the Universe. Einstein's General Relativity (GR) gives a description of gravity, and there are various reasons for extending it. One such extension refers to unifying the other forces in Nature with gravity in the framework of GR. The very first approach in this direction was due to Born and Infeld who have tried to unify electromagnetism with gravity. It is a generalization of metric tensor to have both symmetric and antisymmetric parts gives rise to a merging of Maxwell's theory with Einstein's theory. In later decades, attempts have been made to unify the other forces as well.

In this thesis study, we extend Born-Infeld gravity to unify gravity with non-Abelian forces in a natural way. This, which we call Born-Infeld-Riemann gravity, is accomplished by devising a gravity theory based on Riemann tensor itself and subsequently generalizing this tensor to naturally involve gauge degrees of freedom. With this method, preserving the successes of Born-Infeld gravity, we are able to combine Yang-Mills fields (W, Z bosons as well as gluons) with gravity. We perform a phenomenological test of our approach by analyzing cosmic inflation generated by non-Abelian gauge fields.

ÖZET

BORN - INFELD - RIEMANN ÇEKİM KURAMI

Evrenimiz büyük patlamayla oluştu. Bilim insanları tarafindan oluşturulan teoriler temelde evrenimizin oluşumunu anlamaya yöneliktir. Bu amaç için oluşturulmuş teoriler çok çeşitlidir ve farklı farklı ele aldığımızda çözmeye çalıştığımız bazı problemlere ancak cevap verebildikleri görülmektedir. A. Einstein da dahil olmak uzere ortak amaç bu teorileri tek bir teori altında toplayabilmek ve bu teori ile evrenimizin oluşumu, ivmelenmesi, kara madde gibi bir çok problem icin çözüm sunabilmektir. Ancak bugüne kadar tam anlamıyla bir teori oluşturulamamıstır. Esas problem çekim kuvvetini diğer temel kuvvetlerle birleştirememememiz ve tum temel kuvvetleri tek bir teori altinda toplayamamamizdir. Çekimi diğer temel kuvvetlerle birleştirme çabalarından en önemlisi Born-Infeld çekim teorisidir. Bu teoride cekim ve elektromanyetizma bir araya getirilmeye calışılmıştır. Yalnızca metrik tensörü modifiye edilerek başarılabilmiştir. Metrik tensörü hem simetrik hem de antisimetrik parçalardan oluşturulmuş ve antisimetrik parça elektromanyetizma ile bağdaştırılmıştır. Teoriye W-Z alanlarını veya gluonu katmak istediğimizde ne olur? Born-Infeld tipi çekim bunun için yeterli midir? Eldeki teoriler bunun için yeterli değildir. Bu yüzden tezimizde yeni bir teori oluşturduk ve bu teoriyi yalnızca Riemann tensörü simetrilerini kullanarak geliştirdik. Teorimiz, 4. dereceden tensörleri icinde barındırdığı için non-Abelian alanlara izin verir ve kozmik patlamayı da sağlar.

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CHAPTER 1

INTRODUCTION

Our Universe, which has started with a ultra high-energy bang, has its entire history controlled by gravitation. Gravitation, as we understand it after Einstein, shows itself as the curving of the spacetime[13, 15, 26, 28]. The spacetime may also twirl or have torsion. All such properties, minimal or extended, provides different layers of understanding about the Universe. It is with astrophysical and cosmological observations that a correct picture of the Universe will eventually emerge.

Since the very first phases of Einstein's General Relativity (GR), unification of gravity and other forces has always been something desired. The problem is to find a natural way of embedding other forces in the realm of gravitation. To this end, the very first approach was due to Born and Infeld [10] who generalized metric tensor to have both symmetric and antisymmetric parts, and identified its antisymmetric part with the field strength tensor of electromagnetism. This way, one finds a way of combining Maxwell's theory with Einstein's theory in a single pot. This gives a complete description of electromagnetic phenomena in a way covering weak and strong field limits together.

One may wonder, if it is possible to unify other theories with gravity. For example, can one unify the W and Z bosons with gravity? How about the gluon which holds quarks together in nucleus? In Born-Infeld gravity as well as Born-Infeld-Einstein [25] gravity this is not possible. The reason is that these theories involve determinants of the generalized metrics of the form $C_1 g_{\alpha\beta} + C_2 F_{\alpha\beta}$, and if the field strength tensor of the vector field carries an index (like $F^a_{\alpha\beta}$ with a being the index of generators) then the argument of the determinant breaks the gauge symmetry. The determinant over the group generators, on the other hand, does not bring any improvement. Hence, it is simply not possible to unify Yang-Mills fields with gravitational theory.

What can be done? The first observation to make is that, gravitational dynamics can be written in terms of the determinant of the Ricci tensor. This is Eddington's approach [29]. However, it is not necessary to limit ourselves to Ricci tensor. The Riemann tensor itself can be used as well. Giving all the necessary details of such a formalism in Appendix A, we switch from a two-index tensor theory to four-index tensor theory by bringing the Riemann tensor in the game. This approach opens up a host of phenomena to be explored. One application, pertaining to the main topic of this thesis work, is the unification of non-Abelian gauge fields with gravity. The point is that, now theory includes double-determinant of a four-index tensor having the symmetries of Riemann tensor, and one readily finds that $F^a_{\alpha\beta}F^a_{\mu\nu}$ is such a tensor field constructed from non-Abelian gauge fields. We can this formalism as Born-Infeld-Riemann gravity as it involves the Riemann tensor[24, 60].

Apart from this very feature of unification, the non-Abelian fields, in homogeneous and isotropic geometries like the Universe itself, possess the crucial aspect that they can facilitate cosmic inflation [15, 67]. To this end, the Born-Infeld-Riemann formalism provides a natural framework to analyze gauge inflation [44].

In Chapter 2, we give an overview of the GR and its known extensions [27, 31, 53, 54, 56, 58, 59]. We review there GR, metric-affine and scalar-vector-tensor theories. This material proves important in further chapters of the thesis.

In Chapter 3, we discuss inflationary cosmology [2, 13, 15, 37, 52, 54, 67], in brief. We give here main properties of the Universe, and the need to cosmic inflation.

After the cosmological background, we try to examine the main idea of combining gravity and electromagnetism with Born-Infeld(BI) Gravity in Chapter 4. In Chapter 5, we discuss how determinant of Ricci tensor can be used for obtaining gravitational field equations. It is Born-Infeld-Einstein gravity. The main idea is to construct a new theory including both Eddington and BI approach.

In Chapter 6 we discuss Born-Infeld-Riemann gravity and gauge inflation in detail. There we find how rich the model to yield the requisite dynamics for inflation.

Finally, in Chapter 7 we conclude the thesis.

CHAPTER 2

THEORIES OF GRAVITATION

Gravitation or dynamical spacetime has been formulated in different contexts with different purposes. Since Einstein's 1916 formulation of General Relativity (GR), there has arisen different generalizations. In this Chapter we briefly summarize them.

2.1. Metric Theory of Gravity

According to Einstein's formulation of GR, spacetime is a differentiable manifold, and metric tensor $g_{\alpha\beta}$ governs curving and twirling of the spacetime. It is the sole field on the manifold [13]. In general, on a smooth manifold, there are two independent dynamical objects: metric tensor (a collection of clocks and rulers needed for measuring distances and angles) and connection (a guiding force for geodesic motion). In GR, connection is not an independent variable; it depends on the metric tensor via

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\rho} \left(\partial_{\alpha} g_{\beta\rho} + \partial_{\beta} g_{\rho\alpha} - \partial_{\rho} g_{\alpha\beta} \right) \tag{2.1}$$

which is known as the Levi-Civita connection. It is symmetric in lower indices since metric tensor is symmetric [13, 67]. Since Levi-Civita connection is expressed in terms of the metric tensor, GR is described by a single variable; the metric tensor. The field equations of GR are obtained by varying the Einstein-Hilbert action

$$\mathcal{S}\left[g\right] = \int \mathrm{d}^4 x \, \left|-g\right|^{1/2} \left\{ \frac{1}{2} M_{Pl}^2 \mathcal{R}(g) + \mathcal{L}_{mat}\left(g, \psi\right) \right\} \tag{2.2}$$

with respect to the metric tensor (See Appendix B for more general features.). Here, $M_{Pl} = (8\pi G_N)^{-1/2}$ is the Planck scale or the fundamental scale of gravity. The action density depends on the curvature scalar

$$\mathcal{R}(g) = g^{\mu\nu} \mathcal{R}_{\mu\nu}(\Gamma) \tag{2.3}$$

where

$$\mathcal{R}_{\mu\nu}(\Gamma) = \mathcal{R}^{\alpha}_{\mu\alpha\nu}(\Gamma) \tag{2.4}$$

is the Ricci tensor, and

$$\mathcal{R}^{\alpha}_{\mu\beta\nu}(\Gamma) = \partial_{\beta}\Gamma^{\alpha}_{\mu\nu} - \partial_{\nu}\Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\beta\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\beta} \tag{2.5}$$

is the Riemann curvature tensor. The spacetime manifold is perfectly flat in the close vicinity of x^0_μ if all the components of Riemann tensor vanishes at that point; $\mathcal{R}^{\alpha}_{\mu\beta\nu}(\Gamma(x^0)) = 0$.

Variation of the action (2.2) gives the Einstein equations of gravitation [4, 13, 15, 28, 41, 46, 49, 51, 52, 53, 54, 66, 67] (Appendix B for details.)

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi G_N T_{\mu\nu} \tag{2.6}$$

whose right-hand side

$$T_{\mu\nu} = -2\frac{\delta\mathcal{L}_{mat}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{mat}(g,\psi) \tag{2.7}$$

is the energy-momentum tensor of matter and radiation. Here,

$$\mathcal{S}_{mat}[g] = \int \mathrm{d}^4x\ |-g|^{1/2}\,\mathcal{L}_{mat}(g,\psi) \tag{2.8}$$

is the action of the matter and radiation fields ψ. The curvature scalar and matter Lagrangian $\mathcal{L}_{mat}$ both involve the same metric tensor $g_{\mu\nu}$.

It is important that the field equations (2.6) arises from the gravitational action (2.2) by adding an extrinsic curvature term. The reason is that, curvature scalar $\mathcal{R}(g)$ involves second derivatives of the metric tensor, and in applying the variational equations it is not sufficient to specify $\delta g_{\mu\nu}$ at the boundary. One must also specify its derivatives $\delta\partial_\alpha g_{\mu\nu}$ at the boundary. This additional piece does not admit construction of the Einstein-

Hilbert action directly; one adds a term (extrinsic curvature) to cancel the excess term.

The metric formalism is the most common approach to gravitation because equivalence principle is automatic, geodesic equation is simple, and tensor algebra is simplified (metric tensor is covariantly constant). Equivalence principle means that gravitational force acting on a point mass can be altered by choosing an accelerated (non-inertial) coordinate system. In other words, there is always a frame where connection can be set to zero. This point corresponds to a locally-flat coordinate system. The curvature tensor involves both Levi-Civita connection and its derivatives, and making the connection to vanish does not mean that curvature vanishes.

2.2. Metric-Affine Theory of Gravity

As we mentioned in the previous section, in metrical theories of gravity, the metric tensor and connection are not independent variables. However, connection does not have to be symmetric. It can be antisymmetric in lower indices, or it can have no symmetry condition which means that it has both symmetric and antisymmetric parts. Such a theory that has metric tensor and affine connection as *a priori* independent variables is called the Metric-Affine theory of gravitation [8]. Indeed, when the field equations are formed for Einsten- Hilbert action by taking metric tensor and connection as *a priori* independent dynamical quantities, it is obviously shown that the resulting connection turns out to be the same as the Levi- Civita connection. This aspect, known as Palatini formalism, overcomes the difficulties of metrical theory by eliminating the need to extrinsic curvature. This dynamical equivalence between GR and Metric-Affine approach holds when matter sector keeps involving only the Levi-Civita connection. Indeed, if the matter Lagrangian only couples to the Levi-Civita connection *i.e.* it does not couple to general connection, Metric-Affine formulation always reduces to metric formulation [17]. For pure Einstein-Hilbert action the Palatini and Metric-Affine formulations are the same while for the other theories they split as explained.

Metric-Affine theory of gravity (we do not refer to Palatini formulation) involves the metric tensor $g_{\mu\nu}$ and a general connection $\hat{\Gamma}^{\alpha}_{\mu\nu}$ where $\hat{\Gamma}^{\alpha}_{\mu\nu} \neq \Gamma^{\alpha}_{\mu\nu}$. It is a non-Riemannian gravitational theory. Indeed, generality of connection enables one to define the new geometro-dynamical structures which are obtained by using general connection definition. These new geometro-dynamical structures are called torsion and non-metricity tensors. Torsion tensor arises from the antisymmetric part of connection. The non-metricity tensor, conversely to the metric theory of gravity which is metric-compatible, is the covariant derivative of metric tensor according to the general connection $\hat{\Gamma}^{\alpha}_{\mu\nu}$. Mathematically,

$$T^{\alpha}_{\mu\nu} = \hat{\Gamma}^{\alpha}_{\mu\nu} - \hat{\Gamma}^{\alpha}_{\nu\mu} \text{ and } Q_{\alpha\mu\nu} = -\nabla_{\alpha} g_{\mu\nu} \,. \tag{2.9}$$

As a dynamical theory, Metric-Affine gravity is a non-trivial theory (does not reduce to GR dynamically) if the matter Lagrangian involves the general connection $\hat{\Gamma}^{\alpha}_{\mu\nu}$ explicitly.

The Metric-Affine gravity has one more branch in addition to the Palatini formalism: It is the Einstein-Cartan theory [39]. This theory is metric-compatible and torsion tensor is non-vanishing. The geometry which Einstein-Cartan theory defined is called the Riemann-Cartan geometry.

2.3. Scalar-Vector-Tensor Theory of Gravity

As we mentioned in Introduction, GR is a gravitational theory which has magnificent success to explain the universe. In this section, we are going to examine such theories whose aim is to answer some contradictions that could not be explained by GR.

Gravitational force is mediated by a spin-2 tensor field (coming from metric tensor in a given background) in the GR. However, the gravitational sector can be expanded to include other spin multiplets as well [17]. Those other fields can be a scalar (spin 0) field, a vector (spin 1) field or both. From this point of view, one figures out that there are extended gravitational theories which include additional spin multiplets and hence additional degrees of freedom. The first type of extended theories, which are well-studied alternative theories of GR, are scalar-and-tensor theories. These theories consist a scalar field ϕ in addition to the metric tensor [16, 17], and their general action is given by

$$\mathcal{S} = \int \mathrm{d}^4x | - \tilde{g}|^{1/2} \left[\tilde{R} - \left(\omega(\phi) + \frac{3}{2} \right) + \phi^{-2} \mathcal{L}_{mat} \left(\Psi, \phi^{-1} \tilde{g} \right) \right] \tag{2.10}$$

where $\tilde{g}_{\mu\nu}$ is Einstein metric because the action given in (2.2) is written in Einstein frame (theory has well-defined Newton's constant). In fact, the equation (2.10) stands like in addition to Einstein-Hilbert action there are some additional terms referring to scalar fields [16, 43]. Einstein frame is useful to discuss general characteristic of such theories. However, it is easily shown that scalar-tensor theory which is defined in Einstein frame is not a metrical theory since its' Matter Lagrangian couple to ϕ^{-1} in addition to the metric tensor. To work on metrical theory one can define Einstein metric in terms of physical metric as in the given form [43];

$$\tilde{g}_{\mu\nu} = \phi g_{\mu\nu} \tag{2.11}$$

and then the action (2.10) takes the form [43].

$$\mathcal{S} = \int \mathrm{d}^4x|g|^{1/2}\left[R\phi - \frac{\omega(\phi)}{\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \mathcal{L}_{mat}(\Psi, g)\right] \tag{2.12}$$

This action of the scalar-tensor theory is a metrical theory, and its frame is called Jordan frame (the theory needs a well-defined Newton's constant).

A well-known example of scalar-tensor theories is Brans-Dicke theory [43]. If ω in the action (2.10) does not depend on ϕ, then the action takes the form of Brans- Dicke theory

$$\mathcal{S} = \int \mathrm{d}^4x|g|^{1/2}\left[R\phi - \frac{\omega}{\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \mathcal{L}_{mat}(\Psi, g)\right] \tag{2.13}$$

As a metrical theory of gravity, the Brans-Dicke theory is a purely dynamical one since the field equations for the metric tensor involve scalar fields and vice versa [16].

The second type of extended theories constructed by using additional fields are vector-tensor theories. In this kind of theory, there is a dynamical 4-vector field u^μ in addition to metric tensor. General action of these theories given as [16]

$$\mathcal{S} = (16\pi G)^{-1}\int \mathrm{d}^4x|g|^{1/2}\Big[R(1+\omega u_\mu u^\mu) - K^{\mu\nu}_{\alpha\beta}\nabla_\mu u^\alpha\nabla_\nu u^\beta + \lambda(1+u_\mu u^\mu) + \mathcal{L}_{mat}(\Psi, g)\Big] \tag{2.14}$$

where

$$K^{\mu\nu}_{\alpha\beta} = c_1 g^{\mu\nu}g_{\alpha\beta} + c_2\delta^\mu_\alpha\delta^\nu_\beta + c_3\delta^\mu_\beta\delta^\nu_\alpha - c_4 u^\mu u^\nu g_{\alpha\beta} \tag{2.15}$$

If the vector field is time-like i.e $(ds^2 < 0)$, vector tensor theories become a constrained theory where $u^\mu u_\mu = -1$ and vector field has unit norm. If there is no condition on vector fields, it is unconstrained theory. The well-known vector tensor theory is Einsten-Aether theory which is constrained one *i.e.* u^μ is time-like, $u^\mu u_\mu = -1$ and has unit norm [16]. vector-tensor theory with given action is also a metrical theory since matter Lagrangian only couples to metric tensor.

The last type is the scalar-vector-tensor theory (TeVeS). TeVeS is the relativistic generalization of Modified Newtonian Dynamics (MOND) [9, 45]. It is proposed to eliminate the dark matter paradigm by explaining galaxy rotational curves via modified Newtonian dynamics. Even if it could not replace dark matter paradigm without any unknown matter, it has a great success to explain galaxy rotational curves [17]. The action for the TeVeS theory is given as

$$\mathcal{S} = \mathcal{S}_{\tilde{g}} + \mathcal{S}_A + \mathcal{S}_\phi + \mathcal{S}_m \tag{2.16}$$

where $\tilde{g}$ is called as Bekenstein metric

$$g_{\mu\nu} = e^{-2\phi}\tilde{g}_{\mu\nu} - sinh(2\phi)A_\mu A_\nu \tag{2.17}$$

which is time-like for

$$\tilde{g}^{\mu\nu} A_\mu A_\nu = -1\,. \tag{2.18}$$

The various parts of the action (2.16) are given by

$$\mathcal{S}_{\tilde{g}} \;=\; (16\pi G)^{-1} \int \mathrm{d}^4x |-\tilde{g}|^{1/2} \tilde{R} \tag{2.19}$$

$$\mathcal{S}_A \;=\; (-32\pi G)^{-1} \int \mathrm{d}^4x |-\tilde{g}|^{1/2} \left[K F^{\mu\nu} F_{\mu\nu} - 2\lambda(A_\mu A^\mu + 1)\right] \tag{2.20}$$

$$\mathcal{S}_\phi \quad = \quad (-16\pi G)^{-1}\int \mathrm{d}^4x|-\tilde{g}|^{1/2}\left[\mu\hat{g}^{\mu\nu}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi + V(\mu)\right] \tag{2.21}$$

where [17]

$$\hat{g}^{\mu\nu} = \tilde{g}^{\mu\nu} - A^\mu A^\nu\,. \tag{2.22}$$

There is one more way to obtain extended gravitational theories. As we know, general relativity has at most second order derivatives. Therefore, we can obtain another type of extended theory by using higher derivatives. The well-known theory is $f(\mathcal{R})$ gravity. $f(\mathcal{R})$ is the function of scalar curvature $\mathcal{R}$ thus this theory generalizes Einstein's GR. Their action is given as [17, 63].

$$\mathcal{S} = \int \mathrm{d}^4x|-g|^{1/2}f(R)\,. \tag{2.23}$$

Finally, as another extended gravitational theory we recall the Born-Infeld theory of gravitation. In Born-Infeld gravity, the main idea is that in constructing invariant actions all we need is determinant of a tensor (see Appendix A) and this tensor does not need to be the metric tensor (as in all the gravitational theories we have mentioned above). Any other rank-2 tensor, for example, the very Ricci curvature tensor, can well do the job. Since this theory is the main topic of this thesis work, it will be detailed in the following chapters.

CHAPTER 3

INFLATIONARY COSMOLOGY

What is cosmology and why is cosmology so important? Can one explain some problems which are unsolved from the viewpoint of GR ? People have asked these type of questions for years. When Einstein discovered the field equations (2.6), which equates the curvature of space-time to directly the energy-momentum source of matter and radiation, he thought that the universe must be static. In other words, it was necessary to add an extra term to have static solutions of

$$G_{\mu\nu} = (-8\pi G_N)\, T_{\mu\nu} \tag{3.1}$$

That extra term is a constant curvature source known as the cosmological constant (CC), (Λ) [21, 23]. This additional Λ term is also a geometrical modification to the field equations. To balance the gravitational attraction this term must yield a repulsive effect.

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = (-8\pi G_N)\, T_{\mu\nu} \tag{3.2}$$

where Λ denotes vacuum energy which means that it is a constant energy density and has negative pressure. At this point we should discuss the meaning of negative pressure. Why does vacuum energy have negative pressure? There is a basic physical explanation to the notion of negative pressure. For a particular conserved system, energy per unit volume is constant [66]. When the energy of the system is changed, volume changes directly in proportion to the rate of energy change. This actually means that the balance between the attraction by gravity and the needed repulsion is supplied by vacuum energy. Considering a universe without CC means that the universe is empty. On the other hand, if we consider vacuum state, it corresponds to a vanishing energy-momentum tensor $T_{\mu\nu} = 0$. Rearranging equation (3.2)

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} = (-8\pi G_N)\, T^{\Lambda}_{\mu\nu} \tag{3.3}$$

Here $T^{\Lambda}_{\mu\nu}$ is energy-momentum tensor of the vacuum state and CC becomes a new source term.

Actually, when we interpret gravity theoretically, a static universe seems impossible. Because of the fact that not only relativistic gravity but also non-relativistic gravity is attractive. Therefore, because of the attraction stationarity of the universe can not be attained.

At the beginning of the twentieth century, with the help of new generation telescopes lots of galaxies and galaxy sets began to be discovered in the visible universe [53]. As a conclusion of the discoveries, the distribution of the galaxies was interpreted and it is understood that the universe is homogeneous and isotropic. Homogeneity means that the universe looks the same from every point. Isotropy also means that there is no centre of the universe. We are going to examine these notions, homogeneity and isotropy, with Friedmann equations [2, 13, 15], again.

In 1929, with Hubble's great discovery it's understood that the universe is not static. On the contrary, the universe is expanding. The main clue was the red-shifting of the light spread by nearby galaxies which concludes that the galaxies are moving away from us radially -and of course from each other-. One may expect that the expansion slows down because the galaxies attract each other due to gravitation. But against the expectation, expansion accelerates. Magically the observed acceleration is compatible with the vacuum state energy density idea which was not even liked by Einstein at first.

How can the dynamics of a spatially homogeneous and isotropic universe be determined? Are Einstein's field equations sufficient for explanation? We are going to try to answer these questions. As a consequence of isotropy one may write a metric that is spherically symmetric. Beside this, generally the metric can take the form as;

$$ds^2 = -dt^2 + a^2(t)\left(\frac{1}{(1-kr^2)}dr^2 + r^2 d\Omega^2\right) \tag{3.4}$$

which is the Friedmann-Robertson-Walker (FRW) metric [15, 66, 67]. In equation (3.4) dr is radial part and $d\Omega$ is azimuthal part of spherically symmetric spaces. There k denotes constant spatial curvature. Due to the geometry of space-time values of k differs: $k \in \{-1, 0, +1\}$ corresponds to open, flat and closed geometries, respectively. Hyperboloid geometries, $k = -1$, which corresponds constant negative curvature expand forever(open). Flat geometries, $k = 0$, which causes the curvature term to vanish expand forever. Spherical geometries, $k = +1$, which corresponds constant positive curvature expansion will be ceased and it turns into a singular state. In our thesis work we are going to assume $k = 0$ and modify the field equations as flat space. Then our metric takes the form

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2 d\Omega^2\right) \tag{3.5}$$

or in cartesian coordinate system

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)\ . \tag{3.6}$$

The matrix form of the metric tensor

$$g_{\mu\nu} \doteq \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{pmatrix} \tag{3.7}$$

eases the identification of the pressure and energy densities for a given cosmological fluid. In the above, the function $a(t)$ is the scale factor that depends only on time. In other words $a(t)$ is a function that is responsible for time evolution and the expansion of the universe. (It is the radius of the evolving sphere.) To construct the dynamics of the homogeneous and isotropic universe we should substitute our chosen metric into Einstein's field equations.

To do that we should calculate the Ricci Tensor components and Ricci Scalar: As seen in equation (3.7) $g_{tt} = -1$, $g_{ti} = 0$, $g_{ij} = a^2(t)\delta_{ij}$. Inserting these components into Levi-Civita connection, surviving terms are

$$\begin{aligned} \Gamma^t_{ij} &= a\dot{a}\delta_{ij} \\ \Gamma^i_{tj} &= \frac{\dot{a}}{a}\delta^i_j \end{aligned} \tag{3.8}$$

from which the Ricci tensor can be directly calculated:

$$\begin{aligned} R_{tt} &= -3\left(\frac{\ddot{a}}{a}\right) \\ R_{ti} &= 0 \\ R_{ij} &= \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 \end{aligned} \tag{3.9}$$

Contracting with metric tensor one gets the Ricci scalar:

$$R = -R_{tt} + 3R_{ij} = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] \tag{3.10}$$

Here, we should interpret the meaning of $\frac{\dot{a}}{a}$. Assume that there are two isotropic observers any time t and the distance between these observers is D. The rate of changing of D is

$$v \equiv \frac{dD}{dt} = \frac{dD}{da}\frac{da}{dt} = \frac{D}{a}\frac{da}{dt} = \frac{D}{a}\dot{a} \tag{3.11}$$

so one can write $\frac{dD}{dt} = HD$ where we defined

$$H = \frac{\dot{a}}{a} \tag{3.12}$$

as the Hubble parameters. Hubble parameter (having the dimension of inverse time or mass) measures the fractional growth rate of the Universe's size.

3.1. Energy-Momentum Tensor

First of all let us examine the meaning of electromagnetic current and its components. One may consider 4-electromagnetic current; $j^\mu = (\rho, \vec{j})$ where ρ is electric charge density and $\vec{j}$ is current density. As we know electric charge density is the electric charge per volume, we can demonstrate mathematically as $\rho = \frac{\Delta q}{\Delta x \Delta y \Delta z}$ and current density is charge flow per unit time and per area, we can demonstrate mathematically as $\vec{j} = \frac{\Delta q}{\Delta t \Delta y \Delta z}$. If we combine them as 4-vector electromagnetic current; $j^\mu = \frac{\Delta q}{\Delta V^\mu}$ where ΔV^μ is volume. With the help of definition of 4-vector electromagnetic current one can construct energy momentum tensor of space-time via 4-momentum which is $p^\mu = (p^0, \vec{p}) = (E, \vec{p})$. Mathematically,

$$T^{\mu\nu} = \frac{\Delta p^\mu}{\Delta V_\nu} \tag{3.13}$$

As seen directly in Einstein's field equations (2.6), energy momentum tensor is the source of gravity. One of the important features of energy momentum tensor for a conservative system is its conservation. If there is an additional source, then energy momentum tensor does not conserve.

What is the physical meaning of the components of energy momentum tensor? One can directly calculate them from the general definition of energy momentum tensor (3.13) as the following:

$$T^{00} = \frac{\Delta p^0}{\Delta V_0} = \frac{\Delta E}{\Delta x \Delta y \Delta z} \tag{3.14}$$

which means that energy per unit volume⇒ energy density.

$$T^{0i} = \frac{\Delta p^0}{\Delta V_i} = \frac{\Delta E}{\Delta t \Delta y \Delta z} \tag{3.15}$$

which means that energy per unit time per area⇒ energy flux.

$$T^{i0} = \frac{\Delta p^i}{\Delta V_0} = \frac{\Delta p^i}{\Delta x \Delta y \Delta z} \tag{3.16}$$

With a little help of algebra of relativity one can easily see that equation (3.15) and equation (3.16) are equal to each other $T^{0i} = T^{i0}$. It means that energy momentum tensor $T^{\mu\nu}$ is a symmetrical rank (2,0) tensor. For diagonal components of energy momentum tensor;

$$T^{ii} = \frac{\Delta p^i}{\Delta V_i} = \frac{\Delta p^i}{\Delta t \underbrace{\Delta y \Delta z}_{area}} = \frac{normal force}{area} \tag{3.17}$$

According to equation (3.17) above momentum flux is directly proportional to pressure. The off-diagonal terms does not supply this feature. In the presence of this section we can say that the energy momentum tensor of a particular system includes all the features of the system such as energy density, momentum flux, energy flux etc.

We can classify particles according to their energy or momentum flux. The simplest type is dust. Dust includes non-interacting particles. If the particles do not interact with each other, pressure is not mentioned for this kind of systems. Such systems have the energy momentum tensor as the following form;

$$T^{\mu\nu} \doteq \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.18}$$

and 4-velocity is of the form; $U^{\mu} = (c, \vec{0})$

To examine the systems which have a huge number of particles, for simplicity, we try to analyse these type of systems as doing approximations. One of the useful approximation is perfect fluid approximation. In our work we also use perfect fluid approximation. It's a continuous system whose elements interact only through a normal force. As mentioned above interacting through normal force allows energy momentum tensor only has diagonal components.

$$T^{\mu\nu} \doteq \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \equiv (\rho + p)\, U^{\mu} U^{\nu} + g^{\mu\nu} p \tag{3.19}$$

For an arbitrary coordinate frame we can use the expression (3.19)

3.2. Friedmann Equations

We examined energy momentum tensor and perfect fluid form in Section 1 in detail. Now, we have sufficient background to derive the Friedmann Equations.

Inserting 4-velocity $U^t = 1$ and $U_t = -1$ $U^i = 0$ and $U_i = 0$ in (3.19) time component takes the form as;

$$T_{tt} = (\rho + p)\, U_t U_t + g_{tt} p = \rho$$

$$G_{tt} = 8\pi G_N \rho \tag{3.20}$$

and spatial components take the form as;

$$T_{ij} = (\rho + p)\, U_i U_j + g_{ij} p = p\delta_{ij}$$

$$G_{ij} = 8\pi G_N \delta_{ij} p \tag{3.21}$$

Combining (3.20), (3.21) and Einstein field equations for empty space together,

$$G_{tt} = R_{tt} - \frac{1}{2} g_{tt} R = 3 \left(\frac{\dot{a}}{a}\right)^2 = 8\pi G_N \rho \tag{3.22}$$

and

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R = -2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = 8\pi G_N \delta_{ij} p \tag{3.23}$$

From equation (3.22);

$$\rho = \frac{3}{8\pi G_N} \left(\frac{\dot{a}}{a}\right)^2 \tag{3.24}$$

From equation (3.23);

$$3\frac{\ddot{a}}{a} = -4\pi G_N \left(3p + \rho\right) \tag{3.25}$$

The equations (3.24) and (3.25) are called Friedmann Equations.

One can extract important information from Friedmann Equations. First of all equation (3.25) indicates that $\ddot{a} < 0$ which is an exact proof of non-stationary of the universe. In an other words the universe either expands ($\dot{a} > 0$) or contracts ($\dot{a} < 0$). Hubble's discovery about redshift shows that the universe expands ($\dot{a} > 0$) which means that the universe expands faster and faster. Under the homogeneous and isotropic universe assumption, GR predicts that at a time approximately less than H^{-1} (when a=0) the universe was in a singular state. The zero size space is a conclusion of homogeneity. In an other words at this t time the distance between the points in the space was zero and the curvature of spacetime was infinite. Essentially, this is the definition of -Big Bang-. The extension of space-time manifold beyond the big bang makes no sense. This interpretation of GR concludes that the universe began with the big bang. These interpretations is compatible with cosmological solutions since the singularities come from the cosmology.

For the mass density evolution, multiplying equation (3.24) by $a^2$and derivating with respect to t, one finds that

$$\dot{\rho} + \frac{\dot{a}}{a}\left(2 + \rho + 3p\right) = 0 \tag{3.26}$$

For dust ($p = 0$);

$$\begin{aligned} \dot{\rho} + \frac{\dot{a}}{a}\left(2 + \rho\right) &= 0 \\ \rho a^3 &= constant \\ \rho &\propto a^{-3} \end{aligned} \tag{3.27}$$

For radiation ($p = \frac{\rho}{3}$);

$$\dot{\rho} + \frac{\dot{a}}{a}(2 + 3\rho) = 0$$
$$\rho a^4 = constant$$
$$\rho \propto a^{-4} \tag{3.28}$$

These ratios show that when a increases, energy density of radiation decreases faster than energy density of dust. Hence, the radiation content of the present universe may be neglected. But as $a(t)$ goes to zero (for the past process) radiation should have been dominated over the ordinary matter.

3.3. Inflation

As seen in Section 3.2, although the early universe is radiation dominated; the present universe is matter dominated [67]. In Friedmann equation vacuum energy does not exist that is called flatness problem. Beside that for FRW cosmologies horizon problem exists[15]. Actually, the motivation comes from the Spontaneously Symmetry Breaking (SSB) for the inflationary models. Basically, the question is that how did the expansion begin? If it is caused by the CC, Λ, then which sort of physics causes this large energy? People have tried to interpret this inflationary phase in the early universe for years. The most common view is that the huge vacuum energy Λ is caused by the potential of a scalar field which is called inflaton. Suppose that a theory which includes a scalar field is constructed. Then, one may derive the dynamical equation for this scalar field in FRW metric as follows

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{3.29}$$

where $V'(\phi)$ is derivative of potential with respect to ϕ and

$$H^2 = N\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{3.30}$$

For inflation models at the early universe potential energy dominates kinetic energy. So the second derivative of ϕ is very small. It'shown mathematically as the following;

$$\dot{\phi}^2 \ll V(\phi) \tag{3.31}$$

$$|\ddot{\phi}| \ll |3H\dot{\phi}|, |V'(\phi)| \tag{3.32}$$

In the presence of that one may define parameters which are called slow-roll parameters that yield conditions of inflation:

$$\epsilon = \frac{1}{2}\left(\frac{V'}{V}\right)^2 \tag{3.33}$$

and

$$\eta = \frac{V''}{V} \tag{3.34}$$

We can also define the parameters in terms of the Hubble parameter. The only difference between these two notations is that 3.33 and 3.34 cause rolling slowly for a while. If we define the parameters via Hubble constant it causes the field to roll slower. The main point is that, the Universe must start with an exceedingly flat inflaton potential, evolve slowly, and finally land to a minimum of the potential where its oscillations reheat the universe to give today's structure.

CHAPTER 4

BORN - INFELD GRAVITY

In 1934 Born and Infeld [10] extended Maxwell's theory of electrodynamics from a linear nature to a non-linear nature. For the weak field limit, Maxwell's Theory works well but for strong field limit it fails. The main idea of B-I Gravity is to modify Lagrangian density. New terms would be added, if one ensures that Lagrangian is still a scalar density with all of the terms in it. (See Appendix A)

To modify Lagrangian density, we consider a rank-2 tensor field which is neither symmetric nor antisymmetric called $a_{\alpha\beta}$. As we know from Appendix A, the Lagrangian density is of the form

$$\mathcal{L} = (-|a_{\mu\nu}|)^{1/2} \tag{4.1}$$

where $|a_{\mu\nu}|$ is the determinant of the tensor field $a_{\mu\nu}$.

An arbitrary tensor field is formed of a symmetrical field and an anti-symmetrical field. According to this let $a_{\mu\nu} = g_{\mu\nu} + f_{\mu\nu}$. It means that the symmetrical part of $a_{\mu\nu}$ is metric tensor and the anti-symmetrical part of $a_{\mu\nu}$ is totally antisymmetric (an it is taken to be proportional to the field strength tensor of electromagnetic field $F^{EM}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$). With this identification,

$$\mathcal{L} \ni (-|g_{\mu\nu} + f_{\mu\nu}|)^{1/2} \tag{4.2}$$

where for the purpose of constructing the Maxwell action we write

$$\mathcal{L} = (-|g_{\mu\nu} + f_{\mu\nu}|)^{1/2} - (-|g_{\mu\nu}|)^{1/2} \tag{4.3}$$

as the complete action. We are going to expand our expression for small $f_{\mu\nu}$:

$$\mathcal{S} = \int d^4x(-|g_{\mu\alpha}|)^{1/2}(|\delta^\alpha_\nu + f^\alpha_\nu|)^{1/2} \tag{4.4}$$

by using the identity (See Appendix D)

$$(|1+A|)^{1/2} = 1 + \frac{1}{2}TrA + \frac{1}{8}(TrA)^2 - \frac{1}{4}Tr[(A)^2] + \mathcal{O}(A^3). \tag{4.5}$$

Then the second term in action (4.4) expands as

$$(|\mathcal{I} + f^{\alpha}_{\nu}|)^{1/2} = 1 + \frac{1}{2}Trf + \frac{1}{8}(Trf)^2 - \frac{1}{4}Tr[(f)^2] + \mathcal{O}(f^3) \tag{4.6}$$

Since f^{α}_{ν} is totally anti- symmetric tensor, both $Tr(f)$ and $(Tr(f))^2$ give vanishing contribution to the expansion. Only one term survives:

$$Tr[(f)^2] = Tr[f^{\alpha}_{\nu} f^{\nu}_{\beta}] = f^{\alpha}_{\nu} f^{\nu}_{\alpha} = f^{\mu\nu} f_{\mu\nu} \tag{4.7}$$

Then, our action takes the form

$$\mathcal{S} = \int d^4x(-|g_{\alpha\beta}|)^{1/2}(-\frac{1}{4}f^{\mu\nu}f_{\mu\nu}) \tag{4.8}$$

which is known action for electromagnetic field. From this action we are able to derive Maxwell's equations for electromagnetic field.

Since its proposal by Born and Infeld, this theory of electromagnetism has been developed and applied to different problems. As a consequence there occured different type of theories with different properties. These theories are called Born-Infeld type theories [5, 6, 11, 19, 20, 33, 36, 40, 47, 68]. The common point of Born-Infeld gravity theories are that when we examine the action (4.4) functional, we realize that it seems impossible to embed non-Abelian gauge fields into the theory. If we force to embed them into the theory, gauge invariance of theory is broken.

The main topic of this thesis work is Born-Infeld-Riemann gravity, and it will be discussed in Chapter 6.

CHAPTER 5

BORN - INFELD - EINSTEIN GRAVITY

Deser and Gibbons [25] suggested to modify BI gravity. Instead of adding field strength tensor of electromagnetic field, it is also a good idea to add Eddington term in action. In other words, the action can contain the curvature tensor and the metric tensor. It makes action purely gravitational and geometrical:

$$\mathcal{S}_{D-G} = \int d^4x(-|ag_{\mu\nu} + bR_{\mu\nu}|)^{1/2} \tag{5.1}$$

where a and b are constants. Applying the procedure in Chapter 4, one gets:

$$\begin{aligned} \mathcal{S}_{D-G} &= \int d^4x(-|ag_{\mu\alpha}(\delta^\alpha_\nu + \frac{b}{a}R^\alpha_\nu)|)^{1/2} \\ &= \int d^4x \left[-|ag_{\mu\alpha}|)^{1/2}(|\delta^\alpha_\nu + \frac{b}{a}R^\alpha_\nu)|)^{1/2}\right] \\ &= \int d^4x \left[a^2(|-g_{\mu\alpha}|)^{1/2}(|\delta^\alpha_\nu + \frac{b}{a}R^\alpha_\nu)|)^{1/2}\right] \end{aligned} \tag{5.2}$$

Now, the second term above can be expanded as

$$|\delta^\alpha_\nu + \frac{b}{a}R^\alpha_\nu|^{1/2} = 1 + \frac{1}{2}\frac{b}{a}TrR + \frac{1}{8}\frac{b}{a}(TrR)^2 - \frac{1}{4}\frac{b}{a}Tr[(R)^2] + \mathcal{O}(R^3) \tag{5.3}$$

in the small curvature limit.

As we know $(TrR)^2$ and $Tr(R)^2$ terms cause ghosts. To break away ghosty terms we should add an arbitrary tensor field $X_{\mu\nu}$ to our theory. In the presence of that, action takes the form;

$$\mathcal{S}_{D-G} = \int d^4x(-|ag_{\mu\nu} + bR_{\mu\nu} + cX_{\mu\nu}|)^{1/2} \tag{5.4}$$

Here, a, b and c are coupling constants. The same expansion procedure gives now:

$$\begin{aligned}
\mathcal{S}_{D-G} &= \int d^4x(-|ag_{\mu\alpha}(\delta^\alpha_\nu + \frac{b}{a}R^\alpha_\nu + \frac{c}{a}X^\alpha_\nu)|)^{1/2} \\
&= \int d^4x\left[-|ag_{\mu\alpha}|)^{1/2}(|\delta^\alpha_\nu + \frac{b}{a}R^\alpha_\nu + \frac{c}{a}X^\alpha_\nu)|)^{1/2}\right] \\
&= \int d^4x\left[a^2(|-g_{\mu\alpha}|)^{1/2}(|\delta^\alpha_\nu + \frac{b}{a}R^\alpha_\nu + \frac{c}{a}X^\alpha_\nu)|)^{1/2}\right] \qquad (5.5)
\end{aligned}$$

whose second term in radical sign is expanded as (See Appendix D)

$$\begin{aligned}
|\delta^\alpha_\nu + \frac{b}{a}R^\alpha_\nu + \frac{c}{a}X^\alpha_\nu|^{1/2} = 1 &+ \frac{1}{2}\frac{b}{a}TrR + \frac{1}{2}\frac{c}{a}TrX + \frac{1}{8}\left(\frac{b}{a}TrR + \frac{c}{a}TrX\right)^2 \\
&- \frac{1}{4}Tr\left(\frac{b}{a}R^\alpha_\nu + \frac{c}{a}X^\alpha_\nu\right)^2 + \mathcal{O}(3) \qquad (5.6)
\end{aligned}$$

Inserting expansion (5.6) in the action (5.5), we get:

$$\begin{aligned}
\mathcal{S}_{D-G} &= \int \mathrm{d}^4x\Big[a^2(|-g_{\mu\alpha})|^{1/2}\Big(1 + \frac{1}{2}\frac{b}{a}TrR + \frac{1}{2}\frac{c}{a}TrX \\
&+ \frac{1}{8}\Big(\frac{b^2}{a^2}(TrR)^2 + \frac{c^2}{a^2}(TrX)^2\Big) + \frac{1}{4}\frac{bc}{a^2}(TrR)(TrX) \\
&- \frac{1}{4}Tr\Big(\frac{b^2}{a^2}R_{\mu\nu}R^{\mu\nu} + \frac{c^2}{a^2}X_{\mu\nu}X^{\mu\nu} + \frac{bc}{a^2}X^{\mu\nu}R_{\mu\nu}\Big)^2\Big)\Big] \\
\mathcal{S}_{D-G} &= \int \mathrm{d}^4x\Big[a^2(|-g_{\mu\alpha})|^{1/2}\Big(1 + \frac{1}{2}\frac{b}{a}TrR + \frac{1}{2}\frac{c}{a}TrX + \frac{1}{8}\frac{b^2}{a^2}(TrR)^2 \\
&+ \frac{1}{8}\frac{c^2}{a^2}(TrX)^2 + \frac{1}{8}\frac{bc}{4a^2}(TrR)(TrX) - \frac{1}{4}\frac{b^2}{a^2}Tr[R_{\mu\nu}R^{\mu\nu}] \\
&+ \frac{c^2}{a^2}Tr[X_{\mu\nu}X^{\mu\nu}] - \frac{1}{2}\frac{bc}{a^2}Tr[X^{\mu\nu}R_{\mu\nu}]\Big)\Big] \\
\mathcal{S}_{D-G} &= \int \mathrm{d}^4x\Big[a^2(|-g_{\mu\alpha})|^{1/2}\Big(1 + \frac{1}{2}\frac{b}{a}TrR + \frac{1}{2}\frac{c}{a}TrX \\
&+ \frac{1}{4}\frac{b^2}{a^2}\Big(\frac{1}{2}(TrR)^2 - Tr[R_{\mu\nu}R^{\mu\nu}]\Big) \\
&+ \frac{1}{4}\frac{c^2}{a^2}\left(\frac{1}{2}(TrX)^2 - Tr[X_{\mu\nu}X^{\mu\nu}] + \frac{bc}{2a^2}(TrR)(TrX)\right) \\
&- \frac{1}{2}\frac{c^2}{a^2}Tr[X^{\mu\nu}R_{\mu\nu}]\Big)\Big] \qquad (5.7)
\end{aligned}$$

To cancel out the ghost term $\frac{1}{2}(TrR)^2 - Tr[R_{\mu\nu}R^{\mu\nu}]$ we make use of the tensor field $X_{\mu\nu}$

to set

$$\frac{1}{4}\frac{b^2}{a^2}\left(\frac{1}{2}(TrR)^2 - Tr[R_{\mu\nu}R^{\mu\nu}]\right) = -\frac{1}{2}\frac{c}{a}TrX \tag{5.8}$$

or

$$TrX = -\frac{1}{2}\frac{b^2}{ac}\left(\frac{1}{2}(TrR)^2 - Tr[R_{\mu\nu}R^{\mu\nu}]\right) \tag{5.9}$$

Letting $X_{\mu\nu} = \frac{1}{4}g_{\mu\nu}X^{\alpha}_{\alpha}$ we get

$$X_{\mu\nu} = -\frac{b^2}{8c}\left(\frac{1}{2}(TrR)^2 - Tr[R_{\alpha\beta}R^{\alpha\beta}]\right)g_{\mu\nu} \tag{5.10}$$

Since $X_{\mu\nu}$ is order of R^2, the terms which have $X^{\mu\nu}R_{\mu\nu}$ mixed terms cancel out. Because these terms are in cubic or higher contributions. Then our action takes the form;

$$\mathcal{S}_{D-G} = \int d^4x a^2(|-g_{\mu\alpha})|^{1/2}\left(1 + \frac{1}{2}\frac{b}{a}R\right) \tag{5.11}$$

This theory is nothing but the GR action with cosmological constant. It is important to note that, if we are to construct a consistent theory we need to introduce some extra tensor field $X_{\mu\nu}$ to cancel the ghost-giving higher curvature terms.

The main advantage of the Born-Infeld-Einstein gravity is that it provides us with a clear rationale for Einstein-Hilbert term. That term arises as the small curvature limit of a general determinant theory. The Eddington theory [7, 29] corresponds to taking $a = 0$, that is, killing the metric tensor. That theory resides in a complete affine space where there is no notion of distance. (One recalls here that, the Ricci tensor does not involve the metric tensor but Ricci scalar does.)

The disadvantage of the Born-Infeld-Einstein gravity is that it does not allow us to embed non-Abelian gauge fields into the theory due to the action functional (5.1). Otherwise the gauge symmetry is broken.

Although the disadvantage of the theory, it is an accomplished theory for Abelian gauge fields included [48, 64, 65].

CHAPTER 6

BORN - INFELD - RIEMANN GRAVITY

In this chapter we give a generalization of Born-Infeld-Einstein gravity. We try to construct a theory which includes directly the Riemann tensor *i. e.* a rank (0,4) tensor field (not a (0,2) one). We are going to see that, constructing theory from rank - (0,4) tensors allows us to expand the determinant to include non-Abelian gauge fields in addition to the electromagnetic field. This method has not been discovered before, and proves highly important in unifying non-Abelian gauge fields and gravity [24]. Also important is that, in this theory vector inflation comes out naturally [60]. This happens with no need to extra degrees of freedom. All these features are going to be the subject matter of this chapter.

From Appendix A, we know that the notion of determinant can be generalized to higher-rank tensors. For instance, if $T_{\alpha\beta}$ and $F_{\alpha\beta\mu\nu}$ are two tensor fields, one can form an invariant volume as $d^4x\,|T_{\alpha\beta}|^{1/2}$ or as $d^4x\,(DDet(F_{\alpha\beta\mu\nu}))^{1/4}$ where $DDet$ stands double-determinant as needed by a rank-4 tensor [61, 62]. The details can be found in Appendix A. Also the reference [22] gives a more general description.

6.1. Born - Infeld - Riemann Gravity

For D dimensions we write the effective action as

$$\mathcal{S}_{eff} = -\int \mathrm{d}^{\mathrm{D}}x M_D^{D/2}\Bigg[- DDet\Big(\kappa^2\tilde{g}_{\mu\alpha\nu\beta} + \lambda'_R R_{\mu\alpha\nu\beta} + \lambda'_F \tilde{F}_{\mu\alpha\nu\beta} + \tilde{\lambda}_F F^a_{\mu\alpha}\tilde{\tilde{F}}^a_{\nu\beta} + X_{\mu\alpha\nu\beta}\Big)\Bigg]^{1/4} \tag{6.1}$$

where M_D is mass parameter. The parameter κ is curvature constant which has mass dimension 2. Constant curvature term $\tilde{g}_{\mu\alpha\nu\beta}$ equals

$$\tilde{g}_{\mu\alpha\nu\beta} = (g_{\mu\nu}g_{\alpha\beta} - g_{\mu\beta}g_{\alpha\nu}) \tag{6.2}$$

which is nothing but the curvature tensor of a spacetime with constant scalar curvature (proportional to κ^2). This term has a different importance. When we expand the double-determinant, it is going to be expanded around this constant-curvature spacetime configuration. In (6.1) $\lambda_{R'}$ is a dimensionless constant and the mass dimensions of λ'_F and $\tilde{\lambda}_F$ are $[\lambda'_F] = [\tilde{\lambda}_F] = -2$. Finally, the field $X_{\mu\alpha\nu\beta}$ is an arbitrary tensor field introduced for canceling ghosts.

Now, let us discuss the term $\tilde{F}_{\mu\alpha\nu\beta}$ which has the same symmetries as $(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\beta}g_{\alpha\nu})$ and the Riemann Curvature Tensor $R_{\mu\alpha\nu\beta}$. It is easy to see that, for electromagnetic field, this tensor field attains the unique form

$$\tilde{F}_{\mu\alpha\nu\beta} = F_{\mu\alpha}F_{\nu\beta}. \tag{6.3}$$

This term unifies Maxwell theory and gravitation in the way Born-Infeld gravity does. In other words, instead of generalizing metric to $g_{\mu\nu} \to g_{\mu\nu} + F_{\mu\nu}$ as in Born-Infeld gravity, we can construct the tensor field (6.3) which has the same symmetries as the Riemann tensor. The results of the two approaches will be the same.

However, the main novelty is not the inclusion of Maxwell theory; the novelty is that the Yang-Mills theories can be unified with gravity [24]. Indeed, the tensor field (6.3) directly generalizes to

$$\tilde{F}_{\mu\alpha\nu\beta} = F^a_{\mu\alpha}F^a_{\nu\beta} \tag{6.4}$$

where a runs over the adjoint of the group. For SU(N) $a = 1, \dots, N^2 - 1$. The main novelty is that the non-Abelian gauge fields cannot be included in the Born-Infeld formalism which is not gauge invariant because of $F^a_{\mu\alpha}$ appearing with index a not contracted. Here, in the Born-Infeld-Riemann formalism, as we call it, it is automatic. This generalization of Born-Infeld theory thus opens a new avenue where one can embed the Yang-Mills dynamics in gravitational dynamics, which was not possible before.

As usual, the dual of $F^a_{\nu\beta}$ is defined as

$$\tilde{\tilde{F}}^a_{\nu\beta} = \epsilon_{\nu\beta}{}^{\tau\kappa}F^a_{\tau\kappa}\,. \tag{6.5}$$

In Born-Infeld-Riemann gravity, gauge fields enter the double-determinant in a gauge-invariant way, and hence, both Abelian (electromagnetism) and non-Abelian (Yang-Mills theories like electroweak theory and chromo-dynamics) theories are naturally included in the formalism [24]. In our work we examine our theory for a homogeneous and isotropic Universe (FRW background) which is explained in Chapter 3, in detail. We rewrite the effective action for FRW background and calculate the equations of motion. Specializing to $D = 4$ we get,

$$
\begin{aligned}
\mathcal{S}_{eff} = -\int \mathrm{d}^4x M_D^2 \Big[& - DDet\big(\kappa^2 \tilde{g}_{\mu\alpha\nu\beta} + \lambda_{R'} R_{\mu\alpha\nu\beta} + X_{\mu\alpha\nu\beta} + \lambda'_F \tilde{F}_{\mu\alpha\nu\beta} \\
& + \tilde{\lambda}_F F^a_{\mu\alpha} \tilde{\tilde{F}}^a_{\nu\beta}\big)\Big]^{1/4} \\
\mathcal{S}_{eff} = -\int \mathrm{d}^4x M_D^2 \Big[& - DDet\big(\kappa^2 (g_{\mu\nu} g_{\alpha\beta} - g_{\mu\beta} g_{\alpha\nu}) + \lambda_{R'} R_{\mu\alpha\nu\beta} \\
& + X_{\mu\alpha\nu\beta} + \lambda'_F \tilde{F}_{\mu\alpha\nu\beta} + \tilde{\lambda}_F F^a_{\mu\alpha} \tilde{\tilde{F}}^a_{\nu\beta}\big)\Big]^{1/4}
\end{aligned}
\tag{6.6}
$$

Now, applying the same procedure we did in Born-Infeld and Born-Infeld-Einstein gravities we get

$$
\begin{aligned}
\mathcal{S}_{eff} = -\int \mathrm{d}^4x M_D^2 \Big(& - DDet\big(\kappa^2 (g_{\mu\nu'} g_{\alpha\beta'} - g_{\mu\beta'} g_{\alpha\nu'})\big)\Big)^{1/4} \\
& \times DDet\Big(\frac{1}{2}(\delta^{\nu'}_{\nu} \delta^{\beta'}_{\beta} - \delta^{\nu'}_{\beta} \delta^{\beta'}_{\nu}) \\
& + \frac{\lambda_{R'}}{\kappa^2} (Inv)^{\mu'\alpha'\nu'\beta'} R_{\mu'\alpha'\nu\beta} \\
& + \frac{1}{\kappa^2} (Inv)^{\mu'\alpha'\nu'\beta'} X_{\mu'\alpha'\nu\beta} \\
& + \frac{\lambda'_F}{\kappa^2} \tilde{F}_{\mu'\alpha'\nu\beta} + \frac{\tilde{\lambda}_F}{\kappa^2} F^a_{\mu'\alpha'} \tilde{\tilde{F}}^a_{\nu\beta}\Big)^{1/4}
\end{aligned}
\tag{6.7}
$$

Let us first compute the first factor in the integral. Using the notion of double-determinant

(See Appendix A) we obtain

$$
\begin{aligned}
&\left(DDet\left(\kappa^2(g_{\mu\nu'}g_{\alpha\beta'}-g_{\mu\beta'}g_{\alpha\nu'})\right)\right)^{1/4}\\
&=\kappa^2\left(DDet\left((g_{\mu\nu'}g_{\alpha\beta'}-g_{\mu\beta'}g_{\alpha\nu'})\right)\right)^{1/4}\\
&=\kappa^2\left(DDet\left(\tfrac{1}{2!^2}\epsilon^{\mu_1\mu_2}{}_{\mu\alpha}\epsilon_{\mu_1\mu_2\nu'\beta'}\right)\right)^{1/4}\\
&=\kappa^2\tfrac{1}{2!^2}\left(DDet\left(\epsilon^{\mu_1\mu_2}{}_{\mu\alpha}\epsilon_{\mu_1\mu_2\nu'\beta'}\right)\right)^{1/4}\\
=\kappa^2(|-g_{\rho\sigma}|)^{1/2}\tfrac{1}{2!^2}&\left[\epsilon^{\tilde\mu_1\tilde\mu_2\tilde\mu_3\tilde\mu_4}\epsilon^{\tilde\alpha_1\tilde\alpha_2\tilde\alpha_3\tilde\alpha_4}\epsilon^{\nu'_1\nu'_2\nu'_3\nu'_4}\epsilon^{\beta'_1\beta'_2\beta'_3\beta'_4}\epsilon^{\mu_1^1\mu_2^1}{}_{\tilde\mu_1\tilde\alpha_1}\epsilon^{\mu_1^2\mu_2^2}{}_{\tilde\mu_2\tilde\alpha_2}\right]^{1/4}\\
\times&\left[\epsilon^{\mu_1^3\mu_2^3}{}_{\tilde\mu_3\tilde\alpha_3}\epsilon^{\mu_1^4\mu_2^4}{}_{\tilde\mu_4\tilde\alpha_4}\epsilon_{\mu_1^1\mu_1^2\nu'_1\beta'_1}\epsilon_{\mu_2^1\mu_1^2\nu'_2\beta'_2}\epsilon_{\mu_1^3\mu_2^3\nu'_3\beta'_3}\epsilon_{\mu_1^4\mu_2^4\nu'_4\beta'_4}\right]^{1/4}
\end{aligned}
\tag{6.8}
$$

For simplicity let us call the coefficient of the term $\kappa^2(|-g_{\rho\sigma}|)^{1/2}$ as C_{DD}. It reads explicitly

$$
\begin{aligned}
C_{DD}\;=&\tfrac{1}{2!^2}\left[\epsilon^{\tilde\mu_1\tilde\mu_2\tilde\mu_3\tilde\mu_4}\epsilon^{\tilde\alpha_1\tilde\alpha_2\tilde\alpha_3\tilde\alpha_4}\epsilon^{\nu'_1\nu'_2\nu'_3\nu'_4}\epsilon^{\beta'_1\beta'_2\beta'_3\beta'_4}\epsilon^{\mu_1^1\mu_2^1}{}_{\tilde\mu_1\tilde\alpha_1}\epsilon^{\mu_1^2\mu_2^2}{}_{\tilde\mu_2\tilde\alpha_2}\right]^{1/4}\\
&\times\left[\epsilon^{\mu_1^3\mu_2^3}{}_{\tilde\mu_3\tilde\alpha_3}\epsilon^{\mu_1^4\mu_2^4}{}_{\tilde\mu_4\tilde\alpha_4}\epsilon_{\mu_1^1\mu_1^2\nu'_1\beta'_1}\epsilon_{\mu_2^1\mu_1^2\nu'_2\beta'_2}\epsilon_{\mu_1^3\mu_2^3\nu'_3\beta'_3}\epsilon_{\mu_1^4\mu_2^4\nu'_4\beta'_4}\right]^{1/4}
\end{aligned}
\tag{6.9}
$$

As a result, we get:

$$
\left(-DDet\left(\kappa^2(g_{\mu\nu'}g_{\alpha\beta'}-g_{\mu\beta'}g_{\alpha\nu'})\right)\right)^{1/4}=C_{DD}\kappa^2(|-g_{\rho\sigma}|)^{1/2}
\tag{6.10}
$$

Hence, we have calculated the first part of (6.8). It is expected that the double-determinant of the Riemann curvature tensor of a constant-curvature spacetime yields the square of the determinant of the metric tensor.

Now, we proceed with the remaining calculation. First of all, $(Inv)^{\mu'\alpha'\nu'\beta'}$ is nothing but the 4-index identity tensor. Its explicit expression as well as various other details are given in Appendix B.

With the calculated pieces replaced in, the effective action takes the form

$$
\begin{aligned}
\mathcal{S}_{eff}=\;&-\int \mathrm{d}^4x C_{DD}M_D^2\kappa^2(|-g|)^{1/2}\Big(DDet\Big(\frac{1}{2}\Big(\delta^{\nu'}_{\nu}\delta^{\beta'}_{\beta}-\delta^{\nu'}_{\beta}\delta^{\beta'}_{\nu}\Big)+\frac{\lambda_{R'}}{\kappa^2}R^{\nu'\beta'}{}_{\nu\beta}\\
&+\frac{1}{\kappa^2}X^{\nu'\beta'}{}_{\nu\beta}+\frac{\lambda'_F}{\kappa^2}F^{a\nu'\beta'}F^a_{\nu\beta}+\frac{\tilde\lambda_F}{\kappa^2}F^{a\nu'\beta'}\epsilon_{\nu\beta}{}^{\tau\kappa}F^a_{\tau\kappa}\Big)\Big)^{1/4}
\end{aligned}
\tag{6.11}
$$

which is of the form

$$\mathcal{S}_{eff} \quad = -\int \mathrm{d}^4 x C_{DD} M_D^2 \kappa^2 (|-g|)^{1/2} \left(DDet\left(I+A\right)\right)^{1/4} \tag{6.12}$$

From Appendix D, we know the expansion of the term $DDet\left(I+A\right)$. Let us now apply the expansion to our tensor fields.

$$\begin{aligned}
DDet(I+A) = 1 \quad &+ \quad \frac{\lambda_{R'}}{\kappa^2} Tr\left[R^{\nu'\beta'}{}_{\nu\beta}\right] + \frac{1}{\kappa^2} Tr\left[X^{\nu'\beta'}{}_{\nu\beta}\right] \\
&+ \quad \frac{\lambda'_F}{\kappa^2} Tr\left[F^{a\nu'\beta'} F^a_{\nu\beta}\right] + \frac{\tilde{\lambda}_F}{\kappa^2} Tr\left[F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa}\right] \\
&- \quad \frac{1}{2} Tr\Bigg[\Bigg(\frac{\lambda_{R'}}{\kappa^2} R^{\nu'\beta'}{}_{\nu\beta} + \frac{1}{\kappa^2} X^{\nu'\beta'}{}_{\nu\beta} + \frac{\lambda'_F}{\kappa^2} F^{a\nu'\beta'} F^a_{\nu\beta} \\
&+ \quad \frac{\tilde{\lambda}_F}{\kappa^2} F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa}\Bigg)^2\Bigg] \\
&+ \quad \frac{1}{2}\Bigg[Tr\Bigg(\frac{\lambda_{R'}}{\kappa^2} R^{\nu'\beta'}{}_{\nu\beta} + \frac{1}{\kappa^2} X^{\nu'\beta'}{}_{\nu\beta} + \frac{\lambda'_F}{\kappa^2} F^{a\nu'\beta'} F^a_{\nu\beta} \\
&+ \quad \frac{\tilde{\lambda}_F}{\kappa^2} F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa}\Bigg)\Bigg]^2
\end{aligned} \tag{6.13}$$

where each of the pieces of which can be calculated as follows:

$$\begin{aligned}
Tr&\left[\left(\tfrac{\lambda_{R'}}{\kappa^2} R^{\nu'\beta'}{}_{\nu\beta} + \tfrac{1}{\kappa^2} X^{\nu'\beta'}{}_{\nu\beta} + \tfrac{\lambda'_F}{\kappa^2} F^{a\nu'\beta'} F^a_{\nu\beta} + \tfrac{\tilde{\lambda}_F}{\kappa^2} F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa}\right)^2\right] \\
&= Tr\Big[\tfrac{\lambda^2_{R'}}{\kappa^4} R^{\nu'\beta'}{}_{\nu\beta} R^{\rho'\sigma'}{}_{\rho\sigma} + \tfrac{\lambda^2_{F'}}{\kappa^4} F^{a\nu'\beta'} F^a_{\nu\beta} F^{b\rho'\sigma'} F^b_{\rho\sigma} \\
&\quad + \tfrac{\tilde{\lambda}^2_F}{\kappa^4} F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa} F^{b\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^b_{\tau'\kappa'}\Big] \\
&\quad + Tr\Big[\tfrac{1}{\kappa^4} X^{\nu'\beta'}{}_{\nu\beta} X^{\rho'\sigma'}{}_{\rho\sigma} + 2\tfrac{\lambda_{R'}\lambda_{F'}}{\kappa^4} R^{\nu'\beta'}{}_{\nu\beta} F^{a\rho'\sigma'} F^a_{\rho\sigma} \\
&\quad + 2\tfrac{\lambda_{R'}\tilde{\lambda}_F}{\kappa^4} R^{\nu'\beta'}{}_{\nu\beta} F^{a\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^a_{\tau'\kappa'}\Big] \\
&\quad + Tr\Big[2\tfrac{\lambda_{R'}}{\kappa^4} R^{\nu'\beta'}{}_{\nu\beta} X^{\rho'\sigma'}{}_{\rho\sigma} + 2\tfrac{\lambda_{F'}\tilde{\lambda}_F}{\kappa^4} F^{a\nu'\beta'} F^a_{\nu\beta} F^{b\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^b_{\tau'\kappa'} \\
&\quad + 2\tfrac{\lambda_{F'}}{\kappa^4} F^{a\nu'\beta'} F^a_{\nu\beta} X^{\rho'\sigma'}{}_{\rho\sigma}\Big] \\
&\quad + Tr\left[X^{\rho'\sigma'}{}_{\rho\sigma} F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa}\right]
\end{aligned} \tag{6.14}$$

$$
\begin{aligned}
&= \tfrac{\lambda_{R'}^2}{\kappa^4} Tr\left[R^{\nu'\beta'}{}_{\nu\beta} R^{\rho'\sigma'}{}_{\rho\sigma}\right] + \tfrac{\lambda_{F'}^2}{\kappa^4} Tr\left[F^{a\nu'\beta'} F^a_{\nu\beta} F^{b\rho'\sigma'} F^b_{\rho\sigma}\right] \\
&+ \tfrac{\tilde{\lambda}_F^2}{\kappa^4} Tr\left[F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa} F^{b\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^b_{\tau'\kappa'}\right] \\
&+ \tfrac{1}{\kappa^4} Tr\left[X^{\nu'\beta'}{}_{\nu\beta} X^{\rho'\sigma'}{}_{\rho\sigma}\right] + 2\tfrac{\lambda_{R'}\lambda_{F'}}{\kappa^4} Tr\left[R^{\nu'\beta'}{}_{\nu\beta} F^{a\rho'\sigma'} F^a_{\rho\sigma}\right] \\
&+ 2\tfrac{\lambda_{R'}\tilde{\lambda}_F}{\kappa^4} Tr\left[R^{\nu'\beta'}{}_{\nu\beta} F^{a\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^a_{\tau'\kappa'}\right] \\
&+ 2\tfrac{\lambda_{R'}}{\kappa^4} Tr\left[R^{\nu'\beta'}{}_{\nu\beta} X^{\rho'\sigma'}{}_{\rho\sigma}\right] + 2\tfrac{\lambda_{F'}\tilde{\lambda}_F}{\kappa^4} Tr\left[F^{a\nu'\beta'} F^a_{\nu\beta} F^{b\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^b_{\tau'\kappa'}\right] \\
&+ 2\tfrac{\lambda_{F'}}{\kappa^4} Tr\left[F^{a\nu'\beta'} F^a_{\nu\beta} X^{\rho'\sigma'}{}_{\rho\sigma}\right] + \tfrac{\tilde{\lambda}_F}{\kappa^4} Tr\left[X^{\rho'\sigma'}{}_{\rho\sigma} F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa}\right] \qquad (6.15)
\end{aligned}
$$

Calculation of the last term;

$$
\begin{aligned}
&\left[Tr\left(\tfrac{\lambda_{R'}}{\kappa^2} R^{\nu'\beta'}{}_{\nu\beta} + \tfrac{1}{\kappa^2} X^{\nu'\beta'}{}_{\nu\beta} + \tfrac{\lambda'_F}{\kappa^2} F^{a\nu'\beta'} F^a_{\nu\beta} + \tfrac{\tilde{\lambda}_F}{\kappa^2} F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa}\right)\right]^2 \\
&= \tfrac{\lambda_{R'}^2}{\kappa^4}\left[Tr\left(R^{\nu'\beta'}{}_{\nu\beta}\right)\right]^2 + \tfrac{\lambda_{F'}^2}{\kappa^4}\left[Tr\left(F^{a\nu'\beta'} F^a_{\nu\beta}\right)\right]^2 \\
&+ \tfrac{\tilde{\lambda}_F^2}{\kappa^4}\left[Tr\left(F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa}\right)\right]^2 + \tfrac{1}{\kappa^4}\left[Tr\left(X^{\nu'\beta'}{}_{\nu\beta}\right)\right]^2 \\
&+ 2\tfrac{\lambda_{R'}\lambda_{F'}}{\kappa^4} Tr\left(R^{\nu'\beta'}{}_{\nu\beta}\right) Tr\left(F^{a\rho'\sigma'} F^a_{\rho\sigma}\right) \\
&+ 2\tfrac{\lambda_{R'}\tilde{\lambda}_F}{\kappa^4} Tr\left(R^{\rho'\sigma'}{}_{\rho\sigma}\right) Tr\left(sF^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa}\right) \\
&+ 2\tfrac{\lambda_{R'}}{\kappa^4} Tr\left(R^{\nu'\beta'}{}_{\nu\beta}\right) Tr\left(X^{\rho'\sigma'}{}_{\rho\sigma}\right) \\
&+ 2\tfrac{\lambda_{F'}\tilde{\lambda}_F}{\kappa^4} Tr\left(F^{a\nu'\beta'} F^a_{\nu\beta}\right) Tr\left(F^{b\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^b_{\tau'\kappa'}\right) \\
&+ 2\tfrac{\lambda_{F'}}{\kappa^4} Tr\left(F^{a\nu'\beta'} F^a_{\nu\beta}\right) Tr\left(X^{\rho'\sigma'}{}_{\rho\sigma}\right) \\
&+ 2\tfrac{\tilde{\lambda}_F^2}{\kappa^4} Tr\left(F^{b\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^b_{\tau'\kappa'}\right) Tr\left(X^{\nu'\beta'}{}_{\nu\beta}\right) \qquad (6.16)
\end{aligned}
$$

With all off the calculated terms equation (6.13) takes the form;

$$\begin{aligned}
DDet(I+A) = 1 &+ \frac{\lambda_{R'}}{\kappa^2} Tr\left[R^{\nu'\beta'}{}_{\nu\beta}\right] + \frac{1}{\kappa^2} Tr\left[X^{\nu'\beta'}{}_{\nu\beta}\right] + \frac{\lambda'_F}{\kappa^2} Tr\left[F^{a\nu'\beta'} F^a_{\nu\beta}\right] \\
&+ \frac{\tilde{\lambda}_F}{\kappa^2} Tr\left[F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa}\right] - \frac{\lambda^2_{R'}}{2\kappa^4} Tr\left[R^{\nu'\beta'}{}_{\nu\beta} R^{\rho'\sigma'}{}_{\rho\sigma}\right] \\
&- \frac{\lambda^2_{F'}}{2\kappa^4} Tr\left[F^{a\nu'\beta'} F^a_{\nu\beta} F^{b\rho'\sigma'} F^b_{\rho\sigma}\right] \\
&- \frac{\tilde{\lambda}^2_F}{2\kappa^4} Tr\left[F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa} F^{b\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^b_{\tau'\kappa'}\right] \\
&- \frac{\tilde{\lambda}^2_F}{2\kappa^4} Tr\left[F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa} F^{b\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^b_{\tau'\kappa'}\right] \\
&- \frac{1}{2\kappa^4} Tr\left[X^{\nu'\beta'}{}_{\nu\beta} X^{\rho'\sigma'}{}_{\rho\sigma}\right] - \frac{\lambda_{R'}\lambda_{F'}}{\kappa^4} Tr\left[R^{\nu'\beta'}{}_{\nu\beta} F^{a\rho'\sigma'} F^a_{\rho\sigma}\right] \\
&- \frac{\lambda_{R'}\tilde{\lambda}_F}{\kappa^4} Tr\left[R^{\nu'\beta'}{}_{\nu\beta} F^{a\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^a_{\tau'\kappa'}\right] \\
&- \frac{\lambda_{R'}}{\kappa^4} Tr\left[R^{\nu'\beta'}{}_{\nu\beta} X^{\rho'\sigma'}{}_{\rho\sigma}\right] \\
&- \frac{\lambda_{F'}\tilde{\lambda}_F}{\kappa^4} Tr\left[F^{a\nu'\beta'} F^a_{\nu\beta} F^{b\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^b_{\tau'\kappa'}\right] \\
&- \frac{\lambda_{F'}}{\kappa^4} Tr\left[F^{a\nu'\beta'} F^a_{\nu\beta} X^{\rho'\sigma'}{}_{\rho\sigma}\right] - \frac{\tilde{\lambda}_F}{\kappa^4} Tr\left[X^{\rho'\sigma'}{}_{\rho\sigma} F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa}\right] \\
&+ \frac{\lambda^2_{R'}}{2\kappa^4}\left[Tr\left(R^{\nu'\beta'}{}_{\nu\beta}\right)\right]^2 + \frac{\lambda^2_{F'}}{2\kappa^4}\left[Tr\left(F^{a\nu'\beta'} F^a_{\nu\beta}\right)\right]^2 \\
&+ \frac{\tilde{\lambda}^2_F}{2\kappa^4}\left[Tr\left(F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa}\right)\right]^2 + \frac{1}{2\kappa^4}\left[Tr\left(X^{\nu'\beta'}{}_{\nu\beta}\right)\right]^2 \\
&+ \frac{\lambda_{R'}\lambda_{F'}}{\kappa^4} Tr\left(R^{\nu'\beta'}{}_{\nu\beta}\right) Tr\left(F^{a\rho'\sigma'} F^a_{\rho\sigma}\right) \\
&+ \frac{\lambda_{R'}\tilde{\lambda}_F}{\kappa^4} Tr\left(R^{\rho'\sigma'}{}_{\rho\sigma}\right) Tr\left(F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa}\right) \\
&+ \frac{\lambda_{R'}}{\kappa^4} Tr\left(R^{\nu'\beta'}{}_{\nu\beta}\right) Tr\left(X^{\rho'\sigma'}{}_{\rho\sigma}\right) \\
&+ \frac{\lambda_{F'}\tilde{\lambda}_F}{\kappa^4} Tr\left(F^{a\nu'\beta'} F^a_{\nu\beta}\right) Tr\left(F^{b\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^b_{\tau'\kappa'}\right) \\
&+ \frac{\lambda_{F'}}{\kappa^4} Tr\left(F^{a\nu'\beta'} F^a_{\nu\beta}\right) Tr\left(X^{\rho'\sigma'}{}_{\rho\sigma}\right) \\
&+ \frac{\tilde{\lambda}^2_F}{\kappa^4} Tr\left(F^{b\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^b_{\tau'\kappa'}\right) Tr\left(X^{\nu'\beta'}{}_{\nu\beta}\right) \qquad (6.17)
\end{aligned}$$

Taking care of binomial expansion;

$$(1+x)^{1/4} = 1 + \frac{x}{4} - \frac{3}{32}x^2 + \mathcal{O}(3) \qquad (6.18)$$

Applying binomial expansion to (6.17);

$$
\begin{aligned}
\left(DDet\left(I+A\right)\right)^{1/4} = 1 + \tfrac{\lambda_{R'}}{4\kappa^2} Tr\left[R^{\nu'\beta'}{}_{\nu\beta}\right] + \tfrac{1}{4\kappa^2} Tr\left[X^{\nu'\beta'}{}_{\nu\beta}\right] \\
+\tfrac{\lambda'_F}{4\kappa^2} Tr\left[F^{a\nu'\beta'}F^a_{\nu\beta}\right] + \tfrac{\tilde{\lambda}_F}{4\kappa^2} Tr\left[F^{a\nu'\beta'}\epsilon_{\nu\beta}{}^{\tau\kappa}F^a_{\tau\kappa}\right] - \tfrac{\lambda^2_{R'}}{8\kappa^4} Tr\left[R^{\nu'\beta'}{}_{\nu\beta}R^{\rho'\sigma'}{}_{\rho\sigma}\right] \\
-\tfrac{\lambda^2_{F'}}{8\kappa^4} Tr\left[F^{a\nu'\beta'}F^a_{\nu\beta}F^{b\rho'\sigma'}F^b_{\rho\sigma}\right] - \tfrac{\tilde{\lambda}^2_F}{8\kappa^4} Tr\left[F^{a\nu'\beta'}\epsilon_{\nu\beta}{}^{\tau\kappa}F^a_{\tau\kappa}F^{b\rho'\sigma'}\epsilon_{\rho\sigma}{}^{\tau'\kappa'}F^b_{\tau'\kappa'}\right] \\
-\tfrac{\tilde{\lambda}^2_F}{8\kappa^4} Tr\left[F^{a\nu'\beta'}\epsilon_{\nu\beta}{}^{\tau\kappa}F^a_{\tau\kappa}F^{b\rho'\sigma'}\epsilon_{\rho\sigma}{}^{\tau'\kappa'}F^b_{\tau'\kappa'}\right] \\
-\tfrac{1}{8\kappa^4} Tr\left[X^{\nu'\beta'}{}_{\nu\beta}X^{\rho'\sigma'}{}_{\rho\sigma}\right] - \tfrac{\lambda_{R'}\lambda_{F'}}{4\kappa^4} Tr\left[R^{\nu'\beta'}{}_{\nu\beta}F^{a\rho'\sigma'}F^a_{\rho\sigma}\right] \\
-\tfrac{\lambda_{R'}\tilde{\lambda}_F}{4\kappa^4} Tr\left[R^{\nu'\beta'}{}_{\nu\beta}F^{a\rho'\sigma'}\epsilon_{\rho\sigma}{}^{\tau'\kappa'}F^a_{\tau'\kappa'}\right] - \tfrac{\lambda_{R'}}{4\kappa^4} Tr\left[R^{\nu'\beta'}{}_{\nu\beta}X^{\rho'\sigma'}{}_{\rho\sigma}\right] \\
-\tfrac{\lambda_{F'}\tilde{\lambda}_F}{4\kappa^4} Tr\left[F^{a\nu'\beta'}F^a_{\nu\beta}F^{b\rho'\sigma'}\epsilon_{\rho\sigma}{}^{\tau'\kappa'}F^b_{\tau'\kappa'}\right] - \tfrac{\lambda_{F'}}{4\kappa^4} Tr\left[F^{a\nu'\beta'}F^a_{\nu\beta}X^{\rho'\sigma'}{}_{\rho\sigma}\right] \\
-\tfrac{\tilde{\lambda}_F}{4\kappa^4} Tr\left[X^{\rho'\sigma'}{}_{\rho\sigma}F^{a\nu'\beta'}\epsilon_{\nu\beta}{}^{\tau\kappa}F^a_{\tau\kappa}\right] + \tfrac{\lambda^2_{R'}}{32\kappa^4}\left[Tr\left(R^{\nu'\beta'}{}_{\nu\beta}\right)\right]^2 \\
+\tfrac{\lambda^2_{F'}}{32\kappa^4}\left[Tr\left(F^{a\nu'\beta'}F^a_{\nu\beta}\right)\right]^2 + \tfrac{\tilde{\lambda}^2_F}{32\kappa^4}\left[Tr\left(F^{a\nu'\beta'}\epsilon_{\nu\beta}{}^{\tau\kappa}F^a_{\tau\kappa}\right)\right]^2 \\
+\tfrac{1}{32\kappa^4}\left[Tr\left(X^{\nu'\beta'}{}_{\nu\beta}\right)\right]^2 + \tfrac{\lambda_{R'}\lambda_{F'}}{16\kappa^4} Tr\left(R^{\nu'\beta'}{}_{\nu\beta}\right) Tr\left(F^{a\rho'\sigma'}F^a_{\rho\sigma}\right) \\
+\tfrac{\lambda_{R'}\tilde{\lambda}_F}{16\kappa^4} Tr\left(R^{\rho'\sigma'}{}_{\rho\sigma}\right) Tr\left(F^{a\nu'\beta'}\epsilon_{\nu\beta}{}^{\tau\kappa}F^a_{\tau\kappa}\right) + \tfrac{\lambda_{R'}}{16\kappa^4} Tr\left(R^{\nu'\beta'}{}_{\nu\beta}\right) Tr\left(X^{\rho'\sigma'}{}_{\rho\sigma}\right) \\
+\tfrac{\lambda_{F'}\tilde{\lambda}_F}{16\kappa^4} Tr\left(F^{a\nu'\beta'}F^a_{\nu\beta}\right) Tr\left(F^{b\rho'\sigma'}\epsilon_{\rho\sigma}{}^{\tau'\kappa'}F^b_{\tau'\kappa'}\right) \\
+\tfrac{\lambda_{F'}}{16\kappa^4} Tr\left(F^{a\nu'\beta'}F^a_{\nu\beta}\right) Tr\left(X^{\rho'\sigma'}{}_{\rho\sigma}\right) \\
+\tfrac{\tilde{\lambda}^2_F}{16\kappa^4} Tr\left(F^{b\rho'\sigma'}\epsilon_{\rho\sigma}{}^{\tau'\kappa'}F^b_{\tau'\kappa'}\right) Tr\left(X^{\nu'\beta'}{}_{\nu\beta}\right)
\end{aligned}
\tag{6.19}
$$

In the presence of these, our effective action (6.7) takes the form;

$$
\begin{aligned}
\mathcal{S}_{eff} = -\int \mathrm{d}^4x C_{DD} M_D^2 \kappa^2 (|-g|)^{1/2} \Big[1 &+ \frac{\lambda_{R'}}{4\kappa^2} Tr\left[R^{\nu'\beta'}{}_{\nu\beta}\right] + \frac{1}{4\kappa^2} Tr\left[X^{\nu'\beta'}{}_{\nu\beta}\right] \\
&+ \frac{\lambda'_F}{4\kappa^2} Tr\left[F^{a\nu'\beta'} F^a_{\nu\beta}\right] + \frac{\tilde{\lambda}_F}{4\kappa^2} Tr\left[F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa}\right] \\
&- \frac{\lambda^2_{R'}}{8\kappa^4} Tr\left[R^{\nu'\beta'}{}_{\nu\beta} R^{\rho'\sigma'}{}_{\rho\sigma}\right] \\
&- \frac{\lambda^2_{F'}}{8\kappa^4} Tr\left[F^{a\nu'\beta'} F^a_{\nu\beta} F^{b\rho'\sigma'} F^b_{\rho\sigma}\right] \\
&- \frac{\tilde{\lambda}^2_F}{8\kappa^4} Tr\left[F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa} F^{b\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^b_{\tau'\kappa'}\right] \\
&- \frac{\tilde{\lambda}^2_F}{8\kappa^4} Tr\left[F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa} F^{b\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^b_{\tau'\kappa'}\right] \\
&- \frac{1}{8\kappa^4} Tr\left[X^{\nu'\beta'}{}_{\nu\beta} X^{\rho'\sigma'}{}_{\rho\sigma}\right] \\
&- \frac{\lambda_{R'}\lambda_{F'}}{4\kappa^4} Tr\left[R^{\nu'\beta'}{}_{\nu\beta} F^{a\rho'\sigma'} F^a_{\rho\sigma}\right] \\
&- \frac{\lambda_{R'}\tilde{\lambda}_F}{4\kappa^4} Tr\left[R^{\nu'\beta'}{}_{\nu\beta} F^{a\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^a_{\tau'\kappa'}\right] \\
&- \frac{\lambda_{R'}}{4\kappa^4} Tr\left[R^{\nu'\beta'}{}_{\nu\beta} X^{\rho'\sigma'}{}_{\rho\sigma}\right] \\
&- \frac{\lambda_{F'}\tilde{\lambda}_F}{4\kappa^4} Tr\left[F^{a\nu'\beta'} F^a_{\nu\beta} F^{b\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^b_{\tau'\kappa'}\right] \\
&- \frac{\lambda_{F'}}{4\kappa^4} Tr\left[F^{a\nu'\beta'} F^a_{\nu\beta} X^{\rho'\sigma'}{}_{\rho\sigma}\right] \\
&- \frac{\tilde{\lambda}_F}{4\kappa^4} Tr\left[X^{\rho'\sigma'}{}_{\rho\sigma} F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa}\right] + \frac{\lambda^2_{R'}}{32\kappa^4}\left[Tr\left(R^{\nu'\beta'}{}_{\nu\beta}\right)\right]^2 \\
&+ \frac{\lambda^2_{F'}}{32\kappa^4}\left[Tr\left(F^{a\nu'\beta'} F^a_{\nu\beta}\right)\right]^2 + \frac{\tilde{\lambda}^2_F}{32\kappa^4}\left[Tr\left(F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa}\right)\right]^2 \\
&+ \frac{1}{32\kappa^4}\left[Tr\left(X^{\nu'\beta'}{}_{\nu\beta}\right)\right]^2 + \frac{\lambda_{R'}\lambda_{F'}}{16\kappa^4} Tr\left(R^{\nu'\beta'}{}_{\nu\beta}\right) Tr\left(F^{a\rho'\sigma'} F^a_{\rho\sigma}\right] \\
&+ \frac{\lambda_{R'}\tilde{\lambda}_F}{16\kappa^4} Tr\left(R^{\rho'\sigma'}{}_{\rho\sigma}\right) Tr\left(F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa}\right) \\
&+ \frac{\lambda_{R'}}{16\kappa^4} Tr\left(R^{\nu'\beta'}{}_{\nu\beta}\right) Tr\left(X^{\rho'\sigma'}{}_{\rho\sigma}\right) \\
&+ \frac{\lambda_{F'}\tilde{\lambda}_F}{16\kappa^4} Tr\left(F^{a\nu'\beta'} F^a_{\nu\beta}\right) Tr\left(F^{b\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^b_{\tau'\kappa'}\right) \\
&+ \frac{\lambda_{F'}}{16\kappa^4} Tr\left(F^{a\nu'\beta'} F^a_{\nu\beta}\right) Tr\left(X^{\rho'\sigma'}{}_{\rho\sigma}\right) \\
&+ \frac{\tilde{\lambda}^2_F}{16\kappa^4} Tr\left(F^{b\rho'\sigma'} \epsilon_{\rho\sigma}{}^{\tau'\kappa'} F^b_{\tau'\kappa'}\right) Tr\left(X^{\nu'\beta'}{}_{\nu\beta}\right) \Big]
\end{aligned} \quad (6.20)
$$

We should cancel out the ghosty terms as in previous chapter via our rank-4 arbitrary tensor field $X_{\mu\alpha\nu\beta}$.

$$\begin{aligned}\frac{1}{4\kappa^2}X &= \frac{1}{8\kappa^4}\left(R^{\nu'\beta'}{}_{\nu\beta}R_{\nu'\beta'}{}^{\nu\beta} - \frac{R^2}{4}\right)\\ X &= \frac{1}{2\kappa^2}\left(R^{\nu'\beta'}{}_{\nu\beta}R_{\nu'\beta'}{}^{\nu\beta} - \frac{R^2}{4}\right)\end{aligned} \tag{6.21}$$

Assume that;

$$X^{\nu'\beta'}{}_{\nu\beta} = A\frac{\lambda_{R'}}{2\kappa^2}\left(R^{\nu'\beta'}{}_{\nu\beta}R_{\nu'\beta'}{}^{\nu\beta} - \frac{R^2}{4}\right)\left(\delta^{\nu'}_{\nu}\delta^{\beta'}_{\beta} - \delta^{\beta'}_{\nu}\delta^{\nu'}_{\beta}\right) \tag{6.22}$$

The trace of $X_{\mu\alpha\nu\beta}$;

$$\begin{aligned}X &= \tfrac{1}{2}X^{\nu'\beta'}{}_{\nu\beta}\left(\delta^{\nu'}_{\nu}\delta^{\beta'}_{\beta} - \delta^{\beta'}_{\nu}\delta^{\nu'}_{\beta}\right)\\ X &= \tfrac{1}{2}A\tfrac{\lambda_{R'}}{2\kappa^2}\left(R^{\nu'\beta'}{}_{\nu\beta}R_{\nu'\beta'}{}^{\nu\beta} - \tfrac{R^2}{4}\right)\left(\delta^{\nu'}_{\nu}\delta^{\beta'}_{\beta} - \delta^{\beta'}_{\nu}\delta^{\nu'}_{\beta}\right)\left(\delta^{\nu}_{\nu'}\delta^{\beta}_{\beta'} - \delta^{\nu}_{\beta'}\delta^{\beta}_{\nu'}\right)\\ A &= \tfrac{1}{D(D-1)}\end{aligned} \tag{6.23}$$

D=4 $\Rightarrow A = \frac{1}{12}$. Hence,

$$X^{\nu'\beta'}{}_{\nu\beta} = \frac{\lambda_{R'}}{24\kappa^2}\left(R^{\nu'\beta'}{}_{\nu\beta}R_{\nu'\beta'}{}^{\nu\beta} - \frac{R^2}{4}\right) \tag{6.24}$$

After inserting equation (6.24) into effective action (6.20) and cutting our expansion at second order, the effective action takes the form as;

$$
\begin{aligned}
\mathcal{S}_{eff} = -\int \mathrm{d}^4x C_{DD} M_D^2 \kappa^2 (|-g|)^{1/2} \Big[1 + \tfrac{\lambda_{R'}}{4\kappa^2} R \\
+ \tfrac{\lambda'_F}{4\kappa^2} F^{a\nu\beta} F^a_{\nu\beta} + \tfrac{\tilde{\lambda}_F}{4\kappa^2} F^{a\nu\beta} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa} \\
- \tfrac{\lambda^2_{F'}}{8\kappa^4} \left[F^{a\nu'\beta'} F^a_{\nu\beta} F_{b\nu'\beta'} F^{b\nu\beta} - \tfrac{1}{4} \left(F^a_{\nu\beta} F^{a\nu\beta} \right)^2 \right] \\
- \tfrac{\tilde{\lambda}^2_F}{8\kappa^4} \left[F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa} F^{b\nu'\beta'} \epsilon^{\nu\beta}{}_{\rho\sigma} F^{b\rho\sigma} - \tfrac{1}{4} \left(F^{a\nu\beta} \epsilon^{\nu\beta}{}_{\tau\kappa} F^a_{\tau\kappa} \right)^2 \right] \\
- \tfrac{\lambda_{R'} \tilde{\lambda}_F}{4\kappa^4} \left[R^{\nu'\beta'}{}_{\nu\beta} F^{a\tau'\kappa'} \epsilon^{\nu\beta}{}_{\tau\kappa} F^{a\tau\kappa} - \tfrac{R}{4} F^{a\nu\beta} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa} \right] \\
- \tfrac{\lambda_{F'} \tilde{\lambda}_F}{4\kappa^4} \left[F^{a\nu'\beta'} F^a_{\nu\beta} F_{b\nu'\beta'} \epsilon^{\nu\beta}{}_{\tau\kappa} F^{b\tau\kappa} - \tfrac{1}{4} F^{a\eta\theta} F^a_{\eta\theta} F^{b\nu\beta} \epsilon_{\nu\beta}{}^{\tau\kappa} F^b_{\tau\kappa} \right] \\
+ \tfrac{\lambda_{R'} \lambda_{F'}}{4\kappa^4} \left(R^{\nu'\beta'}{}_{\nu\beta} F^a_{\nu'\beta'} F^{a\nu\beta} F^a_{\nu'\beta'} - \tfrac{R}{4} F^{a\nu\beta} F_{a\nu\beta} \right) \Big] \\
+ \mathcal{O}(A^3)
\end{aligned}
\tag{6.25}
$$

There are two possibilities of gravitational constant and constant curvature. When they are equal to each other $M_D = \kappa$;

$$
\begin{aligned}
\mathcal{S}_{eff} = -\int \mathrm{d}^4x C_{DD} (|-g|)^{1/2} \Big[\kappa^4 + \kappa^2 \tfrac{\lambda_{R'}}{4} R \\
+ \kappa^2 \tfrac{\lambda'_F}{4} F^{a\nu\beta} F^a_{\nu\beta} + \kappa^2 \tfrac{\tilde{\lambda}_F}{4} F^{a\nu\beta} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa} \\
- \tfrac{\lambda^2_{F'}}{8} \left[F^{a\nu'\beta'} F^a_{\nu\beta} F_{b\nu'\beta'} F^{b\nu\beta} - \tfrac{1}{4} \left(F^a_{\nu\beta} F^{a\nu\beta} \right)^2 \right] \\
- \tfrac{\tilde{\lambda}^2_F}{8} \left[F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa} F^{b\nu'\beta'} \epsilon^{\nu\beta}{}_{\rho\sigma} F^{b\rho\sigma} - \tfrac{1}{4} \left(F^{a\nu\beta} \epsilon^{\nu\beta}{}_{\tau\kappa} F^a_{\tau\kappa} \right)^2 \right] \\
- \tfrac{\lambda_{R'} \tilde{\lambda}_F}{4\kappa^4} \left[R^{\nu'\beta'}{}_{\nu\beta} F^{a\tau'\kappa'} \epsilon^{\nu\beta}{}_{\tau\kappa} F^{a\tau\kappa} - \tfrac{R}{4} F^{a\nu\beta} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa} \right] \\
- \tfrac{\lambda_{F'} \tilde{\lambda}_F}{4\kappa^4} \left[F^{a\nu'\beta'} F^a_{\nu\beta} F_{b\nu'\beta'} \epsilon^{\nu\beta}{}_{\tau\kappa} F^{b\tau\kappa} - \tfrac{1}{4} F^{a\eta\theta} F^a_{\eta\theta} F^{b\nu\beta} \epsilon_{\nu\beta}{}^{\tau\kappa} F^b_{\tau\kappa} \right] \\
+ \tfrac{\lambda_{R'} \lambda_{F'}}{4\kappa^4} \left(R^{\nu'\beta'}{}_{\nu\beta} F^a_{\nu'\beta'} F^{a\nu\beta} F^a_{\nu'\beta'} - \tfrac{R}{4} F^{a\nu\beta} F_{a\nu\beta} \right) \Big] \\
+ \mathcal{O}(A^3)
\end{aligned}
\tag{6.26}
$$

When they do not equal to each other $M_D \neq \kappa$;

$$\begin{aligned}\mathcal{S}_{eff} = -\int \mathrm{d}^4 x C_{DD} M_D^2 (|-g|)^{1/2} \Big[\kappa^2 + \tfrac{\lambda_{R'}}{4} R \\ + \tfrac{\lambda'_F}{4} F^{a\nu\beta} F^a_{\nu\beta} + \tfrac{\tilde{\lambda}_F}{4} F^{a\nu\beta} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa} \\ - \tfrac{\lambda^2_{F'}}{8\kappa^2} \left[F^{a\nu'\beta'} F^a_{\nu\beta} F_{b\nu'\beta'} F^{b\nu\beta} - \tfrac{1}{4} \left(F^a_{\nu\beta} F^{a\nu\beta} \right)^2 \right] \\ - \tfrac{\tilde{\lambda}^2_F}{8\kappa^2} \left[F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa} F^{b\nu'\beta'} \epsilon^{\nu\beta}{}_{\rho\sigma} F^{b\rho\sigma} - \tfrac{1}{4} \left(F^{a\nu\beta} \epsilon^{\nu\beta}{}_{\tau\kappa} F^a_{\tau\kappa} \right)^2 \right] \\ - \tfrac{\lambda_{R'} \tilde{\lambda}_F}{4\kappa^2} \left[R^{\nu'\beta'}{}_{\nu\beta} F^{a\tau'\kappa'} \epsilon^{\nu\beta}{}_{\tau\kappa} F^{a\tau\kappa} - \tfrac{R}{4} F^{a\nu\beta} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa} \right] \\ - \tfrac{\lambda_{F'} \tilde{\lambda}_F}{4\kappa^2} \left[F^{a\nu'\beta'} F^a_{\nu\beta} F_{b\nu'\beta'} \epsilon^{\nu\beta}{}_{\tau\kappa} F^{b\tau\kappa} - \tfrac{1}{4} F^{a\eta\theta} F^a_{\eta\theta} F^{b\nu\beta} \epsilon_{\nu\beta}{}^{\tau\kappa} F^b_{\tau\kappa} \right] \\ + \tfrac{\lambda_{R'} \lambda_{F'}}{4\kappa^2} \left(R^{\nu'\beta'}{}_{\nu\beta} F^a_{\nu'\beta'} F^{a\nu\beta} F^a_{\nu'\beta'} - \tfrac{R}{4} F^{a\nu\beta} F_{a\nu\beta} \right) \Big] \\ + \mathcal{O}(A^3)\end{aligned} \tag{6.27}$$

We are going to consider (6.27) in our work.

6.2. Relation to Vector Inflation

Our goal is to derive dynamical equations for homogeneous and isotropic universe, for FRW background and to demonstrate inflation comes naturally from our effective action. To do that we should calculate traces of tensor fields in FRW background [44] as a second step.

$$\mathcal{S}_{eff} = -\int \mathrm{d}^4x C_{DD} M_D^2 (|-g|)^{1/2} \Big[\kappa^2 + \frac{\lambda_{R'}}{4} R + \frac{\lambda'_F}{4} \underbrace{F^{a\nu\beta} F^a_{\nu\beta}}_{1} + \frac{\tilde{\lambda}_F}{4} \underbrace{F^{a\nu\beta} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa}}_{2}$$
$$-\frac{\lambda^2_{F'}}{8\kappa^2} \left[\underbrace{F^{a\nu'\beta'} F^a_{\nu\beta} F_{b\nu'\beta'} F^{b\nu\beta}}_{3} - \frac{1}{4} \left(F^a_{\nu\beta} F^{a\nu\beta} \right)^2 \right]$$
$$-\frac{\tilde{\lambda}^2_F}{8\kappa^2} \left[\underbrace{F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa} F^{b\nu'\beta'} \epsilon^{\nu\beta}{}_{\rho\sigma} F^{b\rho\sigma}}_{4} - \frac{1}{4} \left(F^{a\nu\beta} \epsilon^{\nu\beta}{}_{\tau\kappa} F^a_{\tau\kappa} \right)^2 \right]$$
$$-\frac{\lambda_{R'} \tilde{\lambda}_F}{4\kappa^2} \left[\underbrace{R^{\nu'\beta'}{}_{\nu\beta} F^{a\tau'\kappa'} \epsilon^{\nu\beta}{}_{\tau\kappa} F^{a\tau\kappa}}_{6} - \frac{R}{4} F^{a\nu\beta} \epsilon_{\nu\beta}{}^{\tau\kappa} F^a_{\tau\kappa} \right]$$
$$-\frac{\lambda_{F'} \tilde{\lambda}_F}{4\kappa^2} \left[\underbrace{F^{a\nu'\beta'} F^a_{\nu\beta} F_{b\nu'\beta'} \epsilon^{\nu\beta}{}_{\tau\kappa} F^{b\tau\kappa}}_{7} - \frac{1}{4} F^{a\eta\theta} F^a_{\eta\theta} F^{b\nu\beta} \epsilon_{\nu\beta}{}^{\tau\kappa} F^b_{\tau\kappa} \right]$$
$$+\frac{\lambda_{R'} \lambda_{F'}}{4\kappa^2} \left(\underbrace{R^{\nu'\beta'}{}_{\nu\beta} F^a_{\nu'\beta'} F^{a\nu\beta}}_{5} - \frac{R}{4} F^{a\nu\beta} F_{a\nu\beta} \right) \Big] + \mathcal{O}(A^3) \tag{6.28}$$

Calculation of the first term;

$$\begin{aligned}
F^{a\nu\beta} F^a_{0\beta} &= F^{a0\beta} F^a_{0\beta} + F^{ai\beta} F^a_{i\beta} \\
&= F^{a00} F^a_{00} + F^{a0i} F^a_{0i} + F^{ai0} F^a_{i0} + F^{aij} F^a_{ij} \\
&= -a^{-2} \dot{\phi} \delta^a_j \delta ji \dot{\phi} \delta^a_i + a^{-2} \dot{\phi} \delta^a_j \delta ji \dot{\phi} \delta^a_i \\
&+ a^{-4} g \phi^2 \delta ki \delta lj \epsilon^a{}_{kl} g \phi^2 \epsilon^a{}_{ij} \\
&= -a^{-2} \dot{\phi}^2 \delta^{aa} - a^{-2} \dot{\phi}^2 \delta^{aa} + a^{-4} g^2 \phi^2 \epsilon^a{}_{kl} \epsilon^a{}_{kl} \\
&= -6a^{-2} \dot{\phi}^2 + 6a^{-4} g^2 \phi^4
\end{aligned} \tag{6.29}$$

Calculation of the second term;

$$\begin{aligned}
F^{a\nu\beta}\epsilon_{\nu\beta}{}^{\tau\kappa}F^{a}_{\tau\kappa} &= F^{a0\beta}\epsilon_{0\beta}{}^{\tau\kappa}F^{a}_{\tau\kappa} + F^{ai\beta}\epsilon_{i\beta}{}^{\tau\kappa}F^{a}_{\tau\kappa} \\
&= F^{a0i}\epsilon_{0i}{}^{\tau\kappa}F^{a}_{\tau\kappa} + F^{a0i}\epsilon_{0i}{}^{\tau\kappa}F^{a}_{\tau\kappa} \\
&+ F^{ai0}\epsilon_{i0}{}^{\tau\kappa}F^{a}_{\tau\kappa} + F^{aij}\epsilon_{ij}{}^{\tau\kappa}F^{a}_{\tau\kappa} \\
&= -a^{-2}\dot{\phi}\delta^{a}_{i}\epsilon_{0i}{}^{j\kappa}F^{a}_{j\kappa} + a^{-2}\dot{\phi}\delta^{a}_{i}\epsilon_{i0}{}^{j\kappa}F^{a}_{j\kappa} \\
&- a^{-4}g\phi^{2}\epsilon^{a}{}_{ij}\left(\epsilon_{ij}{}^{0\kappa}F^{a}_{0\kappa} + \epsilon_{ij}{}^{k\kappa}F^{a}_{k\kappa}\right) \\
&= -a^{-2}\dot{\phi}\delta^{a}_{i}\left(\epsilon_{0i}{}^{jk}F^{a}_{jk} - \epsilon_{i0}{}^{jk}F^{a}_{jk}\right) \\
&- a^{-4}g\phi^{2}\epsilon^{a}{}_{ij}\left(\epsilon_{ij}{}^{0k}F^{a}_{0k} + \epsilon_{ij}{}^{k0}F^{a}_{k0} + \epsilon_{ij}{}^{kl}F^{a}_{kl}\right) \\
&= 2a^{-2}g\dot{\phi}\phi^{2}\epsilon_{i}^{jk}\left(a^{-2}\epsilon_{0i}{}^{jk} + a^{-4}\epsilon_{jk}{}^{i0}\right) \\
&= 2g\dot{\phi}\phi^{2}\epsilon_{i}^{jk}\left(a^{-6}\epsilon_{0ijk} + a^{-6}\epsilon_{jki0}\right) \\
&= 24g\dot{\phi}\phi^{2}a^{-6} \qquad (6.30)
\end{aligned}$$

Calculation of the third term;

$$\begin{aligned}
F^{a\nu'\beta'}F^{a}_{\nu\beta}F_{b\nu'\beta'}F^{b\nu\beta} &= \left(F^{a\nu'0}F^{b}_{\nu'0} + F^{a\nu'i}F^{b}_{\nu'i}\right)F^{b\nu\beta}F^{a}_{\nu\beta} \\
&= F^{a}_{\nu\beta}F^{b\nu\beta}\left(F^{ai0}F^{b}_{i0} + F^{a0i}F^{b}_{0i} + F^{aji}F^{b}_{ji}\right) \\
&= F^{a}_{\nu\beta}F^{b\nu\beta}\left(-2a^{-2}g\dot{\phi}^{2}\delta^{ab} + a^{-4}g\phi^{4}\epsilon^{a}{}_{ji}\epsilon^{a}{}_{ji}\right) \\
&= 2\delta^{ab}a^{-4}\left(-a^{2}\dot{\phi}^{2} + g^{2}\phi^{4}\right)\left(F^{a}_{0\beta}F^{b0\beta} + F^{a}_{i\beta}F^{bi\beta}\right) \\
&= 2\delta^{ab}a^{-4}\left(-a^{2}\dot{\phi}^{2} + g^{2}\phi^{4}\right)\left(-2a^{-2}\dot{\phi}^{2}\delta^{ab} + g^{2}\phi^{4}a^{-4}2\delta^{ab}\right) \\
&= 12a^{-8}\left(g^{2}\phi^{4} - a^{2}\dot{\phi}^{2}\right)^{2} \qquad (6.31)
\end{aligned}$$

Calculation of the fourth term;

$$
\begin{aligned}
F^{a\nu'\beta'}\epsilon_{\nu\beta}{}^{\tau\kappa}F^{a}_{\tau\kappa}F^{b}_{\nu'\beta'}\epsilon^{\nu\beta}{}_{\rho\sigma}F^{b\rho\sigma} &= F^{a0\beta'}\epsilon_{\nu\beta}{}^{\tau\kappa}F^{a}_{\tau\kappa}F^{b}_{0\beta'}\epsilon^{\nu\beta}{}_{\rho\sigma}F^{b\rho\sigma} \\
&+ F^{ai\beta'}\epsilon_{\nu\beta}{}^{\tau\kappa}F^{a}_{\tau\kappa}F^{b}_{i\beta'}\epsilon^{\nu\beta}{}_{\rho\sigma}F^{b\rho\sigma} \\
&= F^{a0i}F^{b}_{0i}\epsilon_{\nu\beta}{}^{\tau\kappa}F^{a}_{\tau\kappa}\epsilon^{\nu\beta}{}_{\rho\sigma}F^{b\rho\sigma} \\
&+ \left(F^{ai0}F^{b}_{i0} + F^{aij}F^{b}_{ij}\right)\epsilon_{\nu\beta}{}^{\tau\kappa}F^{a}_{\tau\kappa}\epsilon^{\nu\beta}{}_{\rho\sigma}F^{b\rho\sigma} \\
&= 2\delta^{ab}a^{-4}\left(-a^{2}\dot{\phi}^{2} + g^{2}\phi^{4}\right)\epsilon_{\nu\beta}{}^{\tau\kappa}F^{a}_{\tau\kappa} \\
&\times \left(\epsilon^{\nu\beta}{}_{0\sigma}F^{b0\sigma} + \epsilon^{\nu\beta}{}_{i\sigma}F^{bi\sigma}\right) \\
&= 2\delta^{ab}a^{-4}\left(-a^{2}\dot{\phi}^{2} + g^{2}\phi^{4}\right) \\
&\times \left(-2a^{-2}\dot{\phi}\delta^{b}_{i}\epsilon^{\nu\beta}{}_{0i} - g\phi^{2}a^{-4}\epsilon^{ba}_{ij}\epsilon^{\nu\beta}{}_{ij}\right) \\
&\times \left(\epsilon^{\nu\beta}{}_{0k}F^{a}_{0k} + \epsilon^{\nu\beta}{}_{k0}F^{a}_{k0} + \epsilon^{\nu\beta}{}_{kl}F^{a}_{kl}\right) \\
&= 2a^{-4}\left(g^{2}\phi^{4} - a^{2}\dot{\phi}^{2}\right) \\
&\times \left(24a^{2}\dot{\phi}^{2} - 24a^{-4}g^{2}\phi^{4}\right)
\end{aligned}
\tag{6.32}
$$

Calculation of the fifth term;

$$
\begin{aligned}
R^{\nu'\beta'}{}_{\nu\beta}F^{a}_{\nu'\beta'}F^{a\nu\beta} &= R^{0\beta'}{}_{\nu\beta}F^{a}_{0\prime\beta'}F^{a\nu\beta} \\
&+ R^{i\beta'}{}_{\nu\beta}F^{a}_{i\beta'}F^{a\nu\beta} \\
&= -4\dot{\phi}^{2}a^{-2}\delta_{ij}R^{0i}{}_{0j} + g^{2}\phi^{4}a^{-4}\epsilon^{a}{}_{ij}\epsilon^{a}{}_{kl}R^{ij}{}_{kl} \\
&= 6a^{-4}\left[\left(\frac{\ddot{a}}{a} + a\left(\frac{\dot{a}}{a}\right)^{2}\right)\left(-2\dot{\phi}^{2}a^{2} + g^{2}\phi^{4}\right)\right]
\end{aligned}
\tag{6.33}
$$

Calculation of the sixth term;

$$\begin{aligned}
R^{\nu'\beta'}{}_{\nu\beta}F^{a}_{\nu'\beta'}\epsilon^{\nu\beta}{}_{\tau\kappa}F^{a\tau\kappa} &= \left(R^{0\beta'}{}_{\nu\beta}F^{a}_{0\beta'} + R^{i\beta'}{}_{\nu\beta}F^{a}_{i\beta'}\right) \\
&\times \left(\epsilon^{\nu\beta}{}_{0\kappa}F^{a0\kappa} + \epsilon^{\nu\beta}{}_{j\kappa}F^{aj\kappa}\right) \\
&= \left(2\dot{\phi}\delta^{a}_{i}R^{0i}{}_{\nu\beta} - g\phi^{2}\epsilon^{a}{}_{ik}R^{ik}{}_{\nu\beta}\right) \\
&\times \left(-2a^{2}\dot{\phi}\delta^{a}_{j}\epsilon^{\nu\beta}{}_{0j} - a^{-4}g\phi^{2}\epsilon^{a}{}_{jl}\epsilon^{\nu\beta}{}_{jl}\right) \\
&= 4a^{-6}g\dot{\phi}\phi^{2}\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^{2}}{a^{2}}\right)\epsilon^{i}{}_{jl}\epsilon_{0ijl} \\
&+ 2a^{-6}g\dot{\phi}\phi^{2}\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^{2}}{a^{2}}\right)\epsilon^{j}{}_{ik}\epsilon_{ik0j} \\
&= 36a^{-6}g\dot{\phi}\phi^{2}\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^{2}}{a^{2}}\right) \qquad (6.34)
\end{aligned}$$

Calculation of the seventh term;

$$F^{a\nu'\beta'}F^{a}_{\nu\beta}F^{b}_{\nu'\beta'}\epsilon^{\nu\beta}{}_{\tau\kappa}F^{b\tau\kappa} = F^{a\nu'\beta'}F^{a}_{\nu\beta}F^{b}_{\nu'\beta'}\left(\epsilon^{\nu\beta}{}_{0\kappa}F^{b0\kappa} + \epsilon^{\nu\beta}{}_{i\kappa}F^{bi\kappa}\right) = 0 \qquad (6.35)$$

Inserting the terms into the equation (6.28);

$$\begin{gathered}
\mathcal{S}_{eff} = -\int \mathrm{d}^{4}xC_{DD}M_{D}^{2}(|-g|)^{1/2}\Big[\kappa^{2} + \tfrac{\lambda_{R'}}{4}R \\
+\tfrac{\lambda'_{F}}{4} - 6a^{-2}\dot{\phi}^{2}\left(+6a^{-4}g^{2}\phi^{4}\right) + \tfrac{\tilde{\lambda}_{F}}{4}24g\dot{\phi}\phi^{2}a^{-6} \\
-\tfrac{\lambda^{2}_{F'}}{8\kappa^{2}}\left[12a^{-8}\left(g^{2}\phi^{4} - a^{2}\dot{\phi}^{2}\right)^{2} - \tfrac{1}{4}\left(F^{a}_{\nu\beta}F^{a\nu\beta}\right)^{2}\right] \\
-\tfrac{\tilde{\lambda}^{2}_{F}}{8\kappa^{2}}\left[48a^{-8}\left(g^{2}\phi^{4} - a^{2}\dot{\phi}^{2}\right)^{2} - \tfrac{1}{4}\left(F^{a\nu\beta}\epsilon^{\nu\beta}{}_{\tau\kappa}F^{a}_{\tau\kappa}\right)^{2}\right] \\
-\tfrac{\lambda_{R'}\tilde{\lambda}_{F}}{4\kappa^{2}}\left[36a^{-6}g\dot{\phi}\phi^{2}\left(\tfrac{\ddot{a}}{a} + 2\tfrac{\dot{a}^{2}}{a^{2}}\right) - \tfrac{R}{4}F^{a\nu\beta}\epsilon_{\nu\beta}{}^{\tau\kappa}F^{a}_{\tau\kappa}\right] \\
-\tfrac{\lambda_{F'}\tilde{\lambda}_{F}}{4\kappa^{2}}\left[-\tfrac{1}{4}F^{a\eta\theta}F^{a}_{\eta\theta}F^{b\nu\beta}\epsilon_{\nu\beta}{}^{\tau\kappa}F^{b}_{\tau\kappa}\right] \\
+\tfrac{\lambda_{R'}\lambda_{F'}}{4\kappa^{2}}6a^{-4}\left[\left(\tfrac{\ddot{a}}{a} + a\left(\tfrac{\dot{a}}{a}\right)^{2}\right)\left(-2\dot{\phi}^{2}a^{2} + g^{2}\phi^{4}\right)\right] \\
-\tfrac{\lambda_{R'}\lambda_{F'}}{4\kappa^{2}}\tfrac{R}{4}F^{a\nu\beta}F_{a\nu\beta}\Big] \qquad (6.36)
\end{gathered}$$

We can rearrange the action;

$$
\begin{aligned}
\mathcal{S}_{eff} = -\int \mathrm{d}^4 x C_{DD} M_D^2 (|-g|)^{1/2} \Big[\kappa^2 + \tfrac{3\lambda_{R'}}{2} \left(\tfrac{\ddot{a}}{a} + a \left(\tfrac{\dot{a}}{a} \right)^2 \right) \\
+ \tfrac{3\lambda'_F}{2} - a^{-4} \left(-a^2 \dot{\phi}^2 + g^2 \phi^4 \right) + 6\tilde{\lambda}_F g \dot{\phi} \phi^2 a^{-6} \\
- \tfrac{\lambda_{F'}^2}{8\kappa^2} \left[12 a^{-8} \left(g^2 \phi^4 - a^2 \dot{\phi}^2 \right)^2 - g a^{-8} \left(g^2 \phi^4 - a^2 \dot{\phi}^2 \right)^2 \right] \\
- \tfrac{\tilde{\lambda}_F^2}{8\kappa^2} \left[48 a^{-8} \left(g^2 \phi^4 - a^2 \dot{\phi}^2 \right)^2 - 144 a^{-12} g^2 \dot{\phi}^2 \phi^4 \right] \\
- \tfrac{\lambda_{R'} \tilde{\lambda}_F}{4\kappa^2} \left[36 a^{-6} g \dot{\phi} \phi^2 \left(\tfrac{\ddot{a}}{a} + 2 \tfrac{\dot{a}^2}{a^2} \right) - 36 a^{-6} g \dot{\phi} \phi^2 \left(\tfrac{\ddot{a}}{a} + \tfrac{\dot{a}^2}{a^2} \right) \right] \\
- \tfrac{\lambda_{F'} \tilde{\lambda}_F}{4\kappa^2} \left[-36 a^{-10} g \dot{\phi} \phi^2 g^2 \phi^4 - a^2 \dot{\phi}^2 \right] \Big]
\end{aligned}
\tag{6.37}
$$

Action takes the form as;

$$
\begin{aligned}
\mathcal{S}_{eff} = -\int \mathrm{d}^4 x 4 M_D^2 (|-g|)^{1/2} \Big[\kappa^2 + \left(\tfrac{3\lambda_{R'}}{2} a^{-1} + \tfrac{3\lambda_{R'}\lambda_{F'}}{4\kappa^2} a^{-5} g^2 \phi^4 + \tfrac{3\lambda_{R'}\lambda_{F'}}{4\kappa^2} a^{-3} \dot{\phi}^2 \right) \ddot{a} \\
\left(\tfrac{3\lambda_{R'}}{2} a^{-2} - \tfrac{3\lambda_{R'}\lambda_{F'}}{4\kappa^2} a^{-6} g^2 \phi^4 + \tfrac{15\lambda_{R'}\lambda_{F'}}{4\kappa^2} a^{-4} \dot{\phi}^2 \right) (\dot{a}^2) \\
\left(\tfrac{3\lambda'_F}{2} - a^{-4} g^2 \phi^4 - \tfrac{3\lambda'_F}{2} a^{-2} \dot{\phi}^2 + 9 \tfrac{\lambda_{F'} \tilde{\lambda}_F}{\kappa^2} a^{-8} g \dot{\phi}^3 \phi^2 \right) \\
-3 \tfrac{\lambda_{F'}^2}{8\kappa^2} a^{-8} g^2 \phi^4 - 6 \tfrac{\lambda_{F'}^2}{\kappa^2} a^{-8} g^2 \phi^4 - 3 \tfrac{\lambda_{F'}^2}{8\kappa^2} a^{-6} \dot{\phi}^2 \Big]
\end{aligned}
\tag{6.38}
$$

To find the dynamics;

$$
\begin{aligned}
\rho &= \frac{\partial \mathcal{L}_{red}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L}_{red} \\
p &= \frac{\partial a^3 \mathcal{L}_{red}}{\partial a^3}
\end{aligned}
\tag{6.39}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}_{red}}{\partial \dot{\phi}} \dot{\phi} &= \frac{3\lambda_{R'}\lambda_{F'}}{2\kappa^2} a^{-3} \dot{\phi} \ddot{a} + \left(\frac{15\lambda_{R'}\lambda_{F'}}{2\kappa^2} a^{-4} \dot{\phi} - \frac{9\lambda_{R'} \tilde{\lambda}_F}{\kappa^2} a^{-8} g \phi^2 \right) \dot{a}^2 \\
&- 3\lambda_{F'} \dot{\phi} a^{-2} - \frac{27\lambda_{F'} \tilde{\lambda}_F}{\kappa^2} g \phi^2 \dot{\phi}^3 a^{-2} + \left(\frac{3\lambda_{F'}}{4\kappa^2} + \frac{12\lambda_{F'} \tilde{\lambda}_F}{\kappa^2} \right) a^{-6} \dot{\phi}^2 \\
&+ 6\tilde{\lambda}_F g \dot{\phi} \phi^2 a^{-6} + \frac{9\lambda_{F'} \tilde{\lambda}_F}{\kappa^2} g^3 \phi^6 \dot{\phi} a^{-10} + \frac{36\tilde{\lambda}_F}{\kappa^2} g \phi^4 \dot{\phi}^2 a^{-12}
\end{aligned}
\tag{6.40}
$$

According to equation (6.39);

$$\begin{aligned}\rho &= -4M_D^2\Big[-\kappa^2-\left(\frac{3\lambda_{R'}}{2}a^{-1}+\frac{3\lambda_{R'}\lambda_{F'}}{4\kappa^2}a^{-5}g^2\phi^4-\frac{3\lambda_{R'}\lambda_{F'}}{4\kappa^2}a^{-3}\dot{\phi}^2\right)\ddot{a}\\ &- \left(\frac{3\lambda_{R'}}{2}a^{-2}-\frac{3\lambda_{R'}\lambda_{F'}}{4\kappa^2}a^{-6}g^2\phi^4+\frac{15\lambda_{R'}\lambda_{F'}}{4\kappa^2}a^{-4}\dot{\phi}^2\right)\dot{a}^2\\ &- \frac{3\lambda_{F'}}{2}a^{-4}g^2\phi^4+\frac{3\lambda_{F'}}{2}a^{-2}\dot{\phi}^2-\frac{18\lambda_{F'}\tilde{\lambda}_F}{\kappa^2}a^{-2}g\phi^2\dot{\phi}^3\\ &+ \left(\frac{3\lambda_{F'}^2}{8\kappa^2}+\frac{6\tilde{\lambda}_F^2}{\kappa^2}\right)\left(a^{-8}g^2\phi^4+a^{-6}\dot{\phi}^2\right)+\frac{18\tilde{\lambda}_F^2}{\kappa^2}a^{-12}\dot{\phi}^2g^2\phi^4\Big] \qquad (6.41)\end{aligned}$$

and

$$\begin{aligned}p &= -4M_D^2\Big[\kappa^2+\left(\lambda_{R'}a^{-1}-\frac{3\lambda_{R'}\lambda_{F'}}{2\kappa^2}a^{-5}g^2\phi^4\right)\ddot{a}\\ &+ \left(\frac{\lambda_{R'}}{2}a^{-2}+\frac{3\lambda_{R'}\lambda_{F'}}{4\kappa^2}a^{-6}g^2\phi^4-\frac{5\lambda_{R'}\lambda_{F'}}{4\kappa^2}a^{-4}\dot{\phi}^2+\frac{15\lambda_{F'}\tilde{\lambda}_F}{\kappa^2}a^{-8}g\phi^2\dot{\phi}\right)\dot{a}^2\\ &- \frac{\lambda_{F'}}{2}a^{-4}g^2\phi^4+\frac{\lambda_{F'}}{2}a^{-2}\dot{\phi}^2-\frac{21\lambda_{F'}\tilde{\lambda}_F}{\kappa^2}a^{-10}g^3\phi^6\dot{\phi}-\frac{15\lambda_{F'}\tilde{\lambda}_F}{\kappa^2}a^{-8}g\phi^2\dot{\phi}^3\\ &+ \left(\frac{5\lambda_{F'}^2}{\kappa^2}+\frac{10\tilde{\lambda}_F^2}{\kappa^2}\right)a^{-8}g^2\phi^4-\left(\frac{3\lambda_{F'}^2}{8\kappa^2}+\frac{6\tilde{\lambda}_F^2}{\kappa^2}\right)a^{-6}\dot{\phi}^2\\ &- 6\tilde{\lambda}_F a^{-6}g\dot{\phi}\phi^2+\frac{54\tilde{\lambda}_F^2}{\kappa^2}a^{-12}\dot{\phi}^2g^2\phi^4\Big] \qquad (6.42)\end{aligned}$$

As mentioned in Chapter 3; to determine inflation conditions we should derive slow-roll parameters. First step is to write down the Hubble Constant H, $\dot{H}$and H^2;

$$\begin{aligned}H^2 = &- 4M_D^2\Big[-\frac{\kappa^2}{3}-\left(\frac{\lambda_{R'}}{2}a^{-1}+\frac{\lambda_{R'}\lambda_{F'}}{4\kappa^2}a^{-5}g^2\phi^4-\frac{\lambda_{R'}\lambda_{F'}}{4\kappa^2}a^{-3}\dot{\phi}^2\right)\ddot{a}\\ &- \left(\frac{\lambda_{R'}}{2}a^{-2}-\frac{\lambda_{R'}\lambda_{F'}}{4\kappa^2}a^{-6}g^2\phi^4+\frac{5\lambda_{R'}\lambda_{F'}}{4\kappa^2}a^{-4}\dot{\phi}^2\right)\dot{a}^2\\ &- \frac{\lambda_{F'}}{2}a^{-4}g^2\phi^4-\frac{\lambda_{F'}}{2}a^{-2}\dot{\phi}^2+\frac{6\lambda_{F'}\tilde{\lambda}_F}{\kappa^2}a^{-2}g\phi^2\dot{\phi}^3\\ &+ \left(\frac{\lambda_{F'}^2}{8\kappa^2}+\frac{2\tilde{\lambda}_F^2}{\kappa^2}\right)\left(a^{-8}g^2\phi^4+a^{-6}\dot{\phi}^2\right)+\frac{6\tilde{\lambda}_F^2}{\kappa^2}a^{-12}\dot{\phi}^2g^2\phi^4\Big] \qquad (6.43)\end{aligned}$$

$$
\begin{aligned}
\dot{H} = \quad & - \quad 4M_D^2\Big[\left(\frac{\lambda_{R'}}{2}a^{-1}+\frac{5\lambda_{R'}\lambda_{F'}}{4\kappa^2}a^{-5}g^2\phi^4-\frac{3\lambda_{R'}\lambda_{F'}}{4\kappa^2}a^{-3}\dot{\phi}^2\right)\ddot{a} \\
& + \quad \left(\lambda_{R'}a^{-2}-\frac{3\lambda_{R'}\lambda_{F'}}{8\kappa^2}a^{-6}g^2\phi^4+\frac{5\lambda_{R'}\lambda_{F'}}{\kappa^2}a^{-4}\dot{\phi}^2-\frac{15\lambda_{R'}\lambda_{F'}}{\kappa^2}a^{-8}g\dot{\phi}\phi^2\right)\dot{a}^2 \\
& + \quad \frac{2\lambda_{F'}}{\kappa^2}\left(a^{-4}g^2\phi^4+a^{-2}\dot{\phi}^2\right)+\frac{21\lambda_{F'}\tilde{\lambda}_F}{\kappa^2}a^{-10}g^3\phi^6\dot{\phi} \\
& - \quad \frac{33\lambda_{F'}\tilde{\lambda}_F}{\kappa^2}a^{-8}g\phi^2\dot{\phi}^3-\left(\frac{\lambda_{F'}^2}{\kappa^2}+\frac{16\tilde{\lambda}_F^2}{\kappa^2}\right)a^{-8}g^2\phi^4 \\
& + \quad 6\tilde{\lambda}_F a^{-6}g\dot{\phi}\phi^2+\frac{36\tilde{\lambda}_F^2}{\kappa^2}a^{-12}\dot{\phi}^2g^2\phi^4\Big]
\end{aligned}
\tag{6.44}
$$

To supply inflation conditions, we should interpret the ratio of $\dot{H}$and H^2 since

$$
\begin{aligned}
\epsilon &= -\frac{\dot{H}}{H^2} \\
-\frac{\dot{H}}{H^2} &\ll 1
\end{aligned}
\tag{6.45}
$$

Then in view of absolute value;

$$
\left|\frac{\dot{H}}{H^2}\right| \gg 1 \tag{6.46}
$$

The Friedmann equations above may seem too complicated to draw a conclusion about slow roll behavior. However, their $a(t)$ dependence already gives some clues on their evolutionary character. Both H^2 and $\dot{H}$ have two pieces; one piece that depends on $\phi(t)$ and another piece that does not. By simply examining those terms up to $\mathcal{O}(a^{-2})$ one sees that $\dot{H}/H^2 \gg 1$. Though a numerical computation might give better view of the solutions, still one concludes that the slow-roll conditions are satisfied for a wide range of parameter values.

CHAPTER 7

CONCLUSION

In this work, we have established a new gravitational theory that we name as Born-Infeld-Riemann gravity. This theory is based on the Riemann tensor in the action. Among other features, it has the important property that, it allows for unification of Yang-Mills fields and gravity in one single formalism. This feature is completely new, and has not been found in other gravitational theories.

For both extracting the physics implications of the model and performing an application to a physical phenomenon, we have discussed, after building the model, the gauge field inflation. In this scenario, cosmic inflation is caused by a non-Abelian gauge field in homogeneous and isotropic background.

This thesis work, supplemented by a number of appendices, concludes that Born-Infeld-Riemann gravity is a physically consistent extension of the GR, and it brings striking novelty in the treatment of non-Abelian gauge fields. Moreover, it covers inflationary epoch for a wide range of parameters.

This theory is an extension of the Born-Infeld theory to non-Abelian fields, and it can explain a number of cosmological phenomena.

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APPENDIX A

INVARIANT VOLUME AND DETERMINANT OF TENSORS

Einstein's Special Relativity Theory shows that we live in a four dimensional world which is called space-time. With the Theory of General Relativity, space-time is considered in a geometrical nature. Basically the theory is based on a geometrical structure which is called manifold. A manifold is smooth and locally flat [13, 66, 67]. Because of its smoothness, manifolds are differentiable. We are going to mention about tensor fields which are defined on manifold, briefly.
First of all is the metric tensor. Metric tensor is a purely mathematical object and benefits to measure space-time intervals. Coordinates can be choosed as $x^\mu \equiv [x^0, x^1, x^2, x^3]$. Here Greek indices label space-time coordinates. For any point of space-time we can find a point that locally inertial: $\zeta^\mu \equiv [\zeta^0, \zeta^1, \zeta^2, \zeta^3]$. The line element is;

$$ds^2 = \eta_{\mu\nu} d\zeta^\mu d\zeta^\nu \tag{A.1}$$

where $\eta_{\mu\nu}$ is Minkowski metric of flat space-time.

$$ds^2 = \eta_{\mu\nu} \frac{d\zeta^\mu}{dx^\tau} \frac{d\zeta^\nu}{dx^\rho} dx^\tau dx^\rho ds^2 = g_{\tau\rho} dx^\tau dx^\rho \tag{A.2}$$

As seen from equation (2) $g_{\tau\rho}$ is $n \times n$ diagonal, symmetric matrix. The symmetric part of the metric tensor is different from zero so that norm can be measured. If the metric tensor is defined in a theory, it is also used for raising or lowering indices. .The discussions still go on about metric tensor: Should a theory include metric tensor, is it necessity? We are going to explain these type of theories in Chapter 2.

In curved space-time how can we parallel transport [13, 41, 53, 67] a vector along a curve? Clearly the transportation is not going to be the same with flat space. Riemann

Curvature Tensor comes from this argument.

$$R^{\mu}{}_{\alpha\nu\beta} = \partial_{\nu}\Gamma^{\mu}_{\beta\alpha} - \partial_{\beta}\Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\nu\lambda}\Gamma^{\lambda}_{\beta\alpha} - \Gamma^{\mu}_{\beta\lambda}\Gamma^{\lambda}_{\nu\alpha} \tag{A.3}$$

Riemann curvature tensor [46] contains everything about the curvature of space-time. We mention flat space if and only if curvature tensor equals zero.

The last structure is connection. It appears in geodesic equation which means that the path of moving object is determined by connection. It is not a tensorial structure that changes with respect to the chosen coordinate system. The symmetrization or anti-symmetrization in lower indices of connection gives us torsion of the space-time. As mathematically we know that the difference between two non-tensorial structure gives a tensorial structure that means torsion is a tensorial structure. Therefore if determined connection is anti-symmetric in its lower indices;

$$\Gamma^{\lambda}_{\alpha\beta} - \Gamma^{\lambda}_{\beta\alpha} = S^{\lambda}_{\alpha\beta} \tag{A.4}$$

If determined connection is symmetric in its lower indices, torsion of the curved space-time is zero.

In classical mechanical systems [41] we use action functional, in general. To understand features of a motion, we take the variation of the action with respect to dynamical variables of motion. In General Relativty(GR) Theory action plays an important role, too. We construct our theories upon Lagrangian Density which is symbolically called $\mathcal{L}$. Beside the great success of GR, it is unfortunately an incomplete theory. It can not explain such big problems that cosmological constant problem, inflation etc. Because of that for years people have tried to modify Einstein's GR Theory in several ways. An appropriate action functional is constructed due to our theory and then equations of motion are derived. Finally due to the equations of motion the results are interpreted. One way to check a theory whether it survives or not, it should be examined that does the theory include GR for limiting cases. The other way, for limiting cases, is to search the results are compatible with the cosmological observations.

First of all the question is that how we can form an invariant action. Basically we define the action;

$$\mathcal{S} = \int \mathrm{d}t L(t) \tag{A.5}$$

where L is called Lagrange function. If it is applied on a space-time manifold for four dimensions;

$$\mathcal{S} = \int \mathrm{d}^4 x \mathcal{L}(g, \Gamma, \phi, \dot{\phi}...) \tag{A.6}$$

Here $\mathcal{L}$ is called Lagrangian density and g is metric tensor, Γ is connection, ϕ is an arbitrary scalar field, $\dot{\phi}$ is the derivation of scalar field with respect to time. As long as we obey some basic rules, we can add inifinite number of terms to Lagrangian density. Now we are going to mention about these basic rules. As we know action is called a scalar-quantity which means it is an invariant under all of the changes. For example; action should be an invariant under coordinate transformations, gauge transformations, conformal transformations.. etc. It makes sense because we don't want the equations of motion to change when the system is in a different frame.

Our primary aim is to make action Lorentz invariant. Suppose that Lagrangian depends only on connection and partial differentiation of connection. On a D dimensional space-time manifold;

$$\mathcal{S} = \int \mathrm{d}^{\mathrm{D}} x \mathcal{L}(\Gamma, \partial\Gamma) \tag{A.7}$$

where

$$\mathrm{d}^{\mathrm{D}} x = \mathrm{d}^{\mu_0} x \wedge \mathrm{d}^{\mu_1} x \wedge \cdots \wedge \mathrm{d}^{\mu_{\mathrm{D}-1}} x \tag{A.8}$$

Changing coordinate system $x \to x'$ [3, 13, 67];

$$\mathrm{d}^{\mathrm{D}} x = \left| \frac{\mathrm{d}x}{\mathrm{d}x'} \right| \mathrm{d}^{\mathrm{D}} x' \tag{A.9}$$

Since $\mathrm{d}^D x \neq \mathrm{d}^\mathrm{D} x'$;$\mathrm{d}^D x$ is not a tensor! We call these type of quantities as tensor density. The coefficient of transformation in the form of determinant$\left|\frac{\mathrm{d}x}{\mathrm{d}x'}\right|$ is called the Jacobian. According to power of the Jacobian the weight of tensor density is defined. $\mathrm{d}^\mathrm{D} x$ is a tensor density of weight "+1". In this situation action is obviously not an invariant. To persist invariance of action, Lagrangian must involve a tensor density of weight "-1". The best way of succeeding that is to use the notion of determinant.

Lorentz coordinate transformation of an arbitrary rank (0,2) tensor field $\mathcal{R}_{\mu\nu}$;

$$\mathcal{R}_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \mathcal{R}_{\mu\nu} \tag{A.10}$$

$$\frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} \mathcal{R}_{\mu'\nu'} \quad = \mathcal{R}_{\mu\nu} \tag{A.11}$$

Taking determinant of both sides;

$$|\mathcal{R}_{\mu\nu}| = \left|\frac{\partial x'}{\partial x}\right|^2 |\mathcal{R}_{\mu'\nu'}| \tag{A.12}$$

where $|\mathcal{R}_{\mu\nu}| = Det\,[\mathcal{R}_{\mu\nu}]$. Since the coefficient of $|\mathcal{R}_{\mu'\nu'}|$ is $\left|\frac{\partial x'}{\partial x}\right|^2$, then the determinant of a rank (0,2) tensor field is a tensor density of weight "-2".

In the same way Lorentz coordinate transformation of an arbitrary rank (1,3) tensor field $\mathcal{Q}_{\mu\alpha\nu\beta}$;

$$\mathcal{Q}^{\mu'}{}_{\alpha'\nu'\beta'} = \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^\beta}{\partial x^{\beta'}} \mathcal{Q}^{\mu}{}_{\alpha\nu\beta} \tag{A.13}$$

$$\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial x^{\beta'}}{\partial x^\beta} \mathcal{Q}^{\mu'}{}_{\alpha'\nu'\beta'} = \mathcal{Q}^{\mu}{}_{\alpha\nu\beta} \tag{A.14}$$

Taking determinant of both sides;

$$\left|\left|\mathcal{Q}^{\mu}{}_{\alpha\nu\beta}\right|\right| = \left|\frac{\partial x'}{\partial x}\right|^4 \left|\left|\mathcal{Q}^{\mu'}{}_{\alpha'\nu'\beta'}\right|\right| \tag{A.15}$$

where $\left|\left|\mathcal{Q}^{\mu}{}_{\alpha\nu\beta}\right|\right| = DDet\left[\mathcal{Q}^{\mu}{}_{\alpha\nu\beta}\right]$. Then the determinant of a rank (1,3) tensor field is a tensor density of weight "-4".(Double Determinant is going to be called DDet in the rest of thesis). As we see in equation, the notion of determinant differs from rank(0,2) tensor field. The reason lies under the definition of determinant.

$$|\mathcal{R}_{\mu\nu}| = \frac{1}{D!} \in^{\mu_0\mu_1\mu_2\mu_3} \in^{\nu_0\nu_1\nu_2\nu_3} \mathcal{R}_{\mu_0\nu_0}\mathcal{R}_{\mu_1\nu_1}\mathcal{R}_{\mu_2\nu_2}\mathcal{R}_{\mu_3\nu_3} \tag{A.16}$$

and

$$\begin{aligned}\left|\left|\mathcal{Q}^{\mu}{}_{\alpha\nu\beta}\right|\right| = & \frac{1}{D!^2} \quad \in_{\mu_0\mu_1\mu_2\mu_3} \in^{\alpha_0\alpha_1\alpha_2\alpha_3} \in^{\nu_0\nu_1\nu_2\nu_3} \in^{\beta_0\beta_1\beta_2\beta_3} \\ & \times \quad \mathcal{Q}^{\mu_0}{}_{\alpha_0\nu_0\beta_0} \mathcal{Q}^{\mu_1}{}_{\alpha_1\nu_1\beta_1} \mathcal{Q}^{\mu_2}{}_{\alpha_2\nu_2\beta_2} \mathcal{Q}^{\mu_3}{}_{\alpha_3\nu_3\beta_3}\end{aligned} \tag{A.17}$$

where $\in_{\mu_0\mu_1\mu_2\mu_3}$ is totally anti-symmetric Levi-Civita Symbol [3, 13]. It's not a tensor or a tensor density. Because under the coordinate transformations, components stay the same.

In any coordinate system;

If $\mu_0\mu_1\mu_2\mu_3$ is an even permutation of 0,1,2,3; then $\in_{\mu_0\mu_1\mu_2\mu_3} = 1$

If $\mu_0\mu_1\mu_2\mu_3$ is an odd permutation of 0,1,2,3; then $\in_{\mu_0\mu_1\mu_2\mu_3} = -1$

Otherwise $\in_{\mu_0\mu_1\mu_2\mu_3} = 0$

Using the definition of determinant [61, 62];

$$\epsilon_{\mu_0\mu_1\ldots\mu_D} = |\mathcal{R}|^{1/2} \in_{\mu_0\mu_1\ldots\mu_D} \tag{A.18}$$

or

$$\epsilon_{\mu_0\mu_1\ldots\mu_D} = ||\mathcal{Q}||^{1/4} \in_{\mu_0\mu_1\ldots\mu_D} \tag{A.19}$$

and

$$\epsilon^{\mu_0\mu_1\ldots\mu_D} = |\mathcal{R}|^{-1/2} \in^{\mu_0\mu_1\ldots\mu_D} \tag{A.20}$$

or

$$\epsilon^{\mu_0\mu_1\ldots\mu_D} = ||\mathcal{Q}||^{-1/4} \in^{\mu_0\mu_1\ldots\mu_D} \tag{A.21}$$

Since $\epsilon^{\mu_0\mu_1\ldots\mu_D}$ is no longer a symbol, indices can be raised or lowered.
Our aim was to make action stay invariant. As mentioned before according to equation (5) $\mathcal{L} \ni$ "a tensor density of weight "-1". After all these discussions;

$$\mathcal{L} \ni |\mathcal{R}|^{1/2}, ||\mathcal{Q}||^{1/4} \tag{A.22}$$

In our work, we use these fundamental concepts and construct our theory based on determinant of rank (0,4) tensor fields and our metric convention is (-+++). Finally, here are some properties used in this work of completely anti-symmetric Levi-Civita Tensor which are exemplified for 3D:

$$\epsilon^{aij}\epsilon^{a}_{ij} \doteq \begin{vmatrix} \delta^a_a & \delta^a_i & \delta^a_j \\ \delta^i_a & \delta^i_i & \delta^i_j \\ \delta^j_a & \delta^j_i & \delta^j_j \end{vmatrix} = 6 \tag{A.23}$$

$$\epsilon^{aij}\epsilon^{b}_{ij} \doteq \begin{vmatrix} \delta^a_b & \delta^a_i & \delta^a_j \\ \delta^i_b & \delta^i_i & \delta^i_j \\ \delta^j_b & \delta^i_j & \delta^j_j \end{vmatrix} = 2\delta^a_b \tag{A.24}$$

and some of them are examplified for 4D:

$$
\begin{aligned}
{\epsilon_{kl}}^{0i}{\epsilon^{kl}}_{0i} &= g_{0k}g_{bl}\epsilon^{ab0i}{\epsilon^{kl}}_{0i} = \epsilon^{ab0i}\epsilon_{ab0i} \\
&\doteq \begin{vmatrix} \delta^k_k & \delta^k_l & \delta^k_0 & \delta^k_i \\ \delta^l_k & \delta^l_l & \delta^l_0 & \delta^l_i \\ \delta^0_k & \delta^0_l & \delta^0_0 & \delta^0_i \\ \delta^i_k & \delta^i_l & \delta^i_0 & \delta^i_i \end{vmatrix} = \begin{vmatrix} 3 & \delta^k_l & 0 & \delta^k_i \\ \delta^l_k & 3 & 0 & \delta^l_i \\ 0 & 0 & 1 & 0 \\ \delta^i_k & \delta^i_l & 0 & 3 \end{vmatrix} = -6 \qquad \text{(A.25)} \\
{\epsilon_{lm}}^{0i}{\epsilon^{lm}}_{kj} &= g_{la}g_{mb}\epsilon^{ab0i}{\epsilon^{lm}}_{kj} = \epsilon^{ab0i}\epsilon_{abkj} \\
&\doteq \begin{vmatrix} \delta^l_k & \delta^l_m & \delta^l_k & \delta^l_j \\ \delta^m_l & \delta^m_m & \delta^m_k & \delta^m_j \\ \delta^0_l & \delta^0_m & \delta^0_k & \delta^0_j \\ \delta^i_l & \delta^i_m & \delta^i_k & \delta^i_j \end{vmatrix} = \begin{vmatrix} 3 & \delta^k_l & 0 & \delta^k_i \\ \delta^l_k & 3 & 0 & \delta^l_i \\ 0 & 0 & 1 & 0 \\ \delta^i_k & \delta^i_l & 0 & 3 \end{vmatrix} = 0 \qquad \text{(A.26)}
\end{aligned}
$$

APPENDIX B

VARIATIONAL APPROACH

In the previous chapter, we explained the meaning of gravity, and the fundamental notions. By using knowledge obtained from introduction, let us continue with explaining variational procedures, to obtain field equations, and extended theories of gravity. Firstly we explain variational method [1, 13, 39, 41, 43, 49].

In classical field theory, the way of finding equations of motion is called variational method. To understand what this method is, let us consider a Lagrangian density $\mathcal{L}(\phi, \dot{\phi})$ and the action is given as;

$$\mathcal{S} = \int \mathrm{d}t L = \int \mathrm{d}^4 x \mathcal{L}\left(\phi^i, \partial_\mu \phi^i\right) \tag{B.1}$$

By considering small variation in this field;

$$\begin{aligned} \phi^i &\rightarrow \phi^i + \delta\phi^i \\ \partial_\mu \phi^i &\rightarrow \partial_\mu \phi^i + \delta(\partial_\mu \phi^i) = \partial_\mu \phi^i + \partial_\mu(\delta\phi^i) \end{aligned} \tag{B.2}$$

Lagrangian Density varies;

$$\mathcal{L}(\phi^i, \partial_\mu \phi^i) \rightarrow \mathcal{L}(\phi^i + \delta\phi^i, \partial_\mu \phi^i + \partial_\mu(\delta\phi^i)) \tag{B.3}$$

thus action varies by virtue of this small variation $\mathcal{S} \rightarrow \mathcal{S} + \delta\mathcal{S}$. Hence;

$$\delta\mathcal{S} = \int \mathrm{d}^4 x \delta\phi^i \left[\frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}\right)\right] \tag{B.4}$$

We assume that ϕ^i is the same at the end points of the integral. Thus, by using this assumption, we conclude that δS should be zero. This leads to field equations [13]. Therefore, by using variational principle, we obtain the eqaution of motion for a Lagrangian which has one dynamical variable.

In general relativity, we use the same procedure to obtain field equations. However, there are some different methods according to dynamical variable in the Lagrangian. We examined these methods in three parts in Chapter 2: Metric formulation, Metric Affine formulation and Affine formulation [7, 12, 13, 15, 30, 43, 51].

APPENDIX C

FIELD-STRENGTH TENSOR

Field Strength Tensor which is an antisymmetric, traceless rank (0,2) tensor field depends on the electromagnetic field of particles. For a 4 dimensional space-time manifold covariant field strength tensor takes the form as;

$$F_{\mu\nu} \doteq \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{C.1}$$

and contravariant field strength tensor which is constructed via two contractions of field strength tensor by metric takes the form as

$$F^{\mu\nu} \doteq \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{C.2}$$

Field strength tensor includes all the information about electromagnetic fields. Since it is an antisymmetric tensor, there are six independent components. The other important notion is the dual of Field Strength Tensor which is;

$$F^*_{\mu\nu} \doteq \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{pmatrix} \tag{C.3}$$

Dual field strength tensor is an atisymmetric tensor field, too. One may associate the duality with rotation. Electric field and Magnetic field are transformative quantities. Dual field strength tensor is a good way to show this transformation. We obtain dual field strength tensor with the help of totally antisymmetric tensor field.

$$F^*_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \tag{C.4}$$

As seen in Appendix A, it is also important to construct invariant volume elements in GR. Dual field strength tensor is one of the ways to construct. Both $F^{\mu\nu}F_{\mu\nu}$ and $F^{\mu\nu}F^*_{\mu\nu}$ are Lorentz invariant quantities. With this analysis we understand the field strength tensor and its dual.

The next step is to examine Field Strength Tensor in the view of Particle Physics. Field Strength Tensor is built up from the four-potential which is $A^\mu = (A^0, \vec{A})$. Here, A^0 is electric potential and$\vec{A}$ is 3D vector potential. Under Gauge Transformation field strength tensor remains unchanged. For abelian theories field strength tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{C.5}$$

If we consider consider non-abelian theories, what does field strength tensor reacts? Would it be remain unchanged? To answer this question, from (C.5) under local gauge transformation $A^\mu \to \frac{i}{g}A^\mu$field strength tensor takes the form as;

$$\begin{aligned}
F'_{\mu\nu} &= GF_{\mu\nu}G^{-1} \\
&= \partial_\mu\left(GA_\nu G^{-1} + \frac{i}{g}(\partial_\nu G)G^{-1}\right) - \partial_\nu\left(GA_\mu G^{-1} + \frac{i}{g}(\partial_\mu G)G^{-1}\right) \\
&= G\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)G^{-1} + \left((\partial_\nu G)A_\mu - (\partial_\mu G)A_\nu\right)G^{-1} \\
&+ G\left(A_\mu(\partial_\nu G^{-1}) + A_\nu(\partial_\mu G^{-1})\right) + \frac{i}{g}\left((\partial_\mu G)(\partial_\nu G^{-1}) - (\partial_\nu G)(\partial_\mu G^{-1})\right) \\
&\neq GF_{\mu\nu}G^{-1}
\end{aligned} \tag{C.6}$$

Field Strength Tensor changes under local gauge transformations. It means that an additional term should be added to field strength tensor. This term comes from the non-abelian group structure. Since $F_{\mu\nu} = \frac{1}{iq}\left[D_\mu, D_\nu\right]$ with $D_\mu = \partial_\mu + iqA_\mu$ Consequently for non-abelian group structures, Field Strength Tensor takes the form as;

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + iq\left[A_\mu, A_\nu\right] \tag{C.7}$$

Field strength tensor is no longer changable in local gauge transformations. If we rearrange equation(C.8), considering group index,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\epsilon^a_{bc} A^b_\mu, A^c_\nu \tag{C.8}$$

In our work, we choose the condition [44] $A^a_0 = 0$ and

$\mu = i \Rightarrow A^a_\mu = \phi(t)\delta^a_\mu$

$\mu = 0 \Rightarrow A^a_\mu = 0$

Then inflation is caused only by spatial terms which depend only on t. In the precense of that Field Sterngth Tensor components are;

$$F^a_{00} = 0 \tag{C.9}$$

$$F^a_{0i} = \dot{\phi}\delta^a_i \tag{C.10}$$

$$F^{a0i} = -a^2\dot{\phi}\delta^{ai} \tag{C.11}$$

APPENDIX D

EXPANSION

In this appendix, we are going to examine the expansion of Determinant expression. It is going to be concluded that the expansion of determinant and the expansion of double determinant expressions are the same. The best way is to examine determinant expression, firstly [3, 67].

$$DetM = e^{Tr(lnM)} \tag{D.1}$$

$$\begin{aligned} Det(\mathcal{I} + A) &= \sum_{k=0}^{\infty} \frac{1}{k!} \left[-\sum_{j=1}^{\infty} \frac{(-1)^j}{j} Tr(A^j) \right]^k \\ &= exp\left[-\sum_{j=1}^{\infty} \frac{(-1)^j}{j} Tr(A^j) \right] \\ = exp\left[Tr\left(-\sum_{j=1}^{\infty} \frac{(-1)^j}{j} Tr(A^j) \right) \right] &= exp\left[Tr(ln(\mathcal{I} + A)) \right] \end{aligned} \tag{D.2}$$

We are going to express the terms of summation on j;

$$\begin{aligned} Det(\mathcal{I} + A) &= \sum_{k=0}^{\infty} \frac{1}{k!} \left[-\sum_{j=1}^{\infty} \frac{(-1)^j}{j} Tr(A^j) \right]^k \\ = \sum_{k=0}^{\infty} \frac{1}{k!} \left[TrA - \frac{1}{2}Tr(A^2) + \frac{1}{3}Tr(A^3) - \frac{1}{4}Tr(A^4) + \frac{1}{5}Tr(A^5) + ... \right]^k \end{aligned} \tag{D.3}$$

Now we are going to express the terms of summation on k;

$$
\begin{aligned}
Det(\mathcal{I}+A) &= \underbrace{1}_{k=0} + \underbrace{TrA - \frac{1}{2}Tr(A^2)}_{k=1} \\
&+ \underbrace{\frac{1}{2}\Big[(TrA)^2 - (TrA)Tr(A^2) + \frac{1}{4}(Tr(A^2))^2\Big]}_{k=2} + ...
\end{aligned} \tag{D.4}
$$

If we cut the expansion in the order of cubic level;

$$
Det(\mathcal{I}+A) = 1 + TrA - \frac{1}{2}Tr(A^2) + \frac{1}{2}(TrA)^2 + \mathcal{O}(A^3) \tag{D.5}
$$

We know binomial series expansion (Arfken,)

$$
(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \tag{D.6}
$$

$$
(Det(\mathcal{I}+A))^{1/2} = \left(1 + \underbrace{TrA - \frac{1}{2}Tr(A^2) + \frac{1}{2}(TrA)^2}_{x}\right)^{1/2} \tag{D.7}
$$

Due to expansion in equation (D.6), one can expand equation (D.7);

$$
\begin{aligned}
(Det(\mathcal{I}+A))^{1/2} &= 1 + \tfrac{1}{2}\left(TrA - \tfrac{1}{2}Tr(A^2) + \tfrac{1}{2}(TrA)^2\right) \\
&\quad - \tfrac{1}{8}\left(TrA - \tfrac{1}{2}Tr(A^2) + \tfrac{1}{2}(TrA)^2\right)^2 \\
(Det(\mathcal{I}+A))^{1/2} &= 1 + \tfrac{1}{2}TrA - \tfrac{1}{4}Tr(A^2) + \tfrac{1}{8}(TrA)^2 + \mathcal{O}(x^3)
\end{aligned} \tag{D.8}
$$

Since Double Determinant expression is the same with Determinant expression as in equation (D.1), the form of expansion must be the same. According to (D.2);

$$DDet(\mathcal{I}+A) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[-\sum_{j=1}^{\infty} \frac{(-1)^j}{j} Tr(A^j) \right]^k$$
$$= 1 + TrA - \tfrac{1}{2}Tr(A^2) + \tfrac{1}{2}(TrA)^2 + \dots \tag{D.9}$$

Due to equation (D.6);

$$(1+x)^{1/4} = 1 + \frac{1}{4}x - \frac{3}{32}x^2 + \mathcal{O}(x^3) \tag{D.10}$$

and arranging equation (D.11)

$$[DDet(\mathcal{I}+A)]^{1/4} = \left[1 + \underbrace{TrA - \frac{1}{2}Tr(A^2) + \frac{1}{2}(TrA)^2}_{x} \right]^{1/4}$$
$$[DDet(\mathcal{I}+A)]^{1/4} = 1 + \tfrac{1}{4}TrA - \tfrac{1}{8}Tr(A^2) + \tfrac{1}{32}(TrA)^2 \tag{D.11}$$

We are going to use these expansions during our work.

APPENDIX E

TRACE OF A MATRIX

In this thesis, we try to construct a new theory. Expansions make us to calculate traces of the rank-4 tensors. In this Appendix, we try to formulate what trace is [3, 50].

Basicly, trace of a matrix is the sum of diagonal elements of matrix. Consider an arbitrary matrix M_{ab}. Trace of M;

$$Tr(M) = M^{a}{}_{a} \tag{E.1}$$

Mathematically we use some features of trace such as:

$$Tr(M - N) = Tr(M) - Tr(N) \tag{E.2}$$

and

$$Tr(MN) \neq Tr(M)Tr(N) \tag{E.3}$$

On the contrary, it should be pointed that

$$\begin{aligned} Tr\left(MN\right) &= \sum_{i}\left(MN\right)_{ii} = \sum_{i}\left(\sum_{j} m_{ij}n_{ji}\right) \\ &= \sum_{j}\sum_{i} n_{ji}m_{ij} = \sum_{j}\left(MN\right)_{jj} \\ &= Tr\left(NM\right) \end{aligned} \tag{E.4}$$

It indicates that we should displace the matrices wit each other in this way.

For tensorial structures, the contraction is caused by metric tensor. For example, trace of an arbitrary rank-2 tensor field, $A\mu\nu$, is

$$Tr(A) = A^{\mu}{}_{\mu} \tag{E.5}$$

It should be used either metric tensor $g_{\mu\nu}$ or (especially for affine theories) δ^{μ}_{ν}

$$Tr(A) = A^{\mu}{}_{\mu} = g^{\mu\nu} A_{\mu\nu} \tag{E.6}$$

or

$$Tr(A) = \delta^{\nu}_{\mu} A^{\mu}{}_{\nu} \tag{E.7}$$

In this thesis, we write down the traces of terms in this way:

$$\begin{aligned}
Tr(F^{a\nu'\beta'} F^{a}_{\nu\beta}) &= I^{\nu\beta}{}_{\nu'\beta'} F^{a\nu'\beta'} F^{a}_{\nu\beta} \\
&= \frac{1}{2}\left(\delta^{\nu}_{\nu'}\delta^{\beta}_{\beta'} - \delta^{\nu}_{\beta'}\delta^{\beta}_{\nu'}\right) F^{a\nu'\beta'} F^{a}_{\nu\beta} \\
&= \frac{1}{2}\left(F^{a\nu\beta} F^{a}_{\nu\beta} - \underbrace{F^{a\beta\nu}}_{-F^{a\nu\beta}} F^{a}_{\nu\beta}\right) \\
&= Tr(F^{a\nu\beta} F^{a}_{\nu\beta})
\end{aligned} \tag{E.8}$$

and

$$\begin{aligned}
Tr(F^{a\nu\prime\beta\prime} \epsilon_{\nu\beta}{}^{\tau\kappa} F^{a}_{\tau\kappa}) &= I^{\nu\beta}{}_{\nu'\beta'} F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^{a}_{\tau\kappa} \\
&= \frac{1}{2}\left(\delta^{\nu}_{\nu'}\delta^{\beta}_{\beta'} - \delta^{\nu}_{\beta'}\delta^{\beta}_{\nu'}\right) F^{a\nu'\beta'} \epsilon_{\nu\beta}{}^{\tau\kappa} F^{a}_{\tau\kappa} \\
&= \frac{1}{2}\left(F^{a\nu\beta} \epsilon_{\nu\beta}{}^{\tau\kappa} F^{a}_{\tau\kappa} - \underbrace{F^{a\beta\nu}}_{-F^{a\nu\beta}} \epsilon_{\nu\beta}{}^{\tau\kappa} F^{a}_{\tau\kappa}\right) \\
&= F^{a\nu\beta} \epsilon_{\nu\beta}{}^{\tau\kappa} F^{a}_{\tau\kappa}
\end{aligned} \tag{E.9}$$